# Quantum-deformation algebra studied as an analytical equivalent of the $s,d$ interacting boson model: Energy spectra

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(Received 27 January 1993)

For a complex $q$-deformation parameter, the analytical energy expression of $SO_q(6)$ symmetry is shown to give all the three limiting symmetries of the U(6) group as well as all the intermediate results of mixed symmetries; i.e., it works equivalently as the numerically solvable general Hamiltonian of the $s$, $d$ interacting boson model (IBM). Also, a symmetry triangle, equivalent to Casten's symbolic symmetry triangle for the IBM, is obtained. The $q$-deformation parameter is found to behave as the symmetry mixing parameter, similar to one in the IBM. Applications are made to a range of nuclei, covering all the possible cases.

PACS number(s): 21.60.Fw, 03.65.Fd

Quantum-deformation algebra ($q$-DA) of the one-dimensional Lie algebra SU(2) group [1–3] as well as the two-dimensional SU(3) group, with SU(2) as its subalgebra [4], have been studied recently [5,6] for looking at the $q$-deformation parameter in this algebra as the dynamical symmetry breaking (symmetry mixing) parameter in the interacting boson model (IBM) [7]. The complex values for the $q$-deformation parameter were also introduced there [5] for the first time.

Very recently, the $q$-DA of the O(6) limiting symmetry of the $s,d$ interacting boson model was constructed by Wang and Yang [8]. In this paper, we show that this $q$-deformed Hamiltonian of the O(6) symmetry gives also the energy spectra of the other two limiting symmetries [U(5) and SU(3)] of the U(6) group as well as all the possible intermediate transitional situations. In other words, we show that *the Hamiltonian of $SO_q(6)$ symmetry represents the general Hamiltonian of the U(6) group of the IBM.* Also, Casten's symmetry triangle [7] [Fig. 1(a)] is obtained for complex $q$ deformation.

For the three subgroup chains (dynamical symmetries) of the U(6) group,

$$\begin{array}{lll} \underset{|n_d}{\mathrm{U}(5)} \supset \underset{\nu',\alpha'}{\mathrm{O}(5)} \supset \underset{J}{O(3)} \supset \underset{M\rangle}{\mathrm{O}(2)}, & & \text{(I)} \\ \underset{[N]}{\mathrm{U}(6)} \rightarrow \underset{|(\lambda,\mu),\bar{\chi}}{\mathrm{SU}(3)} \supset \underset{J}{\mathrm{O}(3)} \supset \underset{M\rangle}{\mathrm{O}(2)}, & & \text{(II)} \\ \underset{|\sigma}{\mathrm{O}(6)} \supset \underset{\nu,\alpha}{\mathrm{O}(5)} \supset \underset{J}{\mathrm{O}(3)} \supset \underset{M\rangle}{\mathrm{O}(2)}, & & \text{(III)} \end{array} \tag{1}$$

the general Hamiltonian, in terms of Casimir operators up to quadratic terms, is [7]

$$\begin{aligned} H = {} & a_0 + a_1 \mathcal{C}_1(\mathrm{U}(6)) + a_2 \mathcal{C}_2(\mathrm{U}(6)) + a_3 \mathcal{C}_2(\mathrm{O}(6)) + a_4 \mathcal{C}_2(\mathrm{O}(5)) \\ & + a_5 \mathcal{C}_2(\mathrm{O}(3)) + a_6 \mathcal{C}_2(\mathrm{SU}(3)) + a_7 \mathcal{C}_1(\mathrm{U}(5)) + a_8 \mathcal{C}_2(\mathrm{U}(5)) \, . \end{aligned} \tag{2}$$

This Hamiltonian, represented symbolically in Fig. 1(a), contains the three limiting symmetries, all possible intermediate situations of mixed symmetries, and can be solved only numerically. The basis states, which diagonalize $H$ in each chain, are also shown in Eq. (1).

