Eikonal expansion for total cross sections of heavy ion reactions

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(Received 30 March 1993)

We study the validity and convergence of the eikonal expansion for the calculation of total σ_t and total reaction σ_R cross sections in the case of low-energy heavy-ion collisions. Investigations are performed by considering few typical optical potentials, varying the incident energy from 10 MeV/nucleon to 300 MeV/nucleon. At low energy, the eikonal expansion does not necessarily converge and faces a violation of flux conservation, especially at low-impact parameter. For well-behaved cases, we find the eikonal approximation to be a convenient starting point. Adding the first correction leads in general to an acceptable accuracy with respect to the quantum mechanical result.

PACS number(s): 25.70.−z, 24.10.Ht

I. INTRODUCTION

The measurements of the total reaction cross section σ_R constitutes a rather unique way of collecting information on the size of unstable nuclei [1–4]. Extracting nuclear radii from σ_R can be achieved by using the impact parameter representation of the scattering amplitude [5]

$$\sigma_R = 2\pi \int_0^\infty [1 - \exp(i2\mathrm{Im}\chi(b))] b\, db \; . \qquad (1)$$

As known from numerous works, at intermediate and high energies the phase-shift function $\chi(b)$ can be calculated in the Glauber model [6]. Its domain of validity is well established and its use at large enough incident energies is quite justified. Specific formulations for nucleus-nucleus scattering have been produced by many authors (see, for instance, Refs. [7,8]).

On the other hand, because of its appealing simplicity and of its geometrical character, there are attempts to apply the same approach at much lower energies [4,9], where it becomes questionable. The problems arising are twofold.

Firstly, the Glauber model asssumes the total interaction to be described in terms of the interaction between individual nucleons. At low energy this cannot be achieved by considering the bare nucleon-nucleon potential, since medium effects are known to be important [10]. Besides the necessity of dealing with effective forces, often density dependent, the coupling to the excited states should also be included. At high energy, this is done by using the closure approximation in the intermediate steps of the multiple scattering.

In other words, at low energy, the construction of the optical potential and the need for coupled channel equations are not obviously compatible with the basic assumptions underlying the Glauber model.

Secondly, this model assumes phase-shift additivity. From potential scattering theory this hypothesis is known to hold asymptotically, at large incident momentum k. In order to check its validity in concrete examples, it is necessary to consider the Glauber phase-shift function as the eikonal approximation of an equivalent optical potential $V(r)$. It is then possible to calculate the corrections to the eikonal phase, and thus to test the phase-shift additivity [11].

On the other hand, in the quantum mechanics approach, supposing the optical potential to be known, calculating total cross sections of heavy-ion reactions requires the summation of numerous high-ℓ partial wave contributions. Such procedures are easily subject to a loss in accuracy. By comparison the eikonal approximation provides us with a very practical tool for computing total cross sections, reducing the accuracy problem. Systematic corrections to the eikonal approximation are available [12,13]. It is thus straightforward to check the eikonal phase or/and to improve its results by evaluating the successive terms of the series expansion.

It is very tempting to apply the same approach at low k. There are, however, two questions to be answered. The available corrections have been derived from a dynamical model valid at medium and high incident energies [13]. They are not necessarily valid in a low-energy regime. Secondly, independently of the potential shape, a rough global criterion for the convergence of the series expansion is provided as by the ratio V_0/E, V_0 being the strength of the potential and E the incident center of mass energy. As the successive contributions are proportional to $(\frac{V_0}{E})^n$, the sum converges for $V_0/E << 1$. At low energy, however, this quantity gets close to unity and the convergence is not ensured.

The purpose of the present work is to discuss the validity of the eikonal approximation and its first few corrections in a low-energy regime. Rather than going through analytical arguments we shall provide few illustrative quantitative estimates.

An attempt along this line has been made by Hussein *et al.* [14]. Within a careful study of the total reaction cross-section calculations, these authors compare WKB and eikonal phase-shift functions for a Woods-Saxon potential. Whereas at high energy the two approximations yield similar results, they differ, as expected, at low energy, the WKB approximation being closer to the quantum mechanical values. For the total reaction cross section, as we shall see, the first correction improves the eikonal result sensibly.

