Unstable infinite nuclear matter in stochastic mean field approach

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(Received 18 November 1993)

In this article, we consider a semiclassical stochastic mean-field approach. In the case of unstable infinite nuclear matter, we calculate the characteristic time of the exponential growing of fluctuations and the diffusion coefficients associated to the unstable modes, in the framework of the Boltzmann-Langevin theory. These two quantities are essential to describe the dynamics of fluctuations and instabilities since, in the unstable regions, the evolution of the system will be dominated by the amplification of fluctuations. In order to make realistic 3D calculations feasible, we suggest to replace the complicated Boltzmann-Langevin theory by a simpler stochastic mean-field approach corresponding to a standard Boltzmann evolution, complemented by a simple noise chosen to reproduce the dynamics of the most unstable modes. Finally we explain how to approximately implement this method by simply tuning the noise associated to the use of a finite number of test particles in Boltzmann-like calculations.

PACS number(s): 21.65.+f, 25.70.Mn, 47.20.−k

I. INTRODUCTION

In recent years many efforts have been concentrated on the study of complex fragment production, which is observed in heavy ion collisions at intermediate energies [1]. In fact, it has been seen that some reactions, in this energy range, lead to the formation of dense and hot drops of nuclear matter, which decay by emission of particles and fragments. This kind of reaction may carry very important information on the properties of nuclear matter at high density and temperature, such as, for instance, its equation of state and, in particular, the nuclear compressibility and the nucleon-nucleon cross section in the medium. Concerning the formation of intermediate mass fragments, a large debate is developing about the responsible mechanism between two main classes of interpretation: (i) statistical decay from an equilibrated source or (ii) direct explosion of the nuclear system, or multifragmentation, expected on the basis of the prediction of a liquid-gas phase transition [2].

The purpose of this paper is to study the growth of nuclear instabilities which are associated to the latter scenario. To achieve this goal we will consider mean-field approaches, which are well suited to study nuclear matter. We will study the properties of all collective modes, present in the considered system, in order to understand if they are unstable or not, and derive the most unstable ones as a function of the various characteristics of the system. We will explicitly achieve this task in the case of infinite nuclear matter at given density and temperature. We will discuss the role of the finite range of the nuclear force and of the nuclear surface energy. We will particularly stress that, in a realistic calculation, the most unstable wavelength is slightly dependent on the position of the system in the spinodal region. Moreover, since we find this wavelength to be of the order of a few fm, the requirement that it must be smaller than the characteristic size of the nucleus is easily verified and therefore does not introduce a strong reduction of the instability region, in the case of a finite size system. Conversely, we will show that the requirement that, in order to get fragments, the instability time must be shorter than the typical time associated to the dynamical evolution of our finite system is a strong condition.

It is important, as well, to know the right value of the amplitude of fluctuations in the unstable region. Indeed pure mean-field equations are not valid in regions where instabilities, bifurcations, or chaos are present. This is the reason why, in the following, we will consider a stochastic mean-field approach to deal with the instability regions. In this sense the recently developed Boltzmann-Langevin (BL) theory [3,4], in which fluctuations are coming from the stochastic part of the collision integral, provides a well-founded method to be applied in the spinodal region.

Nevertheless, since, up to now, three-dimensional (3D) calculations of the BL type are not feasible, we will try in the following to propose a simpler stochastic mean-field approach in which the Langevin term is replaced by a simple noise tuned to give the correct dynamics for the most unstable modes. In particular, considering ordinary simulations of the Boltzmann-like dynamics within the test particle approach, it is possible to take advantage of the noise associated to the use of a finite number of test particles and to adapt these numerical fluctuations to the predictions of the BL theory in the considered situation. This possibility has been already tested on some simplified cases in two dimensions [5] making a comparison between the results of calculations with a suitable number of test particles per nucleon and an exact solution of the BL equation on a lattice [6]. In this work we will derive the strength of the fluctuations projected on the unstable modes and we will estimate the number of test particle to be used in order to mimic a stochastic mean-field calculation.

II. UNSTABLE MODES AND GROWING TIME

Let us consider first, for the sake of simplicity, a cubic box filled with uniform nuclear matter, at given density and temperature. In such a simplified situation, for symmetry reasons, the collective motions are associated to plane waves, characterized by a wave number $\mathbf{k}$. For each mode $\mathbf{k}$, we need to study the corresponding dispersion relation and to define its possible instability region. Indeed we are interested in unstable modes because we want to study if the fragmentation mechanism can be associated to the growing of the related fluctuations. Therefore we will define a $\mathbf{k}$ dependent spinodal region [7] and we will give the values of the instability growing time.

A. Dispersion relation

In order to gain insight into the definition of an unstable mode, let us consider the linear response theory, starting from a pure mean-field approach. Let us discuss this approximation in some detail because some imprecision can be found in the literature about it [8]. We will consider a semiclassical approach describing the time evolution of the one-body phase-space density $f(\mathbf{r},\mathbf{p},t)$ by the mean of the Vlasov equation:

$$\frac{\partial f}{\partial t} + \{H,f\} = 0. \tag{1}$$

Here $H = p^2/2m + U$ is the effective one-body Hamiltonian and $\{\cdot,\cdot\}$ the Poisson bracket, so that Eq. (1) describes the Vlasov evolution for f in its own self-consistent mean field $U = U[\rho]$ generated by the density $\rho(\mathbf{r}) = \int d\mathbf{p} f(\mathbf{r},\mathbf{p})/h^3$. If we are interested in the small fluctuations around the mean density ρ_0, assumed to be independent of $\mathbf{r}$, we can decompose the one-body distribution function as follows: $f(\mathbf{r},\mathbf{p},t) = f_0(\mathbf{p},t) + \delta f(\mathbf{r},\mathbf{p},t)$, where δf represents a small amplitude perturbation on the mean value f_0. Then, it is possible to demonstrate that δf fulfills the following equation:

