Comparison of the Porter-Thomas distribution with neutron resonance data of even-even nuclei

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The low-energy neutron resonance data of the even-even nuclei ^{152}Sm, ^{158}Gd, ^{162}Dy, ^{166,168}Er, ^{182}W, ^{232}Th, and ^{236,238}U have been examined in order to test the validity of the Porter-Thomas distribution of the reduced neutron widths—a chi-squared distribution with one degree of freedom ($v = 1$). In an attempt to circumvent the ever-present problems of missed or spurious s wave levels as well as extra p wave levels, a maximum likelihood statistic was employed which used only measured widths greater than some minimum value. A Bayes-theory test applied to the data helped to ensure that p wave contamination of the s wave level population was not significant. The error-weighted value of the number of degrees of freedom for the nine nuclei studied, $\langle v\rangle = 0.98 \pm 0.10$, is consistent with the theoretical expectation of $v = 1$.

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INTRODUCTION

When a low-energy, ≤ 5 keV, neutron interacts with a heavy nucleus, compound nuclear states are formed. The signatures of these states are the narrow resonances observed in the total neutron-nucleus cross section as a function of incident neutron energy. For most nuclei with $A \geq 150$ (exceptions occur in the actinide region), the compound nuclear state decays via two open channels, namely, elastic neutron emission characterized by a neutron width Γ_n, and the emission of gamma rays leading to neutron capture with the width Γ_γ. The neutron width, capture width, and resonance energy are the parameters which appear in the Breit-Wigner expression for the resonance cross section. At these low energies the neutron initiating the reaction is basically either s wave ($l = 0$) or p wave ($l = 1$), and a measured neutron width is related to an intrinsic or reduced width (Γ_n^l) through different ($l = 0, 1$) penetrability factors, namely $\Gamma_n = P_0\Gamma_n^0$, or $\Gamma_n = P_1\Gamma_n^1$, where $P_0 = (E/1\text{ eV})^{1/2}$ and $P_1 = P_0x^2/(x^2+1)$. $x = kR$ and E, k, and R are the resonance energy, neutron wave number, and effective nuclear radius, respectively. At low energies $kR \ll 1$ and, assuming the *reduced* s and p wave widths are of the same order, the measured p wave resonance widths are much smaller than the *average* s wave widths. Many years ago Porter and Thomas [1] suggested that the reduced neutron widths of resonances characterized by the same quantum numbers obeyed a chi-squared distribution of one degree of freedom (hereafter referred to as the Porter-Thomas Distribution—PTD). The theoretical argument for this was based on the proportionality of the reduced widths to γ^2, where γ is a resonance amplitude defined in the R-matrix theory [2] of nuclear reactions as an overlap integral between a channel wave function and a compound nuclear state function. γ was argued to have a Gaussian distribution about zero which leads to the PTD for the reduced neutron widths. The PTD distribution emerges naturally in random matrix theory, introduced by Wigner [3]. The PTD has the form

$$P(y,v,\bar{y})dy = Ky^{(v/2-1)}\exp(-vy/2\bar{y})dy\ . \qquad (1)$$

K is a normalization constant, y is the reduced neutron width (e.g., Γ_n^0), v is the number of degrees of freedom, and $\bar{y}$ is the average reduced width. For $v = 1$ the PTD results. The nature of the PTD is such that the reduced widths vary in strength over several orders of magnitude and the smallest widths are the most probable. Although generally believed to be correct it is difficult to verify, from neutron resonance data, that $v = 1$ experimentally. Some of the reasons for this are as follows: (i) Due to the limiting effects of finite experimental resolution, in any given experiment in which a statistically significant number of resonances is detected, a number of weak resonances remain undetected. Under these conditions a test of the PTD would lead to values of $v > 1$. (ii) The difficulty of distinguishing weak resonances from statistical fluctuations. Spurious levels are sometimes introduced in this manner. Extra small widths lead to estimates of $v < 1$. (iii) For many nuclei weak s wave widths are indistinguishable from strong p wave resonances. This is a function of the relative strength of the s and p wave strength functions, S_0 and S_1, as well as the resonance energy and the radius, R, of the target nucleus. Extra p wave resonances in an otherwise complete s wave population of levels lead to estimates of $v < 1$. (iv) Because the widths obeying the PTD have values spanning several orders of magnitude, the inherent fluctuations in the determination of v due to finite sample size are large. For this reason it is not feasible to determine v with sufficient accuracy by examining the resonances of just *one* nucleus—experimental data of the necessary quality and completeness do not exist.