II. CHECKING THE EIKONAL EXPANSION

For the sake of clarity, we recall that in the impact parameter representation, the elastic scattering amplitude is given by

$$F(q) = ik \int J_0(qb)[1 - \exp(i\chi(b))]b\, db \ , \tag{2}$$

where k is the incident momentum in the c.m. system and $q = 2k\sin(\theta/2)$ is the momentum transfer.

Assuming unitarity, the total cross section is obtained by applying the optical theorem

$$\sigma_t = \frac{4\pi}{k}\mathrm{Im}F(0) \ . \tag{3}$$

The total elastic cross section can be calculated by integrating the squared amplitude over the whole angular domain (see Appendix):

$$\begin{aligned}\sigma_{\mathrm{el}} &= \int |F(\theta)|^2\, d\Omega \\ &= 2\pi\int_0^\infty |1 - e^{i\chi(b)}|^2 b\, db \ .\end{aligned} \tag{4}$$

The total reaction cross section is then given by the difference $\sigma_R = \sigma_t - \sigma_{\mathrm{el}}$.

Separating the real and imaginary parts of the phase-shift function,

$$\chi(b) = \chi_R(b) + i\chi_I(b) \ ,$$

we can write

$$\sigma_t = 4\pi\int_0^\infty [1 - \cos(\chi_R(b))e^{-\chi_I(b)}]b\, db, \tag{5}$$

$$\sigma_R = 2\pi\int_0^\infty [1 - e^{-2\chi_I(b)}]b\, db \ . \tag{6}$$

For potential scattering the eikonal expansion has been derived by Wallace [12], and it has been written in a compact form by Waxman *et al.* [13]. It is given by

$$\chi(b) = \sum_n -\frac{\mu^{n+1}}{k(n+1)!}\left(\frac{b}{k}\frac{\partial}{\partial b} - \frac{\partial}{\partial k}\frac{1}{k}\right)^n \int_0^\infty V^{n+1}(r)dz \ , \tag{7}$$

where the differentiation with respect to $\frac{\partial}{\partial b}$ and $\frac{\partial}{\partial k}$ are carried out at fixed k and b, respectively. Derivatives with respect to k occur only for potential depending explicitly on velocity. Note that μ is the reduced mass and that we have set $\hbar = 1$. The kinematic is nonrelativistic.

The zero order term in (7) gives the eikonal phase

$$\chi_0(b) = -\frac{\mu}{k}\int_{-\infty}^{+\infty} V(r)dz \ . \tag{8}$$

For local potentials the first and second order corrections are given, respectively, by

$$\chi_1(b) = \frac{-\mu^2}{2k^3}\left(1 + b\frac{\partial}{\partial b}\right)\int_{-\infty}^{+\infty} V^2(r)dz \tag{9}$$

and

$$\chi_2(b) = -\frac{\mu^3}{6k^5}\left(3 + 5b\frac{\partial}{\partial b} + b^2\frac{\partial^2}{\partial b^2}\right)\int_{-\infty}^{+\infty} V^3(r)dz \ . \tag{10}$$

In order to obtain some insight into the problem, we have studied a particular example. We consider an incident ^{6}Li beam scattered by a ^{12}C target, in the range of 10 MeV/nucleon to 100 MeV/nucleon. Note that we consider only light elements so that neglecting the Coulomb phase is not too unreasonable specially since we shall merely study the total reaction cross section. The optical potential has a very simple Gaussian shape

$$V_{\mathrm{opt}}(r) = V_0 e^{-\alpha^2 r^2} + iW_0 e^{-\beta^2 r^2} \ . \tag{11}$$

The strength and ranges are kept independent of the incident energy. Although not realistic this is sufficient for the present purpose, since we merely want to check the merit of the eikonal expansion in given cases rather than describe experimental situations. Spin effects and Coulomb phases are ignored.

The first point to discuss concerns the flux conservation; it requires $|e^{-2\chi_I(b)}| \leq 1$. This inequality is not necessarily fulfilled at small impact parameters and low-incident energy. Therefore it imposes conditions and restricts the physical domain of application of the eikonal epxansion. This is clearly shown, for instance, by looking at $b = 0$.

