Analytic expressions for the Λ energy in the lower nodeless Λ single particle states

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A Λ-nucleus single particle potential, mainly suitable for relatively light hypernuclei, which gives analytic expressions for the Λ energies in the lower nodeless Λ orbits is used for an analysis of the experimental data. The results are discussed and compared with those obtained, in the same way, for a Woods-Saxon potential. It turns out that the analytic expressions for the B_Λ energies, in the region of their validity, give results which are close to the experimental values and compare favorably with the ones obtained with the (more realistic) Woods-Saxon hypernuclear potential. Our analysis also shows a state dependence of the depth of the Λ-nucleus potential which, however, is rather weak. Finally, the use of an effective Λ mass is also discussed.

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I. INTRODUCTION

The production of hypernuclei via the associated production reaction (π^+, K^+) has been shown to be a powerful tool for hypernuclear research [1–5]. The (π^+, K^+) reaction offers the possibility of studying both light and heavy hypernuclei and especially the deeply bound Λ states since it is effective in populating such states because K^+ is only weakly distorted in the nucleus. The reaction mechanism has been extensively discussed by Dover *et al.* [6] and later on by Bando and Motoba [7]. The practicality of this reaction was first demonstrated at Brookhaven for the case of ${}^{12}_{\Lambda}$C [8]. Recently the Λ energies in heavy nuclear systems have been measured by means of this reaction with nuclear targets ranging from ^{9}Be to ^{89}Y [1,2]. More recently the Λ binding energies and ground-state production cross sections for hypernuclei in this region were reported [9].

These new data enable us, for the first time in a rather systematic way, to follow the evolution of the Λ binding energies over an appreciable range of nuclear core mass number and also in a number of cases provides us with a complete set of bound and first unbound energy levels for nodeless Λ orbits at fixed A. This outcome has increased the interest for the derivation of analytic expressions which reproduce Λ binding energies in various Λ states as function of the mass number A.

It is noted that a local hypernuclear potential of Woods-Saxon form

$$V_{\Lambda-A}(r) = -\frac{D}{1+\exp(\frac{r-R}{a})}, \quad R = r_0(A_c)A_c^{1/3} \qquad (1)$$

with a depth $D = 28$ MeV, a radius parameter of the form $r_0(A_c) = 1.128 + 0.439A_c^{-2/3}$ fm (A_c being the mass number of the core nucleus), and a fixed diffusivity $a = 0.54$ fm describes the data very well, though some density dependence or nonlocality is necessary for a better fit [10].

It is desirable, however, to have analytic expressions for the Λ energy in various single particle states which offer an easy and direct estimate and also give evidence about the analytic dependence of the Λ energy on the mass number. In this work the data are analyzed by means of a two-parameter single particle hypernuclear potential which has the advantage of giving rather simple analytic expressions for the Λ energies in the lower nodeless Λ orbits. This potential model has recently been used for the study of the mass number dependence of the oscillator spacing $\hbar\omega_\Lambda$ together with other related quantities [11] and also for an analysis of Ξ^- hypernuclei [12]. In Sec. II the main features of the potential model are briefly described and the eigenvalue problem is given in general form, since in [11] we had restricted ourselves only to the ground state. In Sec. III the proposed expressions for the Λ energies are presented and discussed, while in Sec. IV our numerical results with the proposed potential model as well as the Woods-Saxon one are reported and commented. In addition the role of the effective m^*_Λ mass, treated in an approximate way, is also discussed. Finally in Sec. V our main results are summarized.

