Sum rules in the proton-neutron interacting boson model: Generalized treatment and specific applications

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(Received 9 July 1993)

A detailed study of sum rules for $E0, E2$ and $M1$ and $M3$ transitions within the framework of the proton-neutron interacting boson model (IBM-2) is carried out. Both the non-energy-weighted as well as linear energy-weighted sum rules are derived for rather general IBM-2 Hamiltonians. Special attention is given to the $M1$ sum rule for which a large amount of data exists. We also show how these more general sum rule results reduce to simple answers when considering the limit of U(5), SU(3), and O(6) dynamical symmetries. Finally, a relation between the $M1$ and $M3$ sum rules is pointed out.

PACS number(s): 21.60.Fw, 23.20.Js

I. INTRODUCTION

Sum rules provide useful tools for measuring quantitatively the degree of collectiveness of a given excited state. The controversy about the real nature of a so-called 1^+ "scissors" excitation at low energy, first discovered by Bohle *et al.* in Darmstadt [1], and in particular of its degree of collectivity, originated from the outcome of subsequent experiments using (e,e') and (γ,γ') [2, 3], where considerable fragmentation of low-lying $M1$ strength was observed.

Several microscopic theoretical studies [4–10] aimed at reproducing this fragmentation, and although all finer details could not always be reproduced in a quantitative way, a quite coherent picture was obtained recently starting for the summed strength and its systematics over a wider range of nuclei [11–14]. Also, qualitatively, some properties were observed that indicated a possible collective nature of the $M1$ strength, e.g., its smooth variation with proton (neutron) number Z (N) [15] which was also reached from microscopic calculations [11], as well as a strong correlation with nuclear deformation [6, 13, 14, 16–20]. Moreover, a striking similarity was observed [17, 18] for the rare-earth region in the behavior of the summed low-lying $M1$ strength and the electric quadrupole transition strength to the first excited 2^+ state versus the collective parameter P, introduced by Casten *et al.* [21], which indicates a measure of the interaction energy of the deformation-driving quadrupole proton-neutron force versus the interaction energy of the like nuclear pairing force and hence, is a unifying parameter for describing deformation dependent properties.

The recent observation of these properties of summed magnetic dipole strength stimulated the use of sum-rule techniques in order to study them in more detail. Within the nuclear shell model [22–24] as well as in the interacting boson model (IBM-2) [13, 14, 24, 25], the correlation and saturation properties of both $B(E2;0_1^+ \rightarrow 2_1^+)$ and $\sum_{E_x \leq 4\text{ MeV}} B(M1;0_1^+ \rightarrow 1_f^+)$ were studied extensively starting from a sum-rule approach. These developments caused a revival of the interest in sum-rule calculations in general within the IBM-2. It is the aim of the present paper to describe in detail a number of more technical aspects and to obtain more general results. Then, we derive and discuss a few specific expressions with interesting physical implications. We also study the limits of good F spin and the dynamical symmetries.

II. SUM RULES IN THE IBM-2

The most commonly used sum rules are the non-energy-weighted (NEW) and the linear energy-weighted (EW) sum rules. Although we have studied higher-order EW sum rules as well, we will restrict ourselves here to the former ones, since these results are becoming very complex and lack transparency to allow for an easy physical interpretation.

In calculating the EW and NEW sum rules, we start from the well-known expression [26]

$$\sum_f B(S\lambda;0_1^+ \rightarrow \lambda_f) = \langle 0_1^+|\hat{T}(S\lambda)\hat{T}(S\lambda)|0_1^+\rangle \quad (1)$$

and

$$\sum_f E(\lambda_f^+)B(S\lambda;0_1^+ \rightarrow \lambda_f) = \frac{\hat{\lambda}(-1)^{\lambda+1}}{2}\langle 0_1^+|[[\hat{H},\hat{T}(S\lambda)],\hat{T}(S\lambda)]^{(0)}|0_1^+\rangle \quad , \quad (2)$$

where $\hat{H}$ is the Hamiltonian describing the interacting system, $\hat{T}(S\lambda)$ is a one-body electromagnetic operator of multipolarity λ, and $\hat{\lambda} \equiv (2\lambda+1)^{1/2}$.

These expressions are model independent, but further evaluation of the right-hand side demands model assumptions for the form of the electromagnetic operators and of the Hamiltonian. We have chosen to use the proton-neutron interacting boson model (IBM-2) in order to evaluate the above sum rules. This choice is motivated by the observation that the $M1$ excitations, which were the first cases to be discussed in this context—and more in particular the summed $M1$ strength is equally well described by this model as by the more complex microscopic models [7,11,12] whereas the analytic calculations, although still elaborate, give rise to relatively simple expressions with a straigthforward physical interpretation [13, 14]. For reasons of completeness, we first briefly introduce the model.

A. Basic model IBM-2 assumptions

In the IBM, the valence nucleons (particles or holes) are treated in pairs as $s(L = 0)$ or $d(L = 2)$ bosons, relying on the pairing and collective, quadrupole properties of the nuclear energy spectra. In this way, a substantial truncation of the configuration space, compared to the usual shell-model space, is introduced, which makes the model extremely useful in regions far away from closed shells, such as the well-deformed rare-earth and actinide nuclei. The single-boson states then span a U(6) basis, and the group-theoretical analysis highly simplifies the calculations and even enables analytic treatment in a large number of cases, known as dynamical symmetries.

In the IBM-2, the charge degree of freedom is taken into account, giving rise to proton and neutron bosons. Reduction of the product group $U_\pi(6) \otimes U_\nu(6)$ then gives rise to two classes of states: totally symmetric boson states and mixed-symmetry states, for which the wave function is antisymmetric under the interchange of charge or spatial (+spin) coordinates, only. It is within the latter class that the low-lying 1^+ states are to be found.

An alternative way to include the charge degree of freedom is to introduce an additional quantum number, F spin, which is analogous in its mathematical properties to spin and isospin. The totally symmetric states then correspond to $F = F_{\max} = \frac{N}{2}$ (N is the total boson number), while the mixed-symmetry states have $F_z \leq F < F_{\max}$ [with $F_z = (N_\pi - N_\nu)/2$].

Many review articles [27–30] and several textbooks [31, 32] give a detailed account of the IBM-2 model. Therefore, we will restrict ourselves to a short description of the Hamiltonian and electromagnetic operators used in the calculations.

B. The Hamiltonian

The more general IBM-2 Hamiltonian is given by

$$H = \epsilon_{d_\pi}\hat{n}_{d_\pi} + \hat{n}_{d_\nu} + \kappa_{\pi\pi}\hat{Q}_\pi \cdot \hat{Q}_\nu + \kappa_{\pi\nu}\hat{Q}_\pi \cdot \hat{Q}_\nu + \kappa_{\nu\nu}\hat{Q}_\nu \cdot \hat{Q}_\nu + \lambda \hat{M}_{\pi\nu} \quad , \tag{3}$$

with $\hat{n}_{d_\rho} \equiv (d^\dagger \cdot \tilde{d})_\rho$ ($\rho \equiv \pi, \nu$) the d-boson number operator,

$$\hat{Q}_\rho \equiv (s^\dagger \tilde{d} + d^\dagger s)^{(2)}_\rho + \chi (d^\dagger \tilde{d})^{(2)}_\rho \quad (\rho \equiv \pi, \nu)$$

the quadrupole operator, and

$$\hat{M}_{\pi,\nu} \equiv \xi_2 (d^\dagger_\pi s^\dagger_\nu - d^\dagger_\nu s^\dagger_\pi) \cdot (\tilde{d}_\pi s_\nu - \tilde{d}_\nu s_\pi) - 2 \sum_{k=1,3} \xi_k (d^\dagger_\pi d^\dagger_\nu)^{(k)} \cdot (\tilde{d}_\pi \tilde{d}_\nu)^{(k)} \tag{4}$$

the Majorana term, accounting for the symmetry energy. For the purposes of the present paper, it is more convenient to rewrite the Majorana operator in a recoupled form:

$$\hat{M}_{\pi\nu} = \xi_2 \left(\left[(s^\dagger s)_\nu \cdot (d^\dagger \tilde{d})^{(0)}_\pi + (s^\dagger s)_\pi \cdot (d^\dagger \tilde{d})^{(0)}_\nu \right] \sqrt{5} - (s^\dagger \tilde{d})^{(2)}_\pi \cdot (d^\dagger s)^{(2)}_\nu - (s^\dagger \tilde{d})^{(2)}_\nu \cdot (d^+ s)^{(2)}_\pi \right) + \sum_L \left(\sum_{k=1,3} (-2\xi_k)(2k+1) \begin{Bmatrix} 2 & 2 & L \\ 2 & 2 & k \end{Bmatrix} \right) \times (d^\dagger \tilde{d})^{(L)}_\pi \cdot (d^\dagger \tilde{d})^{(L)}_\nu \quad . \tag{5}$$

It is common to use the parameter choice $\xi_1 = \xi_2 = \xi_3 = 1$, such that the operator reduces to the F-spin invariant form

$$\hat{M}_{\pi\nu} = \left[\frac{\hat{N}}{2} \left(\frac{\hat{N}}{2} + 1 \right) - \hat{F}^2 \right] \quad ,$$

or, alternatively,

$$\hat{M}_{\pi\nu} = \hat{n}_{s_\nu}\hat{n}_{d_\pi} + \hat{n}_{s_\pi}\hat{n}_{d_\nu} - (s^\dagger \tilde{d})^{(2)}_\pi \cdot (d^\dagger s)^{(2)}_\nu - (s^\dagger \tilde{d})^{(2)}_\nu \cdot (d^\dagger s)^{(2)}_\pi + 4 (d^\dagger \tilde{d})^{(0)}_\pi \cdot (d^\dagger \tilde{d})^{(0)}_\nu - \sum_{L=1}^{4} (d^\dagger \tilde{d})^{(L)}_\pi \cdot (d^\dagger \tilde{d})^{L}_\nu \quad . \tag{6}$$

When fitting experimental spectra of vibrational over transitional towards well-deformed rotational nuclei, in the rare-earth region, some anharmonicity terms are included as well [33], but since those have zero or negligible contribution to the sum rules, we do not consider them for the present analytic treatment, in order not to complicate the expressions unnecessarily.

C. Electromagnetic operators

The electromagnetic operators for which we have derived and studied the corresponding sum rules are

$$\hat{T}(E0) = \alpha_\pi \left[(d^\dagger \tilde{d})^{(0)}_\pi + \gamma_\pi (s^\dagger s)_\pi \right] + \alpha_\nu \left[(d^\dagger \tilde{d})^{(0)}_\nu + \gamma_\pi (s^\dagger s)_\nu \right] \quad , \tag{7}$$

$$\hat{T}(M1) = \sqrt{\frac{3}{4\pi}} (g_\pi \hat{L}_\pi + g_\nu \hat{L}_\nu) \quad , \tag{8}$$

with $\hat{L}_\rho \equiv \sqrt{10}(d^\dagger \tilde{d})^{(1)}_\rho$, the angular momentum operator in boson space and g_ρ the boson gyromagnetic factors,

$$\hat{T}(E2) = e_\pi \hat{Q}_\pi + e_\nu \hat{Q}_\nu \quad , \tag{9}$$

with $\hat{Q}_\rho$ the quadrupole operator and e_ρ the effective charge, and

$$\hat{T}(M3) = \sqrt{\frac{35}{8\pi}}\left[\Omega_\pi (d^\dagger\tilde{d})^{(3)}_\pi + \Omega_\nu (d^\dagger\tilde{d})^{(3)}_\nu\right] . \quad (10)$$

One can rewrite the $E0$ operator, making use of the boson number operators $\hat{N} \equiv \hat{n}_s + \hat{n}_d$, in the following way:

$$\begin{aligned}\hat{T}(E0) &= \alpha_\pi\left[\frac{1}{\sqrt{5}}\hat{n}_{d_\pi} + \gamma_\pi \hat{n}_{s_\pi}\right] + \alpha_\nu\left[\frac{1}{\sqrt{5}}\hat{n}_{d_\nu} + \gamma_\nu \hat{n}_{s_\nu}\right] \\ &= \alpha_\pi\left(\frac{1}{\sqrt{5}} - \gamma_\pi\right)\hat{n}_{d_\pi} + \alpha_\nu\left(\frac{1}{\sqrt{5}} - \gamma_\nu\right)\hat{n}_{d_\nu} \\ &\quad + \alpha_\pi\gamma_\pi \hat{N}_\pi + \alpha_\nu\gamma_\nu\hat{N}_\nu . \end{aligned} \quad (11)$$

The last two terms will not contribute to $E0$ transitions and, hence, one finally obtains

$$\hat{T}(E0) = \alpha'_\pi \hat{n}_{d_\pi} + \alpha'_\nu \hat{n}_{d_\nu} . \quad (12)$$

The above allows us to write the $E0$, $M1$, and $M3$ operators using a single form as

$$\hat{T}(S\lambda) = A^\lambda_\pi (d^\dagger\tilde{d})^{(\lambda)}_\pi + A^\lambda_\nu (d^\dagger\tilde{d})^{(\lambda)}_\nu , \quad (13)$$

and calculate a general sum-rule expression for multipolarity λ, which can be extended in a straightforward way to higher multipolarities $\lambda > 3$. Due to its particular form, the $E2$ operator has to be studied separately and will be treated in Sec. V.

III. NON-ENERGY-WEIGHTED SUM RULES FOR $E0$, $M1$, AND $M3$ TRANSITIONS

The calculation of the NEW sum rule is quite straightforward and only in very few specific cases, recoupling can lead towards a simplified expression. One obtains

$$\begin{aligned}\sum_f B(S\lambda; 0^+_1 \to \lambda_f) &= (A^\lambda_\pi)^2 \langle 0^+_1|(d^\dagger\tilde{d})^{(\lambda)}_\pi \cdot (d^\dagger\tilde{d})^{(\lambda)}_\pi|0^+_1\rangle + (A^\lambda_\nu)^2 \langle 0^+_1|(d^\dagger\tilde{d})^{(\lambda)}_\nu \cdot (d^\dagger\tilde{d})^{(\lambda)}_\nu|0^+_1\rangle \\ &\quad + 2A^\lambda_\pi A^\lambda_\nu \langle 0^+_1|(d^\dagger\tilde{d})^{(\lambda)}_\pi \cdot (d^\dagger\tilde{d})^{(\lambda)}_\nu|0^+_1\rangle , \end{aligned} \quad (14)$$

and, using the general recoupling formula

$$\begin{aligned}\left(b^\dagger_{\lambda_1}\tilde{b}_{\lambda_2}\right)^{(\lambda)} \cdot \left(b^\dagger_{\lambda_3}\tilde{b}_{\lambda_4}\right)^{(\lambda)} &= \sum_k (2\lambda+1)\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda \\ \lambda_4 & \lambda_3 & k\end{Bmatrix}\left(b^\dagger_{\lambda_1} b^\dagger_{\lambda_3}\right)^{(k)} \cdot \left(\tilde{b}_{\lambda_2}\tilde{b}_{\lambda_4}\right)^{(k)} \\ &\quad + \left(\frac{2\lambda+1}{2\lambda_1+1}\right)\hat{n}_{\lambda_1}\,\delta_{\lambda_1\lambda_4}\cdot\delta_{\lambda_2\lambda_3} , \end{aligned} \quad (15)$$

with λ_i indicating the charge character (π or ν boson) and angular momentum (s or d boson). This leads to the final expression

$$\begin{aligned}\sum_f B(S\lambda; 0^+_1 \to \lambda_f) &= (A^\lambda_\pi)^2\,\frac{2\lambda+1}{5}\,\langle 0^+_1 \mid \hat{n}_{d_\pi} \mid 0^+_1\rangle + (A^\lambda_\nu)^2\,\frac{2\lambda+1}{5}\,\langle 0^+_1 \mid \hat{n}_{d_\nu} \mid 0^+_1\rangle \\ &\quad + \sum_k (2\lambda+1)\begin{Bmatrix}2 & 2 & \lambda \\ 2 & 2 & k\end{Bmatrix}\Big(\langle 0^+_1 \mid (d^\dagger d^\dagger)^{(k)}_\pi \cdot (\tilde{d}\tilde{d})^{(k)}_\pi|0^+_1\rangle \left(A^\lambda_\pi\right)^2 \\ &\quad + \langle 0^+_1 \mid (d^\dagger d^\dagger)^{(k)}_\nu \cdot (\tilde{d}\tilde{d})^{(k)}_\nu|0^+_1\rangle \left(A^\lambda_\nu\right)^2 + \langle 0^+_1 \mid (d^\dagger_\pi d^\dagger_\nu)^{(k)} \cdot (\tilde{d}_\pi\tilde{d}_\nu)^{(k)}|0^+_1\rangle 2A^\lambda_\pi\cdot A^\lambda_\nu\Big) . \end{aligned} \quad (16)$$

A. The $E0$ sum rule

The NEW sum rule is given by

$$\sum_f B(E0; 0^+_1 \to 0^+_f) = \frac{1}{5}\langle 0^+_1|\left(\alpha'_\pi\hat{n}_{d_\pi} + \alpha'_\nu\hat{n}_{d_\nu}\right)^2|0^+_1\rangle . \quad (17)$$

For an F-spin invariant system, with the ground state a pure $F = F_{\max} = (N_\pi + N_\nu)/2$ totally symmetric boson state, we can use the relationship

$$\langle F_{\max}, F_z|\hat{O}_\pi N_\nu - \hat{O}_\nu N_\pi|F_{\max}, F_z\rangle = 0 , \quad (18)$$

where $\hat{O}_\rho$ is a one-body operator acting in the $\pi(\nu)$ space and obtain [27]

$$\sum_f B(E0; 0^+_1 \to 0^+_f) = \frac{1}{5}\left(\alpha'_\pi\,\frac{N_\pi}{N} + \alpha'_\nu\,\frac{N_\nu}{N}\right)^2 \times \langle 0^+_1|\hat{n}^2_d|0^+_1\rangle \quad (19)$$

with $\hat{n}_d \equiv \hat{n}_{d_\pi} + \hat{n}_{d_\nu}$.