A determination of v with one of the smallest uncertainties was made by Harney [4] ($v = 0.93 \pm 0.11$) using a technique based on the ideas of Krieger and Porter [5], wherein correlations between different channel am-

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plitudes were examined for a unique set of proton data taken at Duke University [6]. A test of the PTD is made here using the neutron resonance data that have been collected over the years [7]. In principle even-even target nuclei with $A \geq 150$ should form good test cases. For s wave neutrons the resonances formed with even-even nuclei have the same spin and parity, $1/2^+$, and form a single population of levels to which the PTD is expected to apply. The larger level spacing and resonance widths of the even-even target nuclei should result in fewer missed levels than neighboring odd A nuclei. The heavier even-even nuclei also possess a statistically significant number of levels where experiment can resolve most levels.

METHOD

The method employed here is intended to determine v in a manner which circumvents the problems (i), (ii), and (iii) mentioned above. Since these difficulties arise from the smallest levels, a statistic was employed which uses only reduced widths larger than some minimum value. In addition, it was necessary to exclude strong p wave widths that are mixed in with the weak s wave levels. An important aid, in this regard, is the Bayes-theory (BT) test suggested by Bollinger and Thomas [8]. The application of this test for even-even target nuclei requires a knowledge of the s wave strength function S_0, the s wave level spacing D_0, the p wave strength function S_1, and the nuclear size. The BT test helps to determine how large the minimum width must be to reduce p wave contamination to negligible proportions. Additional assumptions and the equations required for the BT analysis are outlined in Appendix A. The finite sample size problem (iv) can only be overcome by examining as much data as possible.

The method of estimating v was developed by considering the probability distribution

$$F(y, y_1, v, \bar{y})dy = P(y, v, \bar{y})dy \bigg/ \int_{y_1}^{\infty} P(y, v, \bar{y})dy \; . \tag{2}$$

This distribution is normalized to unity for values of the reduced widths $y_1 \leq y \leq \infty$. $P(y, v, \bar{y})$ is the distribution defined by Eq. (1) and y_1 is some minimum reduced width above which all levels have been detected and are all s wave levels. For n widths $y_1, \ldots, y_n$, the likelihood function

$$L(y_1, v, \bar{y}) = \prod_{i=1}^{n} F(y_i, y_1, v, \bar{y}) \tag{3}$$

is constructed and the maxima conditions $\partial L/\partial v = 0$, $\partial L/\partial \bar{y} = 0$ lead to the equations

$$\langle y \rangle / \langle 1 \rangle = \sum_{i=1}^{n} y_i/n \; , \tag{4a}$$

$$\langle \ln y \rangle / \langle 1 \rangle = \sum_{i=1}^{n} \ln(y_i)/n \; , \tag{4b}$$

where $\langle a \rangle = \int_{y_1}^{\infty} aP(y, v, \bar{y})dy$, with $a = y$, 1, and $\ln y$. Equations (4a) and (4b) are two transcendental equations in the two unknowns v, $\bar{y}$. The right-hand side of these equations has the fixed values determined by a particular set of experimental data. The integrals are calculated numerically for a range of v and $\bar{y}$ until unique values satisfying the equations are found. Monte Carlo calculations carried out using Eq. (1) with known values of v, $\bar{y}$ indicated that Eqs. (4a) and (4b) can be successfully applied to determine these parameters. These same calculations indicated that the value obtained for v is biased (larger than the true value) and a typical correction is reducing v from, e.g., 1.22 to 1.10. All quoted values of v have been corrected for this bias. The correction is mainly a function of the number of levels, n, and the ratio of the minimum width to the average width.

The uncertainty in the determination of v, $\bar{y}$ was assumed to arise from two sources, finite sampling errors and the uncertainty of the individual measured widths. The finite sampling error was determined in the usual way when maximum likelihood estimators are used. For example, for a given value of $\bar{y}$, it was assumed that v is Gaussian distributed about the optimum value; thus it follows from this that $\sigma_v^2 = -[\partial^2 L/\partial v^2]^{-1}$, where σ_v is the standard deviation of v due to finite sample size. Similar remarks apply to the uncertainty of the average width, $\bar{y}$. The other uncertainty, associated with the experimental error of each individual width, was found to be much smaller than the finite sampling error. These two errors were added in quadrature.