II. THE POTENTIAL MODEL

In our approach the Λ-nucleus interaction is approximated by a (spin-averaged) local Λ-nucleus potential of the form

$$V_{\Lambda-A}(r) = -D/\cosh^2\frac{r}{R}, \qquad (2)$$

where $D > 0$ is the potential depth and R the distance from the origin, at which the value of the potential well becomes $-0.42D$. It is therefore a little larger than the "half-depth" radius. This potential model which seems to be suitable mainly for relatively light hypernuclei has some worth-mentioning features: It approximates well the harmonic oscillator potential for $r \ll R$ but unlike this potential its range is finite. Its volume integral can be obtained analytically and, because of this, we can express the radius parameter r_0 in terms of the volume integral of the Λ-nucleon potential $|\bar{V}_{\Lambda N}|$ and of the depth D by assuming the rigid-core model [13,14]. We have

therefore

$$r_0 = \frac{1}{\pi}\left(3|\bar{V}_{\Lambda N}|/D\right)^{1/3}, \quad R = r_0 A_c^{1/3}. \tag{3}$$

It is noted that in the case of the square well the expression of r_0 in the relation $R' = r_0 A_c^{1/3}$ is $r_0 = \left(3|\bar{V}_{\Lambda N}|/4\pi D\right)^{1/3}$.

We may note that potential (2) falls off exponentially for large r, as does the usual Woods-Saxon one, but its surface region is very extended. This fact makes it suitable for comparatively light hypernuclei, but it is less appropriate for the heavy ones. The main advantage, however, in using this potential is that the Schrödinger eigenvalue problem can be solved analytically for states with $l = 0$. The radial wave functions $\Psi_{n0}(r) = rR_{n0}(r)$ are written by means of the Jacobi polynomials in the following form:

$$\Psi_{n0}(y) = N_{n0}(y-1)^{1/2}(y+1)^{-\lambda}P(y)_n^{(1/2,-2\lambda-1/2)}, \tag{4}$$

where $y = \cosh\frac{2r}{R}$. The parameter λ is given by the expression

$$\lambda = \frac{1}{4}\left[\left(1 + 8\mu D R^2/\hbar^2\right)^{1/2} - 1\right], \tag{5}$$

where μ is the Λ-core reduced mass. The normalization constant N_{n0} is expressed in terms of the Γ function (for details see Appendix A) as follows:

$$N_{n0} = \left[\frac{2^{2\lambda-1}\Gamma(n+1)\,\Gamma(2\lambda-n+1/2)\,(2\lambda-2n-1)}{\Gamma(n+3/2)\,\Gamma(2\lambda-n)}\right]^{1/2}. \tag{6}$$

The corresponding energy eigenvalues are given by the expression

$$E_{n0} = \frac{-\hbar^2}{2\mu R^2}\left[\frac{1}{2}\sqrt{\frac{8\mu D R^2}{\hbar^2}+1} - \left(2n+\frac{3}{2}\right)\right]^2 \tag{7}$$

with $n = 0, 1, 2, \ldots$ and $n_{\max} < \lambda - 1/2$.

Finally, another advantage in using the above potential model is the possibility of obtaining easily, as one should expect, analytic expressions for the expectation values of the potential and kinetic energies of a Λ in its ground state [11]. Thus, one finds analytically the dependence of these quantities on A. The expression for the kinetic energy

$$\begin{aligned}\langle T_\Lambda\rangle &= D\frac{(2\lambda-1)(3\lambda+1/2)}{2\lambda(2\lambda+1/2)(2\lambda+1)} \\ &= D\left[3dA_c^{-1/3}\left(2(1+d^2A_c^{-2/3})^{1/2} - (1+d^2A_c^{-2/3})^{-1/2}\right) - 10d^2A_c^{-2/3}\right]\end{aligned} \tag{8}$$

is quite simple and particularly useful in deriving an approximate formula for the oscillator spacing $\hbar\omega_\Lambda$ by applying the virial theorem [11]. The corresponding expression for the square well is [14,15]

$$\langle T_\Lambda\rangle \simeq \frac{\pi^2\hbar^2}{2\mu r_0^2}A^{-2/3} \quad (r_0 \simeq 1.1 - 1.4 \text{ fm}). \tag{9}$$

This expression (or the improved one with higher-order corrections [16]), however, is appropriate only for very large A, since it ignores certain surface effects, quite important in lighter systems, which are expected to give rise to a term proportional to $A_c^{-1/3}$, as is the case with expression (8).

