Shape dependence of the deuteron radius

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The expansion of the square of the ratio of the deuteron radius to the triplet neutron-proton scattering length is generalized to include the D-state component of the deuteron wave function. Although more than 98% of this ratio is independent of the potential model, the remainder is discussed in terms of the dependence on the deuteron D state and the nonlocality of the potential. The convergence for different expansion parameters, all of which can be directly obtained from experimental data, is investigated and an optimal choice is proposed.

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I. INTRODUCTION

Several recent papers discuss the expansion of the ratio of the radius of the deuteron to the triplet neutron-proton scattering length in terms of a dimensionless parameter. This parameter is related to one of the triplet effective ranges multiplied by the square root of the deuteron binding energy or the inverse triplet scattering length [1–3]. A similar expansion holds for a relativistic model involving the Dirac equation [4]. The study of such expansions is motivated by the observation some years ago by Klarsfeld *et al.* [5] that the deuteron radius and the triplet scattering length for a variety of realistic nucleon-nucleon interactions exhibit a linear relation. The experimental values of the deuteron radius and the triplet scattering length determine a region that lies away from the line by a significant amount (three standard deviations). By introducing sufficiently large nonlocal components in the interaction it is possible to remove the discrepancy between theory and the experimental data [6,3]. Wong [7] considers models of quark and relativistic contributions to the deuteron radius and finds that both effects are too small to account for the difference. Recently new elastic electron-deuteron scattering data were used to obtain a revised estimate of the deuteron radius, reducing the discrepancy between theory and experiment [8]. However, the authors of Ref. [8] suggest that the new data are not consistent with earlier data, and so a more definite statement awaits further experimental results.

TABLE I. Experimental values for the deuteron binding energy, mixing parameter, asymptotic S-state amplitude, the triplet scattering length, and effective range (at zero energy).

E_b(MeV)	η	$A_s(\text{fm}^{-1/2})$	a_t(fm)	r_0(fm)
2.224575(9)	0.0268(7)	0.8846(8)	5.419(7)	1.759(5)

The expansions of the radius-to-scattering-length ratio bring about a clear indication of the model dependence of the deuteron radius. The first three terms of the expansion (the second has a coefficient of zero) are independent of the model of interaction; the subsequent terms give the shape dependence. The coefficients of the shape-dependent terms can be expressed in terms of shape parameters of the effective-range expansion of the triplet eigenphase shifts and integrals involving the zero-energy wave function of the system. The model dependence manifests itself in those terms of the expansion that cannot be expressed in terms of low-energy proton-neutron scattering data or quantities describing the asymptotic behavior of the deuteron wave function, which are known experimentally. The experimental quantities we use are listed in Table I.

In this paper we introduce a generalization of an earlier expansion [3], so that it now includes the D state. This is discussed in Sec. II along with some estimates of the effect of the D state on the deuteron radius. In Sec. III the choice of the expansion parameter of the ratio of the deuteron radius to the triplet scattering length is investigated. Differences in effects due to equivalent potentials are studied in Sec. IV. The final section is a discussion of the shape dependence of the deuteron radius in terms of experimental data.

II. GENERAL EXPANSION

In earlier work [1–3] the ratio of the deuteron radius to the scattering length is written as an expansion of a parameter proportional to an effective range multiplied by α, where the bound-state energy of the deuteron is

$E_b = -\hbar^2\alpha^2/m$, m being the nucleon mass, or as an expansion of an effective range times the inverse of the triplet scattering length. There are three different definitions of the effective range, which, although nearly the same, have differences between them of the order of α^2. In Ref. [3], in which the D state was not included, we used $x = \frac{1}{2}r_0\alpha$, where r_0 is the effective-range parameter in the effective-range expansion

$$k\cot\delta = -\frac{1}{a_t} + \frac{1}{2}r_0k^2 - P_e r_0^3 k^4 + \cdots . \tag{2.1}$$

In this paper we include the D state and take δ to be the eigenphase shift of Biedenharn and Blatt [9]. We therefore label the shape parameter P_e rather than P. This is an important consideration, since we will note later that the shape dependence of the expansions is in part due to the dependence on P_e. Although the scattering length and effective range are the same in the eigenphase or "bar" phase representations, the quantity P_e and its counterpart in the "bar" phase representation are not the same. (See, for example, Ref. [10].) The effective range in this case can be expressed in terms of the zero-energy wave function

$$r_0 = 2\int_0^\infty (\bar{u}_0^2 - u_0^2 - w_0^2)dr, \tag{2.2}$$

where u_0 and w_0 are the S- and D-state components of the zero-energy wave function, and $\bar{u}_0$ is the free zero-energy wave function with the same asymptotic form as u_0. The normalization of the wave function is chosen so that $\bar{u}_0 = 1$ when $r = 0$.