For the Gaussian potential (11), the three first contributions to $\mathrm{Im}\chi(0)$ from the eikonal expansion are given by

$$\mathrm{Im}\chi_0 = -\frac{k}{E} W_0 \frac{\sqrt{\pi}}{2\beta} ,$$

$$\mathrm{Im}\chi_1 = -\frac{k}{4E^2} V_0 W_0 \frac{\sqrt{\pi}}{\sqrt{\alpha^2+\beta^2}} , \quad (12)$$

$$\mathrm{Im}\chi_2 = -\frac{k}{24E^3}\left(V_0^2 W_0 \, 9 \frac{\sqrt{\pi}}{2\sqrt{2\alpha^2+\beta^2}} - W_0^3 \frac{3\sqrt{\pi}}{2\sqrt{3}\beta}\right) .$$

Here E is the center-of-mass energy. We introduce the following quantities:

$$x = \frac{W_0}{V_0} , \quad y = \frac{V_0}{E} , k_0 = \frac{k\sqrt{\pi}}{2} , t = \frac{\alpha}{\beta} , z = \frac{k_0}{\alpha} . \quad (13)$$

It allows us to write the sum of the three contributions (12) as

$$\mathrm{Im}\chi(b=0) \simeq -xyzt\left[1 + \frac{y}{2\sqrt{t^2+1}} + \frac{1}{8}y^2\left(\frac{3}{\sqrt{2t^2+1}} - \frac{x^2}{\sqrt{3}}\right)\right] . \quad (14)$$

We have verified that, under the conditions considered in this paper, the higher-order terms do not alter sensibly the argument. The flux conservation requires $\mathrm{Im}\chi(0) \geq 0$. The parameter t being obviously positive, the sign of $\mathrm{Im}\chi(0)$ is determined by the sign of $f(y)$:

$$f \equiv 1 + By + Ay^2 , \quad (15)$$

where

$$B = \frac{1}{2\sqrt{t^2+1}} \quad \text{and} \quad A = \frac{1}{8}\left(\frac{3}{\sqrt{2t^2+1}} - \frac{x^2}{\sqrt{3}}\right)$$

according to (14).

In the parameter space (x,t) we introduce two particular trajectories:

$$\Delta(x,t) = B^2 - 4A = 0 \quad \text{and} \quad A(x,t) = 0 .$$

They define three domains in the (x,t) plan as displayed in Fig. 1. In region I the function $f(y)$ is positive for any $y \in R$, which ensures the flux conservation for all y in this domain.

In regions II and III, the function $f(y)$ has real roots: two negative roots in II, and one positive and one negative in III. Accordingly $f(y)$ has different signs depending on the particular y value, and consequently a subspace of y values lie in a nonphysical domain.

This can be made more transparent by looking at Fig. 2, where the roots have been plotted against x for representative fixed t values, delimiting in this case the physical from the nonphysical domains.

Note that for each fixed t value, there exists 2 critical

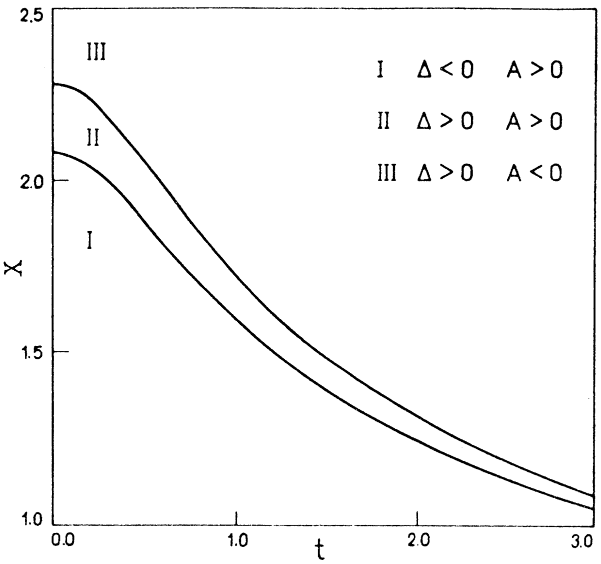

FIG. 1. The 3 domains of the (x,t) plan delimited according to the roots of Eq. (15) (see text).