B. $M1$ sum rule

The NEW sum rule has been worked out recently by Ginocchio [34] in the limit of good F spin. He thereby obtained the result

$$\sum_f B(M1;0_1^+ \rightarrow 1_f^+) = \frac{9}{4\pi}(g_\pi - g_\nu)^2 \frac{P}{N-1} \langle 0_1^+|\hat{n}_d|0_1^+\rangle \quad . \tag{20}$$

Calculating the NEW sum rule, in the more general case, we obtain

$$\begin{aligned}\sum_f B(M1;0_1^+ \rightarrow 1_f^+) &= \frac{3}{4\pi}\langle 0_1^+|\left(g_\pi \hat{L}_\pi + g_\nu \hat{L}_\nu\right)^2|0_1^+\rangle \\ &= \frac{30}{4\pi}\Big[g_\pi^2 \langle 0_1^+|(d^\dagger\tilde{d})^{(1)}_\pi \cdot (d^\dagger\tilde{d})^{(1)}_\pi|0_1^+\rangle + g_\nu^2 \langle 0_1^+|(d^\dagger\tilde{d})^{(1)}_\nu \cdot (d^\dagger\tilde{d})^{(1)}_\nu|0_1^+\rangle \\ &\quad +2g_\pi g_\nu \langle 0_1^+|(d^\dagger\tilde{d})^{(1)}_\pi \cdot (d^\dagger\tilde{d})^{(1)}_\nu|0_1^+\rangle\Big] \quad .\end{aligned} \tag{21}$$

Making use of the recoupling formula (15), we can extract an $\hat{n}_d$ dependence as follows:

$$\sum_f B(M1;0_1^+ \rightarrow 1_f^+) = \frac{9}{2\pi}\left[g_\pi^2 \langle 0_1^+|\hat{n}_{d_\pi}|0_1^+\rangle + g_\nu^2 \langle 0_1^+|\hat{n}_{d_\nu}|0_1^+\rangle\right] + \frac{45}{2\pi}\left[g_\pi^2 \kappa_{\pi\pi} + g_\nu^2 \kappa_{\nu\nu} + 2g_\pi g_\nu \kappa_{\pi\nu}\right] \quad , \tag{22}$$

with

$$\begin{aligned}\kappa_{\rho\rho'} &= \frac{1}{30}\langle 0_1^+|\hat{L}_\rho \cdot \hat{L}_{\rho'}|0_1^+\rangle \\ &\quad -\frac{1}{5}\langle 0_1^+|\hat{n}_{d_\rho}|0_1^+\rangle \delta_{\rho\rho'} (\rho,\rho' = \pi,\nu) \\ &= \sum_k \begin{Bmatrix} 2 & 2 & k \\ 2 & 2 & 1 \end{Bmatrix} \langle 0_1^+|\left(d_\rho^\dagger d_{\rho'}^\dagger\right)^{(k)} \cdot \left(\tilde{d}_\rho \tilde{d}_{\rho'}\right)^{(k)}|0_1^+\rangle.\end{aligned} \tag{23}$$

It is now also possible to rewrite the $M1$ operator as follows:

$$\begin{aligned}\hat{T}(M1) = \sqrt{\frac{3}{16\pi}}\Big((g_\pi + g_\nu)(\hat{L}_\pi + \hat{L}_\nu) \\ +(g_\pi - g_\nu)(\hat{L}_\pi - \hat{L}_\nu)\Big)\end{aligned} \tag{24}$$

or

$$\begin{aligned}\hat{T}(M1) = \sqrt{\frac{3}{4\pi}}\Big(\frac{g_\pi N_\pi + g_\nu N_\nu}{N}(\hat{L}_\pi + \hat{L}_\nu) \\ + \frac{g_\pi - g_\nu}{N}(N_\nu \hat{L}_\pi - N_\pi \hat{L}_\nu)\Big) \quad .\end{aligned} \tag{25}$$

The total angular momentum operator $\hat{L} \equiv \hat{L}_\pi + \hat{L}_\nu$ gives zero contribution to the transition matrix element, so one obtains for the NEW sum rule the following expressions, respectively:

$$\sum_f B(M1;0_1^+ \rightarrow 1_f^+) = \frac{9}{8\pi}(g_\pi - g_\nu)^2 \left(\langle 0_1^+|\hat{n}_{d_\pi} + \hat{n}_{d_\nu}|0_1^+\rangle\right) + \frac{45}{8\pi}(g_\pi - g_\nu)^2 \left(\kappa_{\pi\pi} + \kappa_{\nu\nu} - 2\kappa_{\pi\nu}\right) \quad , \tag{26}$$

$$\begin{aligned}\sum_f B(M1;0_1^+ \rightarrow 1_f^+) &= \frac{9}{2\pi}\frac{(g_\pi - g_\nu)^2}{N^2}\left[N_\nu^2 \langle 0_1^+|\hat{n}_{d_\pi}|0_1^+\rangle + N_\pi^2 \langle 0_1^+|\hat{n}_{d_\nu}|0_1^+\rangle\right] \\ &\quad +\frac{45}{2\pi}\frac{(g_\pi - g_\nu)^2}{N^2}\left[N_\nu^2 \kappa_{\pi\pi} + N_\pi^2 \kappa_{\nu\nu} - 2N_\pi N_\nu \kappa_{\pi\nu}\right] \quad .\end{aligned} \tag{27}$$

Comparing expressions (22), (26), and (27), one finds, equating terms in g_π^2, g_ν^2, and $g_\pi g_\nu$ the interesting results

$$\begin{aligned}\kappa_{\pi\pi} &= -\frac{1}{5}\langle 0_1^+|\hat{n}_{d_\pi}|0_1^+\rangle - \kappa_{\pi\nu}, \\ \kappa_{\nu\nu} &= -\frac{1}{5}\langle 0_1^+|\hat{n}_{d_\nu}|0_1^+\rangle - \kappa_{\pi\nu} \quad ,\end{aligned} \tag{28}$$

from which we obtain the quite simplified expression for the NEW sum rule as

$$\begin{aligned}\sum_f B(M1;0_1^+ \rightarrow 1_f^+) &= -\frac{45}{2\pi}(g_\pi - g_\nu)^2 \kappa_{\pi\nu} \\ &= -\frac{3}{4\pi}(g_\pi - g_\nu)^2 \langle 0_1^+|\hat{L}_\pi \cdot \hat{L}_\nu|0_1^+\rangle \quad ,\end{aligned} \tag{29}$$

or, alternatively, using Eqs. (23) and (28),

$$\sum_f B(M1;0_1^+ \to 1_f^+) = \frac{3}{4\pi}(g_\pi - g_\nu)^2 \langle 0_1^+|\hat{L}_\rho \cdot \hat{L}_\rho|0_1^+\rangle \quad (\rho = \pi,\nu) \quad . \tag{30}$$

Hence, the summed $M1$ strength becomes proportional to the expectation value of the magnitude of the $\pi(\nu)$ total angular momentum.

In order to compare this general result with the results obtained by Ginnocchio [34], we have to assume good F spin. Then, we can apply relation (18) to Eq. (27) and obtain the result

$$\begin{aligned}\sum_f B(M1;0_1^+ \to 1_f^+) &= \frac{9}{2\pi}(g_\pi - g_\nu)^2 \left(\frac{N_\nu^2 N_\pi}{N^3} + \frac{N_\pi^2 N_\nu}{N^3}\right) \langle 0_1^+|\hat{n}_d|0_1^+\rangle \\ &\quad + \frac{45}{2\pi}\frac{(g_\pi - g_\nu)^2}{N^2}\left(N_\nu^2 \kappa_{\pi\pi} + N_\pi^2 \kappa_{\nu\nu} - 2N_\pi N_\nu\, \kappa_{\pi\nu}\right) \\ &= \frac{9}{2\pi}(g_\pi - g_\nu)^2 \frac{N_\pi N_\nu}{N^2}\langle 0_1^+|\hat{n}_d|0_1^+\rangle + \cdots \quad .\end{aligned} \tag{31}$$

Comparing this result with Ginnochio's sum rule, we obtain other equalities in the $\kappa_{\rho\rho'}$ values, i.e.,

$$\begin{aligned}\kappa_{\pi\nu} &= -\frac{1}{5}\frac{N_\pi N_\nu}{N(N-1)}\langle 0_1^+|\hat{n}_d|0_1^+\rangle \quad , \\ \kappa_{\rho\rho} &= -\frac{1}{5}\frac{N_\rho(N_\rho - 1)}{N(N-1)}\langle 0_1^+|\hat{n}_d|0_1^+\rangle \quad (\rho \equiv \pi,\nu) \quad ,\end{aligned} \tag{32}$$

and finally, a relationship between the expectation value of the angular momentum operator and the d-boson number operator results

$$\begin{aligned}\langle 0_1^+|\hat{L}_\pi \cdot \hat{L}_\pi|0_1^+\rangle &= \langle 0_1^+|\hat{L}_\nu \cdot \hat{L}_\nu|0_1^+\rangle \\ &= -\langle 0_1^+|\hat{L}_\pi \cdot \hat{L}_\nu|0_1^+\rangle \\ &= \frac{6N_\pi N_\nu}{N(N-1)}\langle 0_1^+|\hat{n}_d|0_1^+\rangle \quad .\end{aligned} \tag{33}$$

Since for realistic cases, e.g., the Nd, Sm, and Gd nuclei, the Hamiltonian is not F-spin invariant [33], we investigate the sensitivity of the sum-rule result to the F-spin purity of the states. For realistic choices of the parameters, for this series of isotopes [33], we carried out numerical IBM-2 calculations using the program NPBOS [35–37] as a function of the F-spin purity in the ground state. Inspecting the results of Fig. 1, although we observe that the F-spin purity of the ground state remains very high, the deviation of the summed $M1$ strength from the corresponding pure F-spin limit increases rapidly. Still, the proportionality of the $\kappa_{\rho\rho'}$ term with $\langle 0_1^+|\hat{n}_d|0_1^+\rangle$ in the limit of good F spin [Eqs. (32)], suggests that this deviation may be resolved by a scaling factor. Indeed, plotting the right-hand side of Eq. (20) versus the left-hand side, we obtain a straight line, however, with a proportionality factor of 0.92 instead of 1 [see Fig. 2(a)]. Therefore, it is physically meaningful to *use* the summed $B(M1)$ strength from the ground state as a measure for the expectation value of the d-boson number, which is intimately linked with the nuclear mean-square radius in the ground state [24]. Indeed, the experimental finding that the radius increases more rapidly compared to the liquid-drop model prediction when approaching midshell, suggests a dependence on nuclear deformation [38] which, in the IBM language, is connected with the d-boson number expectation value [34, 39].

When plotting the excess of the experimental value of the mean-square radius over the liquid-drop model predictions versus $\langle 0_1^+|\hat{n}_d|0_1^+\rangle$ [Figs. 2(a) and 2(b)], we do not find a linear relationship though. Starting from Eq. (20), the $\hat{n}_d$ matrix element is connected to the nuclear radius expectation value $\langle r^2\rangle$ and one can thus derive the relationship

$$\sum_f B\left(M1;0_1^+ \to 1_f^+\right) = \frac{9}{4\pi}(g_\pi - g_\nu)^2 \frac{P}{N-1}\frac{1}{\alpha'} \times \left[\langle r^2\rangle - \gamma' N\right] \tag{34}$$

(with $\gamma' = \alpha\gamma$). For the variation over rather small spans

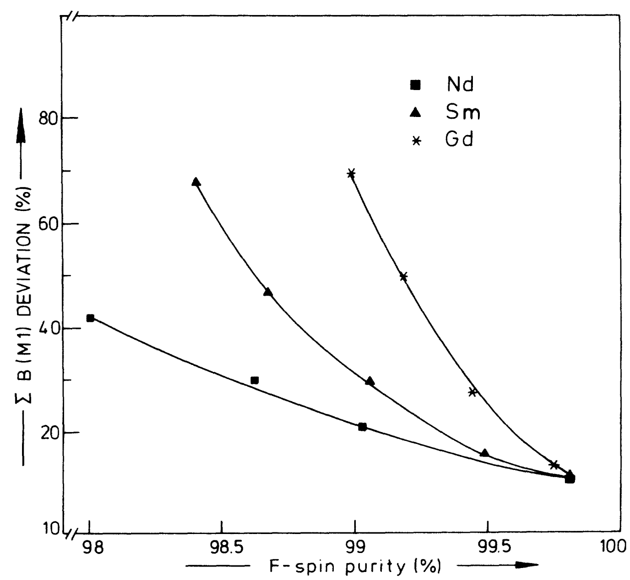

FIG. 1. The procentual deviation between the calculated NEW $M1$ sum rule and the summed $M1$ strength constraining to good F spin for the 0^+ ground state, as a function of the F-spin purity in the 0^+ ground state. The IBM-2 Hamiltonian parameters for Nd, Sm, and Gd nuclei are taken from Scholten [33].

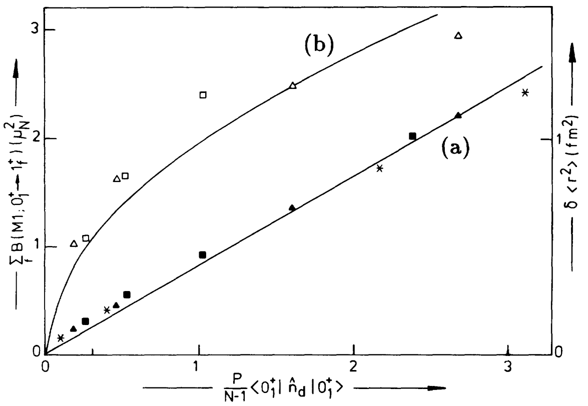

FIG. 2. (a) Relation between the theoretical summed $M1$ strength (NEW) (in units μ_N^2) and the expectation value of $\frac{P}{N-1}\langle 0_1^+|\hat{n}_d|0_1^+\rangle$ for the Nd (■), Sm (▲), and Gd (*) nuclei and this for the N=86, 88, 90, and 92 nuclei (left-hand scale). (b) Variation of the experimental charge radii [40] (in units fm^2) for the nuclei $^{146-150}$Nd (□), $^{148-154}$Sm (△) as a function of the d-boson number ground-state expectation value $\frac{P}{N-1}\langle 0_1^+|\hat{n}_d|0_1^+\rangle$ (right-hand scale).