$$\frac{\partial \delta f}{\partial t} + \frac{\partial \delta f}{\partial \mathbf{r}}\frac{\mathbf{p}}{m} + \frac{\partial f_0}{\partial \mathbf{p}}\frac{\partial \delta U}{\partial \mathbf{r}} = 0, \tag{2}$$

where $\delta U[\rho]$ is the variation of the self-consistent mean field

$$\delta U = -\frac{\partial U}{\partial \rho}\delta\rho. \tag{3}$$

This product must be considered as a folding if U is nonlocal: $\delta U = -\partial U/\partial\rho \otimes \delta\rho$. Introducing the $\mathbf{k}$ representation of δf (i.e., performing a Fourier transformation on the $\mathbf{r}$ variable) the linear response equation (2) can be reduced to the eigenfrequency problem:

$$(\hbar\omega + \mathbf{p}\cdot\mathbf{k}/m)\delta f(\mathbf{k},\mathbf{p},\omega) = 2\,\frac{\partial U(\mathbf{k})}{\partial \rho}\frac{\partial f_0}{\partial p^2}\mathbf{p}\cdot\mathbf{k}\;\delta\rho(\mathbf{k},\omega). \tag{4}$$

The different wave numbers $\mathbf{k}$ are decoupled as we expect from symmetry considerations. In the region where $\mathbf{k}\cdot\mathbf{p} \neq m\hbar\omega$, i.e., in the whole complex ω plane, except on the real axis between $-kv_F/\hbar$ and $kv_F/\hbar$, if we are at zero temperature, we can solve this relation formally:

$$\delta f(\mathbf{k},\mathbf{p},\omega) = \frac{\partial U(\mathbf{k})}{\partial\rho}\frac{\mathbf{k}\cdot\mathbf{p}}{\mathbf{k}\cdot\mathbf{p}+m\hbar\omega}\frac{\partial f_0}{\partial\varepsilon}\delta\rho(\mathbf{k},\omega)\;. \tag{5}$$

The dispersion relation is then obtained by expressing the self-consistency condition $\rho(\mathbf{k}) = \int d\mathbf{p}\; f(\mathbf{k},\mathbf{p})/h^3$. This integration over the momentum $\mathbf{p}$ yields the following condition on $E_k = \hbar\omega$:

$$\left(\frac{\partial U(\mathbf{k})}{\partial\rho}\right)^{-1} = \int \frac{d\mathbf{p}}{h^3}\,\frac{\partial f_0}{\partial p^2}\,\frac{2\;\mathbf{p}\cdot\mathbf{k}}{E_k + \mathbf{p}\cdot\mathbf{k}/m} \equiv G(\omega). \tag{6}$$

Since the integral G is invariant under the transformation $\mathbf{p}\to-\mathbf{p}$, the solutions of Eq. (6) always appear by pairs ω and $-\omega$. Moreover, we observe that $[G(\omega)]^* = G(\omega^*)$ and since, to verify Eq. (6) we need $G(\omega)$ to be real, it comes out that, if ω is a solution, ω^* is a solution too.

B. Zero temperature

Let us solve first the previous equation at zero temperature ($T = 0$). After solving the integral of Eq. (6), we obtain the following equation:

$$\frac{s}{2}\ln\left(\frac{s+1}{s-1}\right) = 1 + \frac{1}{F_0}, \tag{7}$$

where $s = -E_k/k\,v_F$ and $F_0(\mathbf{k}) = [\partial U(\mathbf{k})/\partial\rho]/(2\varepsilon_F/3\rho)$, $\varepsilon_F = mv_F^2/2$ being the Fermi energy and ρ the density. In Eq. (7) the logarithm is defined in the complex plane with a cut on the negative real axis. It should be noticed that Eq. (7) is completely analogous to the dispersion relation obtained using the Fermi liquid formalism [8,9].

As already discussed, Eq. (7) is valid in the whole complex plane except for s on the real axis between -1 and $+1$. It can be seen that Eq. (7) has only real or purely imaginary solutions and that they always come by pairs (s and $-s$). Real solutions for s^2 greater than 1 are found for positive F_0. In this case all modes are stable, their phase velocity is larger than the Fermi velocity, and they are neither damped nor amplified (they only propagate). The corresponding energies are shown in Fig. 1.

If we look for imaginary solutions of the dispersion relation, $s = i\gamma$, we come to the relation

$$1 + 1/F_0 = \gamma\;\arctan(1/\gamma). \tag{8}$$

This equation can be solved if $F_0 < -1$, i.e., if the following condition is fulfilled:

$$\partial U(k)/\partial\rho + \frac{2}{3}\varepsilon_F/\rho < 0. \tag{9}$$

Equation (8) always has two opposite solutions $\pm\gamma$ [9]. Therefore it corresponds to an unstable regime. The corresponding imaginary energies are shown in Fig. 1.