One could also use the parameter $y = \frac{1}{2}\alpha r_d$, where r_d is the effective range at the the deuteron bound-state energy. An expression similar to that of Eq. (2.2) holds for r_d [11]. That is,

$$r_d = 2\int_0^\infty \left(e^{-2\alpha r} - \frac{u_\alpha^2 + w_\alpha^2}{A_s^2(1+\eta^2)}\right) dr \tag{2.3}$$

$$= \frac{1}{\alpha} - \frac{2}{A_s^2(1+\eta^2)}, \tag{2.4}$$

where now the subscript α of u and w refers to the fact that the wave functions are those at the deuteron binding energy. The quantity η is the usual mixing parameter, and the wave function is normalized so that

$$\int_0^\infty (u_\alpha^2 + w_\alpha^2)dr = 1. \tag{2.5}$$

A third possible expansion parameter is $z = \frac{1}{2}\alpha r_m$ [12], where r_m can be expressed in terms of the scattering length a_t, i.e.,

$$(\alpha a_t)^{-1} = 1 - \frac{1}{2}\alpha r_m = 1 - z. \tag{2.6}$$

Bhaduri *et al.* [1] used a fourth alternative as the expansion parameter, which is the triplet zero-energy effective range divided by the triplet scattering length.

The deuteron root-mean-square radius is given by r_D, where

$$r_D^2 = \frac{1}{4}\int_0^\infty (u_\alpha^2 + w_\alpha^2) r^2 dr. \tag{2.7}$$

Let us define

$$\Upsilon \equiv 8r_D^2/a_t^2. \tag{2.8}$$

We shall show that the expression for this quantity for a pure S-state deuteron, as given in Ref. [3], can be generalized to the case in which there is a D-state component in the wave function.

We define the quantity

$$\rho(\alpha, r) = \frac{u_\alpha^2 + w_\alpha^2}{A_s^2(1+\eta^2)}. \tag{2.9}$$

This quantity is the generalization of $u^2(r)$ in the pure S-state case. Then the deuteron-pole effective range can be written

$$\frac{1}{2}r_d = \int_0^\infty \left[e^{-2\alpha r} - \rho(\alpha, r)\right] dr \equiv I_0(\alpha). \tag{2.10}$$

We also need

$$I_2(\alpha) \equiv \int_0^\infty \left[e^{-2\alpha r} - \rho(\alpha, r)\right] r^2 dr. \tag{2.11}$$

The deuteron radius can then be expressed in terms of $I_0(\alpha)$ and $I_2(\alpha)$,i.e.,

$$r_D^2 = \frac{1}{4}\frac{\int_0^\infty \rho(\alpha,r)r^2dr}{\int_0^\infty \rho(\alpha,r)dr} \tag{2.12}$$

$$= \frac{1}{4}\frac{\int_0^\infty e^{-2\alpha r}r^2dr - I_2(\alpha)}{\int_0^\infty e^{-2\alpha r}dr - I_0(\alpha)} \tag{2.13}$$

$$= \frac{1}{8\alpha^2}\frac{1 - 4\alpha^3 I_2(\alpha)}{1 - 2\alpha I_0(\alpha)}. \tag{2.14}$$

Thus

$$\Upsilon = \frac{1 - 4\alpha^3 J(\alpha) I_0^3(\alpha)}{\alpha^2 a_t^2[1 - 2\alpha I_0(\alpha)]}, \tag{2.15}$$

where we have defined the dimensionless quantity

$$J(\alpha) = I_2(\alpha)/I_0^3(\alpha). \tag{2.16}$$

Equation (2.15) is exact and has the same form as the single-channel expression. Using Eqs. (2.10) and (2.6) in Eq. (2.15), we obtain

$$\Upsilon = \frac{1 - \frac{1}{2}\alpha^3 r_d^3 J(\alpha)}{1 - \alpha r_d}(1 - \tfrac{1}{2}\alpha r_m)^2. \tag{2.17}$$