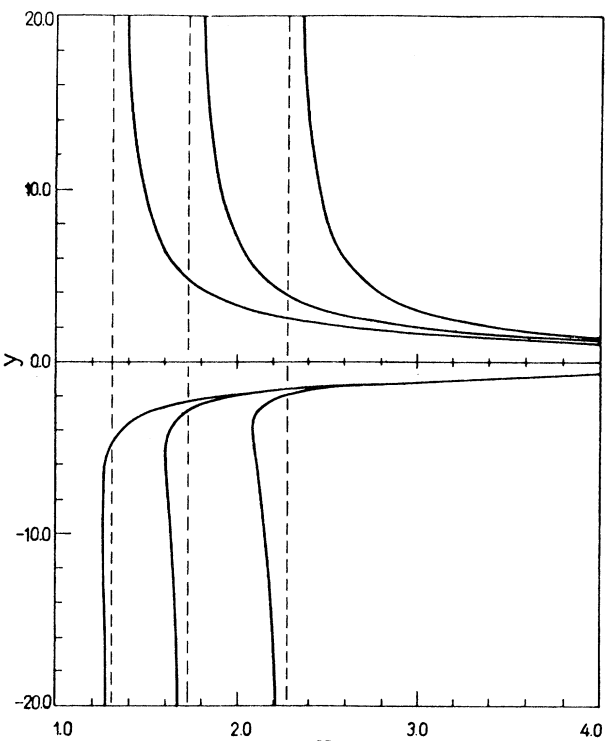

FIG. 2. Physical domains in the (y,x) plane according to the flux conservation condition at $b=0$. The successive asymptotes correspond to $t = 0.0, 1.0$, and 2.0, respectively. The unphysical region, in each case, is the right-hand side of the solid curve, above the minimal value for $y > 0$, and below the maximal value for $y < 0$.

x ($x_{\rm crit}$) given by

$$A = 0 \quad \Rightarrow x_{\rm crit} = \pm\sqrt{\frac{3\sqrt{3}}{\sqrt{2t^2+1}}} \, .$$

Because A is symmetric with respect to $x \to -x$, we may consider the right-half plane only with $x > 0$. Remember that the absorbtive nature of the imaginary part of the potential requires $xy > 0$. Thus the curves of Fig. 2 represent the case of a repulsive interaction. The case for an attractive interaction is its symmetry with respect to the origin.

In the first quadrant, if $x < x_{\rm crit}$ there is no restriction to the value of y. Above $x_{\rm crit}$, however, the physical domain is bounded by the following restrictions:

$$y \leq y_{\rm crit} = \frac{-B + \sqrt{\Delta}}{2A} \, .$$

Beyond $y_{\rm crit}$, the flux conservation condition is violated.

Consequently at low-incident energy, where y can be large, the eikonal expansion may face a violation of the flux conservation. How severe is this deficiency depends on the specific case considered. The present discussion has been limited at $b = 0$, which has very little weight in the calculation of total cross sections, but the same kind of troubles are expected at finite b. Whether a drastic subsidiary condition excluding the nonphysical values of $\chi(b)$ could lead to a reasonable approximation has not been investigated.

The second point to discuss concerns the convergence of the series and the validity of the approximation with respect to the quantum mechanical results. For this later we concentrate our effort on the total reaction cross section. Indeed the total cross section depending on the scattering amplitude at $\theta = 0$, the eikonal approximation works always better for σ_{tot} than for σ_R, which represents only an average.

For illustrative purposes, we take $V_0 = -100$ MeV, $W_0 = -50$ MeV, $\alpha^2 = \beta_2 = 0.1$ fm^{-2}, i.e., values that are typical for low-incident energies. In Table I, we summarize the results. As a function of the incident energy, we quote $|y|$, σ_R calculated at the eikonal approximation and then including successively its two first-order corrections, and finally the exact quantum mechanical result.

We remark that even at 10 MeV/nucleon the rapid decrease of the successive contributions from the eikonal expansion ensures the convergence. Indeed, we have verified that the next order terms become negligibly small. It means that the first criterion based on $|y| < 1$ is too crude, and somewhat too pessimistic, the magnitude of the "derivative" terms playing a decisive role. For this reason one has to be aware that expansions involving a Gaussian and its derivatives exhibit particular convergence properties [15]. The present results cannot be extrapolated to a Woods-Saxon shape, for instance, without care [16].