COMPARISON WITH EXPERIMENTAL DATA

A cursory examination of the neutron resonance data available [7] suggests there exists a significant amount of data for even-even nuclei with $A \geq 150$. However, resonance data were included in the analysis only if the number of levels, of a given nucleus, with widths above the appropriate minimum value was $\gtrsim 45$, the minimum width was less than 10% of the average width, and the value of v was relatively stable to variations of the minimum width. There are nine nuclei meeting these criteria for which the statistical analysis described above was carried out and an estimate of v made. Generally speaking, since v is sensitive to the smallest widths, the minimum width y_1 should be chosen as small as possible, yet at the same time, be sufficiently large that no levels have been missed in the given energy range, nor should there be a significant number of p wave levels.

The three nuclei studied in the actinide region had the largest number of levels per nucleus. The results for these nuclei will be discussed first. The difficulty in this region of the periodic table is that $S_1 > S_0$ so that p wave contamination of the small s wave widths is usually a serious problem. Typical are the ^{232}Th data of Rahn *et al.* [9], shown in Fig. 1, where the reduced neutron widths, assumed to be s wave, are plotted versus the resonance energy. At energies less than 100 eV reduced widths as small as 0.001 meV have been detected, while at an energy of 1500 eV the minimum width detected

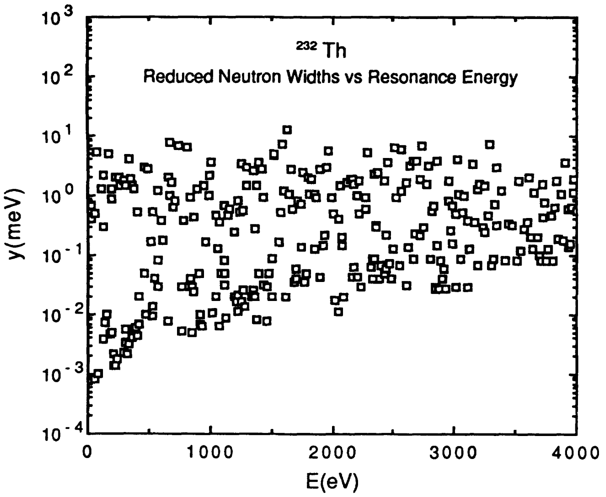

FIG. 1. Plot of the ^{232}Th reduced neutron widths, of Rahn *et al.* [9], as a function of resonance energy. At energies below 100 eV neutron widths as small as 0.001 meV have been detected, while at the highest energies near 4000 eV the smallest width detected is close to 0.08 meV. In contrast, the largest widths measured fluctuate about a value near 9 meV without any noticeable energy dependence. This is strong evidence that weak levels are being missed as the incident neutron energy increases. A plot of this nature can be used as a guide to determine the minimum width above which all levels have been detected in a given energy range. For example, for ^{232}Th it is reasonable to assume that all widths greater than 0.01 meV have been detected in the energy range 0–1500 eV. Plots of the reduced neutron widths vs resonance energy for all the nuclei examined had the same qualitative features as ^{232}Th. These plots help to determine, for each nucleus, the minimum reduced width above which all levels have been detected in a given energy range.

has increased to 0.01 meV. At the highest energies near 4000 eV, the minimum width detected is about 0.08 meV. In contrast, the largest widths measured hover around 9 meV and, allowing for fluctuations, are independent of energy. This is compelling evidence that small levels are being missed with increased frequency as the incident neutron energy increases. For each nucleus studied the plot of the reduced neutron widths vs resonance energy had the same qualitative features as Fig. 1. This type of plot was used, for each nucleus, to help determine y_1 for a given energy range. In order to keep the minimum width for ^{232}Th as small as possible and still include a reasonable number of levels, y_1 could be chosen equal to 0.01 meV and then the data between 0.0 and 1500 eV could be used to determine $v, \bar{y}$. Unfortunately, this approach is not possible because many of the levels in this energy range with widths > 0.01 meV are p wave. Several measurements [10–12] of the ^{232}Th p wave strength function cluster about the value $10^4 S_1 = 1.5$, and this value along with the s wave parameters [9], $10^4 S_0 = 0.84$, $D_0 = 16.7$ eV, were used in the BT analysis. The BT test indicated that if $y_1 = 0.14$, 0.15, and 0.17 meV for the data up to $E = 1900$, 2419, and 2664 eV respectively, the number of possible p wave levels present, out of an average of $\simeq 100$ levels, has been reduced to less than 1/2. For these widths, solving Eqs. (4a) and (4b) yielded $v = 1.25$, 1.43, and 1.27, while $\bar{y} = 1.65$, 1.65, and 1.57 meV. The choices of $v, \bar{y} = 1.32 \pm 0.30$, 1.62 ± 0.18 meV are given in Table I. The quoted errors are dominated by the finite sampling uncertainty.