Let us come back to the special case $|E_k| \leq kv_F$. In such a case Eq. (4) cannot be simply inverted in the sense of the functions because of the presence of a zero in the quantity to be inverted. The simplest way to avoid this

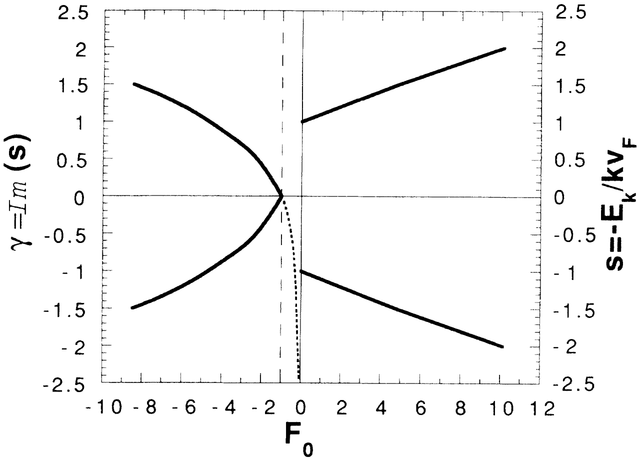

FIG. 1. The roots of the dispersion relation as a function of F_0.

problem is to add a small imaginary part $i\eta$ to the energy and to, later on, take the limit $\eta \longrightarrow 0$. In this case Eq. (6) is still valid if the logarithm of a negative real number α is considered as being $\ln(-\alpha) + i\pi$. Therefore, there are no actual solutions of Eq. (7) with a real s of modulus lower than 1, since the left side has an imaginary part while the right side is just real. We have gotten the known result that the collective mode is dissolved in the noncollective background: The singularity $\omega = -\mathbf{k}\cdot\mathbf{p}/(m\hbar)$ gives rise to a damping of the considered modes, which corresponds to a coherent interaction between the collective mode and those particles which "surf ride" on the crests of the running wave. Since, in equilibrium conditions, there are more particles going slightly slower rather than slightly faster than the wave, there will be an energy transfer from the mode to the individual particles which will result in a damping of this mode. This damping is usually called "Landau damping." It should be remembered that it does not correspond to an actual solution of Eq. (7). However, in order to explain the shape of the response function [9] of the system we can *analytically* continue the dispersion equation, starting from the negative real axis, into the lower half of the complex plane. This is equivalent to consider that the logarithm is cut along the positive real axis. In this case a root will be found on the negative imaginary axis associated to the condition $-1 < F_0 < 0$. Indeed on the imaginary s axis the dispersion condition is now given by

$$\gamma\left(\frac{\pi}{2} - \arctan(\gamma)\right) = 1 + \frac{1}{F_0}. \tag{10}$$

We observe that, since this condition is the same as Eq. (8) in the half plane of the positive imaginary part, the Landau damping branch is the continuation of the unstable branch we discussed previously. The only difference is that, if the interaction F_0 is too attractive ($F_0 < -1$), the modes become unstable, and give rise to the appropriate exponential growing of fluctuations in the system. However, it shall be remembered that in this case the collective modes do exist and that they appear by pairs at $+\omega$ and $-\omega$.

In Fig.1 the roots of Eq. (7) are displayed as a function of F_0, including also the "Landau-damping" contribution $(-1 < F_0 < 0)$.

C. Finite temperature

Let us now study the condition to get unstable collective motions and let us illustrate the results in the low-temperature limit ($T \ll \varepsilon_F$). In such a case we can integrate the dispersion condition (6) which leads to an extension of the instability condition (9):

$$\rho\left(\frac{\partial U(k)}{\partial \rho} + \frac{2}{3}\frac{\varepsilon_F}{\rho} + \frac{\pi^2}{18\varepsilon_F}\frac{T^2}{\rho}\right) < 0. \tag{11}$$

In the limit of long wavelengths, $k \to 0$, or for zero-range interaction the relation (11) reduces to the condition for mechanical instability,

$$\left.\frac{\partial P}{\partial \rho}\right|_T < 0, \tag{12}$$

where P is the pressure.

D. Finite-range interaction, surface energy and instability condition

The dispersion relation (8) predicts an instability time $\tau_k = \hbar/E_k$ proportional to the wavelength $\lambda = \frac{\hbar}{k}$. This is evidently an unphysical result since the smaller the wavelength the faster their growing time. This result is coming from the fact that a zero-range interaction does not introduce any scale.

A physical situation can be recovered if we introduce a finite range for the mean field. This finite range will generate a surface energy which will prevent from getting too small clusters and will introduce a natural cutoff in the unstable lengths. As we will see it will also make the spinodal region k dependent.[1]

The pedagogical simplest way to introduce a finite range mean field is to fold the zero-range one with a function g (such as a Gaussian), the range of this function being selected under the requirement of getting a good surface energy:

$$U \to U \otimes g. \tag{13}$$

In such a case the instability condition simply reads

[1]It is also possible to introduce a gradient term in the potential. This may simulate some quantum effects because in quantum mechanics the kinetic energy corresponds to a Laplacian term and therefore we get a term of the type $\frac{-\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}$ in addition to the potential. Strong surface gradients will therefore introduce some surface energy. In particular, in the Fourier representation, the Laplacian term introduces a term proportional to k^2, so that F_0 can still be formally represented as in Eq. (15). However, most of the surface energy is due to the finite range of the interaction.

$$\left(\frac{\partial U}{\partial \rho} g(k) + \frac{2}{3}\frac{\varepsilon}{\rho} + \frac{\pi^2}{18\varepsilon_F}\frac{T^2}{\rho}\right) < 0. \tag{14}$$

This condition and, therefore, the growing time will depend on k. In fact, it means that we have a k dependent F_0 parameter:

$$F_0 \longrightarrow g(k)\ F_0. \tag{15}$$

Therefore, for fixed conditions of the system (ρ and T), inside the static spinodal region (related to $k = 0$), there will be a limiting k, k_{lim}, corresponding to the condition $-g(k_{\text{lim}})\ F_0 = 1$, above which the dispersion relation is no more unstable. Moreover, between 0 and k_{lim}, there will be a maximum of the instability parameter γ which will define the most unstable mode $k_{\max}$ and its corresponding maximum value of the imaginary frequency, or minimum value of the instability time τ_k.

