Approximation for the algebraic S matrix with an angular momentum dependent potential parameter

L. J. Allen, K. Amos, and L. Berge
School of Physics, University of Melbourne, Parkville, Victoria 3052, Australia

H. Fiedeldey
Department of Physics, University of South Africa, P.O. Box 392, Pretoria, 0001 South Africa
(Received 1 April 1993)

Algebraic scattering theory is developed and the usual algebraic "potential parameters" are shown to involve a second approximation to that by which the non-Coulombic interactions modify the exact S functions found by Euclidean connections of the SO(3,N) group algebras. Elastic scattering data (cross sections) from heavy ion collisions, typical of most and epitomized by strong absorption, lead to algebraic potential parameters that can be described by simple exponential forms.

PACS number(s): 11.55.−m, 25.70.Bc

Recent studies [1–4] have considered a group theoretic approach to the analysis of scattering of heavy ions. The groups SO(3,N) for $N = 1, 2$ are particularly interesting in that context since the dynamical symmetries inherent within them yield S functions that equate to quantum scattering from (modified) Coulomb potentials. Quantum scattering that is dominated by Coulomb effects might then be treated by postulating algebraic "potential parameters"[1] to modify the arguments of the S-function forms from those of the exact realizations of the SO(3,N) groups. The scattering of two heavy nuclear ions is such a suitable system as, in general, it is strongly absorptive in character. Indeed for kinetic energies in excess of 10–20 MeV/nucleon the scattering is usually so strongly absorptive that the process is quite peripheral and the Coulomb interaction dominates any hadronic term. For lower energies the "classical" turning radius is well outside the summed nuclear radii and Coulomb scattering again dominates the elastic events. In both circumstances only large angular momentum channels are essential to explain observed data.

Algebraic scattering theory for the elastic collision of two heavy ions having atomic numbers Z_1 and Z_2 and reduced mass μ gives algebraic S functions in terms of algebraic Jost functions A_l by

$$S_l(k) = (-1)^{(l+1)} \frac{A_l(k)}{A_l(-k)} , \tag{1}$$

wherein k is the wave number. The realization of the SO(3,1) group determines a recursion relation for these elements, namely,

$$S_{l+1}(k) = \left(\frac{l+1+z_l+i\eta}{l+1-z_l-i\eta} \right) S_l(k) , \tag{2}$$

where η is the Sommerfeld parameter,

$$\eta = \frac{\mu Z_1 Z_2 e^2}{\hbar^2 k} , \tag{3}$$

and $z_l(k)$ is the algebraic potential parameter caused by the hadronic interaction between the two heavy ions. To within a constant phase (which we take as zero) the S function is then

$$S_l(k) = \prod_{j=1}^{l} \left[\frac{j+z_j+i\eta}{j-z_j-i\eta} \right] . \tag{4}$$

For heavy ion scattering so many partial waves are involved usually that one may consider the S function to be a continuous function of angular momentum, whence

$$S_{l+1} \approx S_l + \frac{dS_l}{dl} . \tag{5}$$

Then, with the deflection function defined by

$$\Theta(l) = -i \frac{d[\ln S(l)]}{dl} , \tag{6}$$

the approximation for the (continuous) function [Eq. (5)] when used with the recurrence result [Eq. (2)] gives

$$1 + i\Theta(l) = \frac{l+1+z_l+i\eta}{l+1-z_l-i\eta} , \tag{7}$$

from which one finds the relation

$$z_l = \left[\frac{i(l+1)\Theta(l)}{2+i\Theta(l)} - i\eta \right] . \tag{8}$$

In the limit $z_l \to 0$, these two equations reduce to

[1]The terminology, "algebraic potential" is a misnomer. The quantities so classified are all arguments in an expression for the S function. But the term has been used so frequently that we retain it herein in the form algebraic "potential parameter."

$$\Theta(l) \rightarrow \frac{2\eta}{(l+1)-i\eta} \underset{l\rightarrow\infty}{\rightarrow} \frac{2\eta}{(l+1)} , \tag{9}$$

the result expected for pure Coulomb scattering. The asymptotic form for z_l [Eq. (8)] should be appropriate therefore in conditions $l \gg \eta \gg 1$.

