Dependence on the mass number of energy quantities of a Λ in hypernuclei with the cosh and the Gaussian potential

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The Λ binding energies obtained from the (π^+, K^+) data are analyzed with the aid of suitable Λ-nucleus potentials. Exact and approximate analytic expressions, as functions of the mass number, for the binding energy and other energy quantities of a Λ hyperon in hypernuclei in ground and excited states are proposed and the region of their validity is discussed. The density dependence of the Λ-nucleus potential is taken into account in an approximate way and it turns out that its role is essential for a reliable description of the recent hypernuclear data. The formulas for the ground state Λ energy are used to derive approximate expressions for the Λ binding energy in double Λ hypernuclei.

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I. INTRODUCTION

The Λ hyperon separation energy B_Λ is a fundamental quantity of the hypernuclear physics and it is defined by the relation

$$M_{HF}c^2 = (M_{\rm core} + M_\Lambda)c^2 - B_\Lambda, \qquad (1)$$

where M_{HF}, $M_{\rm core}$, and M_Λ are the masses of the hypernucleus, its core, and the Λ hyperon, respectively. In the "early years" the experimental determination of B_Λ was mainly performed using nuclear emulsions. In such experiments the B_Λ values of ground-state hypernuclei have been determined exclusively from π^- mesonic decays. The main reasons for this choice are (a) the energy release is small allowing energies to be accurately determined from range measurements, (b) neutron emission is rare, and (c) the π^- is readily identified from the appearance of its track [1]. Unfortunately, the data are limited in a narrow region of mass numbers ($A_{\rm core} < 15$) where the mesonic decay is dominant. A study of these data [1–3] shows that the Λ hyperon is inhibited by the Pauli principle and that there is an orderly increase of about 1 MeV/nucleon in B_Λ with the mass number of light hypernuclei, there being no evidence of the large fluctuations observed for separation energies of individual nucleons in ordinary nuclei [1].

Later on, the counter technique was adopted and the strangeness exchange reaction was used for the production of hypernuclei (for excellent reviews see [4]). The (K^-, π^-) reactions with low momentum transfer have given data on substitutional orbits, that is, l_Λ is the same with that of the valence neutron orbit. Several target nuclei were used (^{12}C, ^{27}Al, ^{40}Ca, ^{51}V, and ^{209}Bi) [5,6] and in some cases the ground-state Λ energies were measured. The low intensity of the kaon beam, however, did not permit clear measurements for hypernuclei with mass number larger than 32 [4].

More recently the associated production reaction (π^+, K^+) has been very useful in producing bound Λ single-particle states in a variety of nuclei [7–11]. The energy levels of this hyperon have been measured by means of this reaction in nuclear targets ranging from ^{9}Be to ^{89}Y. The peaks observed in the excitation spectra of the (π^+, K^+) reaction in nuclei can be identified with the various orbital angular momentum states s, p, d, f,... of the Λ hyperon. For the case of the ^{12}C$(\pi^+, K^+)^{12}_\Lambda$C reaction where the angular distribution was measured, the identification of observed peaks with s and p Λ single-particle strength was verified. The basic properties and feasibility of the (π^+, K^+) reaction were theoretically studied firstly by Dover *et al.* [12] and then by Bando and Motoba [13]. We recall, that contrary to the (K^-, π^-) reaction, it has the feature to excite nonsubstitutional stretched-spin states preferentially, since it involves a substantial momentum transfer of $q \simeq 340$ MeV/c (at $P_\pi = 1040$ MeV/c) and gives the possibility for the study of the medium and heavy hypernuclei.

The usefulness of the new experimental data is that they provide a much better description of Λ binding energies as a function of the mass number A. For the first time we have Λ binding energies for a wide range of mass numbers and in some cases complete sets of bound and first unbound energy levels for nodeless Λ orbitals at fixed A are available [14]. This fact gives the possibility for a better overall fit of the data for the determination of the parameters of properly chosen Λ-nucleus potentials.

It is noted, that the typical experimental resolution of the above-mentioned experiments is about 3 MeV. It is expected, however, that much better precision would be accomplished at PILAC (Pion Linear Accelerator) [15,16]. Therefore, a better determination of the Λ spacing (especially of heavy hypernuclei, e.g., $^{208}_\Lambda$Pb) will determine more accurately the depth and the range of the Λ-nucleus potential. In addition, one would more stringently test the standard mean-field picture [16].

The experimental data are usually analyzed using proper phenomenological Λ-nucleus potentials such as

the Woods-Saxon potential (see, for example, [14,17–21]) or by means of self-consistent calculations [14,22–23] as, for example, the work of Millener *et al.* [14] where density-dependent and nonlocal Λ-nucleus potentials were derived by means of the Skyrme-Hartree-Fock approximation. This study [14] has shown that the density dependence or nonlocality of the Λ-nucleus potential is quite important for a reliable description of the experimental data.

In this work we propose several approximate analytic expressions for various energy quantities of a Λ in hypernuclei as function of the mass number. These expressions are derived on the basis of the $-V_0/\cosh^2\frac{r}{R}$ (cosh potential) and the Gaussian potential. The radius parameters of these potentials are taken A dependent, which is somewhat equivalent to assuming density dependent potentials.

Nowadays, it is quite straightforward for one to numerically obtain the Λ energies on the basis of the abovementioned methods. However, this does not imply that analytic expressions are no longer attractive and of no interest. Analytic formulas (approximate or, if possible, exact) for the binding energies and other important energy quantities of a Λ hyperon in direct connection with the mass number are very convenient for practical use and show explicitly the analytic dependence of these quantities on the mass number. In addition, our approach offers the possibility to study the role of the additional A dependence which is introduced by the radius parameter $r_0(A_c)$.