Figure 2 displays the contour of the spinodal region for infinite nuclear matter for $k = 0$ and for different modes corresponding to different values of k. In this calculation we have used a folding function $g \alpha \exp(-r^2/2a^2)$. A Skyrme-like parametrization has been considered for the mean-field potential with $A = -356$ MeV, $B = 303$ MeV, and $\sigma = 7/6$.

It is possible to observe that this instability region is smaller for smaller values of λ: this can be explained by the fact that oscillations characterized by a small wavelength, shorter than the range a of the forces, cannot be unstable. In this sense the value of λ for which the associated mode is not unstable anymore, for the given values of density and temperature, depends on the parameter a. We use $a = 0.87$ fm in order to reproduce the interaction radius of the nucleon ($r_0 = 1.2$ fm) [5]. It should be noticed that this value of a gives a surface energy close to the liquid drop value.

Let us consider for instance the mode corresponding to $\lambda = 10$ fm. This mode results to be approximately the most unstable mode as soon as the system lies deeply enough inside the spinodal region. Therefore the most unstable modes happen to be of a rather short wavelength and so in realistic collisions the resulting partitions will favor small nuclei such as Ne.

From the dispersion relation, in the Vlasov case, it is possible to calculate the characteristic time τ responsible for the growing of fluctuations. In Fig. 3 we represent the isocontour curves for the time τ in the (ρ, T) plane. The curve corresponding to $\tau = \infty$ and $\lambda = 10$ fm is the limit of the instability region for the considered value of λ (the dashed line corresponds to $\tau = \infty$, $\lambda = \infty$).

The time τ is related to the "degree of instability" of the system, since it is smaller when the system is more unstable. One can see that when the mode starts to be unstable its instability time is infinite. Therefore it is actually dynamically stable. In fact, in the realistic case of a nuclear reaction, a given mode can be considered as dynamically unstable only when its instability time becomes shorter than a characteristic time for the global evolution of the system. If we consider that the good time scales in nucleus-nucleus collisions are around 50 fm/c, we realize that the dynamically unstable region is much smaller than the actual static spinodal one.

E. Collisional damping

The introduction of the collision term in the Vlasov equation mainly adds a damping (i.e., a positive imaginary part) to the frequency:

$$1/\tau_k \longrightarrow 1/\tau_k - 1/\tau_{\text{relax}}. \tag{16}$$

This can be more clearly seen in the relaxation-time approximation, where $\bar{I}[f]$ is replaced by $-\delta f/\tau_{\text{relax}}$, so that the average effect of the collisions is to change the frequencies $\omega_{\mathbf{k}}^{\pm}$, which are the opposite imaginary solutions of Eq. (6), into $\omega_{\mathbf{k}}^{\pm} + i/\tau_{\text{relax}}$. When the system is very unstable, this correction is not very important and the growing of fluctuations is essentially dominated by the unstable mean field. Therefore the inclusion of the collision term does not change the main conclusions of the previous section.

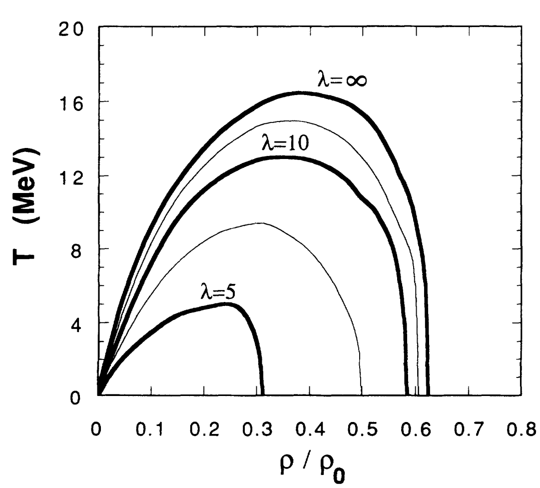

FIG. 2. Contours of the spinodal region for different values of the parameter $\lambda = \frac{\hbar}{k}$.

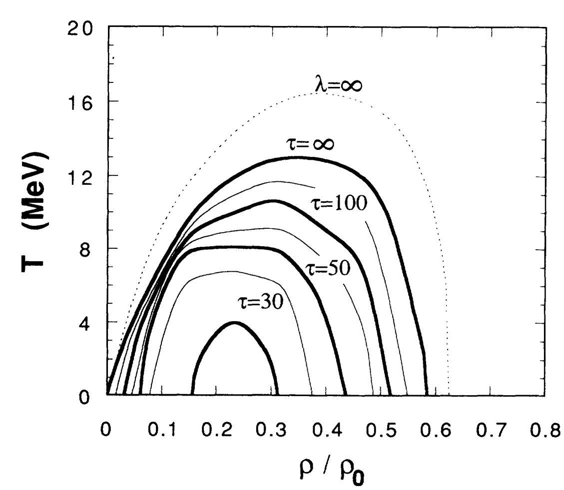

FIG. 3. Isocontour curves of the growing time τ, in the (ρ, T) plane, for $\lambda = 10$ fm. The dashed line defines the spinodal region for $\lambda = \infty$.