The ^{236}U data of Carrao and Brusegan [13] were used up to $E = 1535$ eV. Over this energy range missing s levels and p wave contamination were a problem. If $y_1 = 0.12$ meV the problem of missing levels is eliminated and a BT analysis with ($10^4 S_0 = 1.0$, $D_0 = 16.2$ eV) [13] and $10^4 S_1 = 2.3$ [14] suggested that less than $1/2p$ wave level was present out of 80 levels (removing a level as p wave did not change the result in any significant way). The values of v obtained by increasing the minimum width from 0.13 to 0.17 meV varied between 0.84 and 0.96 while $\bar{y}$ varied between 1.89 and 1.97 meV. The final values are given in Table I.

The analysis of the ^{238}U data [9] was similar to that of thorium in that p wave contamination was the dominant problem. Reported values of the p wave strength function range from $10^4 S_1 = 1.4$ [9] to 2.44 [11]. For the BT test the value of $10^4 S_1$ was assumed to lie between 1.7 and 2.0. The s wave parameters ($10^4 S_0 = 1.08$, $D_0 = 20.8$ eV) were taken from Rahn *et al.* [9]. When $y_1 = 0.23$ to 0.27 meV was selected for the data to $E = 2671$ eV, the BT test indicated that for each value of y_1, 1/2 to 1 level out of $\simeq 90$ were p wave. Solving Eqs. (4a) and (4b) with and without one possible p level removed led to values of v between 0.84 and 0.98 and $\bar{y}$ between 2.40 and 2.58 meV. The compromise values are $v = 0.92$ and $\bar{y} = 2.50$ meV.

Six nuclei between $A = 152$ and 182 were found to be good candidates for this analysis. The final values obtained for v and $\bar{y}$ are given in Table I. Remarks about these nuclei follow.

For the ^{152}Sm data [15], the sole problem of missing s wave levels was overcome by choosing $y_1 = 0.3$ meV for the data to $E = 2657$ eV. By varying the minimum width from 0.3 to 0.5 meV, $v, \bar{y}$ varied from 1.08, 11.12 meV to 1.20, 11.62 meV. The final values for the $\simeq 46$ widths used are given in Table I.

For the BT test of the ^{158}Gd data [16], the s wave parameters $10^4 S_0 = 1.5$ and $D_0 = 86$ eV of Rahn *et al.* [16] were used. These same authors found, for ^{160}Gd, a

TABLE I. Listed are the values of the number of degrees of freedom v and the average neutron width $\bar{y}$ determined from the neutron resonance data of the nine even-even nuclei analyzed as described in the text.

Element	v	$\bar{y}$ (meV)
^{152}Sm	1.13±0.37	11.37±2.11
^{158}Gd	1.28±0.43	15.52±2.72
^{162}Dy	0.64±0.28	11.08±2.36
^{166}Er	0.85±0.24	6.76±1.15
^{168}Er	1.10±0.42	15.10±2.97
^{182}W	1.12±0.38	14.66±2.37
^{232}Th	1.32±0.30	1.62±0.18
^{236}U	0.90±0.29	1.92±0.30
^{238}U	0.92±0.26	2.50±0.36

p wave strength function of $10^4 S_1 = 1.7$, and this value was used for ^{158}Gd. For energies up to 4985 eV all levels with widths greater than 0.42 meV were detected. In order to reduce p wave contamination and still keep a small minimum width, y_1 was chosen equal to 0.75, 0.82, and 1.1 meV. For the first two values of y_1 one possible p wave level was removed out of the 44 levels analyzed; for $y_1 = 1.1$ meV it was not necessary to remove any p levels. Solving Eqs. (4a) and (4b) yielded v between 1.23 and 1.34 and $\bar{y}$ between 15.24 and 15.75 meV. The final choices of 1.28 ± 0.43 and 15.52 ± 2.72 are listed in Table I.