The formulas for the ground state B_Λ may also be used for the derivation of approximate analytic expressions for the Λ binding energy $B_{\Lambda\Lambda}$ in double Λ hypernuclei. It is noted that the study of ΛΛ hypernuclei attracts interest because studying the level schemes and binding energies of such systems one expects to get more information about the nature of Λ-Λ interaction and explore better the structure of the baryon-baryon strong interaction. In addition, ΛΛ hypernuclei may be used to rule out the existence of the H dibaryon in a certain mass region [24,25].

In Sec. II the proposed expressions for the various energy quantities of a Λ in Λ hypernuclei are introduced and discussed, while in Sec. III the approximate formulas for the Λ binding energy in ΛΛ hypernuclei are given. In Sec. IV the numerical results are reported and commented on, and, finally, Sec. V summarizes the main results of the paper.

II. ANALYTIC EXPRESSIONS FOR THE VARIOUS ENERGY QUANTITIES OF A Λ HYPERON IN HYPERNUCLEI

In this section we present analytic exact and approximate expressions of various energy quantities of a Λ in hypernuclei. First we present approximate expressions based on a Gaussian Λ-nucleus potential:

$$V_{\Lambda\text{-}A} = -D\exp\left(\frac{-r^2}{R^2}\right), \tag{2}$$

where $D > 0$ is the depth of the potential and R is the radius parameter which is taken A dependent [$R = r_0(A_c)A_c^{1/3}$] and parametrized in the spirit of [14,21] (see also [26]):

$$r_0(A_c) = r_0 + r_0' A_c^{-2/3}. \tag{3}$$

A_c is the mass number of the host nucleus. For this potential model the eigenvalue problem is treated numerically. It turns out, however, that approximate solutions for the eigenenergies may also be used [27,28] with reasonable accuracy. In [28] large coupling expansions of the eigenenergies, wave functions, and Regge trajectories of any even power potential

$$V(r) = -g^2\sum_{j=0}^{\infty} N_{2j} r^{2j} \tag{4}$$

have been obtained. These general expansions are then used to obtain eigenenergy expansions and Regge trajectories for various potentials. In the case of the Gauss potential, N_{2j} has the form

$$N_{2j} = \frac{(-1)^j a^{2j}}{j!} \tag{5}$$

while combining (1) and (3), g^2 and a are written as follows :

$$g^2 = D, \quad a = 1/R. \tag{6}$$

Then for large coupling constants g^2 the following expressions for the eigenenergies hold [27,28] :

$$\begin{aligned}(K^2+g^2) = {} & ga(2l+q) - \frac{a^2}{2^4}[3(q^2+1)+4(3q-1)l+8l^2] \\ & -\frac{a^3}{3.2^8 g}[q(11q^2+1)+2(33q^2-6q+1)l+24(5q-1)l^2+64l^3] \\ & -\frac{a^4}{3.2^{15}g^2}[4(85q^4+2q^2-423)+l(2720q^3-71q^2+32q+2796) \\ & +32l^2(252q^2-12q+64)+256l^3(41q-9)+4096l^4]+O(1/g^3),\end{aligned} \tag{7}$$

where q=4n+3 with n=0,1,2,..., and $2\mu E_{nl}/\hbar^2 = K^2$, μ being the Λ-core reduced mass. Note, that in (7) g^2 is defined

as $2\mu D/\hbar^2=g^2$. Using formula (6) for a one can write an expression for the Λ energy in any state nl as a function of the core mass number:

$$\begin{aligned}E_{nl} = -D\{&1 - c\epsilon^{-1}(2l+q)A_c^{-1/3} + c^2\epsilon^{-2}2^{-4}[3(q^2+1)+4(3q-1)l+8l^2]A_c^{-2/3}\\ &+c^3\epsilon^{-3}3^{-1}2^{-8}[9(11q^2+1)+2(33q^2-6q+1)l+24(5q-1)l^2+64l^3]A_c^{-1}\\ &+c^4\epsilon^{-4}3^{-1}2^{-15}[4(85q^4+2q^2-423)+l(2720q^3-71q^2+32q+2796)\\ &+32l^2(252q^2-12q+64)+256l^3(41q-9)+4096l^4]A_c^{-4/3}\cdots\}\end{aligned} \tag{8}$$

with

$$c = \left(\frac{\hbar^2}{2\mu D r_0^2}\right)^{1/2} \tag{9}$$

and

$$\epsilon = 1 + \beta A_c^{-2/3}, \qquad \beta = r_0'/r_0. \tag{10}$$

With the aid of formula (8) one can also calculate the "energy level spacing" of two neighboring level ($\Delta l = 1$):

$$\begin{aligned}\Delta_{n,l}^{n,l+1} = D\big(&c\epsilon^{-1}[2(l+1)+q-2l-9]A_c^{-1/3} + c^2\epsilon^{-2}2^4[4(3q-1)(l+1)-4(3q-1)l+(l+1)^2-8l^2]A_c^{-2/3}\\ &+c^3\epsilon^{-3}3^{-1}2^{-8}\{2(33q^2-6q+1)(l+1)-2(33q^2-6q+1)l\\ &+24(5q-1)[(l+1)^2-l^2]+64[(l+1)^3-l^3]\}A_c^{-1}\\ &+c^4\epsilon^{-4}3^{-1}2^{-15}\big\{(2720q^3-71q^2+32q+2796)(l+1)-\{(2720q^3-71q^2+32q+2976)l\\ &+32(252q^2-12q+64)[(l+1)^2-l^2]\\ &+256(41q-9)[(l+1)^3-l^3]+4096[(l+1)^4-l^4]\}A_c^{-4/3}+\cdots\big).\end{aligned} \tag{11}$$

