Weakly bound states of a three-body system

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(Received 13 September 1993)

The equivalent two-body method [H. Feshbach and S. I. Rubinow, Phys. Rev. **98**, 188 (1955)] has been applied to investigate the weakly bound states of a three-body system. An analytical expression for the long-range effective potential is derived if two-body interactions are short-range potentials with Yukawa forms. The weakly bound excited state for the system with two heavy particles and a light particle is first obtained if there is no bound state for two-body subsystems.

PACS number(s): 21.45.+v, 21.30.+y

I. INTRODUCTION

The Efimov effect is one of the most interesting phenomena in a quantum three-body system. It was first pointed out [1,2] by Efimov in 1970s that a three-body system may have a lot of weakly bound states if the two-body interactions are resonant and their strengths approach the critical value which is just necessary to keep two particles bound with zero energy. The physical cause of the effect is in the emergence of the effective attractive long-range potential in the three-body system. The effect is also considered as a generalization of the Thomas theorem [3] which states that three spinless particles do have a weakly bound state when the strengths of two-body interactions approach the critical value.

Thereafter, to search for the long-range effective potential and the bound states in the weakly bound three-body system attracts a lot of physicists [4–8]. Although some people have obtained [6–8] the long-range behavior of the effective potential under various approximations, they cannot give the binding energy and the wave function of the three-body system by their long-range effective potential. There is also no one to answer whether there exist the bound excited states in weakly bound three-body system if there is no bound state for two-body subsystems. In this paper we intend to solve these problems.

II. FORMALISM

In order to simplify the problem, we assume the three-body system is composed of three spinless particles and at least two particles are identical. The particles are denoted as 1, 2, and 3. The masses of particles are M_1, M_2, and M_2 where 2 and 3 are taken as identical particles. It is convenient to discuss the three-body problem in triangular coordinate systems [9–11]. After the center-of-mass motion is removed, the wave functions of the S states depend on three radial variables r_1, r_2, and r_3 which are, respectively, the distances of 1-2, 1-3, and 2-3.

The variation for the Schrödinger equation can be written as follows:

$$\begin{aligned}0 &= \delta\langle\Psi|H-E|\Psi\rangle\\ &= \delta 8\pi^2\int_0^\infty dr_1\int_0^\infty dr_3\int_{|r_1-r_3|}^{r_1+r_3} dr_2 r_1 r_2 r_3\\ &\quad\times\left\{\frac{h^2(M_1+M_2)}{2M_1M_2}\left[\sum_{i=1}^{3}\left(\frac{\partial\Psi}{\partial r_1}\right)^2+t(1,2,3)+t(2,3,1)+t(3,1,2)\right]\right.\\ &\quad\left.+\frac{h^2(M_1-M_2)}{2M_1M_2}\left[\left(\frac{\partial\Psi}{\partial r_3}\right)^2-t(1,2,3)+t(2,3,1)+t(3,1,2)\right]+[U(r_1)+U(r_2)+V(r_3)]\Psi^2-E\Psi^2\right\},\end{aligned} \tag{1}$$

where $t(i,j,k)\equiv(r_i^2+r_j^2-r_k^2)/2r_ir_j(\partial\Psi/\partial r_i)\partial\Psi/\partial r_j\equiv s(i,j,k)(\partial\Psi/\partial r_i)(\partial\Psi/\partial r_j)$, $U(r_1)$, $U(r_2)$, and $V(r_3)$ are the two-body interactions and will be given afterwards.

As Feshbach and Rubinow have done [9], we assume wave functions in S states for the three-body system are

$$\Psi=\Phi(R), R(\eta)\equiv\frac{1}{2}(r_1+r_2+\eta r_3) \tag{2}$$

where the scaling parameter η is used to describe the different dependence of wave functions on r_1, r_2, and r_3.

We define the new coordinate variables R, R_2, and R_3 as follows:

$$\begin{aligned}R &\equiv \frac{1}{2}(r_1+r_2+\eta r_3)\ ,\\ R_2 &\equiv r_2\ ,\\ R_3 &\equiv \frac{1}{2}(1+\eta)r_3\ .\end{aligned} \tag{3}$$

Since the wave function $\Phi(R)$ is only a function of R, the integration over the other two variables (R_2, R_3) can be performed. Then Eq. (1) becomes

$$0 = \delta \int_0^\infty dR R^5 \left\{ \left(\frac{\partial \Phi}{\partial R}\right)^2 D + \left[V_{\rm eff} - \frac{8+5\eta+\eta^2}{15(1+\eta)^3} E\right] \Phi^2 \right\} \quad (4)$$

where

$$D \equiv \frac{1}{60(1+\eta)^3}\{[2\mu + (\mu+\mu')\eta^2](8+5\eta+\eta^2) + \eta(5+\eta)(3\mu+\mu')\} \equiv \frac{D'}{60(1+\eta)^3} , \quad (5)$$

$$\mu = \frac{\hbar^2(M_1+M_2)}{2M_1M_2} \text{ and } \mu' = \frac{\hbar^2(M_1-M_2)}{2M_1M_2} . \quad (6)$$