III. FLUCTUATIONS AND DIFFUSION COEFFICIENT

Up to now we have studied the instabilities and spontaneous symmetry breaking in the framework of mean-field approximations eventually complemented by the average of a Boltzmann-like collision term. This means that we have projected the many-body dynamics onto a reduced many fold retaining only a small part of the dynamical information: a one-body density corresponds to an entire ensemble of many-body states. This lack of full information about the state of the system, both initially and in the course of time, can be taken into account by adding a stochastic term in the effective equation of motion for the one-body variables. In this case one is forced to consider not only one single trajectory, but an ensemble of trajectories.

A. Boltzmann-Langevin approach

In the Boltzmann-Langevin approximation [3] the stochastic nature of the individual collisions is considered to be the dominant source of fluctuations, the collision integral playing the role of the fluctuating Langevin force. This approach considers an ensemble of N identical systems, labeled by $n = 1, ..., N$. Each one is described by its reduced one-body phase-space density $f^{(n)}(\mathbf{r}, \mathbf{p}, t)$. The time evolution of each of these functions is given by an equation of the form

$$\frac{\partial f^{(n)}}{\partial t} + \left\{ H^{(n)}, f^{(n)} \right\} = \bar{I}[f^{(n)}] + \delta I^{(n)}[f^{(n)}], \quad (17)$$

where the left-hand side describes the Vlasov evolution for $f^{(n)}$ while the term on the right-hand side represents the effect of the stochastic two-body collisions. In this equation the stochastic collision integral was separated into two pieces: the average collision term usually considered ($\bar{I}$) and a fluctuating part (δI). The collision term $\bar{I}$ can be obtained from the mean number of transitions $d\nu$, in which nucleons are scattered from two phase-space elements around the locations s_1, s_2 [$s = (\mathbf{r}, \mathbf{p})$] into two other phase-space elements around $s_{1'}$, $s_{2'}$:

$$d\nu_{1,2;1',2'} = f_1 f_2 \bar{f}'_1 \bar{f}'_2 \delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(\mathbf{r}_{1'} - \mathbf{r}_{2'})\delta(\mathbf{r}_1 - \mathbf{r}_{1'}) \times\omega(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_{1'}, \mathbf{p}_{2'}) ds_1 ds_2 ds_{1'} ds_{2'} \; . \quad (18)$$

Here $f_i \equiv f(s_i)$ is the phase-space occupancy at one of the initial locations and $\bar{f}'_i = 1 - f(s_{i'})$ is the Pauli blocking factor expressing the availability at the final locations. A phase-space element is represented by $ds = d\mathbf{r}d\mathbf{p}/h^3$. The elementary transition rate $\omega(\mathbf{p}_1, \mathbf{p}_2; \mathbf{p}_{1'}, \mathbf{p}_{2'})$ incorporates energy-momentum conservation and can be related to the in-medium nucleon-nucleon cross section.

The fluctuating term δI is defined through its correlations which is assumed to be of Markovian type,

$$\langle \delta I(\mathbf{r}, \mathbf{p}, t) \delta I(\mathbf{r}', \mathbf{p}', t') \rangle = 2D(\mathbf{r}, \mathbf{p}; \mathbf{r}', \mathbf{p}', t)\delta(t - t') \; , \quad (19)$$

where D, the so-called diffusion coefficient, is related to the elementary collision number $d\nu$.

B. Small amplitude fluctuations

Considering an unstable situation, the stochastic term δI will test the spontaneous symmetry breaking. Let us illustrate this point by studying the linear response in the Boltzmann-Langevin approximation. If δf_ν are the eigenmodes of the density fluctuations, we can write

$$\delta f = \sum_\nu A_\nu \delta f_\nu . \quad (20)$$

If we define the fluctuation $\sigma^2_{\nu,\mu} = \langle A_\nu A^*_\mu \rangle$, it is possible to demonstrate that the time evolution of this quantity follows the so-called Lalime equation [10]:

$$\frac{d}{dt}\sigma_{\nu\mu} = -i\omega_{\nu\mu}\sigma_{\nu\mu} + 2D_{\nu\mu}, \quad (21)$$

where the frequency $\omega_{\nu\mu} = \omega_\nu - \omega^*_\mu$ is the difference of the eigenvalues associated to the considered mode ν and μ while the diffusion coefficient $D_{\nu\mu}$ represents the fluctuation source term.

It was shown in Ref. [10] that the diffusion coefficient can be related to the elementary collision rate as follows:

$$D_{\nu\mu} = \sum_{\nu'\mu'} O_{\nu\nu'} \; \Delta_{\nu'\mu'} O_{\mu'\mu} \; , \quad (22)$$

where we have introduced the inverse of the overlap matrix of the eigenmodes,

$$O^{-1}_{\nu\mu} = \int \frac{d\mathbf{r}d\mathbf{p}}{h^3} f_\nu(\mathbf{r}, \mathbf{p})^* f_\mu(\mathbf{r}, \mathbf{p}) \; , \quad (23)$$

and the expectation value $\Delta_{\nu\mu}$ is related to the elementary collision rate:

$$\Delta_{\nu\mu} = \frac{1}{2} \int d\nu_{1,2;1',2'} [f_\nu(1)^* f_\mu(1) + f_\nu(1')^* f_\mu(1') + f_\nu(1)^* f_\mu(2) + f_\nu(1')^* f_\mu(2') - 2 f_\nu(1)^* f_\mu(1') - 2 f_\nu(1')^* f_\mu(1)]. \quad (24)$$