Wang and Yang [8] have used the quantum groups $\mathrm{SU}_q(1,1)$ and $\mathrm{SU}_q(2)$ for $q$-deforming the O(6) chain [III in (1)]. This is possible because for the $s,d$ boson system, the algebra of SU(1,1)⊗SO(6) has the chain of subalgebras

$$\mathrm{SU}^{sd}(1,1) \otimes \mathrm{SO}(6) \supset \mathrm{SU}^{d}(1,1) \otimes \mathrm{SO}(5) \supset \mathrm{O}(3) \supset \mathrm{O}(2) \, , \tag{3}$$

with SU(1,1) sharing the same quantum numbers as in the O(6) chain. Here, $\mathrm{SU}^{sd}(1,1)$ and $\mathrm{SU}^{d}(1,1)$ are the corresponding algebras generated by paired $s,d$ and $d$ operators, respectively, along with the total boson number $N$. Equation (3) means that, instead of using the first six terms of Eq. (2) for the $q$-deformed Hamiltonian of the O(6) chain, we can write the same as [SO(3)≈SU(2)]

$$\begin{aligned} H_q^{\mathrm{III}} = {} & H_0 + A \mathcal{C}_2(\mathrm{SU}_q^{sd}(1,1)) \\ & + B \mathcal{C}_2(\mathrm{SU}_q^{d}(1,1)) + \mathcal{C}_2(\mathrm{SU}_q(2)) \, . \end{aligned} \tag{4}$$

The first three terms of Eq. (2) are all included in $H_0$.

The three generators of $\mathrm{SU}_q^{sd}(1,1)$, which satisfy the commutation relations of $\mathrm{SU}_q(1,1)$, namely,

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$[L_3, L_\pm] = \pm L_\pm$ and $[L_+, L_-] = -[2L_3]$, can be written in terms of $s$, $s^\dagger$, $\tilde{d}$, $d^\dagger$ boson operators as

$$L_+ = \frac{\sqrt{5}}{2}(d^\dagger d^\dagger)_0 + \frac{1}{2}s^\dagger s^\dagger ,$$
$$L_- = \frac{\sqrt{5}}{2}(\tilde{d}\tilde{d})_0 + \frac{1}{2}ss , \qquad (5)$$
$$L_3 = \frac{\hat{N}}{2} + \frac{3}{2} ,$$