This last expression can be expanded in powers of α, or in terms of a more convenient dimensionless parameter such as x, y, or z, defined above. In a later section we show that an expansion in z converges at least as rapidly as the other possibilities. Thus we make the expansion in terms of z, although formally we could choose either of the other two parameters just as well. We note that

$$\alpha r_d = \alpha r_m[1 + 2P_e(\alpha r_m)^2 + \cdots] \quad (2.18)$$

$$= 2z(1 + 8P_e z^2 + \cdots), \quad (2.19)$$

where the next term is of the order of z^5. We also make expansions of J and P_e in z:

$$J = J_0 + J_1 z + \cdots, \quad (2.20)$$

$$P_e = P_0 + P_1 z + \cdots. \quad (2.21)$$

Writing a power series in z for Υ,

$$\Upsilon = \sum_{n=0}^{\infty} d_n z^n, \quad (2.22)$$

we obtain the coefficients of the first few terms explicitly,

$$\Upsilon = 1 + z^2 + (2 + 16P_0 - 4J_0)z^3 + (4 + 32P_0 + 16P_1 - 4J_1)z^4 + \cdots. \quad (2.23)$$

This expansion gives exactly the same formula as the one for S-state interactions [12], but the P_i and J_i will, of course, have different values because of the D state. Even with the D state included the first two terms of the expansion of Υ remain independent of the model of interaction.

It is of interest to determine the magnitude of the effect of the D state on the expansion. In Ref. [6] it was shown that the inclusion of the D state, while keeping the binding energy fixed, tends to increase the deuteron radius. We use Eq. (2.17) to analyze this in greater detail. Let us separate the S- and D-state dependence of ρ of Eq. (2.9) by writing

$$\rho(\alpha, r) = \frac{u_\alpha^2}{A_s^2} + \frac{w_\alpha^2 - \eta^2 u_\alpha^2}{A_S^2(1+\eta^2)} \quad (2.24)$$

$$= \rho_s(\alpha, r) + \Delta\rho(\alpha, r), \quad (2.25)$$

where

$$\rho_s(\alpha, r) = \frac{u_\alpha^2}{A_s^2} \quad (2.26)$$

and

$$\Delta\rho(\alpha, r) = \frac{w_\alpha^2 - \eta^2 u_\alpha^2}{A_S^2(1+\eta^2)}. \quad (2.27)$$

The function $\rho_s(\alpha, r)$ can be used to provide an estimate of effects due to the S-wave component of the deuteron wave function, whereas $\Delta\rho(\alpha, r)$ yields D-state effects through η and w_α. We write

$$I_n(\alpha) = I_{n,s}(\alpha) + \Delta I_n(\alpha), \qquad n = 0, 2, \quad (2.28)$$

where

$$I_{n,s} = \int_0^\infty \left[e^{-2\alpha r} - \rho_s(\alpha, r)\right] r^n dr. \quad (2.29)$$

The D-state contribution to I_0 is

$$\Delta I_0 = \int_0^\infty \Delta\rho(\alpha, r) dr \quad (2.30)$$

$$= \frac{1}{A_s^2(1+\eta^2)}(\eta^2 P_S - P_D), \quad (2.31)$$

where P_S and P_D are the S- and D-state probabilities. Similarly we have

$$\Delta I_2 = \int_0^\infty \Delta\rho(\alpha, r) r^2 dr \quad (2.32)$$

$$= \frac{1}{A_s^2(1+\eta^2)}(4\eta^2 r_u^2 - 4r_w^2), \quad (2.33)$$

where

$$r_u^2 = \frac{1}{4}\int_0^\infty r^2 u_\alpha^2 dr \quad \text{and} \quad r_w^2 = \frac{1}{4}\int_0^\infty r^2 w_\alpha^2 dr. \quad (2.34)$$

Obviously the sum of r_u^2 and r_w^2 is r_D^2.