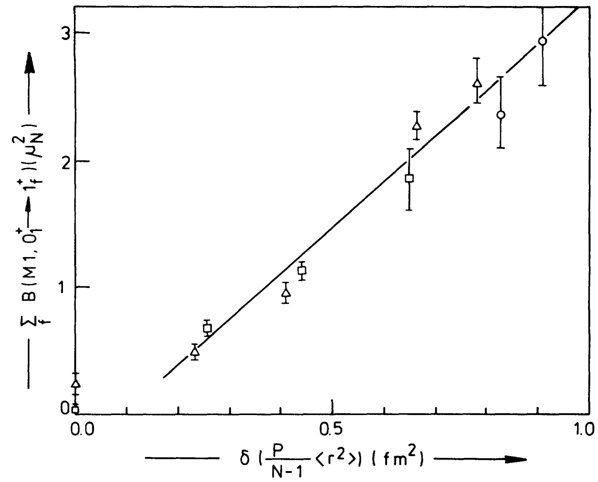

FIG. 3. Relation between the experimental summed $M1$ strength $\sum_f B\left(M1;0_1^+ \to 1_f^+\right)$ ($E_x\tilde{<}4$ MeV) [open symbols for $^{142,146-150}$Nd (□), $^{144,148-154}$Sm (△), and ^{160,162}Dy (O)] and the variation in the quantity $\delta\left(\frac{P}{N-1}\langle r^2\rangle\right)$ (fm^2) which is related to the isotopic shift $\delta\langle r^2\rangle$. The data on $M1$ strength are taken from [17] with the exception for ^{150}Nd where the data point is obtained from [18]. Values for the radii are taken from compilation by Otten [40].

of nuclei for a given isotope and in the light of earlier $E0$ studies in the rare-earth region [38] indicating small values of γ' compared with α', the $\gamma'N$ term might well be neglected. Using now the summed $M1$ strength value as taken from [17], except for the ^{150}Nd data point [18] and the isotopic shifts from a review article by Otten [40], the final results are drawn in Fig. 3. Thereby relation (34) is corroborated to hold rather well for the transitional and deformed Nd, Sm, and Dy nuclei. Approaching the $N = 82$ closed shell, clear deviations from a straight line appear signaling some deficiencies in the simple relationship of Eq. (34).

C. The $M3$ sum rule

For the magnetic octupole transitions, the NEW sum rule becomes

$$\sum_f B(M3;0_1^+ \to 3_f^+) = (\Omega_\pi)^2 \langle 0_1^+|(d^\dagger\tilde{d})_\pi^{(3)} \cdot (d^\dagger\tilde{d})_\pi^{(3)}|0_1^+\rangle + (\Omega_\nu)^2 \langle 0_1^+|(d^\dagger\tilde{d})_\nu^{(3)} \cdot (d^\dagger\tilde{d})_\nu^{(3)}|0_1^+\rangle + 2\,\Omega_\pi\Omega_\nu \langle 0_1^+|(d^\dagger\tilde{d})_\pi^{(3)} \cdot (d^\dagger\tilde{d})_\nu^{(3)}|0_1^+\rangle \quad . \tag{35}$$

In contrast to the $M1$ sum rule where nearly all strength is contained within the lowest mixed-symmetry 1^+ state, here some strength also shows up in the symmetric 3^+ state belonging to the γ band and in the mixed-symmetric 3^+ member of the 2^+ band. From numerical calculations, for the Nd, Sm, and Gd nuclei, one finds that the fragmentation becomes important only for the strongly deformed nuclei ^{152}Nd, ^{154}Sm, and ^{156}Gd. Thereby, we can make use of microscopically determined values for the Ω_ρ ($\rho \equiv \pi,\nu$) parameter in the $M3$ operator. The NEW sum rule is plotted in Fig. 4(a). We observe a deviating behavior for the Gd nuclei which might well be due to the much lower value of Ω_π pointing towards shell effects at $Z = 64$. For the other isotopes, the systematics remains remarkably similar to those for the NEW $M1$ sum rule. Nuclear deformation clearly is an important factor for the $M3$ strength too.

The magnetic $M3$ strength has also been studied within the shell model for medium-light nuclei [41] obtaining a larger fragmentation of $M3$ strength compared to the $M1$ transition strength. Within the $1f_{7/2}^n$ model space, a NEW sum rule has been calculated. For strongly deformed nuclei, studies have been carried out within the two-rotor model (TRM) and schematic random phase approximation (RPA) [42–44] and within the quasiparticle phonon nuclear model [45]. From the TRM, one obtains a strength which is proportional to the $B(M1)$ also with a δ^2 variation. This would suggest a very small strength, except for the strongly deformed nuclei.

IV. ENERGY-WEIGHTED SUM RULES FOR $E0$, $M1$, AND $M3$ TRANSITIONS

For the energy-weighted (EW) sum rule, one has to evaluate the right-hand side of Eq. (2) which involves commutators of products of group generators, coupled to a specific total angular momentum. The Hamiltonian contains one- and two-body terms only, so one obtains

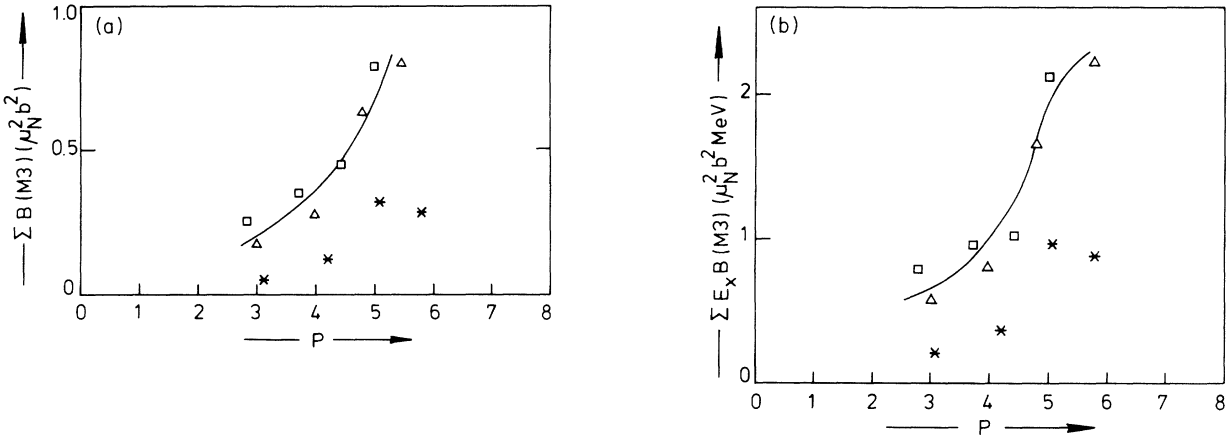

FIG. 4. (a) The NEW $M3$ sum rule for the Nd (□), Sm (△), and Gd (∗) nuclei ($86 \leq N \leq 92$) using the IBM-2 Hamiltonian of Scholten [33]. (b) The EW $M3$ sum rule for the Nd (□), Sm (△), and Gd (∗) nuclei ($86 \leq N \leq 92$) using the IBM-2 Hamitonian of Scholten [33].

commutators of the following type:

$$\left[\left[\hat{A}_\rho(\lambda), \hat{B}_{\rho'}(\lambda')\right] \ , \ \hat{B}_{\rho''}(\lambda')\right]^{(0)}$$

and

$$\left[\left[\hat{A}_\rho(\lambda) \cdot \hat{C}'_\rho(\lambda), \hat{B}_{\rho''}(\lambda')\right] \ , \ \hat{B}_{\rho'''}(\lambda')\right]^{(0)} \quad , \tag{36}$$

where one has $\rho, \rho', \rho'', \rho''' = \pi, \nu$; and $\hat{A}(\lambda)$, $\hat{B}(\lambda)$, and $\hat{C}(\lambda)$ are operators of the type $\left(b^\dagger_{\lambda_1} \tilde{b}_{\lambda_2}\right)^{(\lambda)}$.

These nested commutators have to be worked out "gradually," using properties of commutators of coupled operators, listed in Appendix A and the basic commutator

$$\left[\left(b^\dagger_{\lambda_1}\tilde{b}_{\lambda_2}\right)^{(k_1)}, \left(b^\dagger_{\lambda_3}\tilde{b}_{\lambda_4}\right)^{(k_2)}\right]^{(k_3)}_{\kappa_3} = \sqrt{(2k_1+1)(2k_2+1)}\left[(-1)^{k_3}\begin{Bmatrix} k_1 & k_2 & k_3 \\ \lambda_4 & \lambda_1 & \lambda_2 \end{Bmatrix}\delta_{\lambda_2\lambda_3}\cdot\left(b^\dagger_{\lambda_1}\tilde{b}_{\lambda_4}\right)^{(k_3)}_{\kappa_3} - (-1)^{k_1+k_2}\begin{Bmatrix} k_1 & k_2 & k_3 \\ \lambda_3 & \lambda_2 & \lambda_1 \end{Bmatrix}\delta_{\lambda_1\lambda_4}\left(b^\dagger_{\lambda_3}\tilde{b}_{\lambda_2}\right)^{(k_3)}_{\kappa_3}\right] \quad . \tag{37}$$

TABLE I. Frequently used commutators for the evaluation of the EW sum rules, following Eq. (37). We use the notation $T^{(\lambda)}_\rho \equiv \left(d^\dagger\tilde{d}\right)^{(\lambda)}_\rho$ ($\rho \equiv \pi,\ \nu$).

$\left[(s^\dagger s)^{(0)}_\rho, (s^\dagger\tilde{d})^{(2)}_{\rho'}\right]^{(2)} = (s^\dagger\tilde{d})^{(2)}_\rho\ \delta_{\rho\rho'}$
$\left[(s^\dagger s)^{(0)}_\rho, (d^\dagger s)^{(2)}_{\rho'}\right]^{(2)} = -\ (d^\dagger s)^{(2)}_\rho\ \delta_{\rho\rho'}$
$\left[T^{(k_1)}_\rho, T^{(k_2)}_{\rho'}\right]^{(k_3)} = \hat{k}_1\hat{k}_2\begin{Bmatrix} k_1 & k_2 & k_3 \\ 2 & 2 & 2 \end{Bmatrix}\left[(-1)^{k_3} - (-1)^{k_1+k_2}\right]T^{(k_3)}_\rho\delta_{\rho\rho'}$
$\left[(s^\dagger\tilde{d})^{(2)}_\rho, T^{(k_1)}_{\rho'}\right]^{(k_1)} = (-1)^{k_1}\frac{\hat{k}_1}{\sqrt{5}}\left(s^\dagger\tilde{d}\right)^{(2)}_\rho\ \delta_{2k_2}\delta_{\rho\rho'}$
$\left[(d^\dagger s)^{(2)}_\rho, T^{(k_1)}_{\rho'}\right]^{(k_1)} = -\frac{\hat{k}_1}{\sqrt{5}}\left(d^\dagger s\right)^{(2)}_\rho\ \delta_{2k_2}\delta_{\rho\rho'}$
$\left[(s^\dagger\tilde{d})^{(2)}_\rho, (d^\dagger s)^{(2)}_{\rho'}\right]^{(k)} = (-1)^k\left[\sqrt{5}\left(s^\dagger s\right)^{(0)}_\rho\delta_{0k} - T^{(k)}_\rho\right]\delta_{\rho\rho'}$
$\left[(ds^\dagger s)^{(2)}_\rho, (s^\dagger\tilde{d})^{(2)}_{\rho'}\right]^{(k)} = -\left[\sqrt{5}\left(s^\dagger s\right)^{(0)}_\rho\delta_{0k} - T^{(k)}_\rho\right]\delta_{\rho\rho'}$
$\left[(s^\dagger\tilde{d})^{(2)}_\rho, (s^\dagger\tilde{d})^{(2)}_\rho\right]^{(k)} = 0$
$\left[(d^\dagger s)^{(2)}_\rho, (d^\dagger s)^{(2)}_\rho\right]^{(k)} = 0$
$\left[(d^\dagger_{\rho'}\tilde{d}_\rho)^{(k_1)}, T^{(k_2)}_\rho\right]^{(k_3)} = \hat{k}_1\hat{k}_2(-1)^{k_3}\begin{Bmatrix} k_1 & k_2 & k_3 \\ 2 & 2 & 2 \end{Bmatrix}\left(d^\dagger_{\rho'}\tilde{d}_\rho\right)^{(k_3)},\quad \rho\neq\rho'$
$\left[(d^\dagger_\rho\tilde{d}_{\rho'})^{(k_1)}, T^{(k_2)}_\rho\right]^{(k_3)} = \hat{k}_1\hat{k}_2(-1)^{k_1+k_2+1}\begin{Bmatrix} k_1 & k_2 & k_3 \\ 2 & 2 & 2 \end{Bmatrix}\left(d^\dagger_\rho\tilde{d}_{\rho'}\right)^{(k_3)},\quad \rho\neq\rho'$

A number of frequently used commutators, derived in this way, are now listed in Table I.

Combining all of these results, one finally gives the important double commutators that are needed to evaluate the $E0$, $M1$, and $M3$ EW sum rules

$$\left[\left[\hat{n}_{d_\rho},\hat{T}(S\lambda)\right],\hat{T}(S\lambda)\right]^{(0)}=0 \quad , \tag{38}$$

$$\left[\left[\hat{Q}_\rho\cdot\hat{Q}_\rho,\hat{T}(S\lambda)\right],\hat{T}(S\lambda)\right]^{(0)} = (A_\rho^\lambda)^2\left[\left[\hat{Q}_\rho\cdot\hat{Q}_\rho,\left(d^\dagger\tilde{d}\right)^{(\lambda)}_\rho\right],\left(d^\dagger\tilde{d}\right)^{(\lambda)}_\rho\right]^{(0)} \quad , \tag{39}$$

$$\left[\left[\hat{Q}_\pi\cdot\hat{Q}_\nu,\hat{T}(S\lambda)\right],\hat{T}(S\lambda)\right]^{(0)} \quad , \tag{40}$$

and

$$\left[\left[\hat{M}_{\pi\nu},\hat{T}(S\lambda)\right],\hat{T}(S\lambda)\right]^{(0)} \quad . \tag{41}$$

A. $E0$ sum rule

For the EW $E0$ sum rule, we obtain, using the general relations (38)–(41)

$$\begin{aligned}\sum_f B(E0;0_1^+\to0_f^+)E_x(0_f^+) = &-\sum_{\rho=\pi,\nu}\sum_{\rho'=\pi,\nu}\frac{1+\delta_{\rho\rho'}}{2}\frac{\kappa_{\rho\rho'}}{10}\langle0_1^+|\alpha'^2_\rho\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\rho\cdot\hat{Q}_{\rho'}\\&+\alpha'^2_{\rho'}\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_{\rho'}\cdot\hat{Q}_\rho+2\alpha'_\rho\alpha'_{\rho'}\left(s^\dagger\tilde{d}-d^\dagger s\right)^{(2)}_\rho\cdot\left(s^\dagger\tilde{d}-d^\dagger s\right)^{(2)}_{\rho'}|0_1^+\rangle\\&+\frac{\xi_2}{10}\left(\alpha'_\pi-\alpha'_\nu\right)^2\langle0_1^+|\left(s^\dagger\tilde{d}\right)^{(2)}_\pi\cdot\left(d^\dagger s\right)^{(2)}_\nu+\left(s^\dagger\tilde{d}\right)^{(2)}_\nu\cdot\left(d^\dagger s\right)^{(2)}_\pi|0_1^+\rangle \quad .\end{aligned} \tag{42}$$

The Majorana contribution is only present within the isovector channel of the $E0$ operator ($\alpha'_\pi-\alpha'_\nu\neq0$).

In the F-spin invariant limit (with $\kappa_{\pi\pi}=\kappa_{\nu\nu}=\frac{\kappa_{\pi\nu}}{2}=\kappa, \xi_i=1$ for $i=1,2,3$), the EW sum rule becomes

$$\begin{aligned}\sum_f B(E0;0_1^+\to0_f^+)E_x(0_f^+) = &\frac{-\kappa}{10}\sum_{\rho=\pi,\nu}\sum_{\rho'=\pi,\nu}(\alpha_\rho+\alpha_{\rho'})^2\langle0_1^+|\left(s^\dagger\tilde{d}\right)^{(2)}_\rho\cdot\left(s^\dagger\tilde{d}\right)^{(2)}_{\rho'}+\left(d^\dagger s\right)^{(2)}_\rho\cdot\left(d^\dagger s\right)^{(2)}_{\rho'}|0_1^+\rangle\\&+\left(\frac{1}{10}-\frac{\kappa}{5}\right)(\alpha_\pi-\alpha_\nu)^2\langle0_1^+|\left(s^\dagger\tilde{d}\right)^{(2)}_\pi\cdot\left(d^\dagger s\right)^{(2)}_\nu+\left(s^\dagger\tilde{d}\right)^{(2)}_\nu\cdot\left(d^\dagger s\right)^{(2)}_\pi|0_1^+\rangle\\&-\frac{\kappa}{5}\langle0_1^+|\left(\alpha'^2_\pi\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\pi+\alpha'^2_\nu\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\nu\right)\cdot\left(\chi_\pi\left(d^\dagger\tilde{d}\right)^{(2)}_\pi+\chi_\nu\left(d^\dagger\tilde{d}\right)^{(2)}_\nu\right)|0_1^+\rangle\,,\end{aligned} \tag{43}$$

where a separation into an isoscalar and isovector contribution becomes clear. For the latter, the expectation value involves an operator that replaces a $\pi(s)d$ boson by a $\nu(s)d$ boson and vice versa. For the former, the $s\rightleftharpoons d$ exchange keeps the balance equal for π and ν bosons.