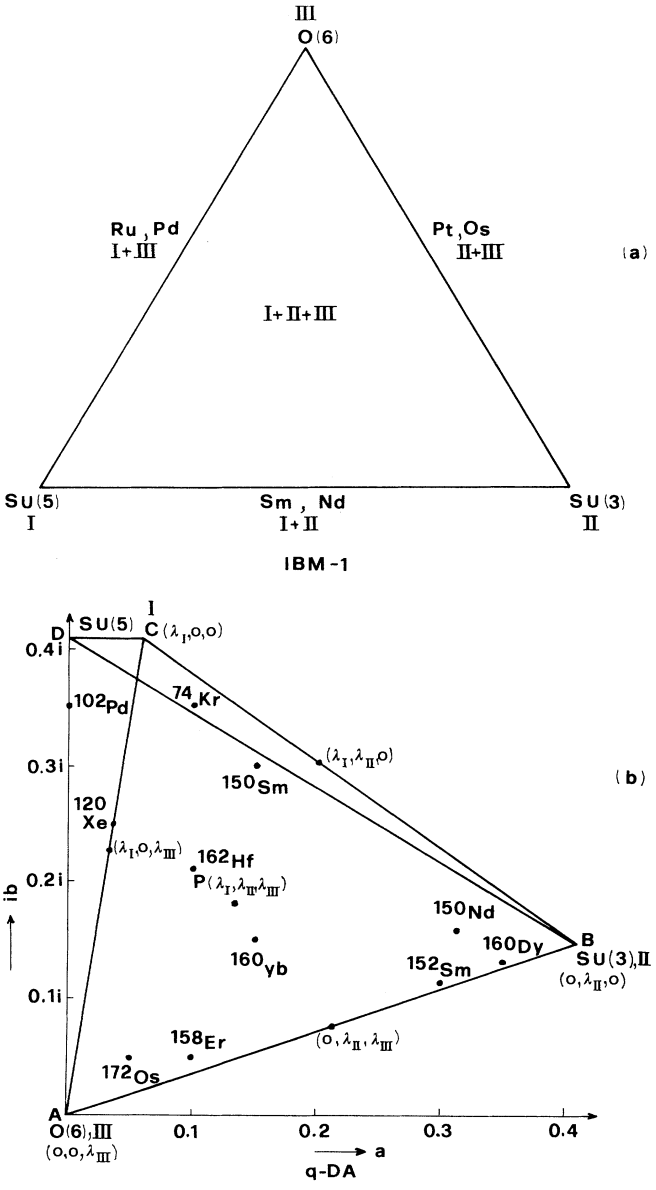


FIG. 1. (a) Symmetry triangle of the IBM-1, symbolizing the three limiting symmetries of the U(6) group and the transitions among them. (b) Symmetry triangle of $q$-DA, given out by the $q$ deformation of the O(6) limiting symmetry in the complex $q$-deformed space. The $(a,b)$ values of nuclei are shown for their best fit to the energy ratio $E_{4^+}/E_{2^+}$. The barycentric coordinates are also indicated.

where $\hat{N} = \hat{n}_s + \hat{n}_d$ is the total $s,d$ boson number operator. Then, analogous to $SU_q(1,1)$, using the basis $|N\sigma\nu\alpha JM\rangle$,

$$\mathcal{C}_2(SU_q^{sd}(1,1)) = \left[\frac{N}{2}\right]\left[\frac{N+4}{2}\right] - L_+L_- , \qquad (6)$$

whose eigenvalues are $[\sigma/2][(\sigma+4)/2]$. Here, the square brackets define $q$ deformation:

$$[x] = \frac{q^x - q^{-x}}{q - q^{-1}} = \frac{e^{\tau x} - e^{-\tau x}}{e^\tau - e^{-\tau}} = \frac{\sinh\tau x}{\sinh\tau} \quad (q = e^\tau) . \qquad (7)$$

Following our earlier works [5,6], $\tau$ is taken to be a complex number ($\tau = a + ib$).

The generators of $SU_q^d(1,1)$ are also given by (5), with $s$ operators neglected, since $SU_q^{sd}(1,1) \supset SU_q^d(1,1)$. Its representation is constructed by noting [9] that the physical basis for $U(5) \supset O(5) \supset O(3)$ and $SU^d(1,1) \otimes SO(5) \supset O(3)$ is the same. Denoting operators by $L'_\pm$ and $L'_3$ and the basis by $|Nn_d\nu\alpha JM\rangle$, we have, for $SU_q^d(1,1)$,

$$\mathcal{C}_2(SU_q^d(1,1)) = \left[\frac{n_d}{2}\right]\left[\frac{n_d+3}{2}\right] - L'_+L'_- , \qquad (8)$$

with eigenvalues $[\nu/2][(\nu+3)/2]$.

Then, knowing that the eigenvalues of the Casimir operator $\mathcal{C}_2(SU_q(2))$ are $[J][J+1]$, we can write the energy eigenvalues of Hamiltonian (4):

$$E_q^{III} = E_0 + A\left[\frac{\sigma}{2}\right]\left[\frac{\sigma+4}{2}\right] + B\left[\frac{\nu}{2}\right]\left[\frac{\nu+3}{2}\right] + C[J][J+1] , \qquad (9)$$