III. EXPRESSIONS FOR THE Λ BINDING ENERGY

Using (7), the g.s. binding energy of the Λ may be written in the form

$$\begin{aligned}B_\Lambda = -E_\Lambda &= \frac{\hbar^2}{2\mu R^2}(2\lambda-1)^2 \\ &= D\left[\left(1+d^2A_c^{-2/3}\right)^{1/2} - 3dA_c^{-1/3}\right]^2,\end{aligned} \tag{10}$$

where

$$d = \left(\hbar^2/8\mu D r_0^2\right)^{1/2}. \tag{11}$$

Expression (10) is a rather simple "semiempirical" formula $B_\Lambda = B_\Lambda(A_c)$ which reproduces the average trend of the variation of B_Λ with A_c and which is obtained from the corresponding energy eigenvalue equation without additional approximation, apart from the assumption that the parameters D and r_0 are independent of A_c. It is of course approximate and it should be clear that it contains only part of a proper hypernuclear formula, as is the case with other semiempirical formulae of this type [14].

An expansion of (10) in powers of A_c can be derived [11]:

$$\begin{aligned}B_\Lambda = D\Big[&1 - 6d_0A_c^{-1/3} + 10d_0^2A_c^{-2/3} \\ &-3d_0^3A_c^{-1} - 3d_0\frac{m_\Lambda}{m_N}A_c^{-4/3} + \cdots\Big]\end{aligned} \tag{12}$$

with

$$d_0 = \left(\frac{\hbar^2}{8m_\Lambda D r_0^2}\right)^{1/2}. \tag{13}$$

The first term dependent on A_c is proportional to $A_c^{-1/3}$ and not to $A_c^{-2/3}$ as for the square well and the Woods-Saxon potentials [14–21]. Such a behavior of B_Λ is observed for the first time, to our knowledge, on the basis of a potential model (with D and r_0 independent of A) [11]. This behavior should be attributed to the shape of the potential which has an extended surface region. It is also seen from (12) that the first four terms of the expansion are independent of the nucleon mass.

It is interesting to note that a formula derived in analogy with the Weizsäcker-Bethe mass formula for ordinary nuclei has the following structure [22]:

$$B_\Lambda = D - FA^{-1/3}, \tag{14}$$

where D and F are constants. Formula (14) has been extrapolated by a straight line in the plot B_Λ versus $A^{-1/3}$ to both $A \to \infty$ and the observed low A region. It turns out, however, that the validity of expression (14) is quite limited [23]. Finally it is noted that in [24] an attempt was made to obtain under some very general considerations a semiempirical formula, which also has a leading term proportional to $A_c^{-1/3}$.

It is interesting, however, to derive an analytic expression which will also include the Λ binding energy in states with $l \neq 0$. As potential (2) is not soluble for $l \neq 0$, for the derivation of such an expression some approximate procedure is necessary. The idea is to approximate $1/r^2$ of the centrifugal term by $1/\sinh^2 r$ (see also [25]). This works well for small values of $\frac{r}{R}$, but the most interesting point is that now the eigenvalue problem is exactly soluble. The expressions for the radial Λ wave functions and the corresponding energy eigenvalues are now

$$\Psi^0_{nl}(y) = N_{nl}(y-1)^{l/2+1/2}(y+1)^{-\lambda}P(y)_n^{(l+1/2,-2\lambda-1/2)}, \tag{15}$$

$$E^0_{nl} = \frac{-\hbar^2}{2\mu R^2}\left[\frac{1}{2}\sqrt{\frac{8\mu D R^2}{\hbar^2}+1} - 2l - \left(2n+\frac{3}{2}\right)\right]^2, \tag{16}$$

while the normalization constant is

$$N_{nl} = \left[\frac{2^{2\lambda-l-1}\Gamma(n+1)\,\Gamma(2\lambda-n+1/2)\,(2\lambda-2n-l-1)}{\Gamma(n+l+3/2)\,\Gamma(2\lambda-n-l)}\right]^{1/2}. \tag{17}$$