B. $M1$ sum rule

The EW magnetic dipole sum rule has been worked out already within the nulcear shell model, properly [22, 23] as well as in the IBM-2 [13, 14, 24, 25]. In the present paper, we start from the more general results, described earlier, and stress a few aspects that have only been very briefly addressed in the former papers.

The general expression for the EW sum rule, after filling out all coefficients, reduces to a rather simple and transparent expression, e.g.,

$$\begin{aligned}\sum_f B(M1;0_1^+\to1_f^+)E_x(1_f^+) = &\frac{9}{4\pi}\kappa_{\pi\nu}(g_\pi-g_\nu)^2\langle0_1^+|\hat{Q}_\pi\cdot\hat{Q}_\nu|0_1^+\rangle\\&+\frac{9}{4\pi}(g_\pi-g_\nu)^2\lambda\Big\{\xi_2\langle0_1^+|\left(s^\dagger\tilde{d}\right)^{(2)}_\pi\cdot\left(d^\dagger s\right)^{(2)}_\nu+\left(s^\dagger\tilde{d}\right)^{(2)}_\nu\cdot\left(d^\dagger s\right)^{(2)}_\pi|0_1^+\rangle\\&+\frac{1}{3}\xi_1\langle0_1^+|\left(d^\dagger\tilde{d}\right)^{(1)}_\pi\cdot\left(d^\dagger\tilde{d}\right)^{(1)}_\nu|0_1^+\rangle+\frac{1}{5}(-3\xi_1+8\xi_3)\langle0_1^+|\left(d^\dagger\tilde{d}\right)^{(2)}_\pi\cdot\left(d^\dagger\tilde{d}\right)^{(2)}_\nu|0_1^+\rangle\\&+2\xi_3\langle0_1^+|\left(d^\dagger\tilde{d}\right)^{(3)}_\pi\cdot\left(d^\dagger\tilde{d}\right)^{(3)}_\nu|0_1^+\rangle+\frac{2}{3}(4\xi_1+\xi_3)\langle0_1^+|\left(d^\dagger\tilde{d}\right)^{(4)}_\pi\cdot\left(d^\dagger\tilde{d}\right)^{(4)}_\nu|0_1^+\rangle\Big\} \quad ,\end{aligned} \tag{44}$$

which for the common choice of the Majorana parameters $\xi_1=\xi_2=\xi_3=1$, i.e., an F-spin invariant Majorana interaction, reduces to

$$\sum_f B(M1;0_1^+ \to 1_f^+)E_x(1_f^+) = \frac{-9}{4\pi}\kappa_{\pi\nu}\,(g_\pi - g_\nu)^2\,\langle 0_1^+|\hat{Q}_\pi\cdot\hat{Q}_\nu|0_1^+\rangle + \frac{9}{4\pi}(g_\pi - g_\nu)^2\,\lambda\Bigg\{\langle 0_1^+|\left(s^\dagger\tilde{d}\right)^{(2)}_\pi\cdot\left(d^\dagger s\right)^{(2)}_\nu + \left(s^\dagger\tilde{d}\right)^{(2)}_\nu\cdot\left(d^\dagger s\right)^{(2)}_\pi|0_1^+\rangle + \sum_k\frac{k(k+1)}{6}\langle 0_1^+|\left(d^\dagger\tilde{d}\right)^{(k)}_\pi\cdot\left(d^\dagger\tilde{d}\right)^{(k)}_\nu|0_1^+\rangle\Bigg\} \quad . \qquad (45)$$

The first term on the right-hand side follows from the contribution of the quadrupole interaction and is proportional to the expectation value of the proton-neutron quadrupole force in the ground state [22]. The latter is a measure of the quadrupole deformation energy which can also be related to the corresponding binding energy in a quadrupole deformed mean field such as the Nilsson model [23]. This close relationship with nuclear deformation has become clear from recent experimental studies [16–18] and a theoretical approach, evaluating the behavior of nuclear deformation and magnetic dipole strength versus the P factor [21].

A reamining contribution is coming from the Majorana interaction. This contribution can amount up to $\simeq 50\%$ of the EW $M1$ sum rule as obtained numerically for the Sm, Gd, and Nd nuclei. For the F-spin invariant form of the Majorana force, however, we can rewrite (45) as follows, making use of expressions (6), (18), and the fact that $\langle 0_1^+|\hat{M}_{\pi\nu}|0_1^+\rangle = 0$ for an $F = F_{\max}$ ground state:

$$\frac{9}{2\pi}\lambda\,(g_\pi - g_\nu)^2\,\frac{N_\pi N_\nu}{N}\langle 0_1^+|\hat{n}_d|0_1^+\rangle + R \quad , \qquad (46)$$

with

$$R = \frac{9}{4\pi}\lambda\,(g_\pi - g_\nu)^2 \times\left\langle 0_1^+\left|\sum_{k=1}^{4}\left(\frac{k(k+1)}{6} - 1\right)\hat{T}^{(k)}_{\pi\nu} - 6\hat{T}^{(0)}_{\pi\nu}\right|0_1^+\right\rangle \quad , \qquad (47)$$

$$\hat{T}^{(k)}_{\pi\nu} = \left(d^\dagger\tilde{d}\right)^{(k)}_\pi\cdot\left(d^\dagger\tilde{d}\right)^{(k)}_\nu \quad .$$

Making use of the recoupling formula (15) and the property $\langle 0_1^+, F = F_{\max}|\left(d^\dagger\tilde{d}\right)^{(k)}_\pi\cdot\left(d^\dagger\tilde{d}\right)^{(k)}_\nu|0_1^+, F = F_{\max}\rangle = 0$ for k odd, one obtains

$$R = -\frac{15}{2\pi}(g_\pi - g_\nu)^2\,\lambda\langle 0_1^+|\left(d^\dagger\tilde{d}\right)^{(1)}_\pi\cdot\left(d^\dagger\tilde{d}\right)^{(1)}_\nu|0_1^+\rangle, \qquad (48)$$

$$R = -\frac{3}{4\pi}(g_\pi - g_\nu)^2\,\lambda\langle 0_1^+|\hat{L}_\pi\cdot\hat{L}_\nu|0_1^+\rangle \quad .$$

Expression (49) is exactly the NEW $M1$ sum rule (29) and hence, in the F-spin invariant limit, reduces to (14) or

$$R = \frac{9}{4\pi}(g_\pi - g_\nu)^2\,\frac{P\cdot\lambda}{N-1}\langle 0_1^+|\hat{n}_d|0_1^+\rangle \quad . \qquad (49)$$

Using (44), (45), and (49), we finally obtain the result

$$\sum_f B(M1;0_1^+ \to 1_f^+)E_x(1_f^+) = \frac{-9}{4\pi}\kappa_{\pi\nu}\,(g_\pi - g_\nu)^2\,\langle 0_1^+|\hat{Q}_\pi\cdot\hat{Q}_\nu|0_1^+\rangle + \frac{9}{4\pi}\lambda\,(g_\pi - g_\nu)^2\,N\cdot\frac{P}{N-1}\langle 0_1^+|\hat{n}_d|0_1^+\rangle \quad , \qquad (50)$$

or, alternatively,

$$\sum_f B(M1;0_1^+ \to 1_f^+)\left(E_x(1_f^+) - \lambda N\right) = \frac{-9}{4\pi}\kappa_{\pi\nu}\,(g_\pi - g_\nu)^2\,\langle 0_1^+|\hat{Q}_\pi\cdot\hat{Q}_\nu|0_1^+\rangle \quad . \qquad (51)$$

This result is particularly interesting since for an F-spin invariant form of the Majorana term, the right-hand side can be worked out inserting an intermediate set of 2^+ states and resulting into [13, 14]

$$\sum_f B(M1;0_1^+ \to 1_f^+)\left(E_x(1_f^+) - \lambda N\right) = c\sum_f B(E2;0_1^+ \to 2_f^+) \quad , \qquad (52)$$

where $\hat{T}(E2) = e_{\text{eff}}\left(\hat{Q}_\pi + \hat{Q}_\nu\right)$, an isoscalar operator,

$$e_{\text{eff}} = \left(-\frac{9}{4\pi}\kappa_{\pi\nu}\,(g_\pi - g_\nu)^2\,\frac{N_\pi N_\nu}{N^2}\right)^{1/2} e\,\text{b} \quad , \qquad (53)$$

and c, a conversion factor μ_N^2 MeV$/(e\,\text{b})^2$ to correct for the dimension mismatch.

We have illustrated this relationship numerically in Fig. 5 for the Sm, Gd, and Nd nuclei. The effective charge deduced from a least-squares fit to a linear relationship is $e_{\text{eff}} = 0.103$. On the same figure, we also plotted the relation of the NEW $B(E2)$ sum rule and the EW $M1$ sum rule. The linearity is maintained, however, one needs to use a renormalization of the effective charge $e^{(r)}_{\text{eff}} = 0.155 \simeq e_{\text{eff}}\sqrt{\frac{E_x}{(E_x - \lambda N)}}$.

C. $M3$ sum rule

For the $M3$ transitions, the EW sum rule reads

$$\sum_f B(M3;0_1^+ \to 3_f^+)E_x(3_f^+) = -\frac{\kappa_{\pi\pi}\Omega_\pi^2}{16\pi}\left(630\hat{Q}_\pi \cdot \left(d^\dagger \tilde{d}\right)^{(2)}_\pi \chi_\pi + 450\chi_\pi^2 \left(d^\dagger \tilde{d}\right)^{(2)}_\pi \cdot \left(d^\dagger \tilde{d}\right)^{(2)}_\pi \right.$$
$$\left. +100\chi_\pi^2 \left(d^\dagger \tilde{d}\right)^{(4)}_\pi \cdot \left(d^\dagger \tilde{d}\right)^{(4)}_\pi \right)$$
$$-\frac{\kappa_{\nu\nu}\Omega_\nu^2}{16\pi}(\pi \to \nu) - \frac{\kappa_{\pi\nu}\left(\Omega_\pi - \Omega_\nu\right)^2}{16\pi}\left(49\hat{Q}_\pi \cdot \hat{Q}_\nu + 105\frac{\Omega_\pi\left(2\Omega_\nu + \Omega_\pi\right)}{\left(\Omega_\pi - \Omega_\nu\right)^2}\chi_\pi \right.$$
$$\times \left(d^\dagger \tilde{d}\right)^{(2)}_\pi \cdot \hat{Q}_\nu + 105\frac{\Omega_\nu\left(2\Omega_\pi + \Omega_\nu\right)}{\left(\Omega_\pi - \Omega_\nu\right)^2}\chi_\nu \left(d^\dagger \tilde{d}\right)^{(2)}_\nu \cdot \hat{Q}_\pi$$
$$-450\frac{\Omega_\pi\Omega_\nu}{\left(\Omega_\pi - \Omega_\nu\right)^2}\left(d^\dagger \tilde{d}\right)^{(2)}_\pi \cdot \left(d^\dagger \tilde{d}\right)^{(2)}_\nu \chi_\pi\chi_\nu$$
$$\left. -100\frac{\Omega_\pi\Omega_\pi}{\left(\Omega_\pi - \Omega_\nu\right)^2}\chi_\pi\chi_\nu \left(d^\dagger \tilde{d}\right)^{(4)}_\pi \cdot \left(d^\dagger \tilde{d}\right)^{(4)}_\nu \right)$$
$$+\frac{49}{16\pi}\lambda\xi_2\left(\Omega_\pi - \Omega_\nu\right)^2 \left(\left(s^\dagger \tilde{d}\right)^{(2)}_\pi \cdot \left(d^\dagger s\right)^{(2)}_\nu + \left(s^\dagger \tilde{d}\right)^{(2)}_\nu \cdot \left(d^\dagger s\right)^{(2)}_\pi\right)$$
$$+\frac{49}{4\pi}\lambda\left(\Omega_\pi - \Omega_\nu\right)^2 \left(\frac{1}{2}\xi_1 \left(d^\dagger \tilde{d}\right)^{(1)}_\pi \cdot \left(d^\dagger \tilde{d}\right)^{(1)}_\nu + \frac{11}{70}\left(-3\xi_1 + 8\xi_3\right)\right.$$
$$\times \left(d^\dagger \tilde{d}\right)^{(2)}_\pi \cdot \left(d^\dagger \tilde{d}\right)^{(2)}_\nu + \frac{9}{28}\xi_3 \left(d^\dagger \tilde{d}\right)^{(3)}_\pi \cdot \left(d^\dagger \tilde{d}\right)^{(3)}_\nu$$
$$\left. +\frac{3}{28}\left(4\xi_1 + \xi_3\right)\left(d^\dagger \tilde{d}\right)^{(4)}_\pi \cdot \left(d^\dagger \tilde{d}\right)^{(4)}_\nu \right) + R \quad , \tag{54}$$

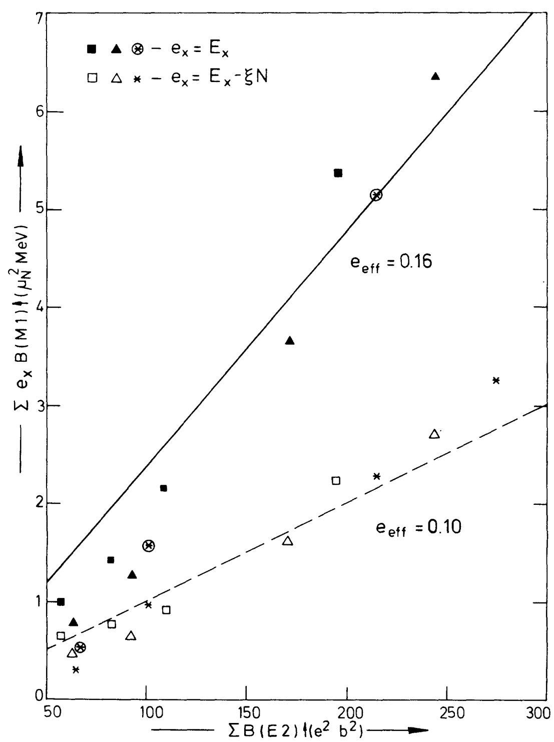

FIG. 5. Illustration of the EW $M1$ sum rule relationship starting from the IBM-2 calculations of $B(E2;0_1^+ \to 2_1^+)$ and $B(M1;0_1^+ \to 1_f^+)$ reduced transition probabilities for the Nd (□), Sm (△), and Gd (∗) nuclei, as discussed in the text. The fit, incorporating the Majorana term and the energy-weighting with $E_x(1_f^+) - \lambda N$, is given by the dashed line. The fit, using $E_x(1_f^+)$ only, also results in an almost linear relationship (full line) but now with a larger (renormalized) effective charge of $e_{\rm eff} \simeq 0.16\ e\,$b.

with

$$R = \frac{49}{4\pi}\lambda\left(-2\Omega_\pi\Omega_\nu\right)\left(\frac{1}{2}\left(\xi_3 - \xi_1\right)\left(d^\dagger \tilde{d}\right)^{(1)}_\pi \cdot \left(d^\dagger \tilde{d}\right)^{(1)}_\nu \right.$$
$$+\frac{1}{14}\left(9\xi_1 - 17\xi_3\right)\left(d^\dagger \tilde{d}\right)^{(2)}_\pi \cdot \left(d^\dagger \tilde{d}\right)^{(2)}_\nu$$
$$+\frac{1}{14}\left(3\xi_1 - 8\xi_3\right)\left(d^\dagger \tilde{d}\right)^{(3)}_\pi \cdot \left(d^\dagger \tilde{d}\right)^{(3)}_\nu$$
$$\left. -\frac{5}{14}\xi_1 \left(d^\dagger \tilde{d}\right)^{(4)}_\pi \cdot \left(d^\dagger \tilde{d}\right)^{(4)}_\nu \right) \quad . \tag{55}$$

Note the similarities with the $M1$ EW sum rule for the leading terms of the quadrupole proton-neutron force and the Majorana contribution. Still, for multipolarity $\lambda = 3$, many additional terms come in, especially the like nucleon quadrupole force seems to give an important contribution.

