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Analysis of the Distribution of the Spacings Between Nuclear Energy Levels. II*

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The statistic $\Lambda(n)$, previously defined for the purpose of comparing empirical distributions of energy-level spacings with theoretical distributions, is applied to the recently published series of neutron-capture levels observed in the even-A erbium isotopes. When the empirical values of $\Lambda(n)$ in the energy range of the highest experimental resolution were averaged over all possible sets of n successive spacings, the resulting value $\Lambda^*(n)$ is found to decrease sharply with increasing values of n. This decrease is consistent with that of the expectation value $\langle\Lambda(n)\rangle$ calculated for Wigner's Gaussian orthogonal ensemble of real symmetric matrices.

I. INTRODUCTION

This study is a continuation of a previous effort[1] to analyze empirical spacing distributions of nuclear energy levels in terms of Wigner's Gaussian orthogonal ensemble (GOE) by means of the Λ statistic introduced in Ref. 1. We were prompted to look into the matter again by the appearance of data of superior quality, namely the neutron-capture levels observed in the erbium isotopes by the Columbia University group[2] at the Nevis synchrocyclotron.

Although quantitative tests of Wigner's random-matrix model[3,4] are still very limited in number, the situation is appreciably better than it was two years ago. The same erbium data, and other data[5] as well, have been subjected to a variety of tests including the Dyson-Mehta[6] Δ statistic and the Dyson F statistic.[5] These quantitative tests are particularly timely in view of the studies in which refinements of the statistical model arising from the Pauli principle and assumptions about the two-body nature of the nuclear force are being investigated.[7,8] It is hoped that studies such as this one and the others referred to above might reveal any systematic deviations from the simplest form of the random-matrix model.

In Sec. II some theoretical properties of the Λ statistic are summarized and a new expression for $\Lambda(n)$ has been derived. This expression is particularly useful for computing the empirical values of $\Lambda(n)$ as a function of n for a set of measured energy values. In Sec. III the empirical values of $\Lambda(n)$ obtained from the s-wave resonances observed in the even-A isotopes of erbium are compared with the expectation values of $\Lambda(n)$ in Wigner's Gaussian orthogonal ensemble. Miscellaneous concluding remarks are contained in Sec. IV.

II. THEORETICAL PROPERTIES OF $\Lambda(n)$

A. Definitions and Analytical Properties

The reader is referred to Ref. 1, hereafter referred to as I, for a more complete discussion of the terminology than will be given here. The values of successive level spacings s_i divided by the mean value D will be denoted by $t_i = s_i/D$ $(i=1, 2, \ldots, n)$. Let $F^*(x; n)$ denote the cumulative distribution of the n observed spacings t_i and let $F(x)$ denote the expectation value of $F^*(x; n)$ in the GOE. The Λ statistic measures the deviation of the empirical F^* from the theoretical expectation F and is defined by the expression

$$\Lambda(t_1, t_2, \ldots, t_n) \equiv n\int_0^\infty [F^*(x, n) - F(x)]^2 dx. \quad (2.1)$$

Our analysis will be based on the assumption that the relevant $F(x)$ is the single spacing distribution implied by the GOE, namely the Gaudin-Mehta[9] distribution. However, throughout this work we shall use the simple Wigner[3,4] distribution

$$F(x) \approx 1 - e^{\pi x^2/4}, \quad (2.2)$$

which is an excellent approximation to $F(x)$. An exact form for $\Lambda(t_1, t_2, \ldots, t_n) \equiv \Lambda(n)$ which is convenient both for computing the empirical values and for theoretical work is

$$\Lambda(n)=\frac{1}{n}\sum_i t_i+\frac{2}{n}\sum_{i<j}\min(t_i,t_j) -2\sum_i\int_0^{t_i}[1-F(x)]dx+n\int_0^\infty[1-F(x)]^2dx, \tag{2.3}$$

where

$$\min(x,y)\equiv\begin{cases}x & \text{if } x\le y\\ y & \text{if } y<x.\end{cases}$$

From Eqs. (2.1) and (2.3) it is clear that $\Lambda(n)$ is a positive symmetric function of the variables $t_i(i=1,2,\ldots,n)$. Keeping n fixed, $\Lambda(n)$ has a unique minimum for the set of spacings

$$t_i=\left(\frac{4}{\pi}\ln\frac{n}{n-i+\frac{1}{2}}\right)^{1/2}. \tag{2.4}$$

The corresponding value of the minimum, plotted as a function of n, is shown as curve (d) in Fig. 1. Actually, of course, the values of $\Lambda(n)$ have a meaning only for integral values of n, and the result is presented as a continuous curve merely as an aid to visualization.