Using the above expressions one can derive approximate analytic expressions for the B_Λ in the lower Λ orbits as well as for the lower Λ spacings:

$$E_{1s} = -D\left[1 - 6d_0A_c^{-1/3} + \frac{15}{2}d_0^2A_c^{-2/3} + \left(\frac{25}{8}d_0^3 + 6\beta d_0\right)A_c^{-1}\right], \tag{12}$$

$$E_{1p} = -D\left[1 - 10d_0A_c^{-1/3} + \frac{35}{2}d_0^2A_c^{-2/3} + \left(\frac{105}{8}d_0^3 + 10\beta d_0\right)A_c^{-1}\right], \tag{13}$$

$$E_{1d} = -D\left[1 - 14d_0A_c^{-1/3} + \frac{63}{2}d_0^2A_c^{-2/3} + \left(\frac{273}{8}d_0^3 + 14\beta d_0\right)A_c^{-1}\right], \tag{14}$$

$$E_{2s} = -D\left[1 - 14d_0A_c^{-1/3} + \frac{75}{2}d_0^2A_c^{-2/3} + \left(\frac{315}{8}d_0^3 + 14\beta d_0\right)A_c^{-1}\right], \tag{15}$$

$$\Delta E_{sp} = 4Dd_0A_c^{-1/3} - 10Dd_0^2A_c^{-2/3} - 2D(5d_0^3 + 2\beta d_0)A_c^{-1}, \tag{16}$$

$$\Delta E_{pd} = 4Dd_0A_c^{-1/3} - 14Dd_0^2A_c^{-2/3} - D(21d_0^3 + 4\beta d_0)A_c^{-1}, \tag{17}$$

where

$$d_0 = \left(\frac{\hbar^2}{8m_\Lambda D r_0^2}\right)^{1/2}. \tag{18}$$

For β=0, one obtains the corresponding expressions for static local case ($R = r_0A_c^{1/3}$). It is seen that the A dependence of the radius parameter affects the coefficient of the A_c^{-1} term and all the higher power terms which also contain the dependence on the nucleon mass. It is also seen that this dependence influences the p and d states more. This is also verified from our numerical calculations (see Sec. IV).

Next by employing the Helmann-Feynman theorem (see, for example, [29]) analytic expressions for the kinetic and potential energy of the Λ orbits may also be obtained:

$$
\begin{aligned}
T_{1s} &= 3Dd_0A_c^{-1/3} - \tfrac{15}{2}Dd_0^2A_c^{-2/3} - D(\tfrac{75}{16}d_0^3 + 3\beta d_0)A_c^{-1}, \\
T_{1p} &= 5Dd_0A_c^{-1/3} - \tfrac{35}{2}Dd_0^2A_c^{-2/3} - D(\tfrac{375}{16}d_0^3 + 5\beta d_0)A_c^{-1}, \\
T_{1d} &= 7Dd_0A_c^{-1/3} - \tfrac{75}{2}Dd_0^2A_c^{-2/3} - D(\tfrac{819}{16}d_0^3 + 7\beta d_0)A_c^{-1}, \\
V_{1s} &= -D[1 - 3d_0A_c^{-1/3} - (\tfrac{25}{16}d_0^3 - 3\beta d_0)A_c^{-1}], \\
V_{1p} &= -D[1 - 5d_0A_c^{-1/3} - (\tfrac{105}{16}d_0^3 - 5\beta d_0)A_c^{-1}], \\
V_{1d} &= -D[1 - 7d_0A_c^{-1/3} - (\tfrac{273}{16}d_0^3 - 7\beta d_0)A_c^{-1}].
\end{aligned}
\tag{19}
$$

We can also approximate the Λ-nucleus interaction by a Λ-nucleus potential of the form [30–33]

$$V_{\Lambda\text{-}A}(r) = -D/\cosh^2\frac{r}{R}, \tag{20}$$

where the same parametrization [expression (3)] is adopted for the radius parameter R. This potential model has the advantage that the Schrödinger eigenvalue problem can be solved analytically for states with l=0. The energy eigenvalues are given by the expression

$$E_{n0} = \frac{-\hbar^2}{2\mu R^2}\left[\frac{1}{2}\sqrt{\frac{8\mu DR^2}{\hbar^2}+1} - \left(2n+\frac{3}{2}\right)\right]^2, \quad n = 0, 1, 2, \ldots . \tag{21}$$

Expression (21) is a rather simple "semiempirical" formula $B_\Lambda = B_\Lambda(A_c)$ which reproduces the average trend of the variation of B_Λ with the mass number A_c. It is, of course, an approximate formula and it should also be clear that it contains only part of a proper hypernuclear formula, as is the case with other semiempirical formulas of this type [34].

Another advantage in using the above potential model is the possibility of easily obtaining analytic expressions for the expectation values of the potential and kinetic energies of a Λ in states with l=0. Thus one has by using the Helmann-Feynman theorem

$$\langle T\rangle_{n0} = E_{n0} + D - \frac{2D(2n+1.5)}{\sqrt{8\mu DR^2/\hbar^2+1}}, \tag{22}$$

$$\langle V\rangle_{n0} = -D + \frac{2D(2n+1.5)}{\sqrt{8\mu DR^2/\hbar^2+1}}. \tag{23}$$