Results for the EW $M3$ sum rule in the rare-earth region are given in Fig. 4(b).

V. SUM RULES FOR THE $E2$ OPERATOR

A. Non-energy-weighted $E2$ sum rule

The NEW $E2$ sum rule can be derived as

$$\sum_f B(E2;0_1^+ \to 2_f^+)$$
$$= \langle 0_1^+ | \left(e_\pi\hat{Q}_\pi + e_\nu\hat{Q}_\nu\right) \cdot \left(e_\pi\hat{Q}_\pi + e_\nu\hat{Q}_\nu\right) | 0_1^+ \rangle$$
$$= e_\pi^2 \langle 0_1^+ | \hat{Q}_\pi \cdot \hat{Q}_\pi | 0_1^+ \rangle + e_\nu^2 \langle 0_1^+ | \hat{Q}_\nu \cdot \hat{Q}_\nu | 0_1^+ \rangle$$
$$+2e_\pi e_\nu \langle 0_1^+ | \hat{Q}_\pi \cdot \hat{Q}_\nu | 0_1^+ \rangle \quad . \tag{56}$$

For $e_\pi = e_\nu = e$, one can insert an intermediate set of states and, applying Eq. (18), assuming further that all isoscalar $E2$ strength is carried by the totally symmetric boson states ($F = F_{\max}$). Hence, one obtains

$$\sum_f B(E2;0_1^+ \to 2_f^+) = e^2 \frac{N^2}{N_\pi N_\nu} \langle 0_1^+|\hat{Q}_\pi \cdot \hat{Q}_\nu|0_1^+\rangle \quad , \tag{57}$$

which establishes the relationship of the summed isoscalar $E2$ strength (mainly carried by the lowes 2_1^+ state) with the expectation value of the proton-neutron quadrupole force in the ground state and hence, with nuclear deformation (see also Sec. IV B).

B. Energy-weighted $E2$ sum rule

The EW $E2$ sum rule for $E2$ transition involves more elaborate calculations, due to the complex form of the $E2$ operator. In principle, the method is the same, making use of the results in Appendix A and Table I.

From numerical calculations for the Nd, Sm, and Gd nuclei, using a realistic Hamiltonian [33] and values for the effective charge $e_\pi = e_\nu = 0.1\ e\,\mathrm{b}$ for simplicity, respectively, $e_\pi = 0.128\ e\,\mathrm{b}$, $e_\nu = 0.057\ e\,\mathrm{b}$ [46, 47], it becomes clear that the strength is carried mainly by the lowest 2_1^+ state, which has the component $F = F_{\max}$ for $\simeq 98\%$, and hence one would not expect the Majorana contribution to be important. Therefore, we leave the latter out at this stage and include its calculation in Appendix B for reasons of completeness.

For the Hamiltonian, including the d-boson number operators and the quadrupole interaction, we obtain the result

$$\begin{aligned}
\sum_f B(E2;0_1^+ \to 2_f^+)E_x(2_f^+) = {} & 5\, e_\pi^2 \epsilon_{d_\pi} \langle 0_1^+|\hat{N}_\pi - \tfrac{6}{5}\hat{n}_{d_\pi}|0_1^+\rangle \\
& +5\, e_\nu^2 \epsilon_{d_\nu} \langle 0_1^+|\hat{N}_\nu - \tfrac{6}{5}\hat{n}_{d_\nu}|0_1^+\rangle + e_\pi^2 \kappa_{\pi\pi} \left(-4+\chi_\pi^2\right) \langle 0_1^+|\hat{Q}_\pi \cdot \hat{Q}_\pi|0_1^+\rangle \\
& +6 e_\pi^2 \kappa_{\pi\pi}\chi_\pi \left(1-\frac{4}{7}\chi_\pi^2\right) \langle 0_1^+|\left(d^\dagger \tilde{d}\right)_\pi^{(2)} \cdot \hat{Q}_\pi|0_1^+\rangle \\
& +4 e_\pi^2 \kappa_{\pi\pi}\left[\left(1+\frac{1}{2}\chi_\pi^2\right)^2 \langle 0_1^+|\left(d^\dagger \tilde{d}\right)_\pi^{(1)} \cdot \left(d^\dagger \tilde{d}\right)_\pi^{(1)}|0_1^+\rangle \right. \\
& \qquad \left. +\left(1-\frac{4}{7}\chi_\pi^2\right)^2 \langle 0_1^+|\left(d^\dagger \tilde{d}\right)_\pi^{(3)} \cdot \left(d^\dagger \tilde{d}\right)_\pi^{(3)}|0_1^+\rangle\right] \\
& +e_\nu^2 \kappa_{\nu\nu} \left(-4+\chi_\nu^2\right) \langle 0_1^+|\hat{Q}_\nu \cdot \hat{Q}_\nu|0_1^+\rangle + 6 e_\nu^2 \kappa_{\nu\nu}\chi_\nu \left(1-\frac{4}{7}\chi_\nu^2\right) \langle 0_1^+|\left(d^\dagger \tilde{d}\right)_\nu^{(2)} \cdot \hat{Q}_\nu|0_1^+\rangle \\
& +4 e_\nu^2 \kappa_{\nu\nu}\left[\left(1+\frac{1}{2}\chi_\nu^2\right)^2 \langle 0_1^+|\left(d^\dagger \tilde{d}\right)_\nu^{(1)} \cdot \left(d^\dagger \tilde{d}\right)_\nu^{(1)}|0_1^+\rangle \right. \\
& \qquad \left. +\left(1-\frac{4}{7}\chi_\nu^2\right)^2 \langle 0_1^+|\left(d^\dagger \tilde{d}\right)_\nu^{(3)} \cdot \left(d^\dagger \tilde{d}\right)_\nu^{(3)}|0_1^+\rangle\right] \\
& +\frac{\kappa_{\pi\nu}}{2}\left[e_\pi^2\left(-4+\chi_\pi^2\right) + e_\nu^2\left(-4+\chi_\nu^2\right)\right] \langle 0_1^+|\hat{Q}_\pi \cdot \hat{Q}_\nu|0_1^+\rangle \\
& +3\kappa_{\pi\nu}\left[e_\pi^2\left(1-\frac{4}{7}\chi_\pi^2\right)\chi_\pi \langle 0_1^+|\left(d^\dagger \tilde{d}\right)_\pi^{(2)} \cdot \hat{Q}_\nu|0_1^+\rangle \right. \\
& \qquad \left. +e_\nu^2\left(1-\frac{4}{7}\chi_\nu^2\right)\chi_\nu \langle 0_1^+|\left(d^\dagger \tilde{d}\right)_\nu^{(2)} \cdot \hat{Q}_\pi|0_1^+\rangle\right] \\
& +4\kappa_{\pi\nu} e_\pi e_\nu \left[\left(1+\frac{1}{2}\chi_\pi^2\right)\left(1+\frac{1}{2}\chi_\nu^2\right) \langle 0_1^+|\left(d^\dagger \tilde{d}\right)_\pi^{(1)} \cdot \left(d^\dagger \tilde{d}\right)_\nu^{(1)}|0_1^+\rangle \right. \\
& \qquad \left. +\left(1-\frac{4}{7}\chi_\pi^2\right)\left(1-\frac{4}{7}\chi_\nu^2\right) \langle 0_1^+|\left(d^\dagger \tilde{d}\right)_\pi^{(3)} \cdot \left(d^\dagger \tilde{d}\right)_\nu^{(3)}|0_1^+\rangle\right] \quad .
\end{aligned} \tag{58}$$

For an F-spin invariant Hamiltonian, with $\epsilon_{d_\pi} = \epsilon_{d_\nu} = \epsilon_d$, $\chi_\pi = \chi_\nu = \chi$, $\kappa_{\pi\pi} = \kappa_{\nu\nu} = \kappa_{\pi\nu}/2 = \kappa/2$, this simplifies into the following form:

$$\sum_f B(E2;0_1^+\to 2_f^+)E_x(2_f^+) = \epsilon_d\left(e_\pi^2N_\pi+e_\nu^2N_\nu\right)\left(5-\frac{6}{N}\langle 0_1^+|\hat{n}_d|0_1^+\rangle\right)$$
$$+\frac{\kappa}{2}\left(-4+\chi^2\right)\langle 0_1^+|\left(e_\pi^2\hat{Q}_\pi+e_\nu^2\hat{Q}_\nu\right)\cdot\left(\hat{Q}_\pi+\hat{Q}_\nu\right)|0_1^+\rangle$$
$$+3\kappa\left(1-\frac{4}{7}\chi^2\right)\chi\langle 0_1^+|\left(e_\pi^2\left(d^\dagger\tilde{d}\right)_\pi^{(2)}+e_\nu^2\left(d^\dagger\tilde{d}\right)_\nu^{(2)}\right)\cdot\left(\hat{Q}_\pi+\hat{Q}_\nu\right)|0_1^+\rangle$$
$$+2\kappa\left[\left(1+\frac{1}{2}\chi^2\right)^2\langle 0_1^+|\left(e_\pi\left(d^\dagger\tilde{d}\right)_\pi^{(1)}+e_\nu\left(d^\dagger\tilde{d}\right)_\nu^{(1)}\right)^2|0_1^+\rangle\right.$$
$$\left.+\left(1-\frac{4}{7}\chi^2\right)^2\langle 0_1^+|\left(e_\pi\left(d^\dagger\tilde{d}\right)_\pi^{(3)}+e_\nu\left(d^\dagger\tilde{d}\right)_\nu^{(3)}\right)^2|0_1^+\rangle\right]\quad. \tag{59}$$

For an isoscalar $E2$ operator with $e_\pi = e_\nu = e$, this results even more with, as the final result,

$$\sum_f B(E2;0_1^+\to 2_f^+)E_x(2_f^+) = e^2\left\{\epsilon_dN\left(5-\frac{6}{N}\langle 0_1^+|\hat{n}_d|0_1^+\rangle\right)\right.$$
$$+\frac{\kappa}{2}\left(-4+\chi^2\right)\langle 0_1^+|\left(\hat{Q}_\pi+\hat{Q}_\nu\right)\cdot\left(\hat{Q}_\pi+\hat{Q}_\nu\right)|0_1^+\rangle$$
$$+3\kappa\left(1-\frac{4}{7}\chi^2\right)\chi\langle 0_1^+|\left(\left(d^\dagger\tilde{d}\right)_\pi^{(2)}+\left(d^\dagger\tilde{d}\right)_\nu^{(2)}\right)\cdot\left(\hat{Q}_\pi+\hat{Q}_\nu\right)|0_1^+\rangle$$
$$+2\kappa\left(\left(1+\frac{1}{2}\chi^2\right)^2\langle 0_1^+|\left(\left(d^\dagger\tilde{d}\right)_\pi^{(1)}+\left(d^\dagger\tilde{d}\right)_\nu^{(1)}\right)^2|0_1^+\rangle\right.$$
$$\left.\left.+\left(1-\frac{4}{7}\chi^2\right)^2\langle 0_1^+|\left(\left(d^\dagger\tilde{d}\right)_\pi^{(3)}+\left(d^\dagger\tilde{d}\right)_\nu^{(3)}\right)^2|0_1^+\rangle\right)\right\}\quad. \tag{60}$$

Note that besides the dependence on terms related to the quadrupole interaction, the d-boson number operator also gives rise to a contribution here.

It is interesting to compare the present EW $E2$ sum-rule expressions and results with some general results derived on sum rules. A very general expression was originally discussed by Lane [48] where for isoscalar $E2$ transtions $[M(E2,\mu)=\sum_i r_i^2Y_2^\mu(\hat{r}_i)]$ a result

$$\sum_f B\left(E2,0_1^+\to 2_f^+\right)\sim\frac{A}{M}\left\langle 0_1^+\left|\sum_{i=1}^{A}r_i^2\right|0_1^+\right\rangle \tag{61}$$

(with A the total nucleon number and M the nucleon mass) was derived. The results expresses a clear dependence on the number of contributing nucleons multiplied with a measure of nuclear deformation in the ground state. A similar, classical sum rule was discussed for the electric $E2$ "multipole operator" $\left(M(E2,\mu)=e\sum_i\left(\frac{1}{2}-t_z(i)\right)r_i^2Y_2^\mu(\hat{r}_i)\right)$ by Bohr and Mottelson [49] giving as the result

$$\sum_f B\left(E2,0_1^+\to 2_f^+\right)\sim\frac{Z}{M}\left\langle 0_1^+\left|\sum_{i=1}^{Z}r_i^2\right|0_1^+\right\rangle\quad, \tag{62}$$

quite similar to the original Lane sum rule. Now, the sum rule depends on the number of protons Z contributing to the motion of the charged system only, multiplied with a radial average of the protons in the ground state. For a charged liquid drop where the mass flow is described by irrotational motion, this classical sum rule (62) is exhausted by a single mode of the surface oscillation. In more practical cases, however, the mass transport can be strongly different from irrotational motion and $B(\text{irrot})/B\simeq 1/10$ indicating that about 10% of the classical sum rule (62) is obtained.

The IBM-2, using the constraint of only treating valence nucleons outside of the closed shells (inert core), presents the result that the EW $E2$ sum rule becomes proportional to the number of valence pairs N ($\equiv N_\pi+N_\nu$). Because of the general Hamiltonians used, expressions (58)–(60) do not look so transparent. In Sec. VI, for dynamical symmetries of the IBM-2, this EW $E2$ sum rule can be evaluated in detail thereby accentuating this particular boson number dependence. There too, it is shown that the EW $E2$ sum rule is largely exhausted by the 2_1^+ level, with contributions from 2_M^+ in the SU(3) and O(6) limit, only if $e_\pi\neq e_\nu$. So, the ratio of the IBM-2 EW $E2$ sum rule to the EW $E2$ sum rules derived by Lane [48] and for the electric multipole moments [49] reads $N_\pi+N_\nu/A$ and $N_\pi+N_\nu/Z$, respectively, where $N_\pi+N_\nu$ denotes the number of valence nucleon pairs, A the total number of nucleons, and Z the total number of protons. These ratios clearly exhibit the "valence model" properties of the IBM-2 model. A precise evaluation of this ratio, using ${}^{156}_{64}\mathrm{Gd}_{92}$ as an example, with a boson effective charge of $e_\pi=e_\nu=0.12\,e\,\mathrm{b}$, $N=12$ bosons, using $\epsilon_d=1$ MeV and the main term in Eq. (60) compared to the value, given by Bohr and Mottelson for the classical $E2$ sum rule [Ref. [49], Eq. (6.177)] [using $\langle r^2\rangle_{\text{protons}}=\frac{3}{5}(1.2A^{1/3})^2\ \mathrm{fm}^2$], gives the result EW $E2$(IBM-2)/EW $E2$(BM) $\simeq$ 10%. Since it is known that low-lying collective $E2$ transitions exhaust about 10% of the EW classical sum rule, it is shown that the collective

$E2$ strength within the IBM-2, properly is of the correct order of magnitude for a valence model space.

In evaluating the EW and NEW $E2$ sum rules explictly one needs to obtain good estimates for the effective charges. The charges as used here e_π (e_ν) denote the boson model proton (neutron) charges which can deviate numerically quite well from the corresponding fermion charges.

Within the nuclear shell model, effective charges for the proton

$$e_p^{\text{eff}} = \left(1 + \frac{Z}{A}\right) e \quad , \qquad e_n^{\text{eff}} = \frac{Z}{A}\, e \tag{63}$$

can be defined taking polarization of the quadrupole distortions of the nucleus (Z, A) into account [49]. General used values are $e_p^{\text{eff}} \simeq 1.5e$, $e_n^{\text{eff}} \simeq 0.5e$.

In the IBM-2, bosons are the basic building blocks and effective boson properties (effective charges, gyromagnetic factors, etc.) have been determined by a mapping procedure. The OAI mapping [50] starts from equating corresponding $E2$ matrix elements for the boson and fermion model space. A rather general outcome gives the result (in magnitude) $e_\pi = e_\nu = 0.1\ e\,\text{b}$ (dimension $e\,\text{b}$). More detailed studies also give the local variations of e_π and e_ν in a given mass region [51, 52] where, in general the proton boson effective is somewhat larger than the neutron boson effective charge. More recently, a slightly different point of view on effective boson charges was presented by Casten *et al.* [53, 54]. Here, effective boson charges are defined as a rate of change in the collective $E2$ matrix element with changing proton boson number (e_π) or with changing neutron boson number (e_ν). Thereby, a detailed derivation of $e_\pi - e_\nu$ was carried out for the $42 \leq Z \leq 98$ nuclei [54].