The difference $1/r^2 - 1/\sinh^2 r$ is then treated as perturbation using the unperturbed wave functions $\Psi^0_{nl}(r)$,

$$\langle V_p\rangle_{nl} = \langle \Psi^0_{nl}|V_p|\Psi^0_{nl}\rangle \tag{18}$$

with

$$V_p = \frac{\hbar^2 l(l+1)}{2\mu R^2}\left(\frac{1}{x^2} - \frac{1}{\sinh^2 x}\right), \quad x = \frac{r}{R}. \tag{19}$$

However, matrix elements of the above form cannot be calculated in closed form. In view of this, the above difference is expanded as follows:

$$x^{-2} - (\sinh x)^{-2} = \frac{1}{3}(\cosh x)^{-2/5} + \sum_{j=2} b_j (\tanh x)^{2j}. \tag{20}$$

Then, keeping the first term of the expansion, which is sufficient to yield stable results (see also [25]), we proceed to the calculation of the matrix elements as described in Appendix B. In the case we are interested in ($n = 0$ and $l \neq 0$) we obtain

$$\langle V_p\rangle_{0,l} = -\frac{\hbar^2 l(l+1)}{6\mu R^2}\frac{\Gamma(2\lambda+1/2)\,\Gamma(2\lambda-l-4/5)}{\Gamma(2\lambda-l-1)\,\Gamma(2\lambda+7/10)}. \tag{21}$$

Then the expression for the Λ energy to first order is

$$E^1_{0l} = E^0_{0l} - \langle V\rangle_{0l}, \tag{22}$$

which gives

$$B^{(l)}_\Lambda = -E^{(l)}_\Lambda = \frac{\hbar^2}{2\mu R^2}\left[[2\lambda-(l+1)]^2 - \frac{l(l+1)\Gamma(2\lambda+0.5)\Gamma(2\lambda-l-0.8)}{3\Gamma(2\lambda+0.7)\Gamma(2\lambda-l-1)}\right]. \tag{23}$$

For $l = 0$ we regain expression (10). For the higher states ($n \neq 0, l \neq 0$) the corresponding expressions become more complicated.

Our numerical results show that the approximate expression (23) works very well: The values obtained with (23) differ from the numerically calculated Λ energy eigenvalues, on the average, less than 1% and therefore (23) may be used instead of treating the eigenvalue problem numerically.

IV. NUMERICAL RESULTS AND COMMENTS

Here, we report the results using the expressions given above and also by numerical integration of the Schrödinger equation for the Woods-Saxon potential. For the determination of D and r_0 of potential (2) we tried two possibilities. First, using (10) we performed a least squares fit [analogous to that discussed by Bodmer and Rote for the ground state B_Λ for the square well poten-

TABLE I. The s, p, and d Λ energies for potential (2) for various hypernuclei. The values in parentheses are for the parameters obtained from the fit to the ground state B_Λ only.

	$B_{1s}^{\rm expt}$	B_{1s}	$B_{1p}^{\rm expt}$	B_{1p}	$B_{1d}^{\rm expt}$	B_{1d}
${}^{12}_{\Lambda}$C	10.75	(10.73) 10.90				
${}^{16}_{\Lambda}$O	12.50	(12.70) 12.89	2.5	(2.35) 2.93		
${}^{28}_{\Lambda}$Si	16.0	(16.13) 16.35	7.0	(6.74) 7.02		
${}^{32}_{\Lambda}$S	17.5	(17.04) 17.05	8.0	(7.64) 7.92		
${}^{40}_{\Lambda}$Ca	18.70	(18.18) 18.17	10.5	(8.97) 9.60		
${}^{51}_{\Lambda}$V	19.90	(19.26) 19.30	12.0	(10.50) 11.05	4.0	(4.0) 4.11
${}^{56}_{\Lambda}$Fe	21.0	(19.70) 19.70		(10.88) 11.70		(4.15) 4.80
${}^{89}_{\Lambda}$Y	22.1	(21.74) 21.61	16.0	(14.01) 14.23	9.5	(7.2) 7.9
${}^{138}_{\Lambda}$Ba		(23.44) 23.04		(16.40) 16.60		(10.30) 10.70
${}^{208}_{\Lambda}$Pb		(24.86) 24.50		(18.30) 18.60		(13.01) 13.16