We now expand $J(\alpha)$,

$$J(\alpha) = \frac{I_{2,s}}{I_{0,s}^3}\left(1 + \frac{\Delta I_2}{I_{2,s}} - 3\frac{\Delta I_0}{I_{0,s}} + \cdots\right). \quad (2.35)$$

Surprisingly the terms involving the function w are dominant in contributing to the changes; the contribution due to u is multiplied by η^2 which is proportional to α^4. ($\alpha = 0.2316$ fm^{-1} for the deuteron.) In ΔI_0, therefore, the quantity $\eta^2 P_S$ goes as α^4 for small α. By model calculations we find that P_D is proportional to α for small α. This is readily seen from the expression of P_D for the Yamaguchi potential with tensor force given in Ref. [13]. One comes to the same conclusion by using the asymptotic form of the deuteron wave function cut off inside a nonzero radius as approximating the deuteron wave function. Finally we used the Reid hard-core potential [14] and reduced the tensor force strength gradually. The resulting variation in the deuteron binding energy and the P_D clearly shows a linear relation between P_D and α. Thus the P_D term is dominant in the expression for ΔI_0. Similarly, in ΔI_2 the quantity $\eta^2 r_u^2$ is proportional to α^2, but r_w^2 is proportional to α.

An estimate of the D-state contribution to Υ can be found by considering the results of the Reid hard-core potential [14]. Including both S and D states, we obtain $\Upsilon = 1.055$. When ρ_s is used instead of ρ to calculate J, $\Upsilon = 1.040$. We emphasize that this difference is about half of that obtained when the r_w^2 part is simply dropped from r_D^2 in Υ. Thus the D state causes an increase of about 1.4% to Υ or about 0.7% to r_D. This result is consistent with the model calculations in Ref. [6].

If the expansion of this ratio, Eq. (2.22), converges rapidly enough so that the term with coefficient d_4 and the following terms can be neglected, the D-state contribution to J and to P_e can be translated as a D-state contribution to the coefficient d_3. We show later that for realistic potentials the truncation of the expansion after the d_3 term is a reasonable approximation. Using the Reid hard-core results to make a typical estimate, we obtain $\Delta d_3(J) = -4\Delta J = 1.22$, where ΔJ is the contribution to J in Eq. (2.35) due to the $\Delta I_{0,2}$ terms.

The contribution to P_e for a zero-energy bound-state system will lead to another contribution to d_3 due to the D state, $\Delta d_3(P_e)$. The value of d_3 for the Reid hard-core potential can be estimated by varying Υ for different but small values of z. Different values of z are obtained by reducing the strength of the tensor component of the Reid potential. Thus for this potential the value of d_3 is 1.77. Clearly the D state contributes significantly to this coefficient. This is consistent with the earlier estimate of 0.7% contribution to r_D, since the third term in the expansion of Υ contributes altogether only 1.5%. Similar contributions arise from nonlocal interactions. Later in this paper we report on d_3 for the Yamaguchi potential without D state ($d_3 = -2.7407$) and with a 4% D state ($d_3 = -1.6964$).

It is remarkable that the D state contributes in the lowest order shape-dependent term, i.e., the d_3 term. The quantities J_0 and P_0 are zero-energy limits, and in this limit both the D-state probability P_D and the asymptotic mixing parameter η vanish. At zero energy, however, the wave function does contain a finite D-state component, but the ratio of D-state to S-state probabilities vanishes, since the S-state wave function becomes unnormalizable. Since the corrections in J and P_e due to the D state vary as α^0 to the lowest order in α, they both contribute to d_3.

III. CHOICE OF EXPANSION PARAMETER

In Sec. II we developed a power series in the parameter z, but we indicated that x and y were also appropriate choices. Besides the expansion given in Eq. (2.22), we introduce the power series

$$\sqrt{\Upsilon} = \frac{\sqrt{8}r_D}{a_t} = \sum_{n=0}^{\infty} a_n x^n \tag{3.1}$$

$$= \sum_{n=0}^{\infty} b_n z^n, \tag{3.2}$$

$$\Upsilon = \frac{8r_D^2}{a_t^2} = \sum_{n=0}^{\infty} c_n x^n \tag{3.3}$$

$$= \sum_{n=0}^{\infty} e_n y^n. \tag{3.4}$$

Bhaduri *et al.* [1] use a different expansion parameter, namely, r_0/a_t. For the sake of comparison with the other series we take $\zeta = r_0/(2a_t)$ and consider also the series

$$\Upsilon = \sum_{n=0}^{\infty} \alpha_n \zeta^n. \tag{3.5}$$

(The α_n's in this paper are 2^n times the α_n's of Bhaduri *et al.*)