VI. REDUCTION OF RULES IN THE LIMIT OF U(5), SU(3), AND O(6) DYNAMICAL SYMMETRIES

In the earlier sections, we have taken into account in the evaluation of both the NEW and EW sum rules rather general IBM-2 Hamiltonians. Reducing those for the particular choice of dynamical symmetries, the more complicated expressions should reduce to some better known expression derived before by Van Isacker *et al.* [27].

A. The U(5) limit

In the U(5) limit, the IBM-2 Hamiltonian is taken to be

$$\hat{H} = \epsilon_d \hat{n}_d + \lambda \hat{M} \quad . \tag{64}$$

Since no d bosons are present in the exact U(5) ground state, the NEW sum rule reduces trivially to 0, or

$$\sum_f B\left(M1; 0_1^+ \to 1_f^+\right) = 0 \quad . \tag{65}$$

The EW $M1$ sum rule [using Eq. (50)] also becomes zero as well as the contribution originating from the Majorana term since that term also becomes proportional to $\langle \hat{n}_d \rangle$. We thus obtain

$$\sum_f B\left(M1; 0_1^+ \to 1_f^+\right) E_x\left(1_f^+\right) = 0 \quad . \tag{66}$$

For the $E2$ EW sum rule, the more general expression (38) only gives contributions coming from the first part in the single-boson contribution with the result

$$\sum_f B\left(E2; 0_1^+ \to 2_f^+\right) E_x\left(2_f^+\right) = 5\epsilon_d \left(e_\pi^2 N_\pi + e_\nu^2 N_\nu\right) \quad . \tag{67}$$

Starting from the $B(E2; 0_1^+ \to 2_1^+)$ and $B(E2; 0_1^+ \to 2_M^+)$ expressions, as derived by Van Isacker *et al.* [27], and the corresponding energy eigenvalues (neglecting the Majorana part for the 2_M^+ state) for these 2_1^+ and 2_M^+ states, it is easily observed that this EW $E2$ sum rule indeed holds.

B. The SU(3) limit

Here, we start from the simplified SU(3) IBM-2 Hamiltonian:

$$\hat{H} = \frac{\kappa_{\pi\nu}}{2}\left(\hat{Q}_\pi + \hat{Q}_\nu\right) \cdot \left(\hat{Q}_\pi + \hat{Q}_\nu\right) + \lambda \hat{M} = \frac{3\kappa_{\pi\nu}}{8} C^{(2)}_{\text{SU(3)}} - \frac{3\kappa_{\pi\nu}}{32} C^{(2)}_{\text{SO(3)}} + \lambda \hat{M} \quad , \tag{68}$$

with the corresponding eigenvalues

$$E = \frac{\kappa_{\pi\nu}}{4}\left(\lambda^2 + \mu^2 + \lambda\mu + 3(\lambda + \mu)\right) - \frac{3\kappa_{\pi\nu}}{16} L(L+1) + \lambda\left(\frac{N}{2} - F\right)\left(\frac{N}{2} + F + 1\right) \quad , \tag{69}$$

since we shall also evaluate the EW sum rules. In Table II we just give a few energy eigenvalues needed in that respect. We also give the expectation values in the SU(3) ground state

$$\left\langle \left(\hat{Q}_\pi + \hat{Q}_\nu\right) \cdot \left(\hat{Q}_\pi + \hat{Q}_\nu\right) \right\rangle = N(2N+3) \quad , \qquad \langle \hat{n}_d \rangle = \frac{4}{3}\,\frac{N(N-1)}{2N-1} \quad . \tag{70}$$

TABLE II. Energy eigenvalues for the 0_1^+, 1_M^+, 2_1^+, and 2_M^+ levels in the exact SU(3) limit, corresponding to the Hamiltonian of Eq. (68).

State	Energy eigenvalue
0_1^+	$\kappa_{\pi\nu} N(N + \frac{3}{2})$
2_1^+	$\kappa_{\pi\nu}\left(N^2 + \frac{3}{2}N - \frac{9}{8}\right)$
1_M^+	$\kappa_{\pi\nu}\left(\frac{3}{8}N^2\right) + \lambda N$
2_M^+	$\kappa_{\pi\nu}\left(N^2 - \frac{9}{8}\right) + \lambda N$

Substitution of the expectation value of $\langle \hat{n}_d \rangle$ in Ginocchio's NEW $M1$ sum rule of Eq. (20) leads to the result

$$\sum_f B\left(M1; 0_1^+ \to 1_f^+\right) = B\left(M1; 0_1^+ \to 1_M^+\right) = \frac{3}{4\pi}\left(g_\pi - g_\nu\right)^2 \frac{8N_\pi N_\nu}{2N-1}\quad, \tag{71}$$

which is the value derived by Van Isacker *et al.* [27]. Starting now from the results given in Eqs. (70) and the energy eigenvalues given in *Table II*, one can deduce from the EW $M1$ sum rule of Eq. (50) the $M1$ strength

$$B\left(M1; 0_1^+ \to 1_M^+\right) = \frac{3}{4\pi}\left(g_\pi - g_\nu\right)^2 \frac{8N_\pi N_\nu}{2N-1} Z\quad, \tag{72}$$

with

$$Z = \frac{\frac{3}{8}\kappa_{\pi\nu}\left(4N^2+4N-3\right) - \lambda N}{\frac{3}{8}\kappa_{\pi\nu}\left(4N^2+N\right) - \lambda N}\quad. \tag{73}$$

For $Z = 1$, this would be identical with the result of Eq. (71). The deviation can be explained by the fact that in deriving Eq. (50), we assumed that besides having good F spin, the $|2_f^+\rangle$ states also had maximal F spin which is clearly not the case (i.e., the $|2_M^+\rangle$ state). Without this constraint the EW $M1$ would have become

$$\sum_f B\left(M1; 0_1^+ \to 1_f^+\right) = \frac{9}{4\pi}\left(g_\pi - g_\nu\right)^2 \left[-\kappa_{\pi\nu}\langle \hat{Q}_\pi \cdot \hat{Q}_\nu\rangle + \frac{2\lambda N_\pi N_\nu}{N^2(N-1)}\langle \hat{n}_d\rangle\right]\quad. \tag{74}$$

The $\langle \hat{Q}_\pi \cdot \hat{Q}_\nu\rangle$ matrix element can be evaluated in the SU(3) limit by summing over the intermediate 2_1^+ and 2_M^+ states. The corresponding reduced $E2$ matrix elements $\langle 0_1^+ || \hat{Q}_\rho || 2_1^+\rangle$ and $\langle 0_1^+ || \hat{Q}_\rho || 2_M^+\rangle$ have been evaluated by Van Isacker *et al.* [27]. Thereby the expectation value $\langle \hat{Q}_\pi \cdot \hat{Q}_\nu\rangle$ becomes

$$\langle \hat{Q}_\pi \cdot \hat{Q}_\nu\rangle = \frac{N_\pi N_\nu (4N+1)}{2N-1}\quad. \tag{75}$$

Substitution of this expectation value (75) in the sum rule of Eq. (74) leads to the result that $Z = 1$.

We have studied the dependence of Z (using $\kappa_{\pi\nu} = 0.08$ MeV and $\lambda = 0.15$ MeV) in Fig. 6 and indeed, $Z \simeq 1$ is a rather good approximation. The deviation becomes largest for $N=3$ but is never larger than $\simeq 10\%$.

Concerning the $E2$ sum rule (EW) and using the SU(3) values $\chi_\pi = \pm\sqrt{\frac{7}{2}}$, $\chi_\nu = \pm\sqrt{\frac{7}{2}}$ with *no* single-boson terms, the sum rule of Eq. (59) reduces into

$$\sum_f B\left(E2; 0_1^+ \to 2_f^+\right) E_x\left(2_f^+\right) = \frac{-\frac{9}{8}\kappa_{\pi\nu}\left(N_\pi e_\pi^2 + N_\nu e_\nu^2\right)}{N} \times \langle 0_1^+ | \left(\hat{Q}_\pi + \hat{Q}_\nu\right)\cdot\left(\hat{Q}_\pi + \hat{Q}_\nu\right) | 0_1^+\rangle + \frac{45}{64}\kappa_{\pi\nu} \times \langle 0_1^+ | \left(e_\pi \hat{L}_\pi + e_\nu \hat{L}_\nu\right)\cdot\left(e_\pi \hat{L}_\pi + e_\nu \hat{L}_\nu\right) | 0_1^+\rangle\quad. \tag{76}$$

The second term on the right-hand side (rhs) can be simplified using Ginocchio's $M1$ sum rule [34] with, as a final result,

$$\sum_f B\left(E2; 0_1^+ \to 2_f^+\right) E_x\left(2_f^+\right) = \frac{-\frac{9}{8}\kappa_{\pi\nu}\left(e_\pi^2 N_\pi + e_\nu^2 N_\nu\right)}{N} \times \langle 0_1^+ | \left(\hat{Q}_\pi + \hat{Q}_\nu\right)\cdot\left(\hat{Q}_\pi + \hat{Q}_\nu\right) | 0_1^+\rangle + \frac{135}{32}\kappa_{\pi\nu}\left(e_\pi - e_\nu\right)^2 \frac{N_\pi N_\nu}{N(N-1)}\langle \hat{n}_d\rangle\quad. \tag{77}$$

This result is an EW $E2$ sum rule for the SU(3) limit in which the Majorana force term is <u>not</u> taken into account. A substitution of the energy eigenvalues (Table II) without the Majorana part and the expectation values of Eqs. (70) leads to the results

$$B\left(E2; 0_1^+ \to 2_1^+\right) = \left(e_\pi N_\pi + e_\nu N_\nu\right)^2 \frac{2N+3}{N}\quad, \tag{78}$$

$$B\left(E2; 0_1^+ \to 2_M^+\right) = \left(e_\pi - e_\nu\right)^2 \frac{3(N-1)}{N(2N-1)} N_\pi N_\nu\quad, \tag{79}$$

which are identical to the results of Van Isacker *et al.* [27].

C. The O(6) limit

Here, the $\hat{n}_d$ d-boson number expectation value has been derived in the O(6) limit with, as a result [27, 28],

$$\langle \hat{n}_d \rangle = \frac{N(N-1)}{2(N+1)} \quad , \tag{80}$$

and so, the NEW $M1$ sum rule of Eq. (20) becomes

$$B\left(M1; 0_1^+ \to 1_M^+\right) = \frac{3}{4\pi} \left(g_\pi - g_\nu\right)^2 \frac{3N_\pi N_\nu}{N+1} \quad . \tag{81}$$

D. Some general sum-rule relations

By neglecting the rest term in the EW $M3$ sum rule R [see Eq. 54], this EW $M3$ sum rule is mainly determined through the quadrupole-quadrupole $\hat{Q}_\pi \cdot \hat{Q}_\nu$ term and the boson number contributions coming from the Majorana term. The final result becomes

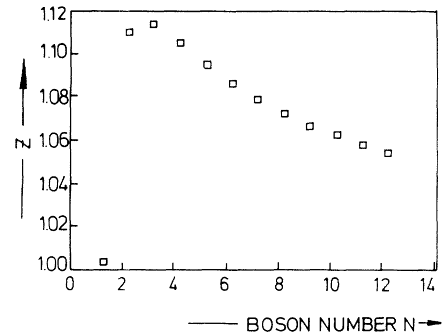

FIG. 6. The Z factor of Eq. (73) indicating the deviation of the $B(M1; 0_1^+ \to 1_M^+)$ value from the exact SU(3) limit result and this as a function of boson number N.

$$\sum_f B\left(M3; 0_1^+ \to 3_f^+\right) E_x\left(3_f^+\right) \simeq \frac{49}{16\pi} \left(\Omega_\pi - \Omega_\nu\right)^2 \left[-\kappa_{\pi\nu} \langle 0_1^+ | \hat{Q}_\pi \cdot \hat{Q}_\nu | 0_1^+ \rangle + \lambda \left(N_\pi \langle 0_1^+ | \hat{n}_{d_\nu} | 0_1^+ \rangle + N_\nu \langle 0_1^+ | \hat{n}_{d_\pi} | 0_1^+ \rangle\right)\right] \quad . \tag{82}$$

Using the same type of approximation in the $M1$ EW sum rule, one obtains the approximate result

$$\sum_f B\left(M1; 0_1^+ \to 1_f^+\right) E_x\left(1_f^+\right) \simeq \frac{9}{4\pi} \left(g_\pi - g_\nu\right)^2 \left[-\kappa_{\pi\nu} \langle 0_1^+ | \hat{Q}_\pi \cdot \hat{Q}_\nu | 0_1^+ \rangle + \lambda \left(N_\pi \langle 0_1^+ | \hat{n}_{d_\nu} | 0_1^+ \rangle + N_\nu \langle 0_1^+ | \hat{n}_{d_\pi} | 0_1^+ \rangle\right)\right] \quad , \tag{83}$$

and so, one obtains the interesting ratio

$$\frac{\sum_f B(M1) E_x(1_f^+)}{\sum_f B(M3) E_x(3_f^+)} \cong \left(\frac{6\left(g_\pi - g_\nu\right)}{7\left(\Omega_\pi - \Omega_\nu\right)}\right)^2 \quad . \tag{84}$$

In Sec. IV B, we have already pointed out the important and close relationship between the EW $M1$ sum rule and the NEW $E2$ sum rule and this particular relation was illustrated in Fig. 5.

In neglecting the rest term R in the $M1$ EW sum rule and putting $e_\pi = e_\nu = e_{\text{eff}}$ also with $\left(g_\pi - g_\nu\right)^2 = 1\mu N^2$, one derives the rather interesting $M1$-$E2$ relationship:

$$\begin{aligned} \frac{1}{e_{\text{eff}}^2} \sum_f B\left(E2; 0_1^+ \to 2_f^+; e_\pi = e_\nu = e_{\text{eff}}\right) = {} & \frac{2\pi}{9} \left(8 - \chi_\pi^2 - \chi_\nu^2\right) \sum_f B\left(M1; 0_1^+ \to 1_f^+\right) E_x\left(1_f^+\right) \\ & + \epsilon_d \left(5N - 6 \langle 0_1^+ | \hat{n}_d | 0_1^+ \rangle\right) + \frac{\lambda}{2} \left(-8 + \chi_\pi^2 + \chi_\nu^2\right) \\ & \times \left(N_\pi \langle 0_1^+ | \hat{n}_{d_\nu} | 0_1^+ \rangle + N_\nu \langle 0_1^+ | \hat{n}_{d_\pi} | 0_1^+ \rangle\right) \quad . \end{aligned} \tag{85}$$

VII. CONCLUSION

In the present work, we have derived both the non-energy-weighted (NEW) and the linear energy-weighted (EW) sum rules within the framework of the proton-neutron interacting boson model (IBM-2) and this, for the electric monopole and quadrupole as well as for the magnetic dipole and octupole transitions. In all cases, we have taken a rather general IBM-2 Hamiltonian and treated, in particular, the effects originating from the Majorana operator which are of importance in the study of the mixed-symmetry states.

The $M1$ NEW sum rule has been studied before by Ginocchio [34], however, with the constraint to good F spin. Here, we extend to calculations in the more general case and point out that the NEW $M1$ strength is proportional to a very good degree (as tested numerically in transitional and deformed Nd, Sm, Gd nuclei) with the d-boson number ground-state expectation value. Thereby we show that an intimate relation exists between the NEW $M1$ summed strength and the nuclear monopole properties (nuclear isotopic shifts with application to the Nd, Sm, and Gd nuclei). A number of rather general results are discussed relating to the NEW $M1$ sum rule, too.

The EW sum rules involve the evaluation of double commutators and some of the technical points needed to arrive at the most concise, final results are amply discussed and elaborated on in Appendices A and B. Here, we discuss in particular the EW $M1$ sum rule for which we present a relation to the NEW $E2$ sum rule, even including the Majorna term. Even though the effect of the latter term is non-negligible, it is shown that the effect can be incorporated into the renormalization of the electric boson charge thereby keeping the almost linear relationship between the EW $M1$ sum rule and the NEW $E2$ sum rule.

We discuss the evaluation of the $E2$ sum rules in a separate section, because of the more complicated nature of this operator, compared to the $E0$, $M1$, and $M3$ operators. Finally, we show that by choosing particularly simple IBM-2 Hamiltonians, i.e., taking U(5), SU(3), and O(6) dynamical symmetry situations, the more general results, derived in Secs. III–V, reduce to very simple results concerning both the EW and NEW sum rules which now contain just a few contributions and can be evaluated fully analytically. We also indicate a relationship between the EW $M1$ and $M3$ sum rules.