Whereas the eikonal approximation is $\sim$ 11% lower than the quantum result at 10 MeV/nucleon, 6% at 30 MeV/nucleon, and 3% at 100 MeV/nucleon, the first eikonal correction brings the agreement to 3% at 10 MeV/nucleon, and 1% already at 30 MeV/nucleon. With the second order included, the quantum value is reached within 3% at 10 MeV/nucleon.

Consequently, we conclude that the eikonal expansion could be a powerful tool to calculate total cross sections, even at low incident energy. It is fairly easy to use, both to check the convergence of the expansion and to approach the quantum mechanical result with 1%.

III. FEW REALISTIC EXAMPLES

To complete the present study of the eikonal expansion, we display in this section few examples somewhat more realistic than the preceding ones. Among many possibilities we have chosen to take few optical potentials describing the elastic scattering of ^{11}Li. This choice is simply motivated by the current interest for this very weakly bound nucleus.

The first potential under consideration is taken from Satchler *et al.* [17]. It has been calculated for ^{11}Li + ^{12}C at 30 MeV/nucleon, using the folding approach with a DDM3Y effective interaction. It can be well fitted by a Gaussian:

$$V_1(r) = (-142.7 - i97.1)e^{-\alpha^2 r^2}$$

($\alpha^2 = 0.084\,166$ fm^{-2}, the intensity are in MeV).

Although designed for 30 MeV/nucleon, this potential has been used at four energies ranging from 20 MeV/nucleon to 300 MeV/nucleon, to study the variation of noneikonal corrections with incident energy. This

TABLE I. Total reaction cross section calculated for ^{6}Li-^{12}C scattering at different energies. The potential is given by Eq. (11) with $V_0 = -100$ MeV, $W_0 = -50$ MeV, $\alpha^2 = \beta^2 = 0.1$ fm^{-2}. The eikonal approximation result and its two first corrections are compared to the quantal value. Cross sections are given in mb, energies in MeV.

E/A	$E_{\rm c.m.}$	$y = \lvert \frac{V_0}{E_{\rm c.m.}} \rvert$	χ_0	$\chi_0 + \chi_1$	$\chi_0 + \chi_1 + \chi_2$	Quantal value
10	40	2.5	1112.75	1200.80	1206.47	1238.84
20	80	1.25	1003.87	1065.45	1076.67	1081.48
30	120	0.83	940.18	988.29	995.29	997.35
40	160	0.625	894.99	935.00	939.47	940.54
50	200	0.50	859.95	894.15	897.26	897.90
60	240	0.42	831.31	861.19	863.46	863.87

TABLE II. Eikonal, first- and second-order contributions to σ_t and σ_R for the potential V_1. Cross sections are given in mb.

		χ_0	$\chi_0 + \chi_1$	$\chi_0 + \chi_1 + \chi_2$	Maximal deviation
20 MeV/nucleon	σ_t	2857.4	2892.9	2888.8	1.1%
	σ_R	1472.7	1522.42	1529.2	3.8%
30 MeV/nucleon	σ_t	2706.0	2733.5	2730.8	0.9%
	σ_R	1397.0	1436.0	1440.8	3.15%
85 MeV/nucleon	σ_t	2317.3	2332.1	2331.7	0.6%
	σ_R	1202.7	1222.9	1224.0	1.75%
300 MeV/nucleon	σ_t	1849.0	1857.0	1857.2	0.5%
	σ_R	967.3	975.0	975.2	0.8%

is not quite realistic, since the optical potential is expected to vary with energy. Nevertheless, it will fix the magnitude of the corrections and their variations with energy in a given situation. The results are summarized in Table II.

As in the preceding section, we have calculated the eikonal approximation and its two first corrections for the total and the total reaction cross sections.

The results for this realistic potential confirm previous findings. The corrections to the total cross section are rather small, and reach hardly 1%. As said above, this is linked to the fact that σ_t depends only on the elastic amplitude at zero momentum transfer. The corrections are larger for σ_R, reaching about 4%, but somewhat smaller than those obtained in Sec. II.

The results show that for both σ_t and σ_R the eikonal expansion converges rather fastly. Very satisfactory values are obtained with a $\chi_0 + \chi_1$ approximation. Finally we note that the corrections decrease slowly with energy. Whereas they are below 1% at 300 MeV/nucleon, they drop only by a factor 2 from 20 MeV/nucleon to 85 MeV/nucleon.