These expressions for the ground state take the simple forms:

$$\langle T_\Lambda\rangle = D\frac{(2\lambda-1)(3\lambda+1/2)}{2\lambda(2\lambda+1/2)(2\lambda+1)}, \tag{24}$$

$$\langle V_\Lambda\rangle = -D\frac{(2\lambda-1)}{(2\lambda+1/2)}, \tag{25}$$

with

$$\lambda = \frac{1}{4}\left[\left(1+8\mu DR^2/\hbar^2\right)^{1/2} - 1\right],$$

which are equivalent to formulas (8) and (7), respectively, of Ref. [31], where, however, r_0 was considered independent of A. The expansions of (21), (24), and (25) as function of A_c have the following form:

$$B_\Lambda = D\left[1 - 6d_0A_c^{-1/3} + 10d_0^2A_c^{-2/3} - (3d_0^3 + 6\beta d_0)A_c^{-1} - 3d_0\frac{m_\Lambda}{m_N}A_c^{-4/3} + \cdots\right], \tag{26}$$

$$V_\Lambda = -D\left[1 - 3d_0A_c^{-1/3} + \left(\frac{3}{2}d_0^3 + 3\beta d_0\right)A_c^{-1} - \frac{3}{2}\frac{m_\Lambda}{m_N}d_0A_c^{-4/3} + \cdots\right], \tag{27}$$

$$T_\Lambda = D\left[3d_0A_c^{-1/3} - 10d_0^2A_c^{-2/3} + \left(\frac{9}{2}d_0^3 - 3\beta d_0\right)A_c^{-1} + \frac{3}{2}d_0\frac{m_\Lambda}{m_N}A_c^{-4/3} + \cdots\right]. \tag{28}$$

It is seen that the forms of the first two terms of the expansions of B_Λ with the two potential models are the same while the third and fourth ones differ a little.

In the framework of the same potential model an approximate analytic expression is derived [33] which also gives the B_Λ in states with $l \neq 0$:

$$B_\Lambda^{(l)} = -E_\Lambda^{(l)} \simeq \frac{\hbar^2}{2\mu R^2}\left[[2\lambda - (l+1)]^2 - \frac{l(l+1)\Gamma(2\lambda+0.5)\Gamma(2\lambda-l-0.8)}{3\Gamma(2\lambda+0.7)\Gamma(2\lambda-l-1)}\right]. \tag{29}$$

Our numerical results have shown that expression (29) works very well: That is, the B_Λ values obtained with (29) differ from the numerically calculated Λ energies less than 1% throughout the periodic table [33]. It should be noted, however, that formula (29) does not express explicitly the dependence of B_Λ on A as is the case with the analytic formulas proposed above.

It is also noted that formula (24) is particularly useful in deriving an expression for the oscillator spacing $\hbar\omega_\Lambda$ which approximates the lower Λ energy level spacing $\Delta E_{\rm sp}$ by applying the virial theorem as proposed in [31]

$$\begin{aligned}\Delta E_{\rm sp} \simeq \hbar\omega_\Lambda &= \frac{4}{3}\langle T\rangle_{HO} \simeq \frac{4}{3}\langle T\rangle_{1s} \\ &= 4Dd_0A_c^{-1/3} - \frac{40}{3}Dd_0^2A_c^{-2/3} \\ &\quad +2D(3d_0^3 - 2\beta d_0)A_c^{-1} \\ &\quad +2Dd_0\frac{m_N}{m_\Lambda}A_c^{-4/3} + \cdots . \end{aligned} \tag{30}$$

It is seen that the first term of the above expression is of the same form with that of expression (16) while the second one is somehow different. However, the third terms of these expressions have opposite contributions and therefore one expects similar results since in addition the parameters of the two potentials do not differ appreciably (see next section).

Finally, in the framework of the analysis of Λ hypernuclei with the aid of $-V_0/\cosh^2(r/R)$ potential it is also possible to derive approximate analytic expressions of the mass number A_c as function of the B_Λ. The simplest expression which may be derived is based on the assumption that $\mu = m_\Lambda$:

$$A_c^{(0)} \equiv A - 1 = \left(\frac{64d_0}{B_\Lambda}\right)\left[\sqrt{1+8\left(\frac{D}{B_\Lambda}\right)} - 3\right]^{-3} . \tag{31}$$

As should be expected, this expression is not accurate for light nuclei. This is obvious if we recall that

$$\mu = m_\Lambda\left(1 + \frac{m_\Lambda}{m_N}A_c^{-1}\right)^{-1}, \tag{32}$$

which shows that we completely ignore the second term of the binomial. However, this is valid only for large A_c. Our numerical results show that for A_c=12 there is a deviation of about 13%. However, as the mass number increases we obtain better results, e.g., for A_c around 90 the deviation is less than 1.5%. In general, our study shows that expression (31) is satisfactory for $A_c > 30$. Next expanding the binomial of (32) in powers of A_c and including now the second term of the expansion one obtains after some algebra

$$A_c^{(1)} \equiv A - 1 = 3\frac{m_\Lambda}{m_N}\left\{-1 + \sqrt{1 + 6\frac{m_\Lambda}{m_N}\left[\frac{1}{8d_0}\left(\frac{B_\Lambda}{D}\right)^{1/2}\left(-3 + \sqrt{1+8\frac{D}{B_\Lambda}}\right)\right]^3}\right\}^{-1} . \tag{33}$$

Our results show that (33) is very satisfactory. For $A_c = 12$ the deviation from the "exact value" is about 0.3%. We note also, that including the other terms of the expansion in an approximate way the following very accurate expression is obtained:

$$A_c^{(2)} = 3\frac{m_\Lambda}{m_N}\left\{-1 + \sqrt{1 + 6\frac{m_\Lambda}{m_N}y^3\left[2 - \left(1 + \frac{m_\Lambda}{m_N}y^3\right)^{3/2} + \frac{3}{2}\frac{m_\Lambda}{m_N}y^3\right]}\right\}^{-1} , \tag{34}$$

where $y^3 \equiv A_c^0$.

