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Radius and radial moments of the deuteron

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New values are extracted for the deuteron rms matter radius $r_D = 1.9547 \pm 0.0019$ fm and the matter radial deuteron moments $\langle r^4 \rangle = 54.2 \pm 0.1$ fm^4 and $\langle r^6 \rangle = 1828 \pm 1$ fm^6 by analyzing the experimental ratio of (e,d) to (e,p) scattering.

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There is increasing interest in the literature [1–7] in extracting the deuteron matter radius r_D by analyzing results of elastic electron scattering experiments. Some of these analyses, and also the analysis of this paper, involve fitting the "experimental" electric charge form factor $C_E(q^2)$ in the low-q^2 region by a polynomial of a certain order in q^2 [5];

$$C_E(q^2) = 1 + a_1 q^2 + a_2 q^4 + \cdots , \tag{1}$$

where $r_D^2 = 6a_1$ or—equally well—by using a continued-fraction method [6]. We recall that the matter radial moment $\langle r^{2k} \rangle$ is the expectation value of $(r/2)^{2k}$:

$$\langle r^{2k} \rangle = 2^{-2k} \int_0^\infty (u^2 + w^2) r^{2k} dr , \tag{2a}$$

e.g., $\langle r^2 \rangle = r_D^2$ for $k = 1$. The coefficients a_k are related to the radial moments $\langle r^{2k} \rangle$ by

$$a_k = \frac{(-)^k}{(2k+1)!} \langle r^{2k} \rangle . \tag{2b}$$

The "experimental" values of $C_E(q^2)$ are obtained from the experimental data of Simon *et al.* [3] for the ratio $R(q^2) = G_{ED}(q^2)/G_{Ep}(q^2)$ of deuteron to proton form factors by using

$$C_E(q^2) = R(q^2)(1+\tau)^{1/2}[1 + G_{En}(q^2)/G_{Ep}(q^2)]^{-1} \tag{3}$$

and the relation

$$G_{En}(q^2)/G_{Ep}(q^2) = \frac{0.1192}{6} q^2 \tag{4}$$

of Isgur *et al.* [8], where $G_{En}(q^2)$ is the neutron form factor, $\tau = q^2/4m_p^2$ is the Darwin-Foldy correction, and $m_p = 938.2786/\hbar c = 4.574\,91$ fm^{-1} is the proton mass. The expansion of $C_E(q^2)$ in powers of q^2 may be written for the present purpose in the form

$$C_E(q^2) - 1 - \sum_{n=m+1}^{n=N} a_n q^{2n} = a_1 q^2 + a_2 q^4 + \cdots + a_m q^{2m} . \tag{5}$$

The values of the coefficients a_n ($m < n \le N$) on the left-hand side (LHS) of Eq. (5) are dominated by the asymptotic normalization. The "correction" $\sum_{n=m+1}^{n=N} a_n q^{2n}$ is calculated using the radial deuteron wave functions of the local potential of Ref. [7]. The local potential of Mustafa, Hassan, Kermode, and Zahran [7] (MHKZ) has the correct asymptotic behavior, i.e., fitting the recent experimental values of the asymptotic S-state amplitude $A_S = 0.8838 \pm 0.0004$ fm$^{-1/2}$ of Stoks *et al.* [9] and the asymptotic D/S ratio $\eta = 0.0273 \pm 0.0005$ of Borbély *et al.* [10]. The value $\eta = 0.0273 \pm 0.0005$ of Borbély *et al.* [10] is consistent, in particular, with our recent prediction $\eta = 0.027\,01 \pm 0.000\,19$ [11]. The MHKZ potential fits too the experimental value of the deuteron quadrupole moment ($Q = 0.2860 \pm 0.0015$ fm^2 [12] and $Q = 0.2859 \pm 0.0003$ fm^2 [13]). Unfortunately, the values listed in Table I of Ref. [7] for the free parameters of this potential are overtruncated; therefore, they are given here in Table I with a larger number of significant figures. The values assumed for m are $m = 2$, 3, 4, and 5 and for N is $N = 70$. The meson exchange current (MEC) contribution is taken into account by using the correction $\Delta r_D = 0.0034 \pm 0.0003$ fm of Kohno [14].