C. Infinite nuclear matter

For example, in the case of an unstable mode of the infinite nuclear matter we have defined $\tau_\nu = i/\omega_\nu$, so that the solution of the previous equation for the diagonal terms $\sigma^2_\nu \equiv \sigma_{\nu\nu}$ reads

$$\sigma^2_\nu(t) = D_\nu \tau_\nu (e^{\frac{2t}{\tau_\nu}} - 1) + \sigma^2_{\nu_0} e^{\frac{2t}{\tau_\nu}} , \quad (25)$$

where $\sigma^2_{\nu_0}$ is the initial value of σ^2_ν.

Using the results of the previous section we can label the eigenmodes by $\mathbf{k}$ and a sign, since, for each $\mathbf{k}$, two solutions of opposite sign were found. By convention we will associate the positive solution either with the positive frequency in the stable regime or with the exponentially growing solution in the unstable regime. In this case Eq. (20) reads

$$\delta f = \sum_{\mathbf{k}} (A^+_{\mathbf{k}} \delta f^+_{\mathbf{k}} + A^-_{\mathbf{k}} \delta f^-_{\mathbf{k}}). \tag{26}$$

For instance, in the unstable regime, the amplitude $A^+_{\mathbf{k}}$ increases following an exponential behavior determined by the characteristic time τ_k, described in the previous section. If we call D^+_k the diffusion coefficient related to the k^+ mode, the corresponding variance $\sigma^2_\nu \equiv \sigma^+_{\mathbf{k}} = \langle A^+_{\mathbf{k}} A^{+*}_{\mathbf{k}} \rangle$ follows the equation $\frac{\partial}{\partial t}\sigma^+_{\mathbf{k}} = \frac{2}{\tau_k}\sigma^+_{\mathbf{k}} + 2D^+_k$ which is analogous to Eq. (21).

Let us now consider the density fluctuation $\delta\rho(\mathbf{r}, t)$ in $\mathbf{r}$ and let us define the Fourier transform of the fluctuations of the density, $F(k,t) = \int d\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}/\hbar} \delta\rho(\mathbf{r},t)$; we find

$$\begin{aligned} \sigma(\mathbf{k}, t) &= \langle |F(k,t)|^2 \rangle \\ &= \int\int d\mathbf{r}\, d\mathbf{r}'\ e^{-i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')/\hbar} \langle \delta\rho(\mathbf{r},t)\delta\rho(\mathbf{r}',t)\ \rangle. \end{aligned} \tag{27}$$

This Fourier transform of the fluctuations in $\mathbf{r}$ space can be easyly developed onto the eigenmodes associated with a given $\mathbf{k}$, leading to the following relation [11]:

$$\begin{aligned} \sigma(\mathbf{k}, t) = {} & D^+_k \tau_k [e^{2t/\tau_k} - 1] + \sigma^+_{\mathbf{k},0} e^{2t/\tau_k} \\ & + D^-_k \tau_k [1 - e^{-2t/\tau_k}] + \sigma^-_{\mathbf{k},0} e^{-2t/\tau_k} \\ & -4\ D^\pm_k t + \sigma^\pm_{\mathbf{k},0}, \end{aligned} \tag{28}$$

where $\sigma_{\mathbf{k},0} = \sigma^+_{\mathbf{k},0} + \sigma^-_{\mathbf{k},0} + \sigma^\pm_{\mathbf{k},0}$ represents the initial value of the variance $\sigma(\mathbf{k})$. From this equation one can see that the dynamics of the fluctuations in $\mathbf{r}$ is fully determined, in the linear regime, by the instability time and the diffusion coefficients which can be analytically derived using Eq. (22). It should be noticed that for a time larger than the instability time the behavior of $\sigma(\mathbf{k}, t)$ can be approximated by

$$\sigma(\mathbf{k}, t) = D^+_k \tau_k [e^{2t/\tau_k} - 1] + \sigma_{\mathbf{k}^+,0} e^{2t/\tau_k} \tag{29}$$

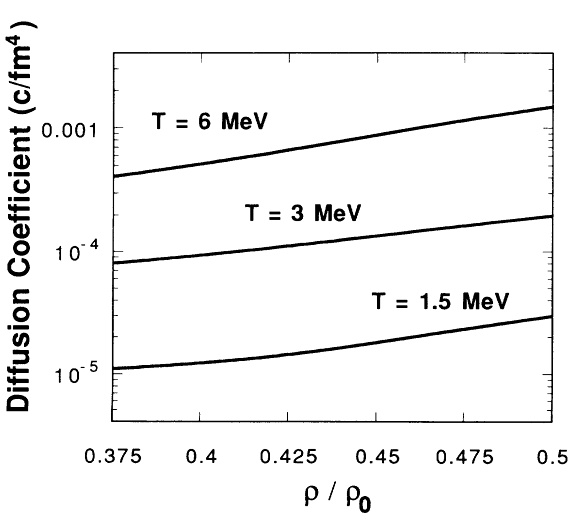

FIG. 4. The diffusion coefficient D as a function of the density, for three different values of the temperature. We consider $\lambda = 10$ fm.

until a chaotic regime is reached and the linear response theory does not apply anymore. In Fig. 4 the coefficient D^+_k, for the mode $\lambda = 10$ fm, is represented corresponding to different values of temperature and density inside the spinodal region (see Fig. 2). We have used the nucleon-nucleon cross section $\sigma_{NN} = 4.1$ fm^2. Since the diffusion coefficient D_k is directly related to the number of collisions suffered by the system, it increases both with temperature (like T^2 at small temperature) and density (like ρ).