where, from first three terms of (2), $E_0 = a_0 + a_1 N + a_2 N(N+5)$, which is constant for a given total boson number $N$. The quantum numbers take the values $\sigma = N, N-2, \ldots, 1$ or $0$ ($N$ odd or even), $\nu = \sigma, \sigma-1, \ldots, 1, 0$, and $J = \lambda, \lambda+1, \ldots, 2\lambda-2, 2\lambda$ ($2\lambda-1$ missing), where $\lambda = \nu - 3\alpha$ with $\alpha = 0, 1, \ldots$ . For $q \to 1$ or $\tau \to 0$, Eq. (9) gives the energy eigenvalues of the (undeformed) O(6) chain (III) of U(6), which establishes the validity of the deformation procedure used here. In the following, we present our calculations for $N = 6$ and only the ground state $(\sigma,\alpha) = (6,0)$ band since this band is a characteristic of the vibrational or rotational spectra.

Figure 2 presents the results of our calculation for energy spectra, using Eq. (9), for pure real and pure imaginary $q$ deformation $\tau$. The quasirotational ($\gamma$-soft) spectrum of the undeformed O(6) symmetry lies at $\tau = 0$. We notice that for real $\tau$ ($= a$) the energy spectrum tends to become more and more rotational [SU(3) symmetry] and for imaginary $\tau$ ($= ib$) it becomes more and more vibrational [SU(5) symmetry]. Furthermore, the transition from vibrational to rotational spectra or vice versa, is continuous, which means reaching not only the limiting SU(5) and SU(3) symmetries but also the mixed cases. It may be remembered here that, in the IBM, there is one-to-one correspondence between the states of, say, O(6) and SU(5) symmetries, etc. [7]. For the rotational and vibrational spectra, respectively, $a \leq 0.05$ and $b \leq 0.35$, since the $12^+$ state then deviates strongly. These limiting

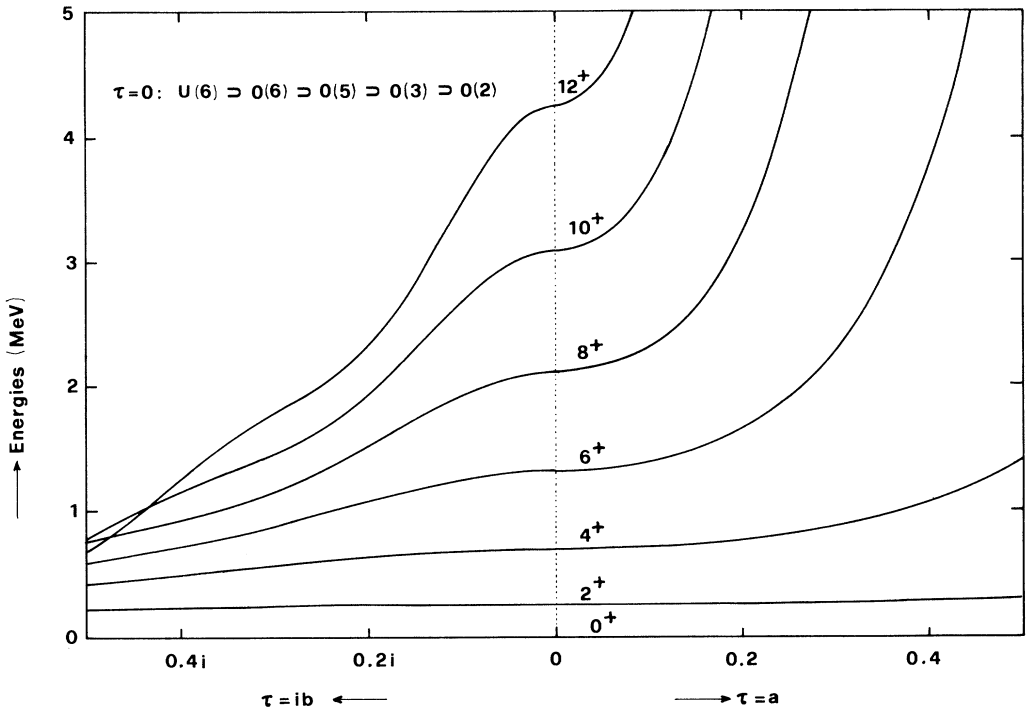


FIG. 2. The energies of the $(\sigma,\alpha)=(6,0)$ band of the O(6) limiting symmetry for the total boson number $N=6$, as a function of the pure real ($\tau=a$) and pure imaginary ($\tau=ib$) $q$-deformation parameter $\tau$ ($=a+ib$).