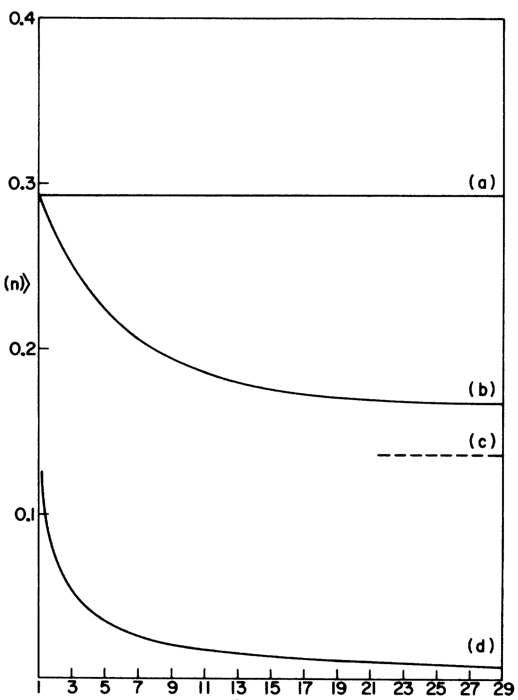

FIG. 1. Plots illustrating various theoretical properties of $\Lambda(n)$ as a function of n: (a) expectation value $\langle\Lambda(n)\rangle$ vs n on the assumption that n successive spacings are statistically independent and follow the Wigner distribution; (b) $\langle\Lambda(n)\rangle$ vs n on the assumption that the statistics of n successive spacings are determined by the GOE; (c) estimate of the most probable value of $\Lambda(n)$ in the GOE; (d) the n dependence of the smallest possible value of $\Lambda(n)$, obtained by substituting the values of t_i given by formula (2.4) into expression (2.3).

B. Statistical Properties of $\Lambda(n)$

Some statistical properties of $\Lambda(n)$ were reported in I. For the most part these were obtained with the help of a Monte Carlo calculation on the basis of the GOE. The most important results are recapitulated in the form of several curves shown in Fig. 1. Curve (b) is a plot of the expectation value of $\Lambda(n)$ in the GOE as a function of n. It should be noted that curve (b) is actually a smoothed version of the Monte Carlo results. (For more details, see the discussion pertaining to Fig. 1 of I.) The broken line (c) similarly marks the most probable value of $\Lambda(n)$ in the GOE.

On the other hand, curve (a) is not based on the GOE. It is a plot of the exact expectation value of $\Lambda(n)$ based on the assumption that successive spacings are statistically independent but follow the Wigner distribution. Thus, the appreciable difference between curves (a) and (b) is the direct consequence of the correlations between the spacings that are implied by the GOE.

In Sec. III we shall examine the degree to which the $\Lambda(n)$ for the erbium data exhibits the n dependence that is predicted by the GOE.

III. ANALYSIS OF NEUTRON-CAPTURE LEVELS OBSERVED IN ERBIUM ISOTOPES

A. Empirical Values of $\Lambda(n)$

The experimental data on which the discussion is based were taken from the recently published work of the Columbia group.[2] We refer the reader to Ref. 2 for a description of the experiment, quotation of results, and analyses of the data. Some of these analyses, it should be noted, serve the same purpose as our present considerations, namely, to ascertain whether or not the series of s-wave resonances observed in targets of the separated isotopes ^{166}Er, ^{168}Er, and ^{170}Er are in agreement with some of the predictions of the GOE.

To begin with, the empirical values of $\Lambda(n)$ were computed by means of expression (2.3) for the sets of successive spacings $t_1, t_2, \ldots, t_n$, with $n=1,2,\ldots$. It should be recalled that $t_1=(E_2-E_1)/D$, $t_2=(E_3-E_2)/D$, etc., where $E_1, E_2, \ldots$ represent the *successive* resonant energies reported in Ref. 2 with E_1 denoting the value of the *lowest* energy level. The value of D used in the computation of $\Lambda(n)$ is the one that is cited by the authors of Ref. 2. Values of $\Lambda(n)$ were obtained, as described,