IV. SIMPLIFIED STOCHASTIC MEAN FIELD

Calculations of the BL type [6] contain the fluctuation term coming from the stochastic part of the collision integral, which has been used in the previous derivation of the coefficients D_k. Unfortunately, realistic simulations in a real three-dimensional space are not feasible. Nevertheless, since we have an analytical estimation of the amount of fluctuations present in the system, we can consider the possibility to replace this correct value of the fluctuation rate by a simple noise. Indeed, we have seen that the dynamics is dominated by a few unstable modes. Therefore we may think to introduce a simple noise, in the mean-field approaches, tuned in such a way to reproduce the amount of fluctuations in the unstable modes predicted by the Langevin theory developed above.

A. Fluctuating mean field

In particular, we may introduce a noise in the one-body Hamiltonian. In Ref. [12] such a possibility was suggested and the special case of a noise commuting with the Hartree-Fock Hamiltonian was studied. It was shown [13] that this method, called iterative-time smoothing, yields very promising results as far as thermalization is concerned. One may even think to extend the previous approach by adding random matrices to the one-body Hamiltonian, building some bridge to the quantum chaos problem. These studies are now in progress both from the quantum mechanical point of view than from the semiclassical approximations.

B. Fluctuations in the test particle methods

A simpler approach has already been tested, namely to use the noise associated to a finite number of test particles, in ordinary semiclassical simulations, as a fluctuation source [5]. In this case, if $N_{\rm test}$ is the number of test particles per nucleon, the equation for $\sigma(\mathbf{k})$ reads

$$\begin{aligned} \sigma(\mathbf{k}) = {} & (D^+_k \tau_k / N_{\rm test})\ [e^{2t/\tau_k} - 1] + \sigma^+_{\mathbf{k},0} e^{2t\tau_k} \\ & + (D^-_k \tau_k / N_{\rm test})\ [1 - e^{-2t/\tau_k}] + \sigma^-_{\mathbf{k},0} e^{-2t\tau_k} \\ & -4\ D^\pm_k / N_{\rm test}\ + \sigma^\pm_{\mathbf{k},0} \end{aligned} \tag{30}$$

because the stochasticity coming from the Monte Carlo estimation of the collision integral corresponds to the diffusion coefficient of the BL approach reduced by a factor $N_{\rm test}$, while an initial noise σ_0, of statistical nature, is present. However, it should be noticed that, since test particles are actually propagated in the simulation, the sources of fluctuations may contain more information than the Boltzmann-Langevin term divided by N_{test}, because a gas of test particles may contain higher order correlations of a classical type.

The statistical noise may be easily calculated starting from the statistical fluctuations of the number of test particles around a mean value $n\ N_{\text{test}}$: $\delta\rho/\rho = 1/\sqrt{n\ N_{\text{test}}}$, where n is the number of nucleons contained in the cells where the density is calculated. This noise is big in comparison with the fluctuations coming from the diffusion coefficient. In such a case, the growing of the fluctuations will be soon dominated by an exponential increase determined by the characteristic time τ_k and by the coefficient $(\sigma_{\mathbf{k},0} + D_k\tau_k/N_{\text{test}})$, while in the calculations of the BL type [6] this coefficient is equal to $D_k\tau_k$, since $\sigma_{\mathbf{k},0} = 0$. On the other hand, since in the test particle case $\sigma_{\mathbf{k},0}$ scales like $1/N_{\text{test}}$, it comes out possible to tune the number of test particles in order to mimic the BL dynamics predicted for the most unstable modes, i.e., in order to fulfill the equation $\sigma_{\mathbf{k},0} + D_k\tau_k/N_{\text{test}} = D_k\tau_k$ [5].

In Fig. 5 this number is given as a function of the density for different temperatures. As the diffusion coefficient D_k increases with temperature and density the plotted number of test particles correspondingly decreases since a larger noise is associated to a smaller number of test particles. Of course, in order to have a correct description of the mean dynamics, this number must not be too small ($N_{\text{test}} > 20$). Therefore if the Boltzmann-Langevin term is too large, this method cannot be applied and one must keep a large number of test particles and introduce an additional noise in the dynamical equation.

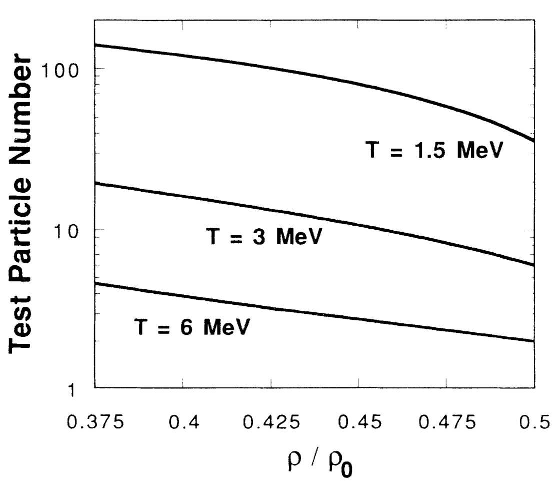

FIG. 5. Test particle number to be used in order to reproduce the physical amount of fluctuations of the BL theory. The figure shows the dependence of this number on the density, for three different values of the temperature T, for $\lambda = 10$ fm.