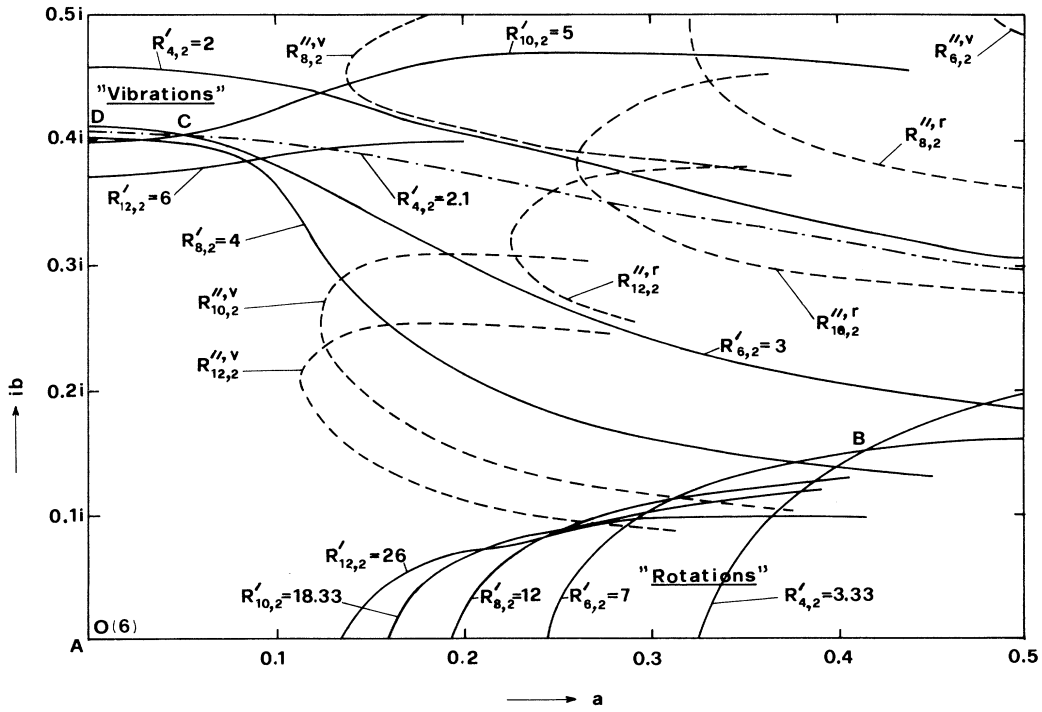


FIG. 3. The constant-value energy ratio plots ($R_{J2}=E_{J+}/E_{2+}$) for the pure vibrational and pure rotational spectra of the O(6) symmetry in a complex $q$-deformed space. The effect of changing the energy ratio $R'_{42}=2.0$ to 2.1 is also illustrated.