tial [17,18] (see also [26])] to the experimental ground state B_Λ for $12 \leq A \leq 89$, reported recently by the Brookhaven group [9]. The best fit values are $D = 35.56$ MeV, $r_0 = 1.075$ fm. Next, using expression (23), we performed a global fit including also the available p and d state Λ energies obtained from the (π^+, K^+) data. The p_Λ and d_Λ energies were those from the figures of Refs. [1,2,10]. As experimental errors for these we used those in [10]. The best fit values are $D = 34.31$ MeV, $r_0 = 1.144$ fm. In both cases the best fit values lead to a shallower but a little wider potential well, compared with those of an earlier study [11] where, however, experimental data for the ground state B_Λ from older experiments with less accuracy had been used. The depth in the second fit is slightly smaller, the relative diference being 3.5%, while the range is a little longer. In Table I the B_Λ values in the s, p, and d states are displayed for hypernuclei covering for the sake of comparison a wide range of A. The available experimental B_Λ values are also shown. The s_Λ energy of ${}^{56}_{\Lambda}$Fe also obtained from the (π^+, K^+) reaction is rather preliminary [27]. The values in parentheses correspond to the first fit, i.e., to the parameters from the fit to only the s_Λ energies. In Fig. 1 the Λ energies in the s, p, and d states, calculated with (23) (global fit), are plotted against $A_c^{-1/3}$ together with the experimental values, in which the s_Λ and p_Λ energies of ${}^{32}_{\Lambda}$S, observed in the (K^-, π^-) reaction [28] are also included.

We also repeated the same procedure for the Woods-Saxon potential, where the Schrödinger equation is solved numerically. To improve the fit, instead of using $R = r_0 A_c^{1/3}$, a more complicated expression, as in [21], is used for the radius of the potential $[R = r_0(A) A_c^{1/3}]$ in which r_0 is A dependent :

$$r_0(A_c) = \frac{1}{2^{1/3}} r_0 \left(\left\{ 1 + \left[1 + \frac{4}{27} \left(\frac{\pi a}{r_0 A_c^{1/3}} \right)^6 \right]^{1/2} \right\}^{1/3} + \left\{ 1 - \left[1 + \frac{4}{27} \left(\frac{\pi a}{r_0 A_c^{1/3}} \right)^6 \right]^{1/2} \right\}^{1/3} \right). \quad (24)$$

This expression is obtained in the framework of the folding model and for a Woods-Saxon potential holds provided that $e^{-R/a} \ll 1$, except for the light hypernuclei [29]. The best fit values are as follows: $1s$ state $D = 26.16$ MeV, $r_0 = 1.255$ fm and $a = 0.38$ fm; global fit $D = 26.84$ MeV, $r_0 = 1.21$, and $a = 0.38$ fm. It is again seen that there is a small difference of about 2.5% between the two depths. In Table II the B_Λ values in the s, p, and d states for the Woods-Saxon potential are shown for the same A_c as in Table I.