Very similar results have been obtained with a second potential derived by Satchler *et al.* [17] for ^{11}Li+^{12}C at 85 MeV/nucleon. They are not displayed here.

The situation is slightly different for another potential again taken from Ref. [17]. It describes ^{11}Li+^{12}C at 30 MeV/nucleon as previously, but a Woods-Saxon imaginary part has been added to the folding potential. It can also be reasonably fitted by Gaussians insisting on the potential tails. In this case, the ranges are different for the real and the imaginary parts. Our parametrization yields

$$V_2(r) = -142.7e^{-\alpha^2 r^2} - i\, 65.67e^{-\beta^2 r^2}$$

($\alpha^2 = 0.084\,166$ fm^{-2}, $\beta^2 = 0.109\,57$ fm^{-2}, the intensities are in MeV).

The results are displayed in Table III. We note that the general features are the same as for potential V_1. The magnitude of the corrections, however, is bigger and reaches about 10% for σ_R. It practically meets the results of the preceding section.

The third example is constituted by two phenomenological Woods-Saxon potentials fitted on recent ^{11}Li+^{28}Si quasielastic scattering data at 29 MeV/nucleon [18]. They both reproduce the data in a satisfactory way. They differ at the first deep minimum at around 3° in the c.m., as well as in their predictions for σ_t and σ_R. For both potentials the diffusitivity of the real part is very large, suggesting a diffuse refractive interaction region induced by a neutron halo. They read

$$V(r) = -V_0\left(1 + \exp\frac{r - r_v}{a_v}\right)^{-1} - iW_0\left(1 + \exp\frac{r - r_w}{a_w}\right)^{-1} .$$

TABLE III. Eikonal, first- and second-order contributions to σ_t and σ_R for the potential V_2. Cross sections are given in mb.

		χ_0	$\chi_0 + \chi_1$	$\chi_0 + \chi_1 + \chi_2$	Maximal deviation
20 MeV/nucleon	σ_t	2677.0	2719.4	2718.8	1.5%
	σ_R	981.3	1066.0	1085.6	10.6%
30 MeV/nucleon	σ_t	2524.2	2557.6	2556.5	1.24%
	σ_R	923.0	988.5	1000.2	8.4%
85 MeV/nucleon	σ_t	2132.2	2150.6	2150.6	0.9%
	σ_R	773.8	803.4	805.4	4.1%
300 MeV/nucleon	σ_t	1669.5	1680.9	1681.2	0.7%
	σ_R	593.7	602.5	602.6	1.5%

TABLE IV. Parameters of the two Woods-Saxon potentials (a and b) fitted to ^{11}Li-^{28}Si elastic scattering data at 29 MeV/nucleon. Intensities are in MeV, radii and diffusivities in fm. Eikonal, first- and second-order contributions to σ_t and σ_R for these two potentials, respectively. Cross sections are given in mb.

	V_0	r_v	a_v	W_0	r_W	a_W
(a)	70.0	4.76	1.315	127.4	4.966	0.507
(b)	104.38	4.908	1.250	340.65	5.197	0.532
		χ_0	$\chi_0+\chi_1$	$\chi_0+\chi_1+\chi_2$		Maximal deviation
(a)	σ_t	4585.0	4657.8	4658.8		1.6%
	σ_R	1493.9	1545.3	1546.7		3.55%
(b)	σ_t	5033.2	5093.6	5092.5		1.2%
	σ_R	1910.2	1968.2	1970.6		3.2%

The values of the parameters are displayed in Table IV, together with the results for σ_t and σ_R. In spite of the fact that the two phenomenological potentials have quite different ranges and intensities for their respective real and imaginary parts, and while they differ by 10% on σ_t and almost 30% on σ_R, the relative corrections due to noneikonal propagation are of the same relative amount.

Again the results are rather similar to those quoted before. It gives confidence that our conclusions express general features and are not distorted by peculiarities of the chosen examples.

IV. CONCLUSIONS

In the present work we have investigated the limitations of the eikonal approximation in the low-energy regime for the calculation of total and total reaction cross sections. This has been done by computing the first few terms of the eikonal series expansion for some specific optical potentials. In spite of the arbitrariness of our choice, the results are coherent enough to reveal a general behavior.