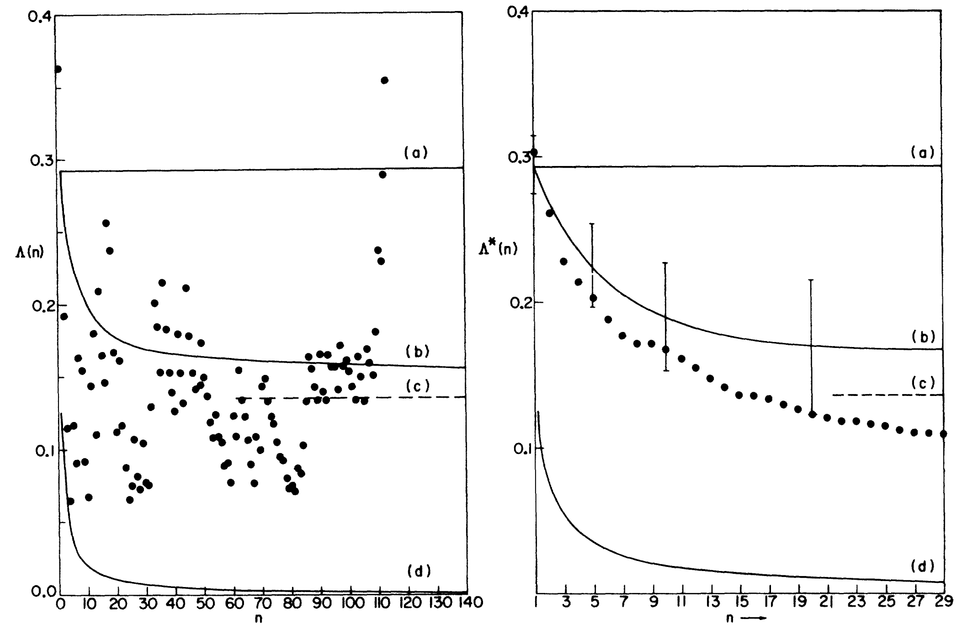

FIG. 2. Plots illustrating the application of the Λ statistic to the s-wave resonances observed in ^{166}Er. The solid circles in the left half represent values of $\Lambda(n)$ as given by substituting the lowest $n+1$ levels (n spacings) and the value $D=37.6$ reported in Ref. 2 into Eq. (2.3). The solid circles in the right half of the figure are the values of $\Lambda^*(n)$ obtained by averaging $\Lambda(n)$ with respect to all sets of n successive spacing which can be formed from the lowest 109 resonance energies quoted in Ref. 2. The error bars represent a rough estimate of the uncertainty in $\Lambda^*(n)$, as discussed in the text. The theoretical curves labeled (a)–(d) have the same meaning as in Fig. 1 of this paper.

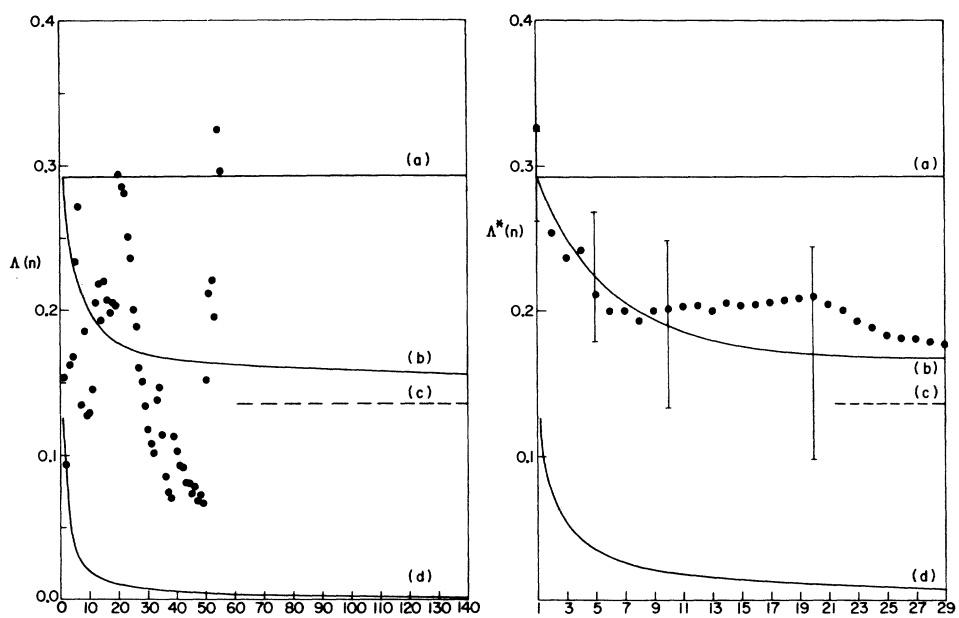

FIG. 3. Plots illustrating the application of the Λ statistic to the s-wave resonances observed in ^{168}Er. The plots have the same meaning as in Fig. 2. The values of $\Lambda^*(n)$ plotted in the right half are based on the lowest 49 levels and the value $D=93.6$ eV reported in Ref. 2.