However, for pure Coulomb scattering or for any algebraic potential parameter that is independent of l, the product form of the S function [Eq. (4)] reduces to the conventional form

$$S_l(k) = \frac{\Gamma(l+1+z+i\eta)}{\Gamma(l+1-z-i\eta)} , \tag{10}$$

where $\Gamma(z)$ represents the gamma function. But to use this form with an algebraic potential parameter that is dependent upon l invokes a second approximation to that of the insertion of an algebraic potential parameter in the first place. We identify that second approximation form for the SO(3,1) case by using Eq. (10) with the substitution

$$z_l(k) \rightarrow v_l(k) . \tag{11}$$

To date that form has been used in applications either with a Woods-Saxon parametric form for v_l [1,2] or by directly matching the algebraic S function to one that fits the measured cross section. The latter was usually of the type defined by McIntyre, Wang, and Becker [5], viz.,

$$|\, S_l^M \,| = \left[1+\exp\left(\frac{l_g-l}{\Delta}\right)\right]^{-1} \tag{12}$$

and

$$\mathrm{Re}(\delta_l^M) = \mu\left[1+\exp\left(\frac{l-l'_g}{\Delta'}\right)\right]^{-1} . \tag{13}$$

For large l values, the McIntyre phase shift is

$$2i\delta_l^{(M)} \underset{l\rightarrow\infty}{\longrightarrow} -\exp\left(\frac{l_g-l}{\Delta}\right) + 2i\mu\exp\left(\frac{l'_g-l}{\Delta'}\right) . \tag{14}$$

But one may anticipate that the form found for $v_l(k)$ by fitting data will be structurally different to that of $z_l(k)$ when Eq. (4) is used to fit the same data. Herein we consider just what those differences are for typical cases. For completeness, we also note that a similar reduction and double approximation gives the SO(3,2) form for the S function

$$S_l = \frac{\Gamma[\frac{1}{2}(l+2+w_l+i\eta)]\;\Gamma[\frac{1}{2}(l+1-w_l+i\eta)]}{\Gamma[\frac{1}{2}(l+2+w_l-i\eta)]\;\Gamma[\frac{1}{2}(l+1-w_l-i\eta)]}\, e^{2i\ln(2\eta)} . \tag{15}$$

Interest in algebraic potential parameters, and especially of simple functional forms of them, initially stemmed from the expectation that they could be determined from underlying microscopic reaction processes without recourse to coordinate space interactions [1]. Although the S functions as given in Eqs. (10) and (15) have a compact analytic form, we see from the foregoing that the S function of Eq. (4) has also that advantage by being an analytic expression in terms of the deflection function $\Theta(l)$. Neverthless, recently, we identified a coordinate space potential corresponding to an algebraic one by using inverse scattering theory [4]. Only the $v_l(k)$ and $w_l(k)$ forms were considered in that study and differences were found between directly fitted forms (mapped to McIntyre functions) and the postulated (Woods-Saxon)

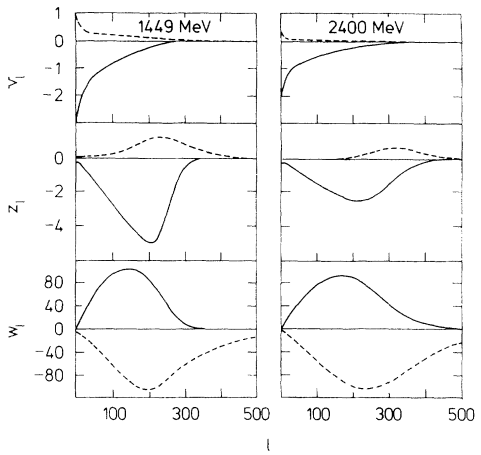

FIG. 1. The diverse algebraic potential parameters obtained from fits to the 1449 MeV (left) and 2400 MeV (right) elastic scattering cross sections of ^{12}C from ^{208}Pb. The real and imaginary parts are represented by the solid and dashed curves, respectively.

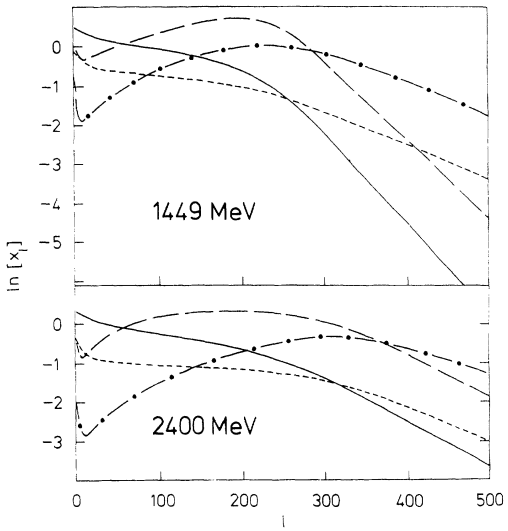

FIG. 2. The logarithms of the real and imaginary components of v_l and z_l algebraic potential parameters from the analyses of ^{12}C-^{208}Pb scattering. The logarithms of the real and imaginary components of v_l are displayed by the solid and short dashed curves, respectively, while those of z_l are given by the long dashed and dot-dashed curves, respectively.