We found the eikonal approximation to be a good starting point, even at 10 MeV/nucleon, to calculate σ_t and σ_R. Corrections are easily incorporated, which bring the result within 1% (or better) of the quantum mechanical value.

As far as Glauber type calculations are concerned we suggest following the same line by first computing the phase equivalent optical potential. This can be done by inverting the phase-shift function in the usual way [6]

$$V_{\text{equ}}(\mathbf{r}) = \frac{\hbar v}{\pi r}\frac{\partial}{\partial r}\int_r^\infty \frac{\chi(b)b\,db}{\sqrt{b^2-r^2}} ,$$

where $\chi(b)$ can take the popular form

$$\chi(\mathbf{b}) = \frac{\sigma}{2}\int dz\, d^3r'\rho_A(\mathbf{r}-\mathbf{r}')\rho_B(\mathbf{r}') .$$

Note that the possibility of going beyond the leading contribution in the Glauber model makes this last more interesting than a semiempirical formula.

As we were completing this manuscript we became aware of a paper by Fäldt, Ingemarsson, and Mahalanabis [19] devoted to a subject similar to the present one. Their conclusions are in good agreement with ours.

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APPENDIX A

As far as the total elastic cross section is concerned, formula (4) is only an approximated relationship [19]. Its derivation implies an integration over the momentum transfer q from 0 to ∞ instead of 0 to $q_{\max} = 2k$. While fighting for the accuracy of σ_R, one may wonder about the corrections to σ_{el} as given by Eq. (4). In general this cannot be investigated without numerical integrations and should be done for each particular case.

However, we shall provide here two crude estimates, sufficient to get a feeling about the order of magnitude of the corrections.

1° *Gaussian case.*—Suppose we take

$$[1 - e^{i\chi(b)}] = e^{-\alpha^2 b^2} . \tag{A1}$$

Then

$$F(q) = \frac{ik}{2\alpha^2}e^{-q^2/4\alpha^2}$$

and

$$\sigma_{el} = \frac{\pi}{2\alpha^2}(1 - e^{-2k^2/\alpha^2}) . \tag{A2}$$

The parameter α^2 is roughly related to the size of the system by $\langle r^2\rangle = \frac{3}{2}\frac{1}{\alpha^2}$. For ^{6}Li on ^{12}C at 60 MeV incident energy, we find $2k^2/\alpha^2 > 100$, which makes the correction totally negligible.

2° *Exponential case.*—A single exponential is not very realistic but the difference of two exponentials can give a reasonable model. So we take

$$1 - e^{i\chi(b)} = Ae^{-\alpha b} - Be^{-\beta b} , \tag{A3}$$

with $\beta > \alpha$ and $A > B$.

Then

$$F(q) = \frac{ik2\Gamma(3/2)}{\sqrt{\pi}}\left(\frac{A\alpha}{(\alpha^2+q^2)^{3/2}} - \frac{B\beta}{(\beta+q^2)^{3/2}}\right)$$

and

$$\sigma_{\mathrm{el}} = \Gamma\left(\frac{3}{2}\right)^2 \left[\frac{A^2}{\alpha^2} - \frac{A^2}{(1+\frac{4k^2}{\alpha^2})^2} + \frac{B^2}{\beta^2} - \frac{B^2}{(1+\frac{4k^2}{\beta^2})^2} - \frac{8AB}{(\alpha^2-\beta^2)^2}\left((\alpha^2+\beta^2) - \frac{\alpha\beta(4k^2+\frac{\alpha^2+\beta^2}{2})}{\sqrt{16k^4+(\alpha^2+\beta^2)4k^2+\alpha^2\beta^2}}\right)\right]. \tag{A4}$$

As it can be checked, corrections with respect to $q_{\mathrm{max}} = \infty$ are of the order of $\frac{\alpha^2}{16k^4}$, $\frac{\beta^2}{16k^4}$, or $\frac{\alpha\beta}{16k^4}$ on ^{12}C; at 60 MeV incident energy these corrections are smaller than 10^{-3}.

Consequently, within the range of energies, beams, and targets considered in the present work formula (4) seems justified for our purposes.

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