To make the values of a_k of Eqs. (1) and (5) more consistent with the values of $C_E(q^2)$ (and hence, more accurate results would have been obtained), the point $C_E(0) = 1$ given by the boundary condition is used as a constraint and the analytic asymptotic contribution $\epsilon = \int_R^\infty (u^2 + w^2) dr$ to the normalization factor $[\int_0^\infty (u^2 + w^2) dr]^{-1/2}$ is neglected by the numerical methods producing the deuteron waves u and w of the local potential MHKZ of Ref. [7]. Deuteron properties, in particular A_S and η, will not change by neglecting ϵ, except for r_D; it negligibly changes from 1.963 16 fm to 1.962 56 fm. A small change in r_D would not affect the results of the analysis (as discussed in conjunction with Table III).

The results of fitting the experimental data are listed in Table II. It is found that the order $m = 3$ gives the best fit to the data. Also, the determined values of r_D are

TABLE I. Values with more significant figures for the free parameters $A_C(n)$, $A_{LS}(n)$, and $A_T(n)$ of the local potential MHKZ of Ref. [7].

n	$A_C(n)$	$A_{LS}(n)$	$A_T(n)$
2	−0.149 132 78(4)	0.593 922 46(3)	−0.421 205 54(3)
3	0.303 243 03(5)	−0.123 513 94(5)	0.694 378 45(4)
4	−0.193 483 26(6)	0.883 782 25(5)	−0.316 960 90(5)
5	0.480 649 93(6)	−0.240 061 96(6)	0.486 753 91(5)
6	−0.405 839 99(6)	0.210 874 90(6)	−0.192 918 95(5)

TABLE II. The values deduced for r_D, $\langle r^4 \rangle$, $\langle r^6 \rangle$, $\langle r^8 \rangle$, and $\langle r^{10} \rangle$. The MEC contribution $\Delta r_D = 0.0034 \pm 0.0003$ fm [14] has been included only in r_D. The corresponding values of the MHKZ potential [7] are also listed in the bottom row for comparison.

m	r_D (fm)	$\langle r^4 \rangle$ (fm^4)	$\langle r^6 \rangle$ (fm^6)	$\langle r^8 \rangle$ (fm^8)	$\langle r^{10} \rangle$ (fm^{10})
2	1.9587±0.0013	54.59±0.04			
3	1.9547±0.0019	54.22±0.14	1828±1		
4	1.9536±0.0028	54.03±0.39	1824±9	104 754±93	
5	1.9542±0.0049	54.18±1.15	1830±42	104 898±984	8 369 000±120 00
	1.962 564	54.751 57	1831.738	104 803.06	8 367 000

stable (i.e., similar) for $m \geq 3$. We quote here

$$r_D = 1.9547 \pm 0.0019 \text{ fm} \tag{6}$$

corresponding to $m=3$ as the result of this method for the rms matter radius of the deuteron. This value of r_D is very similar to our previous one [7] $r_D = 1.9546 \pm 0.0021$ fm, which uses a different method of analysis. This new value of r_D is also in agreement—within the quoted errors—with the determinations of Allen *et al.* [4] $r_D = 1.952 \pm 0.004$ fm, of McTavish [5] $r_D = 1.956 \pm 0.005$ fm, and of Klarsfeld *et al.* [6] $r_D = 1.953 \pm 0.003$ fm.