for the three isotopes and are plotted as the solid circles in the left halves of Figs. 2–4. The following comments apply to all three cases: (1) The solid circles lie, generally speaking, within 1 standard deviation $\sigma(\Lambda(n))$ of the mean value $\langle\Lambda(n)\rangle$ obtained from the GOE up to a certain value $n=n_0$. The values of n_0 are approximately 109, 49, and 31 for ^{166}Er, ^{168}Er, and ^{170}Er, respectively. Although the scatter of the points about the mean is fairly large, it is nevertheless clear that the empirical values are in disagreement with curves (a) and (d) but may be compatible with curve (b), which is based on the GOE. (2) For $n>n_0$, the empirical values of $\Lambda(n)$ rise steeply and monotonically to very large (off-scale) values with increasing values of n. This behavior is a result of the systematic deterioration of the experimental resolution with increasing energy. The value of n_0, beyond which there is a marked deterioration of the quality of the data, agrees with the conclusions reached by the authors of Ref. 2 by other means.

B. Averaging with Respect to all Sets of n Successive Spacings

The values of $\Lambda(n)$ plotted in the left halves of Figs. 2–4 are based on the n successive spacings ($n=1, 2, \ldots$) that are formed from the $n+1$ energy levels beginning with the resonance observed at *lowest* energy. Clearly the analysis of Sec. III A did not make full use of the available data, since there exist many other sets of n successive spacings based on the *second, third,* etc., *lowest* energy levels. If altogether N successive energy levels are observed (with a reasonably good presumption that very few, if any, levels have been missed and very few, if any, spurious levels have been included) there will be $N-1$ nearest-neighbor energy-level spacings and $N-n$ distinct sets of n successive spacings. These $N-n$ sets may be used to define $\Lambda^*(n)$, the average value of $\Lambda(n)$, by means of the relation

$$\Lambda^*(n)\equiv\frac{1}{N-n}\sum_{k=0}^{N-n-1}\Lambda_k(n), \tag{3.1}$$

where

$$\Lambda_k(n)\equiv\Lambda(t_{k+1}, t_{k+2}, \ldots, t_{k+n}),$$
$$k=0, 1, \ldots, N-n-1. \tag{3.2}$$

In this notation $\Lambda_0(n)$ is the quantity that is plotted in the left halves of Figs. 2–4.

The average values $\Lambda^*(n)$ for the three targets ^{166}Er, ^{168}Er, and ^{170}Er are plotted as solid circles on the right halves of Figs. 2–4, respectively. In carrying out the averaging, the lowest $N=n_0$ levels of each spectrum were used, i.e., the averaging was confined to the levels lying in the energy interval of highest experimental resolution (as explained in Sec. III A). The following conclusion applies to all three cases. The empirically observed variation of $\Lambda^*(n)$ is definitely in disagree-