III. APPROXIMATE EXPRESSIONS FOR THE Λ BINDING ENERGY OF ΛΛ HYPERNUCLEI

The existing data about double Λ hypernuclei are quite few. Old experiments with nuclear emulsions gave evidence for the existence of ${}^{6}_{\Lambda\Lambda}$He and ${}^{10}_{\Lambda\Lambda}$Be [35,36]. The overall binding of the Λ hyperons $B_{\Lambda\Lambda}$ and the contribution $\Delta B_{\Lambda\Lambda}$ from the Λ-Λ interaction is (17.7±0.4) MeV, (4.3±0.4) MeV and (10.9±0.5) MeV, (4.7±0.5) MeV, respectively. The values of $\Delta B_{\Lambda\Lambda}$ gave rise to many theoretical studies regarding their compatibility to be accounted for the same $V_{\Lambda\Lambda}$ potential (see, for example, [37]). Recent experiments [38,39] carried out at KEK renewed the interest. A new double hypernuclear event was reported and it was intepreted as either ${}^{10}_{\Lambda\Lambda}$Be or ${}^{13}_{\Lambda\Lambda}$B, the $B_{\Lambda\Lambda}$ energies being (8.5±0.7) and (27.6 ±0.7) MeV, respectively. It is noted, however, that the first choice corresponds to a repulsive Λ-Λ interaction ($\Delta B_{\Lambda\Lambda}$ = −4.9 ±0.7) while the latter to an attractive one ($\Delta B_{\Lambda\Lambda}$ = 4.9 ±0.7). This event was analyzed theoretically by Dover *et al.* [24] and by Yamamoto *et al.* [40]. Both studies concluded that the interpretation of the event as ${}^{13}_{\Lambda\Lambda}$B is the most probable. This is consistent with the old data and in accord with a recent reinterpretation of the ${}^{10}_{\Lambda\Lambda}$Be event by Dalitz *et al.* [41]. Dover *et al.* [24] state that a $\Delta B_{\Lambda\Lambda} > 0$ should be expected in any reasonable model. Finally, Himeno *et al.* [42] calculated the $B_{\Lambda\Lambda}$ of the three events on the basis of the G-matrix theory in finite nuclei by adopting as bare interaction the variants D

and F of the Nijmegen potential [43]. It was shown that, when interpreting the new event as ${}^{13}_{\Lambda\Lambda}$B, the D model reproduces the data very well, while the model F does not.

Using formula (21), one may derive an approximate expression for the $B_{\Lambda\Lambda}$ by making the necessary assumption that double Λ hypernuclei are stable against the strong interaction conversion $\Lambda\Lambda \to \Xi N$. Specifically, having in mind that

$$B_{\Lambda\Lambda}({}_{\Lambda\Lambda}Z^A) \simeq 2B_\Lambda(Z^{A-1}) + \Delta B_{\Lambda\Lambda} \tag{35}$$

one can write

$$B_{\Lambda\Lambda} \simeq 2D\left\{\left[1+\left(\frac{d}{\epsilon}\right)^2 A_c^{-2/3}\right]^{1/2} -3\left(\frac{d}{\epsilon}\right)A_c^{-1/3}\right\}^2 + {}^1S_0 \tag{36}$$

with

$$d = \left(\frac{\hbar^2}{8\mu D r_0^2}\right)^{1/2}, \tag{37}$$

where, however, $A_c \equiv A-2$ and ${}^1S_0$ is the expectation value of the Λ-Λ interaction in the ground state. The latter can be calculated using a proper Λ-Λ potential. In this work we assume that the Λ-Λ interaction is attractive. The s_Λ^2 matrix element is approximated by the empirical $\Delta B_{\Lambda\Lambda}$ value (i.e., $s_\Lambda^2 \simeq 4.5$ MeV) which, in addition, is considered fixed for all nuclei. The latter means that the A dependence of the Λ-Λ matrix element in the s state is omitted.

Expansions (26) and (12) may also be used, in the same way, for rough estimates of $B_{\Lambda\Lambda}$. Hence one can write

$$B_{\Lambda\Lambda} \simeq D\left[C - 12d_0A_c^{-1/3} + 20d_0^2A_c^{-2/3} -(6d_0^3+12\beta d_0)A_c^{-1} + \cdots\right], \tag{38}$$

$$B_{\Lambda\Lambda} \simeq D\left[C - 12d_0A_c^{-1/3} + 15d_0^2A_c^{-2/3} + \left(\frac{25}{4}d_0^3 + 12\beta d_0\right)A_c^{-1} + \cdots\right], \tag{39}$$

where $C = 2 + {}^1S_0/D$.

It is also noted that formula (37) may be easily extended for the triple Λ hypernuclei. In such a case, however, the first term of (37) is multiplied by 3, $A_c \equiv A-3$, and one also has to add the expectation value of the Λ-Λ interaction between the s- and p-state lambdas.

One should keep in mind, however, that multi-Λ hypernuclei are generally unstable to $\Lambda\Lambda \to \Xi N$ strong interaction decays. This problem has been very recently investigated in detail by Dover and Gal [44]. The authors have derived a generalized Bethe-Weizsäcker mass formula for strange hadronic matter, where cascade particles have also been included in order to ensure the stability against the strong conversion and various possibilities were considered regarding the strength of the hyperon-hyperon interaction. Therefore, it is clear that our simple formulas for the $B_{\Lambda\Lambda}$ values should be considered within the limits imposed by our assumptions, that is stability against strong decays and a strong $\Lambda\Lambda$ interaction implied by the few empirical data, which, in addition, is considered independent of the mass number.

IV. NUMERICAL RESULTS AND COMMENTS

The determination of the parameters of the two potentials is performed by a least-squares fit (in a way analogous to that used in [45] and discussed in detail by Bodmer and Rote for the ground state B_Λ for the square well potential [46,47]) to the experimental B_Λ values. At this point we note that in a very recent publication [33] it was shown that a fitting procedure to only ground-state Λ energies leads to rather poor results. The ground state B_Λ are very well reproduced but the Λ energies in the other states do not, in general, agree well with the data. It was also shown that the quality of the fit is improved if one takes into account all the available data. In addition, it is expected that an overall fit to all available data leads perhaps to more reliable results instead of adjusting the parameters to reproduce the binding states of chosen hypernuclei.