It is interesting that the results of the analyses of McTavish [5] $r_D = 1.956 \pm 0.005$ fm and Allen *et al.* [4] $r_D = 1.952 \pm 0.004$ fm could have been changed to be $r_D = 1.955 \pm 0.005$ fm and $r_D = 1.955 \pm 0.004$ fm, respectively, which are very consistent with our results of (6), if a more accurate approximation of the deuteron quadrupole form factor $C_Q(q^2)$ had been used by McTavish (see the footnote of Ref. [5]) and, the MEC contribution $\Delta r_D = 0.0034 \pm 0.003$ fm [14] would not have been ignored for its smallness [4] by Allen *et al.* Klarsfeld *et al.* [6] have studied the effect of a change in A_S on the deuteron radius. Their value $r_D = 1.9532$ fm ($= 1.9498 +$MEC [14]) could have also been changed to be $r_D = 1.9539$ fm, which is very consistent with the result of (6), if they used (as in our case) the value $A_S = 0.8838 \pm 0.0004$ fm$^{-1/2}$ of Stoks *et al.* [9] instead of $A_S = 0.8800 \pm 0.0060$ fm$^{-1/2}$ [6].

We quote too, as the result of this method, the following values for the deuteron matter radial moments:

$$\langle r^4 \rangle = 54.2 \pm 0.1 \text{ fm}^4 , \tag{7a}$$

$$\langle r^6 \rangle = 1828 \pm 1 \text{ fm}^6 , \tag{7b}$$

$$\langle r^8 \rangle = 104\,754 \pm 93 \text{ fm}^8 , \tag{7c}$$

$$\langle r^{10} \rangle = 8\,369\,000 \pm 12\,000 \text{ fm}^{10} . \tag{7d}$$

The MEC corrections are not included in these values of Eq. (7). The values for $\langle r^4 \rangle$ and $\langle r^6 \rangle$ are consistent with the corresponding values of Klarsfeld *et al.* [6] $\langle r^4 \rangle = 54.5 \pm 0.3$ fm^4 and $\langle r^6 \rangle = 1914 \pm 20$ fm^6. The values of $\langle r^8 \rangle$ and $\langle r^{10} \rangle$ (which are mostly determined by the asymptotic normalization) are from the $m=4$ and $m=5$ solutions, respectively.

The effect of the variation of the nonasymptotic parts of the deuteron wave functions on the prediction of r_D, $\langle r^4 \rangle$, and $\langle r^6 \rangle$ is investigated (Table III) by calculating the "correction" $\sum_{n=m+1}^{n=N} a_n q^{2n}$ of Eq. (5) using transformed radial deuteron wave functions having different interior shapes but the same asymptotic radial dependences. These waves are produced by applying unitary transformations of the form used by Kermode *et al.* [15] to the deuteron waves of the local potential MHKZ of Ref. [7]. The nonlocality parameters $(\alpha,\beta) = (2.5, 1.2)$, (3.0,0.6), and (3.5,0.6) produce the transformed waves of Fig. 4 of Ref. [7]. The large difference in the shapes at small radii of these transformed deuteron wave functions

TABLE III. The effect of the variation of the nonasymptotic parts of the deuteron wave functions. The correction $\sum_{n=m+1}^{n=N} a_n q^{2n}$ (with $m=3$) of Eq. (5) is calculated for phase equivalent potentials produced by unitary transformations which use the potential MHKZ as a reference potential. The first three pairs (α,β) of parameters of the unitary transformations produce the transformed deuteron waves of Fig. 4 of Ref. [7]; the last three produce transformed waves having unrealistic model values of $\bar{r}_D$ and Q.

α (fm^{-1})	β (fm^{-1})	$\bar{r}_D$ (fm)	Q (fm^2)	r_D (fm)	$\langle r^4 \rangle$ (fm^4)	$\langle r^6 \rangle$ (fm^6)
2.5	1.2	1.952	0.289	1.9546±0.0019	54.2±0.1	1828±1
3.0	0.6	1.953	0.280	1.9547±0.0019	54.2±0.1	1829±1
3.5	0.6	1.952	0.279	1.9547±0.0019	54.2±0.1	1828±1
1.7	0.7	2.087	0.367	1.9556±0.0019	54.4±0.1	1833±1
2.5	1.5	1.828	0.224	1.9545±0.0019	54.2±0.1	1828±1
1.7	1.5	1.657	0.168	1.9514±0.0019	53.6±0.1	1811±1

TABLE IV. The effect produced mostly by the variation of the asymptotic parts of the radial deuteron wave functions. The correction $\sum_{n=m+1}^{n=N} a_n q^{2n}$ (with $m=3$) of Eq. (5) is calculated for standard deuteron potential models having various values of A_S and η.