ACKNOWLEDGMENTS

The authors are grateful to A. Richter for many stimulating discussions in the early phases and for the extra motivation to study sum rules in a broader context. They are indebted to R. F. Casten, S. Kuyacak, O. Scholten, J. Wood, L. Zamick, W. Ziegler, H. Wörtche, and W. Nazarewicz at various stages of this study. The authors wish to thank the NFWO and IIKW for financial support which made this research possible. This work was also supported by a NATO Research Grant No. CRG92/0011.

APPENDIX A

The calculations of the basic three terms originating from the IBM-2 Hamiltonian [Eq. (3)] and contributing, in general, to the energy-weighted (EW) sum rule for the $E0$, $M1$, and $M3$ operators, i.e., the $\hat{Q}_\rho \cdot \hat{Q}_\rho$ ($\rho = \pi, \nu$), $\hat{Q}_\pi \cdot \hat{Q}_\nu$, and the Majorana $\hat{M}_{\pi\nu}$ terms turns down to the evaluation of the double commutators

$$\left[\left[\hat{Q}_\rho . \hat{Q}_\rho, \left(d^\dagger \tilde{d}\right)^{(\lambda)}_\rho\right], \left(d^\dagger \tilde{d}\right)^{(\lambda)}_\rho\right]^{(0)} \quad , \tag{A1}$$

$$\left[\left[\hat{Q}_\pi . \hat{Q}_\nu, \left(d^\dagger \tilde{d}\right)^{(\lambda)}_\rho\right], \left(d^\dagger \tilde{d}\right)^{(\lambda)}_\rho\right]^{(0)} \quad , \tag{A2}$$

$$\left[\left[\hat{M}_{\pi\nu}, \left(d^\dagger \tilde{d}\right)^{(\lambda)}_\rho\right], \left(d^\dagger \tilde{d}\right)^{(\lambda)}_\rho\right]^{(0)} \quad . \tag{A3}$$

We have to make use of the following important, general results:

$$\left[\hat{T}^{(\lambda)}_\rho \cdot \hat{T}^{(\lambda)}_{\rho'}, \left(d^\dagger \tilde{d}\right)^{(k)}_{\rho''}\right]^{(k)} = \sum_L \sqrt{\frac{2L+1}{2k+1}} (-1)^\lambda \left(\left[\hat{T}^{(\lambda)}_\rho, \left(d^\dagger \tilde{d}\right)^{(k)}_{\rho''}\right]^{(L)} \otimes \hat{T}^{(\lambda)}_{\rho'} \right)^{(k)} + \sum_L \sqrt{\frac{2L+1}{2k+1}} (-1)^{L+k} \left(\hat{T}^{(\lambda)}_\rho \otimes \left[\hat{T}^{(\lambda)}_{\rho'}, \left(d^\dagger \tilde{d}\right)^{(k)}_{\rho''}\right]^{(L)} \right)^{(k)} \quad . \tag{A4}$$

Calling now

$$\hat{X}^{(L)}_{\rho'} \delta_{\rho'\rho''} = \left[\hat{T}^{(\lambda)}_{\rho'}, \left(d^\dagger \tilde{d}\right)^{(k)}_{\rho''}\right]^{(L)} , \tag{A5}$$

the more complicated commutator

$$\left[\left[\hat{T}^{(\lambda)}_\rho \cdot \hat{T}^{(\lambda)}_{\rho'}, \left(d^\dagger \tilde{d}\right)^{(k)}_{\rho''}\right]^{(k)}, \left(d^\dagger \tilde{d}\right)^{(k)}_{\rho'''}\right]^{(0)} \tag{A6}$$

reduces into the result

$$\sum_L \sqrt{\frac{2L+1}{2k+1}} \Bigg[(-1)^\lambda \Bigg((-1)^{k+L+\lambda} \left(\left[\hat{X}^{(L)}_\rho, \left(d^\dagger \tilde{d}\right)^{(k)}_{\rho'''}\right]^{(\lambda)} \otimes \hat{T}^{(\lambda)}_{\rho'} \right)^{(0)} \delta_{\rho\rho''}\delta_{\rho\rho'''} + \left(\hat{X}^{(L)}_\rho \otimes \left[\hat{T}^{(\lambda)}_{\rho'}, \left(d^\dagger \tilde{d}\right)^{(k)}_{\rho'''}\right]^{(L)} \right)^{(0)} \delta_{\rho\rho''}\delta_{\rho'\rho'''} \Bigg) + (-1)^{L+k} \Bigg((-1)^{k+L+\lambda} \left(\left[\hat{T}^{(\lambda)}_\rho, \left(d^\dagger \tilde{d}\right)^{(k)}_{\rho'''}\right]^{(L)} \otimes \hat{X}^{(L)}_{\rho'} \right)^{(0)} \delta_{\rho'\rho''}\delta_{\rho\rho'''} + \left(\hat{T}^{(\lambda)}_\rho \otimes \left[\hat{X}^{(L)}_{\rho'}, \left(d^\dagger \tilde{d}\right)^{(k)}_{\rho'''}\right]^{(\lambda)} \right)^{(0)} \delta_{\rho'\rho''}\delta_{\rho'\rho'''} \Bigg) \Bigg] \quad . \tag{A7}$$

We now give a number of detailed, intermediate results needed to calculate the EW sum rule for the $E0$, $M1$, and $M3$ operators.

First, the result for (A6) [or (A7)] for the quadrupole operator $\hat{Q}_\rho \cdot \hat{Q}_\rho$ becomes

$$\left[\left[\hat{Q}_\rho.\hat{Q}_\rho, \left(d^\dagger \tilde{d}\right)^{(\lambda)}_\rho\right]^{(\lambda)}, \left(d^\dagger \tilde{d}\right)^{(\lambda)}_\rho\right]^{(0)} = \sum_L \sqrt{\frac{2L+1}{2\lambda+1}} \left[(-1)^{\lambda+L}\left(\left[\left[\hat{Q}, \left(d^\dagger \tilde{d}\right)^{(\lambda)}\right]^{(L)}, \left(d^\dagger \tilde{d}\right)^{(\lambda)}\right]^{(2)} \otimes \hat{Q}\right)^{(0)} \right.$$
$$+2\left(\left[\hat{Q}, \left(d^\dagger \tilde{d}\right)^{(\lambda)}\right]^{(L)} \otimes \left[\hat{Q}, \left(d^\dagger \tilde{d}\right)^{(\lambda)}\right]^{(L)}\right)^{(0)}$$
$$\left. + (-1)^{\lambda+L}\left(\hat{Q} \otimes \left[\left[\hat{Q}, \left(d^\dagger \tilde{d}\right)^{(\lambda)}\right]^{(L)}, \left(d^\dagger \tilde{d}\right)^{(\lambda)}\right]^{(2)}\right)^{(0)}\right] \quad . \tag{A8}$$

Starting from the explicit value of the double commutator

$$\left[\left[\hat{Q}, \left(d^\dagger \tilde{d}\right)^{(\lambda)}\right]^{(L)}, \left(d^\dagger \tilde{d}\right)^{(\lambda)}\right]^{(2)} = \frac{2\lambda+1}{5}\,\delta_{L,2}\left(s^\dagger \tilde{d} + d^\dagger s\right)^{(2)}$$
$$+\chi_\rho(2\lambda+1)\sqrt{5(2L+1)}\begin{Bmatrix} L & \lambda & 2 \\ 2 & 2 & 2 \end{Bmatrix}^2 2(-1)^\lambda\left((-1)^{L+\lambda}-1\right)\left(d^\dagger \tilde{d}\right)^{(2)} \quad , \tag{A9}$$

the total contribution for the *first* and *third* terms on the rhs of Eq. (A8) becomes

$$2(-1)^\lambda\,\frac{\sqrt{2\lambda+1}}{5}\left(s^\dagger \tilde{d} + d^\dagger s\right)^{(2)} \cdot \hat{Q} + 4\chi_\rho\,\sqrt{2\lambda+1}(-1)^\lambda\left(\frac{1}{5} - (-1)^\lambda \begin{Bmatrix} 2 & 2 & 2 \\ \lambda & 2 & 2 \end{Bmatrix}\right)\left(d^\dagger \tilde{d}\right)^{(2)} \cdot \hat{Q} \quad . \tag{A10}$$

We have made use of the following equality in obtaining the second term of (A10), i.e.,

$$\sum_L (2L+1)\left(1-(-1)^{L+\lambda}\right)\begin{Bmatrix} L & \lambda & 2 \\ 2 & 2 & 2 \end{Bmatrix}^2 = \frac{1}{5} - (-1)^\lambda \begin{Bmatrix} 2 & 2 & 2 \\ \lambda & 2 & 2 \end{Bmatrix} \quad . \tag{A11}$$

The *second* term on the rhs of Eq. (A8) leads to the result (using similar methods)

$$2\frac{\sqrt{2\lambda+1}}{5}\left((-1)^\lambda s^\dagger \tilde{d} - d^\dagger s\right)^{(2)} \cdot \left((-1)^\lambda s^\dagger \tilde{d} - d^\dagger s\right)^{(2)}$$
$$+4\sqrt{2\lambda+1}\chi\begin{Bmatrix} 2 & \lambda & 2 \\ 2 & 2 & 2 \end{Bmatrix}\left(1-(-1)^\lambda\right)\left((-1)^\lambda s^\dagger \tilde{d} - d^\dagger s\right)^{(2)} \cdot \left(d^\dagger \tilde{d}\right)^{(2)}$$
$$+20\sqrt{2\lambda+1}\chi^2\sum_L (-1)^L \begin{Bmatrix} 2 & \lambda & L \\ 2 & 2 & 2 \end{Bmatrix}^2\left(1-(-1)^{L+\lambda}\right)\left(d^\dagger \tilde{d}\right)^{(L)} \cdot \left(d^\dagger \tilde{d}\right)^{(L)} \quad . \tag{A12}$$

For the contribution from the $\hat{Q}_\pi \cdot \hat{Q}_\nu$ term from the IBM-2 Hamiltonian to the EW $E0$, $M1$, $M3$ sum rule one needs to evaluate the double commutator

$$\left[\left[\hat{Q}_\pi \cdot \hat{Q}_\nu, A^\lambda_\pi\left(d^\dagger \tilde{d}\right)^{(\lambda)}_\pi + A^\lambda_\nu\left(d^\dagger \tilde{d}\right)^{(\lambda)}_\nu\right], A^\lambda_\pi\left(d^\dagger \tilde{d}\right)^{(\lambda)}_\pi + A^\lambda_\nu\left(d^\dagger \tilde{d}\right)^{(\lambda)}_\nu\right]^{(0)}$$
$$= \left(A^\lambda_\pi\right)^2\left[\left[\hat{Q}_\pi \cdot \hat{Q}_\nu, \left(d^\dagger \tilde{d}\right)^{(\lambda)}_\pi\right], \left(d^\dagger \tilde{d}\right)^{(\lambda)}_\pi\right]^{(0)} + (\pi \rightleftharpoons \nu) + A^\lambda_\pi \cdot A^\lambda_\nu\left[\left[\hat{Q}_\pi \cdot \hat{Q}_\nu, \left(d^\dagger \tilde{d}\right)^{(\lambda)}_\pi\right], \left(d^\dagger \tilde{d}\right)^{(\lambda)}_\nu\right]^{(0)} + (\pi \rightleftharpoons \nu) \quad . \tag{A13}$$

We now evaluate the terms in $\left(A^\lambda_\pi\right)^2$ and in $A^\lambda_\pi\,A^\lambda_\nu$, respectively, with the following results.

(1) $\left(A^\lambda_\pi\right)^2$ term:

$$(-1)^\lambda\sqrt{2\lambda+1}\left[\frac{1}{5}\left(s^\dagger \tilde{d} + d^\dagger s\right)^{(2)}_\pi + 2\chi_\pi\left(\frac{1}{5} - (-1)^\lambda \begin{Bmatrix} 2 & 2 & 2 \\ \lambda & 2 & 2 \end{Bmatrix}\right) \cdot \left(d^\dagger \tilde{d}\right)^{(2)}_\pi\right] \cdot \hat{Q}_\nu \quad . \tag{A14}$$

(2) $A_\pi^\lambda\, A_\nu^\lambda$ term:

$$\frac{\sqrt{2\lambda+1}}{5}\left((-1)^\lambda s^\dagger\tilde{d}-d^\dagger s\right)_\pi^{(2)}\cdot\left((-1)^\lambda s^\dagger\tilde{d}-d^\dagger s\right)_\pi^{(2)}$$
$$-\chi_\pi\sqrt{2\lambda+1}\left(1-(-1)^\lambda\right)\begin{Bmatrix}2\,\lambda\,2\\2\,2\,2\end{Bmatrix}\left(d^\dagger\tilde{d}\right)_\pi^{(2)}\cdot\left(s^\dagger\tilde{d}+d^\dagger s\right)_\nu^{(2)}$$
$$-\chi_\nu\sqrt{2\lambda+1}\left(1-(-1)^\lambda\right)\begin{Bmatrix}2\,\lambda\,2\\2\,2\,2\end{Bmatrix}\left(s^\dagger\tilde{d}+d^\dagger s\right)_\pi^{(2)}\cdot\left(d^\dagger\tilde{d}\right)_\nu^{(2)}$$
$$+10\chi_\pi\chi_\nu\sqrt{2\lambda+1}\sum_L(-1)^L\begin{Bmatrix}2\,\lambda\,L\\2\,2\,2\end{Bmatrix}^2\left[1-(-1)^{\lambda+L}\right]\left(d^\dagger\tilde{d}\right)_\pi^{(L)}\cdot\left(d^\dagger\tilde{d}\right)_\nu^{(L)}\quad. \tag{A15}$$

The contribution, coming from the Majorana term, to the double commutator gives the terms

$$\left[\left[\hat{M}_{\pi\nu},A_\pi^\lambda\left(d^\dagger\tilde{d}\right)_\pi^{(\lambda)}+A_\nu^\lambda\left(d^\dagger\tilde{d}\right)_\nu^{(\lambda)}\right],A_\pi^\lambda\left(d^\dagger\tilde{d}\right)_\pi^{(\lambda)}+A_\nu^\lambda\left(d^\dagger\tilde{d}\right)_\nu^{(\lambda)}\right]^{(0)}$$
$$=\left(A_\pi^\lambda\right)^2\left[\left[\hat{M}_{\pi\nu},\left(d^\dagger\tilde{d}\right)_\pi^{(\lambda)}\right],\left(d^\dagger\tilde{d}\right)_\pi^{(\lambda)}\right]^{(0)}+(\pi\rightleftharpoons\nu)+A_\pi^\lambda A_\nu^\lambda\left[\left[\hat{M}_{\pi\nu},\left(d^\dagger\tilde{d}\right)_\pi^{(\lambda)}\right],\left(d^\dagger\tilde{d}\right)_\nu^{(\lambda)}\right]^{(0)}+(\pi\rightleftharpoons\nu)\quad. \tag{A16}$$

Since the Majorana term in the expansion given in Eq. (5), contains terms with various structure, i.e., the $\hat{n}_s\hat{n}_d$ term (term I), the quadrupole $(s^\dagger\tilde{d})^{(2)}\cdot(d^\dagger s)^{(2)}$ term (term II) and the sum over multipoles $(d^\dagger\tilde{d})^{(L)}\cdot(d^\dagger\tilde{d})^{(L)}$ (term III), we give the following separate contributions.