First consider a purely S-state deuteron. In Refs. [3,12] we showed that for the sticky core potential the expression $\Upsilon = 1 + z^2$ is exact. This potential can therefore be represented by the series (2.22) with $d_0 = d_2 = 1$ and all the other d_n are zero. Series expansions of Υ for this potential in the other variables yield nonzero higher coefficients. Clearly the variable z provides the best convergence. In order to test this for a different potential, we use the central Yamaguchi potential [15] for which all the coefficients can be calculated explicitly. The results are given in Tables II and III.

TABLE II. Coefficients of the series given in Eqs. (3.1) and (3.2) for the Yamaguchi potential. For both cases the coefficients for $n = 0$ are 1, for $n = 1$ are 0, and for $n = 2$ are $\frac{1}{2}$.

n	a_n	b_n
3	−1.3704	−1.3704
4	0.0046	0.1960
5	1.1900	0.7785
6	−0.4463	−0.9996
7	−0.2896	−0.1028

Since we have given an explicit expression for the expansion up to the z^4 term in Eq. (2.23), we consider the convergence of the series beyond the x^4, y^4, or z^4 term. From Tables II and III it is evident that the series expansion of Υ in z gives the smallest contribution from the higher-order terms. Note that for $\alpha = 0.2316$ fm^{-1} (from the binding energy of the deuteron), the values of x, y, and z are almost equal, i.e., $x = 0.199$, $y = 0.201$, and $z = 0.200$.

The series in ζ converges much less rapidly than the one in z (its convergence is comparable to the series in y), although it must noted that if experimental data are used to obtain values for the expansion parameters, ζ ($= 0.16$) is smaller than z ($= 0.20$). Furthermore it appears that from the third term on all the terms are negative, so that the series truncated after the second term gives an upper bound.

In Ref. [12] we considered a number of different model central potentials and found that, with the exception of the Hulthen potential, all the potentials yielded $|d_4| < |d_3|$. The reason for the exception is discussed in Ref. [12]. The Eckart potential was not included in that study, and provides another exception to the rule. In general, however, we can say that for most model interactions the series expansion in z for Υ converges at least as rapidly as the power series in x , and more rapidly than the series in y or ζ.

To investigate the effect of the D state on the expansion we use the Yamaguchi potential with the tensor force [13]. Even with the tensor force, analytical expressions are given for all the quantities needed to evaluate

TABLE III. Coefficients of the series given in Eqs. (3.3), (2.22), (3.4), and (3.5) for the Yamaguchi potential. For all cases the coefficients for $n = 0$ and $n = 2$ are 1 and for $n = 1$ are 0.

n	c_n	d_n	e_n	α_n
3	−2.7407	−2.7407	−0.7407	−0.7407
4	0.3457	0.6420	−2.2840	−2.7531
5	1.0096	0.1866	−7.2538	−38.5679
6	1.0332	0.0748	−24.1127	−24.7334
7	0.4796	0.0358	−83.1479	−74.1737

TABLE IV. Numerical values of coefficients in series (2.22) for the Yamaguchi potential with tensor component [13].

d_3	d_4	d_5	d_6	d_7
−1.6964	1.0696	0.0390	−0.7774	−0.8450

the coefficients. Using z as the expansion parameter, we find again that $d_0 = d_2 = 1$ and $d_1 = 0$. Further d_n's are listed in Table IV. These coefficients are functions of β, γ, and t, the central force inverse range, the tensor force inverse range, and the tensor force strength, respectively. We have used the values of Ref. [13], which fit the deuteron to old data. The magnitude of the coefficients in this case also indicate good convergence of the expansion.