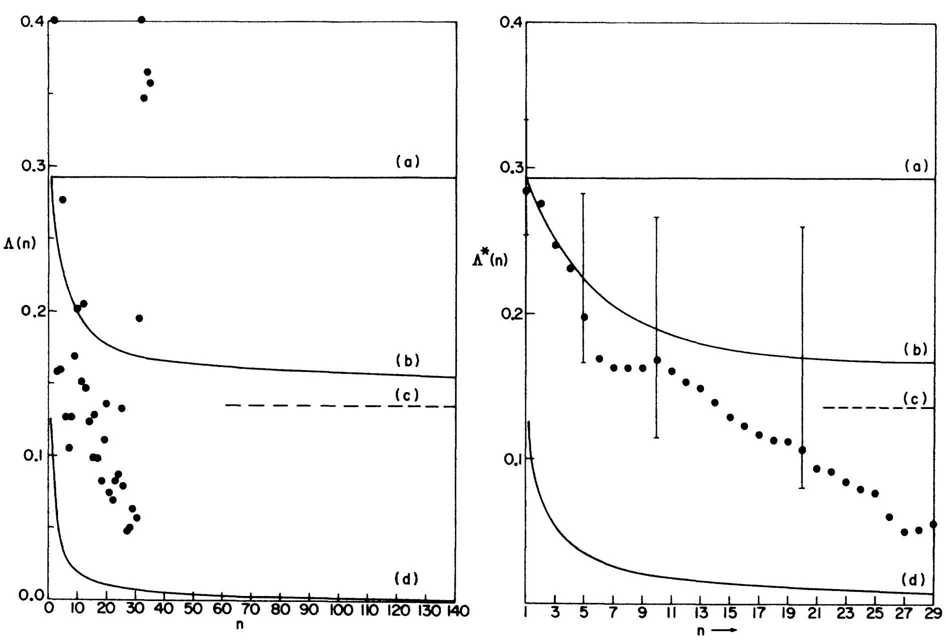

FIG. 4. Plots illustrating the application of the Λ statistic to the s-wave resonance observed in ^{170}Er. The plots have the same meaning as in Fig. 2. The values of $\Lambda^*(n)$ plotted in the right half are based on the lowest 31 resonance energies and the value $D=149.0$ eV reported in Ref. 2.

ment with curves (a) and (d), and clearly exhibits (at least approximately) the sharp decrease predicted by the GOE as n increases from 1 to 10.

Unfortunately, we are unable to state precisely whether the observed deviations of the empirical quantities $\Lambda^*(n)$ from curve (b) are within the limits to be expected from the GOE, because we are unable at this time to assess the correlations that enter into the evaluation $\sigma(\Lambda^*(n))$, the standard deviation of $\Lambda^*(n)$, the square of which is given by the expression

$$\sigma^2(\Lambda^*(n)) = \frac{\sigma^2(\Lambda(n))}{N-n} \times \left[1 + \frac{2}{N-n} \sum_{k=1}^{N-n-1} C_n(0,k)(N-k-n)\right], \tag{3.3}$$

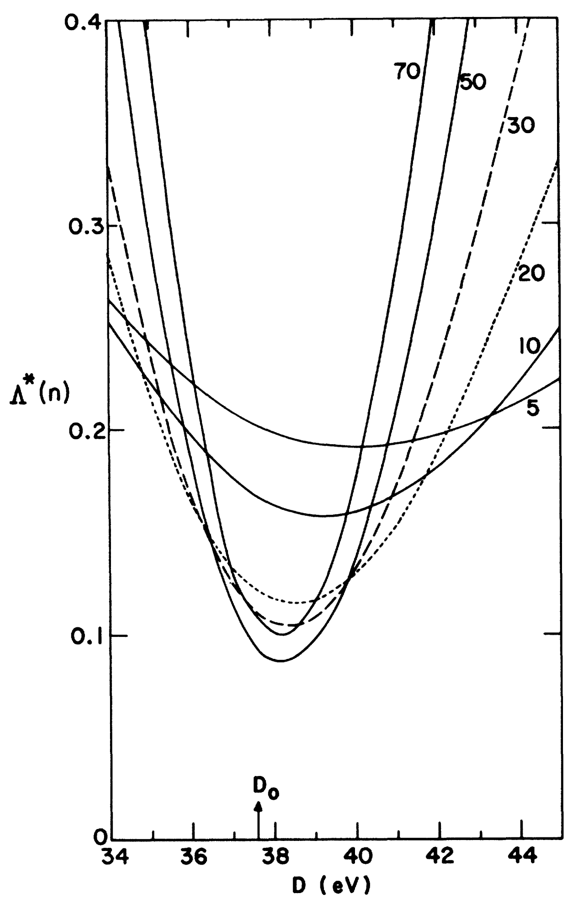

FIG. 5. Plots illustrating the dependence of $\Lambda^*(n)$ on the mean value D of the spacings. The values of $\Lambda^*(n)$ were obtained by averaging $\Lambda(n)$ with respect to all possible sets of n successive spacings that can be formed from the lowest 109 s-wave resonances reported in Ref. 2 for the target ^{166}Er. The various curves correspond to the value $n=5$, 10, 20, 30, 50, and 70, as indicated. The value $D_0 = 37.6$ is the mean value quoted in Ref. 2.

where

$$C_n(0,k) = \frac{\langle \Lambda_0(n)\Lambda_k(n)\rangle - \langle \Lambda_0(n)\rangle^2}{\sigma^2(\Lambda(n))}, \tag{3.4}$$

and, of course,

$$\sigma^2(\Lambda(n)) = \langle \Lambda^2(n)\rangle - \langle \Lambda(n)\rangle^2. \tag{3.5}$$

It should be noted that the n dependence of the standard deviation of $\sigma(\Lambda(n))$, defined by Eq. (3.5), is at least approximately known from the Monte Carlo calculations reported in I. [See the set of points (a) of Fig. 2 of I.] What is not known are the correlation coefficients $C_n(0,k)$ defined by Eq. (3.4).