In the present work the parameters of the potential models are determined by global fits to the s, p, and d energies of ${}^{16}_{\Lambda}$O, ${}^{28}_{\Lambda}$Si, ${}^{40}_{\Lambda}$Ca, ${}^{51}_{\Lambda}$V, and ${}^{89}_{\Lambda}$Y hypernuclei. The experimental ground state B_Λ together with their experimental errors are taken from [11]. The p_Λ and d_Λ energies are those from figures of Refs. [8,9,14]. As experimental errors for these we used those in [14]. Table I contains the experimental values obtained from the (π^+, K^+) data together with the quoted errors used in the fitting procedures. The data for ${}^{32}_{\Lambda}$S and ${}^{56}_{\Lambda}$Fe are also included. It is noted that the s and p Λ energies of ${}^{32}_{\Lambda}$S, were observed in (K^-, π^-) reaction [6] while the s_Λ energy of ${}^{56}_{\Lambda}$Fe was obtained from a (π^+, K^+) reaction and is rather preliminary [48]. Finally, for the sake of comparison with our results, Table I also includes the theoretical predictions of a Woods-Saxon potential obtained with the parametrization of [14], which gives an excellent fit to the data.

In order to investigate the role of the density dependence, which is introduced effectively by the A dependence of the radius parameter, we performed two fits for each potential. The first one was for the local static case (β=0) while in the second formula (3) for the radius parameter was used. For the Gaussian potential the Schrödinger equation is solved numerically while for the cosh potential the analytic formula (29) is used. The best fit values for the Gaussian potential are first fit D= 33 MeV, r_0 = 1.273 fm, second fit D= 32.45 MeV, r_0=1.438, r_0' = −1.118 fm while for the cosh potential they are first fit D= 34.38 MeV and r_0= 1.142 fm, second fit D= 33.68 MeV, r_0= 1.311 fm, r_0' = −1.133 fm. The results of the second fit give, for both potential models, a somewhat

TABLE I. The observed Λ single-particle binding energies and the theoretical values obtained from a Woods-Saxon (WS) potential using the parametrization of Ref. [14].

	$B_{1s}^{\rm exp}$	$B_{1s}^{\rm WS}$	$B_{1p}^{\rm exp}$	$B_{1p}^{\rm WS}$	$B_{1d}^{\rm exp}$	$B_{1d}^{\rm WS}$
$^{16}_{\Lambda}$O	12.5±0.35	12.98	2.5±0.5	2.53		
$^{28}_{\Lambda}$Si	16.0±0.28	16.44	7.0±1.0	7.18		
$^{32}_{\Lambda}$S	17.5±0.50	17.18	8.0±0.5	8.31		
$^{40}_{\Lambda}$Ca	18.7±1.10	18.36	10.5±1.0	10.16		
$^{51}_{\Lambda}$V	19.9±1.00	19.54	12.0±1.0	12.10	4.0±1.0	3.98
$^{56}_{\Lambda}$Fe	21.0±1.50	19.96		12.81		4.92
$^{89}_{\Lambda}$Y	22.1±1.60	21.83	16.0±1.0	16.07	9.5±1.0	9.42
$^{138}_{\Lambda}$Ba		23.25		18.67		13.23
$^{208}_{\Lambda}$Pb		24.31		20.67		16.28

shallower but wider potential well.

In Table II the Λ binding energies in s, p, and d state for the Gaussian potential, calculated numerically and by means of the proposed approximate expressions, are displayed. It is seen that the s_Λ energies in all cases are very close and agree well with the experimental B_Λ. For the other Λ states the results of the first fit (values in parentheses) show that as A_c increases the agreement with the data becomes worse especially for A >50. On the contrary, the Λ energies obtained from the second fit are very much improved and all of them are within the quoted error bars. This is due to the corrections introduced by the A dependence of the radius parameter, which, in addition, increases the region of validity of the potential model. The same is the case with the cosh potential as it is seen from Table III. The values in parentheses again correspond to the first fit.

Our analysis has shown that the comparison of the results of the two potentials (second fit) with those of the more realistic Woods-Saxon potential (Table I) shows that the values of E_{1s} agree well for $16 \leq A \leq 208$. For the other nodeless Λ single particle states the comparison is quite satisfactory for $A \leq 100$. For larger A the p_Λ and d_Λ energies differ from those of the Woods-Saxon potential systematically by about 1.5 MeV.

Having in mind that the proposed potential models are not expected to be valid for the heavy system and that for systems with $A \leq 12$ the depth parameter should also depend rather strongly on A, the region of their validity should probably be $16 \leq A \leq 100$.

In Tables IV and V the theoretical predictions for the Λ kinetic and potential energy for both potentials using the parameters of the second fit are shown. For the Gaussian potential the Λ kinetic and potential energies were calculated numerically and by means of the approximate expressions (19). For the cosh potential the ground-state Λ kinetic and potential energy was calculated using expressions (24) and (25) respectively, while the p and d state kinetic and potential energies were calculated numerically. It is seen that the approximate analytic expressions give results which are usually close to those calculated numerically. For $A \leq 28$, however, and for the p and d states, the convergence is not good. Therefore, the corresponding expressions should be used for hypernuclei in the region $28 \leq A \leq 100$.