The "goodness" of the z expansion is tested by comparing the deuteron radius calculated from the truncated series with the exact value. The results are listed in Table V. The expansion parameters, z, ζ, and x were calculated by using the experimental α and a_t and fitting the potential parameters accordingly, except in the cases of the Reid and Moscow [16] potentials for which the values of α and a_t determined by the potential were used. Table V shows that truncation after the z^4 term gives excellent results, and truncation after the z^3 term already gives nearly exact results. The results with the two Yamaguchi potentials illustrate the D state pushing the deuteron wave function outward (thus increasing the value of the deuteron radius). The deep square well with the excited state at the deuteron energy has a wave function that is pulled toward the origin in contrast to the D-state effect which pushes the wave function outward. For the Reid and Moscow potentials the coefficients d_3 and d_4 are obtained by making a least-mean-squares fit to $\Upsilon - 1 - z^2$ for different z. Different values of z and Υ are obtained by reducing the strength of the tensor part of the potential. The last two rows of the Table V allow a comparison of these results when different expansion parameters are used.

IV. SIMPLE POTENTIAL MODELS

In this section we study the expansion of Υ in z for potentials that are equivalent in some sense. The first pair of potentials is the Eckart and the separable Yukawa, which are phase shift equivalent; i.e., their phase shifts are exactly the same at all energies. The second pair, the Hulthen and the central Yamaguchi, have identical bound-state wave functions. Finally, the square-well potential and the separable square-well equivalent potential have the same zero-energy wave functions. For each potential we calculate P_0, P_1, J_0, J_1, d_3, and d_4. For all but the Eckart potential the exact expressions are contained in Ref. [12]. For the Eckart potential $P_0 = P_1 = 0$, but we have not been able to obtain expressions in closed form for J_0 and J_1. These quantities can be expressed in terms of integrals which we evaluated numerically. The results are given in Table VI.

We are comparing in each case the results of a separable and of a local potential, i.e., extreme limits of nonlocality. The quantities P_0 and P_1 are asymptotic properties which will only give information on the energy shell. The quantities J_0 and J_1 on the other hand are dependent on the details of the wave function, and consequently will depend on the off-shell properties of the interaction. Thus with the separable Yukawa and the Eckart potentials we note that they give the same scattering results, but their bound-state wave functions are different. These two potentials give identical on-shell results, but their off-shell properties differ. The Yamaguchi and the Hulthen potentials have identical bound-state wave functions, but they are not phase equivalent. Hence the P's differ whereas the J's are identical. In the case of the square well and the separable square-well equivalent potentials, the zero-energy wave function is the same. This make the J_0's the same for both potentials, and the J_1's very close, indicating that at short range the deuteron bound-state wave function does not differ much from the zero-energy wave function. The P's are close in values but not quite equal. Although the shape parameter P_e can be obtained from the zero-energy wave functions [9,10], which are the same for the two potentials, P_e also depends on the derivative of the wave function with respect to energy evaluated at zero energy, which is different for the two potentials.

V. DISCUSSION

The expansion of Υ in terms of the parameter z, which can be directly related to experimental data, provides a transparent method of delineating the model dependence of the deuteron radius. Since the expansion of Υ in z con-

TABLE V. r_D in fm calculated from Υ truncated after the z^n term. The last column gives d_3 for the potential mentioned. Note that in the last two rows the expansion parameter is ζ or x, as indicated, rather than z.

Potential	z^0	z^2	z^3	z^4	exact	d_3
Square well (central)	1.916	1.955	1.950	1.950	1.950	−0.614
Deep square well	1.916	1.955	1.922	1.943	1.934	−4.216
Reid hard core	1.908	1.946	1.959	1.960	1.960	1.77
Moscow	1.908	1.946	1.960	1.959	1.959	1.96
Yamaguchi (central)	1.916	1.955	1.933	1.934	1.934	−2.74
Yamaguchi (with tensor)	1.916	1.955	1.942	1.943	1.943	−1.70
Yamaguchi (ζ)	1.916	1.941	1.938	1.936	1.934	
Yamaguchi (x)	1.916	1.956	1.933	1.933	1.934	

TABLE VI. The coefficients of P_e, J, and Υ with z as the expansion parameter for "equivalent" potentials.