It is seen from Table I that the $1s$ Λ energies in both cases are very close and agree well with the experimental B_Λ. For the other states the results of the second fit are slightly better for both potentials. However, as A_c increases the agreement (apart from s_Λ energies) becomes worse especially for $A \gtrsim 50$. In addition the comparison of the results for potential (2) and the more realistic (for $A \gtrsim 16$) Woods-Saxon potential (Table II) shows that the values of E_{1s} obtained with these two potentials agree fairly well for $16 \lesssim A \lesssim 208$. For $A \lesssim 12$ the various expressions obtained with our model are not expected to be reliable enough because their D and r_0 should depend rather strongly on A. Thus, and in view of the comments made above, formula (10) for the ground state B_Λ seems to be rather accurate almost throughout most of the Periodic Table ($12 \lesssim A \lesssim 208$). This is interesting, because the relevant formula (apart from its simplicity) is valid for both light and heavy hypernuclei, contrary to those of Walecka type [14–20] which are more appropriate for the heavier systems and those potentials which are valid only for light hypernuclei [22,30–32].

On the other hand, the two potentials do not compare well for the other nodeless Λ single particle states for $A \gtrsim 50$. Therefore, expression (23) does not seem appropriate for heavier hypernuclei. This should be attributed to the shape of potential (2) which, though it falls off exponentially, as the Woods-Saxon potential is not so "flat" with the appearance of an extended rather "unrealistic" surface compared with the Woods-Saxon potential, which is well established as a fair discription for $A \gtrsim 16$. Thus,

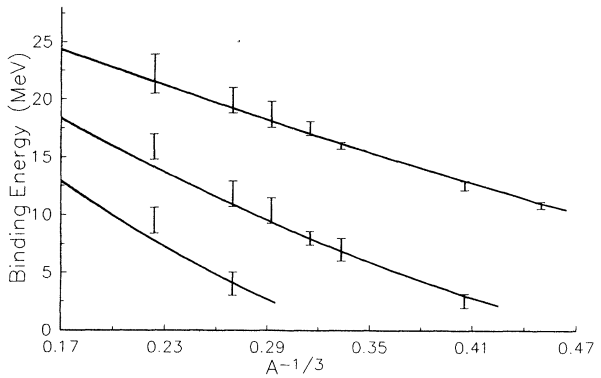

FIG. 1. The observed Λ single particle binding energies and the theoretical values obtained from expression (23).

the large surface thickness does not have a serious effect on the ground state energies even for large A, but affects more the Λ energies in the other single particle states as A increases.

We also fitted the p_Λ energies with both potentials using data for $16 \leq A \leq 89$. Our results show that the depths are a little different from the ones obtained using the s_Λ energies. The relative differences are 7.5% and 12% respectively. The calculated p state energies reproduce very well all the experimental B_Λ for both potentials, also the E_{1p} energies calculated with potential (2) and the Woods-Saxon potential are very close throughout the Periodic Table. However, in this case the Λ energies in the s and d states are not reproduced well.

The above analysis of the data indicates that there is a state dependence of the potential which is rather weak. This is in accordance with a recent study [29].

The state dependence of the depth is more apparent for potential (2), and a similar comment could be made for a square well potential [29]. This could be explained by the fact that the above potentials are not suitable for all A. However, potential (2) gives a more satisfactory description of the lighter hypernuclei ($A \lesssim 18$) than the Woods-Saxon potential. For the latter our analysis shows that an excellent fit is obtained if data in the s, p, and d states with $A \geq 28$ are used. The parameters in this case are $D = 27.25$ MeV, $r_0 = 1.189$, $a = 0.4$ fm. This fit also reproduces rather well the f_Λ energy in ${}^{89}_{\Lambda}$Y ($E_{1f} = 2.80$ MeV, $E_{1f}^{\rm expt} \simeq 2.5$ MeV). One should keep in mind, however, the limited number of the data and their relatively large errors. More data, especially for heavier elements, with much better precision (expected to be accomplished at PILAC [33,34]) will determine more accurately the depth and the range of the Λ-nucleus potential.

Finally, though our analysis is purely phenomenological, we also considered approximately an effective Λ mass $m_\Lambda^* \neq m_\Lambda$. This problem has been discussed in [10] (see also [35,36]) by employing density-dependent and nonlocal Λ-nucleus potentials by means of the Skyrme-Hartree-Fock approximation. Such an approach overcomes the limitation of a local Λ-nucleus potential which cannot satisfactorily fit simultaneously light and heavy hypernuclei. These studies indicate that the data are better described for $m_\Lambda^*/m_\Lambda \simeq 0.8$.