For the ^{162}Dy data [17] the BT test ($10^4 S_0 = 1.88$, $D_0 = 64.6$ eV and $10^4 S_1 = 1.1$ from Liou *et al.* [17]) suggested a few p waves were detected, but with $y_1 = 0.62$ meV, p wave contamination to $E = 4000$ eV is reduced to no more than 1/2 level out of 50; in addition, no larger widths were missed. Increasing y_1 from 0.62 to 0.84 meV for the data to $E = 4000$ eV led to values of v between 0.60 and 0.68 and $\bar{y}$ between 10.7 and 11.3 meV. The final values are listed in Table I.

For the ^{166}Er data of Liou *et al.* [18] the BT analysis ($10^4 S_0 = 1.7$, $D_0 = 37.6$ eV, $10^4 S_1 = 0.7$) [18] suggested four possible p waves to $E = 2984$ eV. Choosing a minimum width of 0.105 meV for this energy range reduced the p wave contamination to less than 1/2 level out of 70, with no missed levels. Varying y_1 between 0.105 and 0.17 meV gave consistent values of v, $\bar{y}$ around the final choices of 0.85 ± 0.24, 6.76 ± 1.15 meV.

For ^{168}Er [18] p wave contamination rather than missed levels was the dominant problem. A BT test ($10^4 S_0 = 1.5$, $D_0 = 93.6$ eV, $10^4 S_1 = 0.7$) [18] indicated that for the data up to $E = 4885$ eV the minimum width had to be larger than 0.60 meV to reduce the number of possible p levels to less than one out of about 45 widths. Using $y_1 = 0.39$, 0.60 meV (with two and one possible p levels removed, respectively), and $y_1 = 0.72$ and 0.93 (with no p levels removed), the solutions to Eqs. (4a) and (4b) for v were found to lie between 1.03 and 1.14, while $\bar{y}$ fell between 14.76 and 15.33. The final choices are $v = 1.10 \pm 0.42$ and $\bar{y} = 15.10 \pm 2.97$.

Missing s levels comprised the only difficulty with the ^{182}W data [19]. For energies up to 4492 eV the minimum width was varied between 1.3 and 2.2 meV. For these approximately 58 levels consistent values around the final choices $v = 1.12 \pm 0.38$ and $\bar{y} = 14.66 \pm 2.37$ were obtained.

CONCLUSION

Using a statistic which attempts to circumvent the difficulties of missing levels, spurious levels, and p wave contamination of an s wave population of levels, nine sets of neutron resonance data were examined to test the correctness of the PTD. The error-weighted average value of v for the nine even-even nuclei listed in Table I is $\langle v \rangle = 0.98 \pm 0.10$. Given the 10% uncertainty, the fact that $\langle v \rangle$ is so close to 1 can be considered somewhat fortuitous. However, the result is completely consistent with the PTD ($v = 1$) and overlaps Harney's [4] result of $\langle v \rangle = 0.93 \pm 0.11$ found using a different technique and determined from data of a different nature.

APPENDIX

Described here are the assumptions and equations used to assign a probability to a measured width as being either s wave or p wave—the so-called Bayes-theory test [8]. When an s wave neutron is incident upon an even-even nucleus only $J = 1/2^+$ compound nuclear states can be formed. For p wave neutrons the presence of the spin-orbit interaction leads to both $J = 3/2^-$ and $J = 1/2^-$ resonant states. If the ratio of p levels to s levels is taken to be 3 to 1 and if there are twice as many $J = 3/2^- p$ levels as $J = 1/2^- p$ levels, then the relative probability that a measured width is p wave can be expressed as

$$\mathrm{Prob}(p) = (2P_p^+ + P_p^-)/(P_s + 2P_p^+ + P_p^-) \ , \qquad \text{(A1)}$$

where the $\mathrm{Prob}(s) = 1.0 - \mathrm{Prob}(p)$. P_s, $P_p^{+,-}$ represent the s and p wave probabilities of detecting a level of a given width and have the functional form of Eq. (1) with $K, v = 1$. The $+$ and $-$ superscripts associated with the p wave probabilities refer to the $J = 3/2^-$ and $1/2^-$ levels, respectively. The measured width (not the reduced width) is substituted into the expressions for $P_{s,p}$. This requires that the average widths (also not reduced) be determined at the resonance energy of the level being examined and are found by multiplying the average *reduced* widths by the s and p wave penetrabilities evaluated at the resonance energy. Thus, given the assumptions above, estimating the average reduced widths is what remains to be done. It follows from the definitions of the s and p wave strength functions that the average reduced widths can be expressed as

$$\begin{aligned} \overline{\Gamma_n^0} &= S_0 D_0 \ , \\ \overline{(g\Gamma_n^1)}^{\,-} &= 3S_1 D_0/(C+1) \ , \\ \overline{(g\Gamma_n^1)}^{\,+} &= C\overline{(g\Gamma_n^1)}^{\,-}/2 \ . \end{aligned} \qquad \text{(A2)}$$