Pot.	Ref.	A_S ($\mathrm{fm}^{-1/2}$)	η	r_{D} (fm)	$\langle r^4 \rangle$ (fm^4)	$\langle r^6 \rangle$ (fm^6)
B	[18]	0.9132	0.0343	1.9626±0.0022	56.4±0.2	1943±1
Bonn	[19]	0.9046	0.0267	1.9600±0.0021	55.6±0.2	1894±1
TS-*A*	[20]	0.9000	0.0240	1.9586±0.0020	55.3±0.2	1875±1
GK7	[21]	0.8983	0.0257	1.9576±0.0020	55.0±0.1	1869±1
TRS	[22]	0.8883	0.0262	1.9557±0.0019	54.5±0.1	1832±1
Paris	[23]	0.8868	0.0261	1.9553±0.0019	53.3±0.1	1826±1
HJ	[24]	0.8851	0.0265	1.9551±0.0019	54.4±0.1	1835±1
Mach-C	[25]	0.8850	0.0266	1.9549±0.0019	54.2±0.1	1819±1
RHC	[17]	0.8803	0.0259	1.9537±0.0019	54.0±0.1	1814±1
RSC	[17]	0.8776	0.0262	1.9529±0.0019	53.7±0.1	1790±1
GK1	[21]	0.8768	0.0271	1.9526±0.0019	53.7±0.1	1796±1
GK9	[21]	0.8767	0.0267	1.9525±0.0019	53.6±0.1	1792±1
MZ	[26]	0.8712	0.0134	1.9510±0.0019	53.2±0.1	1777±1

(as can be seen from Fig. 4 of Ref. [7]) do not affect the results obtained because these waves have "reasonable" model values of the quadrupole moment Q and the deuteron radius r_{D}. The values of Q and r_{D} are partly dependent on the "inside" part of the deuteron waves, e.g., a large change could be produced in each value of Q [15] and r_{D} [16] of the Reid hard-core potential [17] by changing only the "inside" parts of the deuteron wave functions. This signifies using in this analysis the MHKZ potential [7] which has the correct value of Q and a "reasonable" value of r_{D}. The nonasymptotic parts of the wave functions which give unrealistic model values of Q and r_D would mostly produce incorrect results as can be seen in the case of the transformed deuteron waves corresponding to $(\alpha,\beta)=(1.7,0.7)$, (2.5,1.5), and (1.7,1.5). These unrealistic choices for the nonasymptotic parts of the deuteron waves produce an uncertainty of 0.0042 fm in the predicted value for r_{D} which is about 25% of the difference between the two extreme values $r_{\mathrm{D}}=1.9635\pm0.0045$ fm and $r_{\mathrm{D}}=1.947\pm0.029$ fm given as determinations of r_{D} by Berard *et al.* [1] and Akimov *et al.* [2], respectively.

The effect of changing the asymptotic normalization on the predicted results for r_{D}, $\langle r^4 \rangle$, and $\langle r^6 \rangle$ is investigated (Table IV) by calculating the "correction" $\sum_{n=m+1}^{n=N} a_n q^{2n}$ of Eq. (5) using the radial deuteron wave functions of standard potential models [17–26] having various values of A_S and η. The values obtained for r_{D} and the higher order moments increase with A_S and are almost not influenced by the change in the value of η. The contribution to the standard error δr_{D} of r_{D} caused by the errors $\pm\Delta A_S=0.0004\ \mathrm{fm}^{-1/2}$ and $\pm\Delta\eta=0.0005$ in the experimental values of A_S and η, respectively, is negligibly small because of the smallness of the values of $\partial r_{\mathrm{D}}=\partial A_S=0.265\ \mathrm{fm}^{-3/2}$, ΔA_S and $\Delta\eta$. Klarsfeld *et al.* [6] used a different approach and obtained $\partial r_{\mathrm{D}}/\partial A_S \simeq 0.2\ \mathrm{fm}^{-3/2}$.

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