Potential	P_0	P_1	J_0	J_1	d_3	d_4
Separable Yukawa	0.0000	0.0000	1.6250	0.3750	−4.5000	2.5000
Eckart	0.0000	0.0000	0.8225	0.4057	−1.2899	2.3773
Yamaguchi	−0.0185	−0.0247	1.1111	0.5926	−2.7407	0.6420
Hulthen	0.0381	0.1287	1.1111	0.5926	−1.8346	4.9085
Separable square-well equivalent	−0.0360	−0.0411	0.5228	0.5586	−0.6667	−0.0427
Square well	−0.0327	−0.0360	0.5228	0.5756	−0.6138	0.0764

verges rapidly, the model dependence can be expressed as the remainder of the series after the first three terms. We define therefore the quantity

$$\Delta = \Upsilon - 1 - z^2, \tag{5.1}$$

where Υ and z are calculated using Eqs. (2.8) and (2.6) with r_D, α, and a_t of the potential or derived from experimental data. Since Υ is approximately 1, Δ gives the fractional model dependence of Υ and is approximately twice the fractional model dependence of r_D.

For all the (semi)phenomenological potentials considered by Klarsfeld *et al.* [5], and from which they obtained the a_t versus r_D line, Δ averages 0.014, the values ranging from 0.0124 to 0.0168. These are "realistic" potentials, and therefore the D-state component is included in the deuteron wave function.

With local S-state potentials, such as the square well or Eckart potential, either fitting the deuteron binding energy, Δ is −0.0054. Combining this result with that of the previous paragraph suggests D-state effects of about 0.019. This estimate is consistent with that of Sec. II. Furthermore the central Yamaguchi potential gives −0.022, demonstrating again that nonlocal potentials fitting the deuteron binding energy give smaller radii for the deuteron. A local square-well potential with two bound states, the excited state at the deuteron energy, also has $\Delta = -0.022$.

The model dependence of the deuteron radius is therefore about 1%. The model dependences due to the D-state component of the deuteron wave function or the nonlocality of the potential are both again of the order of 1%. These two effects may cancel each other since a larger D-state probability tends to increase Υ, whereas increased nonlocality causes Υ to decrease. In any case experimental data need to have at least a precision of this magnitude in order to help resolve the model dependence.

The quantity Δ can be obtained directly from experimental data, that is, from the values of r_D, α, and a_t. Using $r_D = 1.950$ fm, given in Ref. [8] as the best value from the analysis of Klarsfeld *et al.* [5], we obtain −0.0054 for Δ. For the estimate of r_D based on more recent data, namely, $r_D = 1.961$ fm [8], we obtain $\Delta = 0.0063$. Thus even if the newer data stand up to future scrutiny, a little less than half of the discrepancy between theory and experiment remains.

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[1] R.K. Bhaduri, W. Leidemann, G. Orlandini, and E.L. Tomusiak, Phys. Rev. C **42**, 1867 (1990).
[2] D.W.L. Sprung, H. Wu, and J. Martorell, Phys. Rev. C **42**, 863 (1990).
[3] M.W. Kermode, S.A. Moszkowski, M.M. Mustafa, and W. van Dijk, Phys. Rev. C **43**, 416 (1991).
[4] F.M. Toyama, D.J. Beachey, Y. Nogami, and W. van Dijk, Phys. Rev. C **44**, 67 (1991).
[5] S. Klarsfeld, J. Martorell, J.A. Oteo, M. Nishimura, and D.W.L. Sprung, Nucl. Phys. **A456**, 373 (1986).
[6] W. van Dijk, Phys. Rev. C **40**, 1437 (1989).
[7] C.W. Wong, Phys. Lett. B **291**, 363 (1992).
[8] D.W.L. Sprung and H. Wu, Acta Phys. Pol. B **24**, 503 (1993).
[9] L.C. Biedenharn and J.M. Blatt, Phys. Rev. **93**, 1387 (1954).
[10] W. van Dijk and M.W. Kermode, Phys. Rev. C **45**, 1528 (1992).
[11] D.W.L. Sprung, M.W. Kermode, and S. Klarsfeld, J. Phys. G **8**, 913 (1982).
[12] S.A. Moszkowski, M.W. Kermode, and W. van Dijk, Acta Phys. Pol. B **24**, 597 (1993).
[13] Y. Yamaguchi and Y. Yamaguchi, Phys. Rev. **95**, 1635 (1954).
[14] R.V. Reid, Ann. Phys. (N.Y.) **50**, 411 (1968).
[15] Y. Yamaguchi, Phys. Rev. **95**, 1628 (1954).
[16] V.I. Kukulin and V.N. Pomerantsev, Prog. Theor. Phys. **88**, 159 (1992).