(1) $\left(A_\pi^\lambda\right)^2$ term: (a) Term I,

vanishing contribution. (A17)

(b) Term II,

$$-\xi_2(-1)^\lambda\,\frac{\sqrt{2\lambda+1}}{5}\left(\left(s^\dagger\tilde{d}\right)_\pi^{(2)}\cdot\left(d^\dagger s\right)_\nu^{(2)}+\left(s^\dagger\tilde{d}\right)_\nu^{(2)}\cdot\left(d^\dagger s\right)_\pi^{(2)}\right)\quad. \tag{A18}$$

(c) Term III,

$$2(-1)^\lambda\sqrt{2\lambda+1}\sum B(L)\left(\frac{1}{5}-(-1)^{\lambda+L}\begin{Bmatrix}2\,2\,2\\\lambda\,2\,2\end{Bmatrix}\right)\times\left(d^\dagger\tilde{d}\right)_\pi^{(L)}\cdot\left(d^\dagger\tilde{d}\right)_\nu^{(L)}\quad, \tag{A19}$$

with

$$B(L)=\sum_{k=1,3}(-2\xi_k)(2k+1)\begin{Bmatrix}2\,2\,L\\2\,2\,k\end{Bmatrix}\quad. \tag{A20}$$

(2) $A_\pi^\lambda A_\nu^\lambda$ term: (a) Term I,

vanishing contribution. (A21)

(b) Term II

$$\xi_2(-1)^\lambda\,\frac{\sqrt{2\lambda+1}}{5}\left(\left(s^\dagger\tilde{d}\right)_\pi^{(2)}\cdot\left(d^\dagger s\right)_\nu^{(2)}+\left(s^\dagger\tilde{d}\right)_\nu^{(2)}\cdot\left(d^\dagger s\right)_\pi^{(2)}\right)\quad. \tag{A22}$$

(c) Term III,

$$2(-1)^\lambda\sqrt{2\lambda+1}\sum_L B(L)\left(\frac{1}{5}-(-1)^{\lambda+L}\begin{Bmatrix}2\,2\,2\\\lambda\,2\,2\end{Bmatrix}\right)\times\left(d^\dagger\tilde{d}\right)_\pi^{(L)}\cdot\left(d^\dagger\tilde{d}\right)_\nu^{(L)}\quad. \tag{A23}$$

Combining these various contributions, originating from the EW and thus from the double commutator evaluation, the various $E0$, $M1$, and $M3$ EW expressions, discussed in Sec. IV can be derived in a straightforward way.

APPENDIX B

For the $E2$ operator, the results derived from the single-boson and the quadrupole $\kappa_{\pi\pi}\hat{Q}_\pi\cdot\hat{Q}_\pi+\kappa_{\nu\nu}\hat{Q}_\nu\cdot\hat{Q}_\nu+\kappa_{\pi\nu}\hat{Q}_\pi\cdot\hat{Q}_\nu$ IBM-2 terms, are given in Eqs. (58)–(60).

Here, we discuss the contribution from the Majorana term to the $E2$ EW sum rule. First of all, we start from the general expression

$$\hat{T}(E2)=e_\pi\hat{Q}_\pi+e_\nu\hat{Q}_\nu\quad, \tag{B1}$$

and evaluate

$$\left[\left[\hat{H},\hat{T}(E2)\right],\hat{T}(E2)\right]^{(0)}\quad. \tag{B2}$$

This becomes, written out in some more detail,

$$e_\pi^2\left[\left[\hat{H},\hat{Q}_\pi\right],\hat{Q}_\pi\right]+(\pi\rightleftharpoons\nu)+2e_\pi e_\nu\left[\left[\hat{H},\hat{Q}_\pi\right],\hat{Q}_\nu\right]^{(0)}\quad. \tag{B3}$$

These two specific contributions lead to the expression (for $\rho=\pi,\nu$)

$$\left[\left[\hat{H},\hat{Q}_\rho\right],\hat{Q}_\rho\right]^{(0)} = \left[\left[\hat{H},\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\rho\right],\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\rho\right]^{(0)} + \left[\left[\hat{H},\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\rho\right],\left(d^\dagger\tilde{d}\right)^{(2)}_\rho\right]^{(0)}\chi_\rho$$
$$+\left[\left[\hat{H},\left(d^\dagger\tilde{d}\right)^{(2)}_\rho\right],\left(d^\dagger s+s^\dagger\tilde{d}\right)^{(2)}_\rho\right]^{(0)}\chi_\rho + \left[\left[\hat{H},\left(d^\dagger\tilde{d}\right)^{(2)}_\rho\right],\left(d^\dagger\tilde{d}\right)^{(2)}_\rho\right]^{(0)}\chi^2_\rho \quad , \tag{B4}$$

and

$$\left[\left[\hat{H},\hat{Q}_\pi\right],\hat{Q}_\nu\right]^{(0)} = \left[\left[\hat{H},\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\pi\right],\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\nu\right]^{(0)} + \left[\left[\hat{H},\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\pi\right],\left(d^\dagger\tilde{d}\right)^{(2)}_\nu\right]^{(0)}\chi_\nu$$
$$+\left[\left[\hat{H},\left(d^\dagger\tilde{d}\right)^{(2)}_\pi\right],\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\nu\right]^{(0)}\chi_\pi + \left[\left[\hat{H},\left(d^\dagger\tilde{d}\right)^{(2)}_\pi\right],\left(d^\dagger\tilde{d}\right)^{(2)}_\nu\right]^{(0)}\chi_\pi\chi_\nu \quad . \tag{B5}$$

Again, calling the various types of terms in the Majorana force (see Appendix A), term I, term II, and term III, respectively, we have the following various separate contributions.

(1) Term I, $\hat{n}_s\hat{n}_s$; (a) Contribution in e^2_π (similar for e^2_ν with $\pi \rightleftharpoons \nu$)

$$-5\ \xi_2\ \hat{N}_\nu\left(\hat{N}_\pi - \frac{6}{5}\hat{n}_{d_\pi}\right) \quad . \tag{B6}$$

(b) Contribution in $2e_\pi e_\nu$

$$\frac{1}{2}\xi_2\left(s^\dagger\tilde{d}-d^\dagger s\right)^{(2)}_\pi\cdot\left(s^\dagger\tilde{d}-d^\dagger s\right)^{(2)}_\nu \quad . \tag{B7}$$

(2) Term II, quadrupole term: (a) Contribution in e^2_π (similar for e^2_ν with $\pi \rightleftharpoons \nu$)

$$-\xi_2\left[\left(s^\dagger\tilde{d}\right)^{(2)}_\pi\cdot\left(s^\dagger\tilde{d}\right)^{(2)}_\nu + \left(d^\dagger s\right)^{(2)}_\pi\cdot\left(d^\dagger s\right)^{(2)}_\nu - \left(3+\frac{1}{2}\chi^2_\pi\right)\left(\left(s^\dagger\tilde{d}\right)^{(2)}_\pi\cdot\left(d^\dagger s\right)^{(2)}_\nu + \left(d^\dagger s\right)^{(2)}_\pi\cdot\left(s^\dagger\tilde{d}\right)^{(2)}_\nu\right) + \chi_\pi\left(d^\dagger\tilde{d}\right)^{(2)}_\pi\cdot\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\nu\right] \quad . \tag{B8}$$

(b) Contribution in $2e_\pi e_\nu$

$$\xi_2\left\{-5\left(\hat{N}_\pi-\hat{n}_{d_\pi}\right)\left(\hat{N}_\nu-\hat{n}_{d_\nu}\right) + \left(\left(\hat{N}_\pi-\hat{n}_{d_\pi}\right)\hat{n}_{d_\nu} + \left(\hat{N}_\nu-\hat{n}_{d_\nu}\right)\hat{n}_{d_\pi}\right)\right.$$
$$+\frac{\chi_\pi}{2}\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\pi\cdot\left(d^\dagger\tilde{d}\right)^{(2)}_\nu + \frac{\chi_\nu}{2}\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\nu\cdot\left(d^\dagger\tilde{d}\right)^{(2)}_\pi$$
$$\left.-\frac{\chi_\pi\chi_\nu}{2}\left(\left(s^\dagger\tilde{d}\right)^{(2)}_\pi\cdot\left(d^\dagger s\right)^{(2)}_\nu + \left(s^\dagger\tilde{d}\right)^{(2)}_\nu\cdot\left(d^\dagger s\right)^{(2)}_\pi\right) - \frac{1}{\sqrt{5}}\sum_L\left(d^\dagger\tilde{d}\right)^{(L)}_\pi\cdot\left(d^\dagger\tilde{d}\right)^{(L)}_\nu\right\} \quad . \tag{B9}$$

(3) Term III, multipole sum: (1) Contribution in e^2_π (similar for e^2_ν with $\pi \rightleftharpoons \nu$)

$$\frac{2}{5}\left(3\xi_1+7\xi_3\right)\left(\hat{N}_\pi-\hat{n}_{d_\pi}\right)\hat{n}_{d_\nu} + \frac{1}{5}\chi_\pi\left(3\xi_1-8\xi_3\right)\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\pi\cdot\left(d^\dagger\tilde{d}\right)^{(2)}_\nu$$
$$+\sum_L B(L)\left[\frac{1}{2}-\chi^2_\pi\left(\frac{1}{\sqrt{5}}-(-1)^L\sqrt{5}\begin{Bmatrix}L&2&2\\2&2&2\end{Bmatrix}\right)\right]\left(d^\dagger\tilde{d}\right)^{(L)}_\pi\cdot\left(d^\dagger\tilde{d}\right)^{(L)}_\nu \quad . \tag{B10}$$

(b) Contribution in $2e_\pi e_\nu$

$$-\frac{1}{10}\left(6\xi_1+14\xi_3\right)\left(\left(s^\dagger\tilde{d}\right)^{(2)}_\pi\cdot\left(d^\dagger s\right)^{(2)}_\nu + \left(d^\dagger s\right)^{(2)}_\pi\cdot\left(s^\dagger\tilde{d}\right)^{(2)}_\nu\right)$$
$$-\frac{1}{10}\left(-3\xi_1+8\xi_3\right)\left(\chi_\nu\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\pi\cdot\left(d^\dagger\tilde{d}\right)^{(2)}_\nu + \chi_\pi\left(s^\dagger\tilde{d}+d^\dagger s\right)^{(2)}_\nu\cdot\left(d^\dagger\tilde{d}\right)^{(2)}_\pi\right)$$
$$-\frac{\sqrt{5}}{2}\chi_\pi\chi_\nu\sum_L\left[\xi_1\left(-\frac{3}{5}\begin{Bmatrix}2&2&1\\2&2&L\end{Bmatrix}+(-1)^L 6\begin{Bmatrix}2&2&2\\2&2&L\\1&2&2\end{Bmatrix}\right)\right.$$
$$\left.+\xi_3\left(\frac{8}{5}\begin{Bmatrix}2&2&3\\2&2&L\end{Bmatrix}+(-1)^L 14\begin{Bmatrix}2&2&2\\2&2&L\\3&2&2\end{Bmatrix}\right)\right]\times\left(d^\dagger\tilde{d}\right)^{(L)}_\pi\cdot\left(d^\dagger\tilde{d}\right)^{(L)}_\nu \quad . \tag{B11}$$

[1] D. Bohle *et al.*, Phys. Lett. **137B**, 27 (1984).
[2] A. Richter, in *Proceedings of the 4th International Spring Seminar on Nuclear Physics*, edited by A. Covello (World Scientific, Singapore, 1993), and references therein.
[3] U. Kneissl, Prog. Part. Nucl. Phys. **24**, 41 (1990), and references therein.
[4] P. Sarriguren, E. Moya de Guerra, R. Nojarov, and A. Faessler, J. Phys. G **19**, 2191 (1993), and references therein.
[5] A. Faessler and R. Nojarov, Phys. Rev. C **41**, 1243 (1990).
[6] I. Hamamoto and C. Magnusson, Phys. Lett. B **260**, 6 (1991).
[7] C. De Coster, K. Heyde, and A. Richter, Nucl. Phys. **A542**, 375 (1992).
[8] D. Zawischa, M. MacFarlane, and J. Speth, Phys. Rev. C **42**, 1461 (1990).
[9] T. Otsuka, Nucl. Phys. **A507**, 129c (1990).
[10] R. R. Hilton, Ann. Phys. (N.Y.) **214**, 258 (1992).
[11] C. De Coster and K. Heyde, Nucl. Phys. **A524**, 441 (1991); **A529**, 507 (1991).
[12] C. De Coster and K. Heyde, Phys. Rev. Lett. **63**, 2797 (1989).
[13] K. Heyde and C. De Coster, Phys. Rev. C **44**, R2262 (1991).
[14] K. Heyde, C. De Coster, A. Richter, and H.-J. Wörtche, Nucl. Phys. **549**, 103 (1992).
[15] U. Hartmann *et al.*, Nucl. Phys. **A465**, 25 (1987).
[16] W. Ziegler, C. Rangacharyulu, A. Richter, and C. Spieler, Phys. Rev. Lett. **65**, 2515 (1990).
[17] C. Rangacharyulu, A. Richter, H.-J. Wörtche, W. Ziegler, and R. F. Casten, Phys. Rev. C **43**, R949 (1991).
[18] J. Margraf *et al.*, Phys. Rev. C **45**, R521 (1992); **47**, 1474 (1993).
[19] E. Garrido, E. Moya de Guerra, P. Sarriguren, and T. M. Udias, Phys. Rev. C **44**, R1250 (1991).
[20] I. Hamamoto and W. Nazarewicz, Phys. Lett. B **297**, 25 (1992).
[21] R. F. Casten, D. S. Brenner, and P. Haustein, Phys. Rev. Lett. **58**, 658 (1987).
[22] L. Zamick and D. C. Zheng, Phys. Rev. C **44**, 2522 (1991); **46**, 2106 (1992).
[23] E. Moya de Guerra and L. Zamick, Phys. Rev. C **47**, 2604 (1993).
[24] K. Heyde, C. De Coster, and D. Ooms (unpublished).
[25] D. Ooms, Lic. thesis, University of Gent, 1992 (unpublished).
[26] D. J. Rowe, *Nuclear Collective Motion* (Methuen, New York, 1970).
[27] P. Van Isacker, K. Heyde, J. Jolie, and A. Sevrin, Ann. Phys. (N.Y.) **171**, 253 (1986).
[28] O. Scholten, K. Heyde, P. Van Isacker, J. Jolie, J. Moreau, M. Waroquier, and J. Sau, Nucl. Phys. **A438**, 41 (1985).
[29] F. Iachello and I. Talmi, Rev. Mod. Phys. **59**, 339 (1987).
[30] P. O. Lipas, P. von Brentano, and A. Gelberg, Rep. Prog. Phys. **11**, 1363 (1990).
[31] F. Iachello and A. Arima, *The Interacting Boson Model* (Cambridge University Press, Cambridge, 1987).
[32] D. Bonatsos, *Interacting Boson Models of Nuclear Structure* (Clarendon Press, Oxford, 1988).
[33] O. Scholten, Ph.D. thesis, University of Groningen, 1980 (unpublished).
[34] J. Ginocchio, Phys. Lett. B **265**, 6 (1991).
[35] T. Otsuka and N. Yoshida, Users Manual of the program NPBOS, JAERI 1985.
[36] H. Harter, A. Gelberg, and P. von Brentano, Phys. Lett. **157B**, 1 (1985).
[37] H. Harter, Ph.D. thesis, University of Köln, 1987 (unpublished).
[38] K. Heyde, C. De Coster, D. Ooms, and A. Richter, Phys. Lett. B **312**, 267 (1993).
[39] J. N. Ginocchio and M. W. Kirson, Nucl. Phys. **A350**, 31 (1980).
[40] E. Otten, in *Treatise on Heavy Ion Science*, edited by D. A. Bromley (Plenum, New York, 1988), Vol. 8, p. 517.
[41] O. Scholten, A. E. L. Dieperink, K. Heyde, and P. Van Isacker, Phys. Lett. **149B**, 279 (1984).
[42] N. Lo Iudice, Phys. Rev. C **38**, 2895 (1988).
[43] L. Zamick, Phys. Rev. C **33**, 691 (1986).
[44] H. Liu and L. Zamick, Phys. Rev. C **36**, 2064 (1987).
[45] G. Soloviev, *Theory of Atomic Nuclei: Quasiparticles and Phonons* (Institute of Physics, Bristol, 1992).
[46] T. Otsuka and J. Ginocchio, Phys. Rev. Lett. **54**, 777 (1985).
[47] G. Wenes, T. Otsuka, and J. Ginocchio, Phys. Rev. C **37**, 1878 (1988).
[48] A. M. Lane, *Nuclear Theory* (W.A. Benjamin, New York, 1964).
[49] A. Bohr and B. Mottelson, *Nuclear Structure* (W.A. Benjamin, New York, 1975), Vol. 20.
[50] T. Otsuka, A. Arima, and F. Iachello, Nucl. Phys. **A309**, 1 (1978).
[51] J. Ginocchio and P. Van Isacker, Phys. Rev. C **33**, 365 (1986).
[52] A. F. Barfield and K. P. Lieb, Phys. Rev. C **41**, 1762 (1990).
[53] R. F. Casten and A. Wolf, Phys. Lett. B **280**, 1 (1992).
[54] A. Wolf and R. F. Casten, Phys. Rev. C **46**, 1323 (1992).