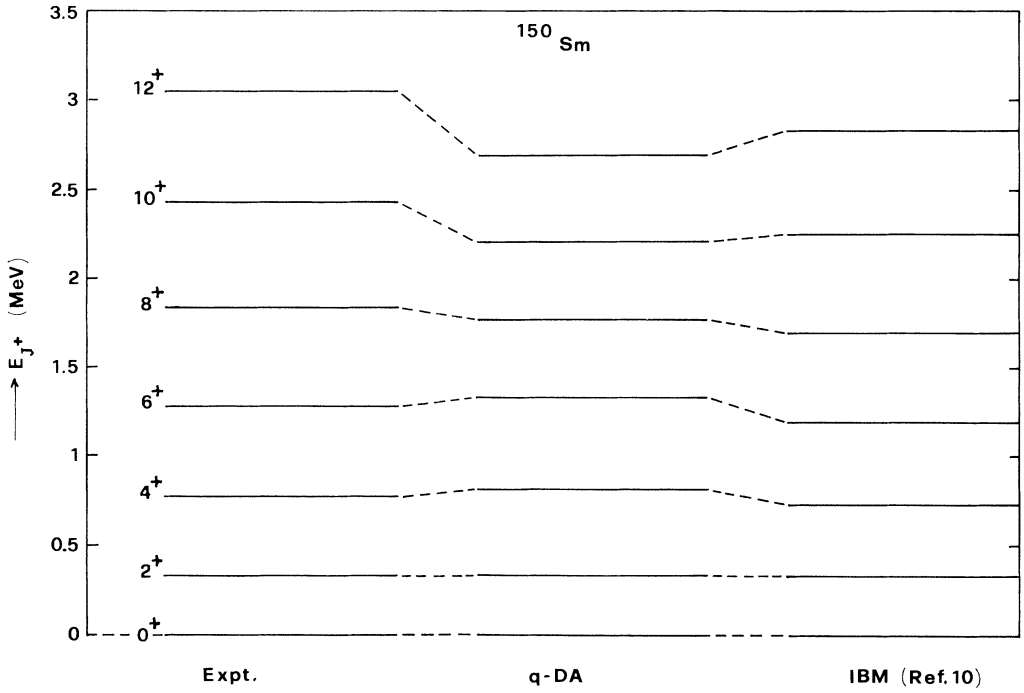


FIG. 4. The calculated $q$-DA and IBM spectra compared with experimental data for $^{150}$Sm. The spectra are normalized to $2^+$ state and for $q$-DA the parameters $(a,b)=(0,0.25)$ for an overall best fit.

values of both $a$ and $b$ parameters, however, get extended if the parameter $\tau$ is taken to be complex ($\tau=a+ib$).

Figure 3 shows the constant-value energy ratios $E_{J^+}/E_{2^+}$ ($=R_{J,2}$) for a pure vibrational, $R_{J,2}=\frac{1}{2}J$, and a pure rotational, $R_{J,2}=\frac{1}{6}J(J+1)$, spectrum in the complex $(a,b)$ plane. Each ratio is shown to have two well-separated wings (evident in contour plots), marked by a single prime and double prime (with $v$ and $r$, respectively, for vibrational and rotational cases). We concentrate only on single primed wings (solid lines) since double primed wings (dashed lines) represent the oscillating parts of the energies. We notice that for each limit the lines can be made to go through a point by relaxing the exact vibrational (or rotational) limit within less than 5%. This is illustrated for the case of the vibrational $R'_{4,2}=2.0$ line. Taking $R'_{4,2}=2.1$, instead of 2, gives the dot-dashed line which coincides exactly with other constant-value lines. Also, it is important to note that the pure vibrational $R'_{4,2}=2$ line never meets the other low-lying vibrational states. This means that only an *anharmonic* vibrational SU(5) limit can be realized in these calculations, which is apparently achievable for $b=0.41$ and $a=0$ to 0.06 ( marked, line $CD$). On the other hand, the axially symmetric rotational SU(3) limit is nicely realized since here $2^+$, $4^+$, and $6^+$ states already meet in a point (marked $B$) and $8^+$ to $12^+$ states can be made to go through $B$ by varying their values slightly. Thus, the $q$ deformation of the O(6) symmetry in Fig. 3 gives both the anharmonic vibrational SU(5) and axially symmetric rotational SU(3) limits, which together with the undeformed O(6) symmetry at the origin $A$ are shown in Fig. 1(b) at the vertices. Hence, admitting different orders of anharmonicity in the SU(5) limit, the symmetry triangle in this model oscillates in between the triangles $ABC$ and $ABD$. Furthermore, the sides and the area of the symmetry triangle, i.e., the region between the vibrational and rotational limits in Fig. 3, can also be determined by actual calculations, which give the transitions between the limiting symmetries. We illustrate this below, in terms of our fitting the model to real nuclei.