We ignore for the sake of simplicity the density dependence of m_Λ^* and treat m_Λ as a free parameter. Thus we repeated our fitting procedures but varying the Λ mass. For potentials (2), as one should expect, using $m_\Lambda^* < m_\Lambda$ does not influence the results. The reason is that, as seen from expressions (10)–(13), the use of m_Λ^* leads to an increase of the range of the potential but the product $m_\Lambda r_0^2$ remains constant, while the depth of the well is practically the same. This is also confirmed by our numerical calculations in all cases we examined. On the other hand, for the Woods-Saxon potential our calculations show that $m_\Lambda^* \sim (0.85-0.9)m_\Lambda$ does somewhat improve our fit. We note that in [21] and for this potential there is an analogous expansion [formula (13)] in powers of A for the ground state B_Λ. It is seen from this expression that apart from the leading term (which corresponds also to the case of an infinite square well), where the quantity $m_\Lambda r_0^2$ enters, the other terms contain additional contribution from m_Λ and perhaps this explains the observed small improvement in the fits. However, such a treatment is rather crude, since the density dependence is ignored and this should be taken into account in assessing these results.

TABLE II. The s, p, and d Λ energies for the Woods-Saxon potential for various hypernuclei. The values in parentheses correspond to the fit for the ground state B_Λ.

	$B_{1s}^{\rm expt}$	B_{1s}	$B_{1p}^{\rm expt}$	B_{1p}	$B_{1d}^{\rm expt}$	B_{1d}
${}^{12}_{\Lambda}$C	10.75	(9.92) 9.60				
${}^{16}_{\Lambda}$O	12.50	(12.33) 12.10	2.5	(1.69) 1.20		
${}^{28}_{\Lambda}$Si	16.0	(16.21) 16.19	7.0	(6.85) 7.36		
${}^{32}_{\Lambda}$S	17.50	(16.98) 17.05	8.0	(8.66) 8.18		
${}^{40}_{\Lambda}$Ca	18.70	18.15 (18.25)	10.5	(10.66) 10.30		
${}^{51}_{\Lambda}$V	19.90	(19.26) 19.43	12.0	(12.66) 12.41	4.0	(5.05) 4.39
${}^{56}_{\Lambda}$Fe	(21.0)	(19.65) 19.85		(13.378) 13.16		(6.06) 5.44
${}^{89}_{\Lambda}$Y	22.1	(21.28) 21.61	16.0	(16.44) 16.42	9.50	(10.63) 10.20
${}^{138}_{\Lambda}$Ba		(22.46) 22.89		(18.73) 18.86		(14.16) 13.98
${}^{208}_{\Lambda}$Pb		(23.31) 23.80		(20.40) 20.65		(16.81) 16.80

V. SUMMARY

In conclusion, a two-parameter Λ-nucleus potential which, in general, is rather suitable for relatively light hypernuclei and leads to analytic expressions for the Λ energy in various Λ orbits has been used for an analysis of the experimental (π^+, K^+) data. Our Λ energies, which have the leading term of their expansion in powers of A_c to be proportional to $A_c^{-1/3}$ (due to "surface effects" introduced by the potential model), give results which are close to the experimental values and compare favorably with the ones obtained with the more realistic Woods-Saxon potential in the region of their validity. The proposed expressions are relatively simple and therefore may easily be used. Our analysis also shows a very weak state dependence of the depth of the Λ-nucleus potential. Finally treating the Λ mass as a free parameter, though we cannot make a firm conclusion, the use of an effective Λ mass ($m_\Lambda^* < m_\Lambda$) does not influence the results for potential (2), while for the Woods-Saxon it slightly improves the fit.