We may, however, estimate the values of the correlation coefficients (3.4) by noting that $C_n(0,0) = 1$, that the coefficient $C_n(0,k)$ is presumably a decreasing function of k, and that it will be small if $k \gg n$. If we use the crude assumption

$$C_n(0,k) = \begin{cases} 1 - k/n, & k \le n \\ 0, & k > n, \end{cases} \tag{3.6}$$

which incorporates these features, together with our quantitative knowledge of $\sigma(\Lambda(n))$ from I, then we obtain values of $\sigma(\Lambda^*(n))$ which are represented by the lengths of the error bars in the right halves of Figs. 2–4. It will be noted that the deviations of $\Lambda^*(n)$ lie within 1 standard deviation of the mean. But it must be emphasized that our knowledge of the standard deviation is based on an unreliable estimate which needs to be improved upon.

C. Dependence of $\Lambda^*(n)$ on D

In the calculations (Secs. III A and III B) that led to the empirical values of $\Lambda(n)$ and $\Lambda^*(n)$, the average spacing D (which enters the formulas through the relation $t_i = s_i/D$) was treated as a known quantity.[10] In fact, we used the values of D determined in Ref. 2. Actually, however, the value of D is known, at best, only to within a few per cent, and it seems worthwhile to inquire how sensitively the values of, for example, $\Lambda^*(n)$ depend on D. An insight into this question was gained by calculating the values of $\Lambda^*(n)$ for ^{166}Er as a function of D for various values of n. As in Sec. III B, values of $\Lambda^*(n)$ were computed on the basis of the best data, namely the lowest 109 levels. The values of $\Lambda^*(n)$ are plotted as a function of D in Fig. 5, where the various curves correspond to values of $n=5$, 10, 20, 30, 50, and 70 as labeled. The authors of Ref. 2 determined the value $D_0 = 37.6$ eV. Inspection of Fig. 5 reveals the following features: $\Lambda^*(n)$ has a minimum in the general vicinity of the arithmetic mean $D_0 = 37.6$. However, the minima lie at values D_1

which are somewhat greater than D_0. The difference $D_1 - D_0$ decreases and the minima become sharper with increasing values of n. It would seem that, for sufficiently large values of n, the value of D for which $\Lambda^*(n)$ is minimized may approach the arithmetic mean. However, a detailed theory is lacking

IV. CONCLUDING REMARKS

In order to appreciate the implications of the preceding results, it must be realized that all three sets of spacings leading, respectively, to curves (a), (b), and (d) of Figs. 1–4 follow the Wigner probability distribution of spacings, Eq. (2.2). The three sets of spacings differ, however, in the distribution of the quantity $\Lambda(n)$ that measures the deviation of the empirical distribution of n successive spacings from the Wigner distribution. As may be seen by comparing curves (a) and (b), the correlations between successive spacings implied by the GOE result in a significant reduction in the values of $\Lambda(n)$.

The considerations contained in this work would be rendered more precise by a reliable calculation of the standard deviation of $\Lambda^*(n)$ in the GOE. Such a calculation could have been carried out as part of the Monte Carlo calculations reported in I, but unfortunately the introduction of the averaged quantity $\Lambda^*(n)$ was not forseen at that time.

Within the limits of our analysis, as noted above, the data for ^{166}Er, ^{168}Er, and ^{170}Er seem to be compatible with the GOE. Needless to say, analyses such as ours cannot establish the validity of a particular statistical model beyond doubt; at best it is possible to eliminate theories, for example the models leading to curves (a) and (d) of Figs. 1–4. On the other hand, the two-body matrix ensembles studied by French and Wong[7] and by Bohigas and Flores[8] cannot be evaluated from our analyses, because the statistical properties of $\Lambda(n)$ have not yet been calculated within the framework of these models.

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