We have fitted a number of nuclei whose determined $(a,b)$ values are shown in Fig. 1(b). Interestingly enough, they all lie within the proposed symmetry triangle. Figure 4 illustrates the nature of the comparisons between our work (the $q$-DA), the IBM [10], and the experiments for $^{150}$Sm. Apparently, the $q$-DA is shown to compare equally well with experiments.

Finally, the labeling of the $q$-DA symmetry triangle in terms of generalized barycentric coordinates [11] is convenient and seems realistic. For a triangle with vertices I, II, and III, one can uniquely assign to any point $P(a,b)$ in the plane of triangle a set of three numbers $(\lambda_{\rm I},\lambda_{\rm II},\lambda_{\rm III})$, satisfying $\mathbf{v}_P=\sqrt{\lambda_{\rm I}}\mathbf{v}_{\rm I}+\sqrt{\lambda_{\rm II}}\mathbf{v}_{\rm II}+\sqrt{\lambda_{\rm III}}\mathbf{v}_{\rm III}$, with $\lambda_{\rm I}+\lambda_{\rm II}+\lambda_{\rm III}=1$. Here, the vectors give the positions of the corresponding points. These coordinates have the property that if $P$ belongs to the side opposite to vertex I then $\lambda_{\rm I}=0$ and if $P$ is the vertex I then $\lambda_{\rm II}=\lambda_{\rm III}=0$. Thus, the vertices are given as $(\lambda_{\rm I},0,0)(0,\lambda_{\rm II},0)(0,0,\lambda_{\rm III})$, the sides as $(\lambda_{\rm I},\lambda_{\rm II},0)(0,\lambda_{\rm II},\lambda_{\rm III})$, and $(\lambda_{\rm I},0,\lambda_{\rm III})$, and any point $P$ within the triangle as $(\lambda_{\rm I},\lambda_{\rm II},\lambda_{\rm III})$. Any point lying in the exterior of the triangle must have at least one negative

value. From a geometrical viewpoint the coordinates $\lambda_i$ are proportional to the lengths of segments joining the point $P(a,b)$ with their projections on the lines opposite to $i$, and can be easily related to the $(a,b)$ coordinates of $P$ and the three vertices. Thus, the state vector of general Hamiltonian (2) is

$$\Psi_{U(6)}=\sqrt{\lambda_I}\Psi_{U(5)}+\sqrt{\lambda_{II}}\Psi_{SU(3)}+\sqrt{\lambda_{III}}\Psi_{O(6)}\ , \quad (10)$$

which shows that the $\lambda$'s are mixing coefficients, similar to ones used in the IBM [7].

Concluding, we have studied the $q$ deformation of the O(6) limiting symmetry of the $s,d$ interacting boson model. The $q$-deformation parameter, for complex values, is found to work as a dynamical symmetry-breaking parameter of giving the energy spectra of all the three limiting symmetries of the U(6) group and also all the possible transitions among them. In other words, the $q$-deformed Hamiltonian of the O(6) symmetry gives all the results of the general Hamiltonian of the IBM. Thus, an equivalent of a large numerical problem is obtained in terms of a simple analytical expression. Furthermore, a representation, similar to the symbolic symmetry triangle of Casten for the IBM, is found to emerge in a natural way, which is shown to be conveniently expressible in barycentric coordinates. This allowed an interpretation of the complex $q$-deformation parameter as a symmetry mixing parameter. A similar study of the $q$-DA of other two limiting symmetries will be interesting, but is not expected to modify the present picture.

The authors are thankful to Professor Abdus Salam, The International Atomic Energy Agency, and UNESCO for hospitality at the International Center for Theoretical Physics, Trieste, Italy, where a part of this work is done.

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