C. Application to multifragmentation: Discussion

The method presented above may apply to nuclear collisions and, in particular, to multifragmentation. Indeed, if we believe that the nuclear fragmentation is due to the fact that the system enters a volume instability region, which is one of the most probable scenarii, at a given point of the reaction the system will be characterized by the presence of an extended participant region. This region is predicted to be approximately equilibrated (with a given temperature T), its mean density ρ being below the normal density. Following Eq. (22) with these values of ρ and T the right value of fluctuations can be determined at this point for all modes, while Eq. (8) allows us to calculate the associated growing time and, then, the most unstable modes.

We would like to stress here that fluctuations are not important before the system enters an instability region since they are not amplified; in a stable situation, in fact, their amplitude is related to the mean value of the density and it evolves according to the mean density behavior. Even if fluctuations in the compression phase are larger than in a situation corresponding to a volume instability, when the system is still stable, they relax toward the instantaneous equilibrium value during the course of the evolution because of the collisional damping. Indeed, due to the collisional damping, the variance σ_f reaches its equilibrium value more rapidly (by a factor of 2) than f reaches f_0 [4]. Few memories of the previous amplitude will be kept and the actual value of fluctuations will reach the one corresponding to the new reduced density, before getting amplified. In our scenario we do not consider the possibility to enter the spinodal region with some clusters (α particles, for instance) already formed. This would change the mechanism of fragmentation in the sense that the effects of the interaction between these clusters should be taken into account to describe fragment formation [14].

The values of density and temperature that the system presents when it reaches the spinodal region may be used to determine the related fluctuation amplitude and then the number of test particles to use if we want to reproduce it by ordinary semiclassical simulations, as explained in the previous section. Actually also in the Boltzmann-Langevin calculations some fluctuations are already present when the system becomes unstable [σ_0 is not equal to zero in Eq. (25)]. This underestimation of the right value of fluctuations introduces just a negligible shift in the fragment formation time [15].

Even if these calculations remain of qualitative nature since they contain many approximations (actually the density of the unstable region formed during a nuclear collision is not uniform, the system is not homogenous and surface effects can be very important), they make it possible to get an idea of the pattern followed by the fragmentation. In fact it must be observed that the fragmentation obtained in semiclassical simulations [16] is not just a "numerical" effect but it may contain a physical meaning. Fragments come up because there is an instability in the system and this is a physical effect. The finite sampling noise just allows the system to show up this instability. The use of an adapted number of test particles and then of a controlled noise may make calculations nearer to the reality by introducing the right time scales for the symmetry breaking processes.

D. Validity of the stochastic mean-field approach

All these conclusions have been obtained within stochastic mean-field approaches. However, one may worry about the limits of validity of this approximation at small densities. In fact it must be observed that, even if we are dealing with a strong force, these approaches are justified by the huge quantum zero-point motion of the nucleons, which is much greater than the typical ranges of the nuclear forces (0.5 fm for the hard repulsive core and 1 fm for the nuclear attraction). In fact these arguments are at the basis of the introduction of the shell model and mean-field approximation in nuclear physics [17]. Some correlations and, in particular, pairing effects, which are known to be important, can still be treated in an extended mean-field framework, such as the Hartree-Fock-Bogoliubov treatment.

These considerations are valid also at low density, except in a small region of density and temperature where p-n correlations start to be important. This region can be roughly estimated by $\rho \approx 3/(4\pi r_d^3) \approx \frac{1}{10}\rho_0$ and $T \approx B_d$, where r_d^2 is the deuteron mean square radius (≈ 5 fm^2) and B_d is the deuteron binding energy (≈ 2 MeV). Therefore, outside of this region, the fragmentation of nuclear matter is driven by the instabilities of the mean field and is far away from the aggregation mechanism characteristic of the molecules (for which the zero-point motion is negligible).

Moreover stochastic approaches, considering ensembles of independent trajectories, are always predictive, in the sense that the ensemble behavior is well defined, even in the spinodal region (see Ref. [15] for more details). Moreover it shall be noticed that, at the limit of classical dynamics, with pointlike particles, the exact many-body density matrix can always be represented as a distribution of independent particle densities. Therefore, as far as the stochastic term is well chosen, stochastic mean-field theories may contain exactly all many-body correlations.

V. CONCLUSIONS

We have presented a general method to identify and to study unstable modes in the framework of the linear response theory. For a cubic box of nuclear matter with boundary conditions, we calculate the time τ, characteristic of the exponential growing of fluctuations, in the realistic case of a finite range force, for systems which lie in the spinodal instability region. In the same framework we are able to derive the diffusion coefficient of the Boltzmann-Langevin theory, which is proportional to the amount of fluctuations present in the system.

Realistic calculations in 3D are feasible adopting the prescription of replacing the correct amount of fluctuations by a simple noise. In particular, using the noise associated to the use of a finite number of test particles in ordinary Boltzmann-like calculations, we derive the number of test particles to be used, as a function of the main characteristics, namely the density and the temperature, of the system. This controlled amplitude of fluctuations, that we introduce in the dynamics, allows us to deal with processes in which fluctuations are very important such as, for instance, multifragmentation events.

ACKNOWLEDGMENTS

Many thanks are due to A. Guarnera for very helpful discussions. We warmly thank also M. Belkacem, A. Bonasera, F. Burgio, M. Di Toro, F. Gulminelli, and J. Randrup. We want also to acknowledge the special role played by P. Fox in the maturation of this work.

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