The $V_{\rm eff}$ in Eq. (5) is defined as follows:

$$V_{\rm eff} = \frac{1}{R^5}\int_0^R dR_3 \int_{R-R_3}^{R-\beta R_3} dR_2 (2R - R_2 - \nu R_3) R_2 R_3 \times [U(r_1) + U(r_2) + V(r_3)] . \quad (7)$$

To make the variation to $\Phi(R)$ in Eq. (4), we have

$$\frac{1}{R^5}\frac{\partial}{\partial R}\left(R^5 \frac{\partial \Phi}{\partial R}\right) - \frac{1}{D}\left[V_{\rm eff} - \frac{8+5\eta+\eta^2}{15(1+\eta)^3}E\right]\Phi = 0 . \quad (8)$$

If $F(R) \equiv R^{5/2}\Phi(R)$ is introduced and inserted into Eq. (8), then Eq. (8) is changed into

$$\frac{d^2F}{dR^2} + \left[\frac{4(8+5\eta+\eta^2)}{D'}E - \frac{V_{\rm eff}}{D} \frac{\frac{3}{2}(\frac{3}{2}+1)}{R^2}\right] F = 0 . \quad (9)$$

So we complete the reduction of the three-body problem to an "effective two-body equation" denoted by Eq. (9). This is also why the method is called as an equivalent two-body method [9]. For a given η, we can obtain the energy and the wave function of the ground state by solving Eq. (9). As the energy is the function of η, finally we shall change the value of η in order to choose a best η which minimizes the energy. For the best η, different eigenvalues of Eq. (9) will correspond to eigenfunctions with different nodes. The least eigenvalue is the energy of the ground state. The next one is the energy of the first bound excited state.

III. NUMERICAL RESULTS AND DISCUSSIONS

In the numerical calculation, the masses of particles are chosen as the integer times of the nucleon mass. $M_1 = m_1 m$ and $M_2 = m_2 m$ where m_1 and m_2 are the integers and m is the nucleon mass. The two-body interactions are taken as S-wave potentials with Yukawa forms.

The potential between 2 and 3 is

$$V(r_3) = (-s)147.585(1/m_2)b^{-2}\frac{b}{r_3}\exp\left(-2.1196\frac{r_3}{b}\right) , \quad (10)$$

where s and b (fm) are, respectively, the parameters of the well depth and the force range. When $s \le 1$, the scattering length is negative and there is no bound state for the two-body subsystem.

The potentials of 1-2 and 1-3 are

$$U(r_i) = (-s_c)\frac{m_1+m_2}{2m_1m_2}147.585 b_c^{-2}\frac{b_c}{r_i}\exp\left(-2.1196\frac{r_i}{b_c}\right) , \quad (11)$$

where $i = 1, 2, b_c$ and s_c are also the parameters of the force range and the well depth, respectively. There is no bound state for the two-body subsystem if the well depth parameter $s_c \le 1$.

Substituting Eqs. (10) and (11) into Eq. (7), we obtain the analytical expression of the effective potential for the two-body interactions with Yukawa forms:

$$\begin{aligned} V_{\rm eff} &= 2V_{\rm eff1} + V_{\rm eff3} , \\ V_{\rm eff1} &= (-V_1)\frac{1}{A^5R^5}\left\{(AR-1)\left[AR - 1 + \exp(-AR) - \frac{1}{\beta^2}(A\beta R\exp(-A\gamma R) - \exp(-A\gamma R) + \exp(-AR))\right]\right. \\ &\quad \left. -\beta[A^2R^2 - 2AR + 2 - 2\exp(-AR)] + \frac{1}{\beta^3}[(A^2\beta^2R^2 - 2A\beta R + 2)\exp(-A\gamma R) - 2\exp(-AR)]\right\} , \\ V_{\rm eff3} &= (-\gamma V_3)\frac{1}{B^5\gamma^5R^5}\left\{\exp(-B\gamma R)\left[-\frac{1}{6}\gamma^2B^3\gamma^3R^3 + \frac{1}{2}\nu\gamma B^2\gamma^2R^2 + (2-\nu^2)B\gamma R - \nu^2 - 2\beta\right]\right. \\ &\quad \left. + B^2\gamma^2R^2 - 2\nu B\gamma R + \nu^2 + 2\beta\right\} , \end{aligned} \quad (12)$$

where $\nu = 2\eta/(\eta+1)$, $\beta = (\eta-1)/(\eta+1)$, $\gamma = 1-\beta$, $A = 2.1196/b_c$, $B = 2.1196/b$, $V_1 = s_c 147.585[(m_1+m_2)/2m_1m_2]b_c^{-2}2.1196$, $V_3 = s147.585(1/m_2)b^{-2}2.1196$. The expression for $V_{\rm eff1}$ in Eq. (14) should be replaced by an expression similar to Eq. (15) for $\eta = 1$, i.e., $\beta = 0$.