In Table VI the theoretical values of the lower energy level spacing $\Delta_{\rm sp}$ for both potentials together with the available experimental values are shown. Columns 3 and 4 contain the results obtained numerically and by means of (16) for the Gaussian potential. In the fifth column are the results for cosh potential using formulas (29), while in the last column the values of $\hbar\omega_\Lambda$ are displayed. These are obtained with the first two terms of the expansion of (32) as in [31], namely,

$$\hbar\omega_\Lambda \simeq 37A_c^{-1/3} - 25.4A_c^{-2/3}. \tag{40}$$

TABLE II. The B_Λ energies in s, p, and d states with the Gaussian potential calculated numerically and with the corresponding approximate expressions (see text) for a number of hypernuclei. The values in parentheses were obtained using for the radius R of the potential the simple expression $R = r_0 A_c^{1/3}$.

	$B_{1s}^{\rm num}$	$B_{1s}^{\rm appr.}$	$B_{1p}^{\rm num}$	$B_{1p}^{\rm appr.}$	$B_{1d}^{\rm num}$	$B_{1d}^{\rm appr.}$
$^{16}_{\Lambda}$O	(12.98) 12.39	12.47	(2.86) 2.39	1.18		
$^{28}_{\Lambda}$Si	(16.19) 16.25	16.30	(6.92) 7.24	6.86		
$^{32}_{\Lambda}$S	(16.88) 17.02	17.06	(7.86) 8.31	8.01		
$^{40}_{\Lambda}$Ca	(17.96) 18.21	18.24	(9.37) 10.00	9.80		
$^{51}_{\Lambda}$V	(19.05) 19.38	19.41	(10.94) 11.70	11.57	(3.98) 5.00	4.48
$^{56}_{\Lambda}$Fe	(19.45) 19.80	19.82	(11.52) 12.32	12.21	(4.66) 5.74	5.29
$^{89}_{\Lambda}$Y	(21.27) 21.66	21.67	(14.23) 15.12	15.07	(7.90) 9.17	8.96
$^{138}_{\Lambda}$Ba	(22.27) 23.14	23.15	(16.52) 17.40	17.38	(10.77) 12.07	11.96
$^{208}_{\Lambda}$Pb	(24.01) 24.31	24.34	(18.44) 19.26	19.25	(13.23) 14.48	14.42

TABLE III. The B_Λ energies in s, p, and d states with the cosh potential using expression (29) for a number of hypernuclei. The values in parentheses were obtained using for the radius R of the potential the simple expression $R = r_0 A_c^{1/3}$.

	B_{1s}	B_{1s}	B_{1p}	B_{1p}	B_{1d}	B_{1d}
${}^{16}_{\Lambda}$O	(12.96)	12.55	(2.98)	2.52		
${}^{28}_{\Lambda}$Si	(16.20)	16.27	(6.87)	7.21		
${}^{32}_{\Lambda}$S	(16.90)	17.08	(7.78)	8.26		
${}^{40}_{\Lambda}$Ca	(18.01)	18.32	(9.23)	9.93		
${}^{51}_{\Lambda}$V	(19.14)	19.54	(10.80)	11.64	(3.97)	5.02
${}^{56}_{\Lambda}$Fe	(19.55)	19.97	(11.36)	12.25	(4.64)	5.77
${}^{89}_{\Lambda}$Y	(21.46)	21.95	(14.07)	15.11	(7.70)	9.11
${}^{138}_{\Lambda}$Ba	(23.05)	23.51	(16.42)	17.45	(10.52)	11.98
${}^{208}_{\Lambda}$Pb	(24.38)	24.79	(18.41)	19.38	(12.99)	14.42

TABLE IV. The kinetic Λ energies in s, p, and d states for the Gaussian and the cosh potentials for various Λ hypernuclei (for details see text).

	T_{1s}^{num}	$T_{1s}^{\text{appr.}}$	T_{1s}^{cosh}	T_{1p}^{num}	$T_{1p}^{\text{appr.}}$	T_{1p}^{cosh}	T_{1d}^{num}	$T_{1d}^{\text{appr.}}$	T_{1d}^{cosh}
${}^{16}_{\Lambda}$O	7.94	8.58	7.87	8.91	12.01	8.11			
${}^{28}_{\Lambda}$Si	6.89	7.14	7.02	9.43	10.41	9.00			
${}^{32}_{\Lambda}$S	6.66	6.84	6.79	9.28	10.06	8.95			
${}^{40}_{\Lambda}$Ca	6.24	6.37	6.42	8.97	9.42	8.77			
${}^{51}_{\Lambda}$V	5.82	5.91	6.03	8.56	8.90	8.48	10.04	10.24	9.50
${}^{56}_{\Lambda}$Fe	5.66	5.74	5.89	8.39	8.68	8.35	10.00	10.08	9.43
${}^{89}_{\Lambda}$Y	4.93	4.96	5.10	7.52	7.63	7.62	9.42	9.22	9.16

TABLE V. The potential Λ energies in s, p, and d states for the Gaussian and the cosh potentials for various Λ hypernuclei (for details see text).

	V_{1s}^{num}	$V_{1s}^{\text{appr.}}$	V_{1s}^{cosh}	V_{1p}^{num}	$V_{1p}^{\text{appr.}}$	V_{1p}^{cosh}	V_{1d}^{num}	$V_{1d}^{\text{appr.}}$	V_{1d}^{cosh}
${}^{16}_{\Lambda}$O	−20.33	−21.04	−20.16	−11.30	−13.30	−10.69			
${}^{28}_{\Lambda}$Si	−23.14	−23.43	−23.28	−16.67	−17.14	−16.22			
${}^{32}_{\Lambda}$S	−23.67	−23.90	−23.86	−17.60	−18.13	−17.21			
${}^{40}_{\Lambda}$Ca	−24.45	−24.62	−24.73	−18.97	−19.34	−18.70			
${}^{51}_{\Lambda}$V	−25.20	−25.31	−25.56	−20.26	−20.51	−20.11	−15.06	−15.65	−14.47
${}^{56}_{\Lambda}$Fe	−25.46	−25.56	−25.86	−20.71	−20.92	−20.60	−15.74	−16.23	−15.20
${}^{89}_{\Lambda}$Y	−26.59	−26.65	−27.12	−22.64	−22.75	−22.73	−18.59	−18.82	−18.27

TABLE VI. The lowest Λ energy level spacing calculated with the Gaussian and the cosh potentials together with the experimental values and the $\hbar\omega_\Lambda$ values from (40) for various Λ hypernuclei.