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APPENDIX A

For the calculation of the normalization constant N_{n0} (the procedure is the same for N_{nl}) it is necessary to calculate the following integral J:

$$J = \int_1^\infty (y-1)^{1/2}(y+1)^{-2\lambda-1/2}|P(y)_n^{(1/2,-2\lambda-1/2)}|^2 dy. \tag{A1}$$

The integral J can be expressed in closed form by exploiting the properties of the Jacobi polynomials (see [37]). Thus, using the orthogonality property of the Jacobi polynomials expression (A1) takes the form

$$J = l_{nn}^{(1/2,-2\lambda-1/2)} \int_1^\infty (y-1)^{1/2}(y+1)^{-2\lambda-1/2} P(y)_n^{(1/2,-2\lambda-1/2)} y^n dy, \tag{A2}$$

where $l_{nn}^{(1/2,-2\lambda-1/2)}$ is the coefficient of y^n in $P(y)_n^{(1/2,-2\lambda-1/2)}$. Then using the Rodrigues formula [38], we obtain

$$J = \frac{l_{nn}^{1/2,-2\lambda-1/2}}{2^n n!} \int_1^\infty y^n \left(\frac{d}{dy}\right)^n \{(y-1)^{n+1/2}(y+1)^{n-2\lambda-1/2}\} dy. \tag{A3}$$

Next integrating by parts n times we get the following form:

$$J = (-1)^n 2^{-2n} \frac{\Gamma(2n-2\lambda+1)}{\Gamma(n+1)\Gamma(n-2\lambda+1)} \int_1^\infty (y-1)^{n+1/2}(y+1)^{n-2\lambda-1/2} dy. \tag{A4}$$

The integral in (A4) can be obtained analytically (see [39]) and therefore (A1) after simple algebra is finally written

$$J = (-1)^n 2^{-2\lambda-1} \frac{\Gamma(2n-2\lambda+1)\Gamma(2\lambda-2n+1)\Gamma(n+3/2)}{\Gamma(n+1)\Gamma(n-2\lambda+1)\Gamma(2\lambda-n+1/2)}. \tag{A5}$$

APPENDIX B

Using the properties of the Jaccobi polynomials [37,38,40] the calculation of the matrix elements of expression (18) using (19) and (20) is reduced to the calculation of the following integral:

$$\int_1^\infty (y-1)^{l+1/2}(y+1)^{-2\lambda-7/10)}|P(y)_n^{(l+1/2,-2\lambda-1/2)}|^2 dy. \tag{B1}$$

Following Miller [40] we expand the polynomials in (B1) in a finite basis of Jacobi polynomials of the form $P(y)_k^{(1/2,-2\lambda-7/10)}$, namely,

$$P(y)_n^{(1/2,-2\lambda-1/2)} = \sum_{k=0}^n C(\lambda)_k^n P(y)_k^{(1/2,-2\lambda-7/10)}, \tag{B2}$$

where the coefficients $C(\lambda)_k^n$ may be expressed in terms of gamma functions (see formula (3.3) of [40]):

$$C(\lambda)_k^n = \frac{(2k+l-2\lambda+0.8)}{(n-k)!} \frac{\Gamma(-4\lambda-0.2)}{\Gamma(-n-4\lambda-0.2)} \times \frac{\Gamma(n+k+l-2\lambda+1)\Gamma(k+l-2\lambda+0.8)\Gamma(n+l+1.5)}{\Gamma(n+l-2\lambda+1)\Gamma(k+l+1.5)\Gamma(n+k+l-2\lambda+1.8)}. \tag{B3}$$

Thus we have now to calculate instead of integral (B1) the following quantity:

$$\sum_{k=0}^{n} |C(\lambda)_k^n|^2 \int_1^\infty (y-1)^{l+1/2}(y+1)^{-2\lambda-7/10} |P(y)_k^{(l+1/2,-2\lambda-7/10)}|^2 dy. \tag{B4}$$

The last integral, however, is the same as the one calculated in Appendix A.

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