It is evident that the effective potential is a long-range potential. Its asymptotic behavior for a very large R is

$$V_{\rm eff} \sim -c_1\frac{b_c}{R^3} - c_2\frac{b}{R^3} \quad (13)$$

TABLE I. Numerical results for three-body systems.

m_1	m_2	s_c	η	E_0 (or E_1) (MeV)	$\overline{R^2}$ (fm^2)	$\overline{r_3{}^2}$ (fm^2)	$\overline{r_1{}^2}$ (fm^2)
1	1	1.00	1.0	−1.23	63.01	30.88	30.88
1000	1	1.00	0.6	−0.99	38.85	31.39	23.74
1	100	1.00	4.3	−1.28	46.96	2.57	14.62
1	1000	1.00	9.0	−1.78	38.62	0.53	11.29
1	1000	1.00	9.0	$-2.01\times10^{-2}(E_1)$	1079.29	14.73	315.55
1	10000	1.00	16.0	−2.06	35.12	0.16	10.11
1	10000	1.00	16.0	$-5.32\times10^{-2}(E_1)$	688.45	3.04	198.20
1	100000	1.00	28.0	−2.17	34.00	0.05	9.74
1	100000	1.00	28.0	$-7.08\times10^{-2}(E_1)$	597.04	0.87	171.02

and this agrees with Zheng and Macek's work [8]. One important difference between this paper and previous works [6–8] on the Efimov effect is that only we can give the long-range effective potential for an arbitrary R. It is this point that we can give energies and wave functions of bound states by the long-range effective potential.

The numerical results are given in Table I. In the calculation, we fix parameters of force range $b = b_c = 5$ fm and parameters of well depth $s = s_c = 1.0$. If there is no bound state for two-body subsystems, we hope to know whether there exist the bound excited states in the three-body system for different mass ratio of particles. (The masses of particles are chosen as the integer times of the nucleon mass and denoted by the integers m_1 and m_2.) In the table, E_0 and E_1 are, respectively, energies of the ground state and first excited state. $\overline{R^2}$ is the mean-square distance of R $[R = \frac{1}{2}(r_1 + r_2 + \eta r_3)]$. $\overline{r_3{}^2}$ and $\overline{r_1{}^2}$ are, respectively, the mean-square distances of 2-3 and 1-2. They are defined as follows:

$$\overline{R^2} = \frac{\int_0^\infty dR R^2 F^2(R)}{\int_0^\infty dR F^2(R)} , \tag{14}$$

$$\overline{r_3{}^2} = \frac{\int_0^\infty dR\Phi^2(R) \int_0^R dR_3 \int_{R-R_3}^{R-\beta R_3} dR_2 (2R - R_2 - \nu R_3) R_2 R_3 r_3{}^2}{\int_0^\infty dR\Phi^2(R) \int_0^R dR_3 \int_{R-R_3}^{R-\beta R_3} dR_2 (2R - R_2 - \nu R_3) R_2 R_3} , \tag{15}$$

$$\overline{r_1{}^2} = \frac{\int_0^\infty dR\Phi^2(R) \int_0^R dR_3 \int_{R-R_3}^{R-\beta R_3} dR_2 (2R - R_2 - \nu R_3) R_2 R_3 r_1{}^2}{\int_0^\infty dR\Phi^2(R) \int_0^R dR_3 \int_{R-R_3}^{R-\beta R_3} dR_2 (2R - R_2 - \nu R_3) R_2 R_3} . \tag{16}$$

It is seen clearly from the first two rows of Table I that there is no bound excited state for the systems with three identical particles and with two light particles and a heavy particle if there is no bound state for two-body subsystems. We also try to search the bound excited states for these systems by changing the mass ratio of particles but we do not find them. The first bound excited state appears when the strengths of two-body interactions increase to the value which leads to a weakly bound state for two-body subsystems. It agrees with the previous work on a three-body system [12].

For the system with two heavy particles and a light particle, if the mass ratio between a heavy particle and a light particle is about greater than 1000, it is seen from the table that there exists a bound excited state although there is no bound state for two-body subsystems. Fonseca, Redish, and Shanley [6] also consider that a large mass ratio between a heavy particle and a light particle is a necessary condition for the Efimov effect and this agrees with the above conclusion. It goes without saying that the bound excited state is an Efimov state. In this state, the mean-square-root distance between the heavy particle and the light particle is greater than the force range of the two-body interactions and means the light particle is mainly outside the force range. This is due to the effect of the long-range effective potential in the three-body system.

IV. CONCLUSION

In summary, the weakly bound states in a three-body system have been studied in detail by the equivalent two-body method for a three-body system. An analytical and complete expression for the long-range effective potential is derived for the two-body interactions with Yukawa forms. The bound excited state in the system with two heavy particles and a light particle is first discovered although there is no bound state for two-body subsystems. This is the first direct evidence on the Efimov effect. The numerical result suggests that maybe one can observe this kind of states in the molecular system with two heavy particles and a light particle.

ACKNOWLEDGMENTS

I would like to acknowledge the help from Prof. W. Mittig and Prof. F. Carstoiu. I also thank Prof. Xu Gong-ou for the discussion. This work was supported by the National Natural Science Foundation of China.

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