	$\Delta_{\text{sp}}^{\text{exp}}$	$\Delta_{\text{sp}}^{\text{Gauss}}$	$\Delta_{\text{sp}}^{\text{Gauss}}$ (appr.)	$\Delta_{\text{sp}}^{\text{cosh}}$	$\hbar\omega_\Lambda$
${}^{16}_{\Lambda}$O	10.0 ±0.5	10.08	11.30	10.03	9.43
${}^{28}_{\Lambda}$Si	9.00 ±1.0	9.01	9.44	9.06	8.58
${}^{32}_{\Lambda}$S	9.50 ±0.5	9.05	9.05	8.82	8.34
${}^{40}_{\Lambda}$Ca	8.20 ±1.0	8.24	8.45	8.39	7.97
${}^{51}_{\Lambda}$V	7.90 ±1.0	7.70	7.84	7.90	7.54
${}^{89}_{\Lambda}$Y	6.10 ±1.5	6.45	6.60	6.84	6.60

Again it is seen that the theoretical spacings calculated with both potentials are close to the experimental spacings.

It is interesting to note that the leading term in all the expansions of the formulae proposed above is $A_c^{-1/3}$ which is an indication of certain surface effects introduced by the potential models. The corrections due to the $A_c^{-1/3}$ term are rather appreciable for hypernuclei with $A \leq 100$ for which these expressions are mainly valid.

Our study has also shown that the use of Λ-nucleus potentials with parameters depended on A, which somehow reflects their density dependence or nonlocality, leads to better results. This is in accord with other more elaborated studies and shows the necessity in using density dependent Λ-nucleus potentials for a reliable description of the modern binding energy data which have invalidated older theoretical approaches based on local static potentials.

Finally, the theoretical predictions for $B_{\Lambda\Lambda}$ for a number of double Λ hypernuclei calculated with the various formulas proposed in the preceding section are displayed in Table VII. It is noted that for the sake of simplicity we set $\beta = 0$. In addition, since the derivation of these expressions is based on the formulas for the ground-state B_Λ we used as parameters the ones which are obtained from a fit to only ground state B_Λ and which reproduce excellently the ground-state data for A≥12. Namely, D=35.56 MeV, r_0=1.075 fm and D= 34.16 MeV, r_0=1.199 fm for the cosh and the Gaussian potential, respectively. Using the empirical $\Delta B_{\Lambda\Lambda}$ value ($\sim$ 4.5 MeV) for the s_Λ^2 matrix element one can calculate the $B_{\Lambda\Lambda}$ values by means of formula (36) (second column). In the same table are also displayed the $B_{\Lambda\Lambda}$ values calculated with the first four terms of the expansions (38) and (39), respectively, namely,

$$B_{\Lambda\Lambda} \simeq 75.6 - 139A_c^{-1/3} + 75.5A_c^{-2/3} - 7.4A_c^{-1}, \quad (41)$$

$$B_{\Lambda\Lambda} \simeq 72.8 - 122.2A_c^{-1/3} + 45.5A_c^{-2/3} + 5.7A_c^{-1}. \quad (42)$$

It is seen from Table VI that the values calculated with (41) and (42) are very close to the ones obtained with expression (36).

In conclusion, we would like to point out that the present analysis is of a more simplified nature compared to certain other more sophisticated approaches. It should be noted, however, that the potentials assumed in the present work are very suitable and serve our purpose very well. Their use has ensured a remarkable simplicity of our approach, thus allowing the derivation of several interesting analytic expressions for the Λ energies as function of the core mass number. Such expressions, derived on the basis of a potential model, are proposed for the first time, to our knowledge, apart perhaps from the ones for the ground state Λ energies for which there are such expressions (see [33] and references cited therein).

TABLE VII. The $B_{\Lambda\Lambda}$ energies calculated with formulas (36), (41), and (42) for various double Λ hypernuclei.

	$B_{\Lambda\Lambda}$ (36)	$B_{\Lambda\Lambda}$ (41)	$B_{\Lambda\Lambda}$ (42)
${}^{13}_{\Lambda\Lambda}$B	28.8	27.7	27.6
${}^{16}_{\Lambda\Lambda}$O	31.3	30.4	30.4
${}^{28}_{\Lambda\Lambda}$Si	37.4	37.0	37.0
${}^{32}_{\Lambda\Lambda}$S	38.7	38.5	38.4
${}^{40}_{\Lambda\Lambda}$Ca	40.8	40.8	40.7
${}^{51}_{\Lambda\Lambda}$V	43.1	43.1	43.0

V. SUMMARY

In this paper two Λ-nucleus potentials suitable mainly for relatively light and intermediate hypernuclei have been used for the derivation of analytic expressions for the various energy quantities of a Λ in Λ hypernuclei. The density dependence of these potentials was taken into account approximately by considering the A dependence of their radius parameters. It turned out that such a choice was essential for a reliable description of the hypernuclear data.

The proposed analytic expressions give values which in most of the cases are fairly close to those calculated numerically. In addition, they are rather simple and hence they are suitable for practical use in various phenomenological studies.

Finally, in the framework of this approach approximate analytic formulas for $B_{\Lambda\Lambda}$ in double Λ hypernuclei have been derived under the assumption that the Λ-Λ interaction is attractive and ignoring the strong ΛΛ conversion.

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