shapes assumed by others. It is worth noting that Amado and Sparrow [2] studied connections between SO(3,2) algebraic potential parameters $w_l(k)$ and coordinate space ones by means of the eikonal approximation, finding that if $w_l(k) \to \exp(-\mu l)$, then $V(r) \to r^{-\frac{5}{2}} \exp(-\mu k r)$. But that connection is highly nonlinear save in the periphery, although Zielke *et al.* [6] recently have found explicit coordinate realizations of the algebraic Hamiltonians with general SO(3,2) dynamical symmetry. Likewise, with a semiclassical approach and the SO(3,1) potential parameter $v_l(k)$, Hussein, Pato, and Iachello [2] found that an exponentially decreasing form with l mapped to an exponentially decreasing coordinate space interaction.

For the 1449 and 2400 MeV ^{12}C-^{208}Pb scattering cross sections, simple S functions suffice to give excellent fits to the data. With the McIntyre form we were able to fit those cross sections with χ^2 per degree of freedom (χ^2/F) of 1.2 – 1.3, signifying a quality fit to the measured values. Those (McIntyre) S functions and the fits to the data were given previously [4]. The algebraic potential parameters obtained by mapping the various forms [Eqs. (4), (10), and (15)] to the "experimental" S functions are given in Figs. 1 and 2. [Similar results for $z_l(k)$ were obtained when the total deflection function (nuclear plus Coulomb) was used in Eq. (8). The imaginary parts are slightly smaller overall.] Clearly, $z_l(k)$, $v_l(k)$, and $w_l(k)$ for these two cases as shown in Fig. 1 have distinctively different shapes with l. But one must recall that only the large l values ($\gtrsim 200$ and $\gtrsim 250$ for 1449 and 2400 MeV, respectively) are of any significance in fitting the cross-section data. For those angular momentum values all three algebraic potential parameters are essentially monotonically decreasing and exponentially so. But each has a different exponent coefficient. That is evident from Fig. 2 wherein the logarithms of the separate real and imaginary parts of the algebraic potential parameters z_l and v_l are shown. Those results are again smooth and asymptote to a linear form with large l. One may expect that simple functional forms of these potential parameters will reproduce the plotted shapes. But it is to be noted that the large l values are parallel (real and imaginary components considered separately) with $\ln(z_l)$ about twice that of $\ln(w_l)$.

We conclude that algebraic scattering theory, when applied to the analysis of heavy ion scattering that is characterized by strong absorption, results in smooth exponential forms for the algebraic potential parameters at the large angular momentum values that are important so far as fits to the cross-section data are concerned. But the exponent values are different according to use of the z_l, v_l, or w_l representations. As the latter two result from a mathematical approximation in addition to the basic algebraic scattering theory concept, presumably the z_l forms should be used and particularly so if an investigation seeks an underlying theory of the algebraic potential parameter.

[1] Y. Alhassid, F. Iachello, and J. Wu, Phys. Rev. Lett. **56**, 271 (1986); J. Wu, F. Iachello, and Y. Alhassid, Ann. Phys. (N.Y.) **173**, 68 (1987); F. Iachello, *ibid.* **192**, 133 (1989).

[2] Y. Alhassid, F. Iachello, and B. Shao, Phys. Lett. B **201**, 183 (1988); M.S. Hussein, M.P. Pato, and F. Iachello, Phys. Rev. C **38**, 1072 (1988); R.D. Amado and D.A. Sparrow, *ibid.* **34**, 1997 (1986); R. Lichtenthaler–Filho, A.C.C. Villari, L.C. Gomes, and P. Carrilho–Soares, Phys. Lett. B **269**, 49 (1991).

[3] K. Amos, L. Berge, H. Fiedeldey, I. Morrison, and L.J. Allen, Phys. Rev. Lett. **64**, 625 (1990).

[4] K. Amos, L. Berge, L.J. Allen, and H. Fiedeldey, Phys. Rev. C **47**, 2827 (1993).

[5] J.A. McIntyre, K.H. Wang, and L.G. Becker, Phys. Rev. **117**, 1337 (1960); S.D. Baker and J.A. McIntyre, *ibid.* **161**, 1200 (1967).

[6] A. Zielke, R. Maass, W. Scheid, and O.L. Weaver, Phys. Rev. A **42**, 1358 (1990).