Given S_0, D_0, and S_1, only the parameter C needs to be determined. C is the ratio of the compound nucleus formation cross section for p wave neutrons in the $J = 3/2^-$ and $1/2^-$ states. The values of C, calculated with an optical model, depend on the mass number A but are relatively insensitive to the parameters of the potential needed to bracket the experimental values of S_1 vs A for $A \geq 150$. The value of C is about 1.5 for ^{152}Sm and increases to 2.87 for ^{238}U.

As an example, consider the ^{232}Th level at 2624.3 eV with a measured width of 7.17 meV. With $10^4 S_0 = 0.84$, $D_0 = 16.7$ eV, $10^4 S_1 = 1.5$, $C = 2.8$, and $R = 1.4A^{1/3}$, the probability that this level is s or p wave is 0.446 and 0.554, respectively.

[1] C. E. Porter and R. G. Thomas, Phys. Rev. **104**, 483 (1956).
[2] A. M. Lane and R. G. Thomas, Rev. Mod. Phys. **30**, 257 (1958).
[3] C. E. Porter, *Statistical Theories of Spectral Fluctuations* (Academic, New York, 1965); T. A. Brody, J. Flores, J. B. French, P. A. Mello, A. Pandey, and S. M. Wong, Rev. Mod. Phys. **53**, 385 (1981); M. L. Mehta, *Random Matrices*, 2nd ed. (Academic, New York, 1991).
[4] H. L. Harney, Phys. Rev. Lett. **53**, 537 (1984); see also J. F. Shriner, Jr., G. E. Mitchell, and E. G. Bilpuch, *ibid.* **59**, 435 (1987).
[5] T. J. Krieger and C. E. Porter, J. Math. Phys. **4**, 1272 (1963).
[6] G. E. Mitchell, E. G. Bilpuch, J. F. Shriner, Jr., and A. M. Lane, Phys. Rep. **7**, 1 (1985).
[7] S. F. Mughabghab, M. Divadeenam, and N. E. Holden, *Neutron Cross Sections*, 4th ed. (Academic, New York, 1981), Vol. I, parts A and B.
[8] L. M. Bollinger and G. E. Thomas, Phys. Rev. **171**, 1293 (1968).
[9] F. Rahn, H. S. Camarda, G. Hacken, W. W. Havens, Jr., H. I. Liou, J. Rainwater, M. Slagowitz, and S. Wynchank, Phys. Rev. C **6**, 1854 (1972).
[10] P. Ribon, Ph.D. thesis, Université de Paris, 1969 (unpublished).
[11] C. A. Uttley, C. M. Newstead, and K. M. Diment, in *Proceedings of the Conference on Nuclear Data for Reactors, Paris, 1966* (International Atomic Energy Agency, Vienna, 1967), Vol. I, p. 165.
[12] H. S. Camarda, Phys. Rev. C **9**, 28 (1974).
[13] G. Carraro and A. Brusegan, Nucl. Phys. **A257**, 333 (1976).
[14] A. D. Carlson *et al.*, Nucl. Phys. **A141**, 577 (1970).
[15] F. Rahn, H. S. Camarda, G. Hacken, W. W. Havens, Jr., H. I. Liou, J. Rainwater, M. Slagowitz, and S. Wynchank, Phys. Rev. C **6**, 251 (1972).
[16] F. Rahn, H. S. Camarda, G. Hacken, W. W. Havens, Jr., H. I. Liou, and J. Rainwater, Phys. Rev. C **10**, 1904 (1974).
[17] H. I. Liou, G. Hacken, J. Rainwater, and U. N. Singh, Phys. Rev. C **11**, 462 (1975).
[18] H. I. Liou, H. S. Camarda, S. Wynchank, M. Slagowitz, G. Hacken, F. Rahn, and J. Rainwater, Phys. Rev. C **5**, 974 (1972).
[19] H. S. Camarda, H. I. Liou, G. Hacken, F. Rahn, W. Makofske, M. Slagowitz, S. Wynchank, and J. Rainwater, Phys. Rev. C **8**, 1813 (1973).