Mean fields: Explicit dispersive and retardation properties of the dynamical polarization potential within a simple model

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The dynamical polarization potential corresponding to a simple and tractable, but physically meaningful, two-channel model is derived and examined, with particular attention to its analytic properties. This enables us to illustrate explicitly the emergence of the general dispersive and retardation features expected from the imposition of causality. The energy dependence of the potential, and the associated nonlocality in time expressed by its Fourier transform, are explored. A new theorem is proposed for the asymptotic behavior of the Fourier transforms.

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I. INTRODUCTION

A particle moving through a medium will, in general, experience interactions with the other particles of which the medium is comprised. These may be experienced as a rapidly fluctuating field as the particle proceeds, particularly if the interparticle forces are strong and have a short range, as in the case of a nucleon moving through a nucleus. However, for the purpose of describing the *average* effect of the medium on the particle's motion, these interactions may be replaced by a smooth *mean field M*. When describing an electron scattering from an atom or a molecule [1], or a nucleon scattering from a nucleus [2], this mean field is called an optical-model potential. In the latter case, if the nucleon has negative energy so that it is bound in the medium, the mean field is called a shell-model potential.

If the medium has internal structure, the incident particle may excite it, thereby losing energy itself. If the energy is above the lowest nonelastic threshold, these nonelastic processes result in incident flux being lost ("absorbed") from the elastic channel. The mean field, which describes the elastic scattering of the particle without explicit reference to the internal degrees of freedom of the "target" medium, is then complex, $M=V+iW$, in close analogy to the appearance of a complex refractive index for the scattering of light [3] by an absorptive material. In addition, the nonelastic processes may be virtual; after the initial excitation, following some delay time ("retardation"), the medium may deexcite, returning the incident particle to the elastic channel. This results in a contribution ΔV to the real part of the mean field, $V=V_0+\Delta V$, over and above the value V_0 it would have in the absence of any coupling to the medium that could produce the excitations. Thus, we have

$$M=V_0+\Delta M\ ,\quad \Delta M=\Delta V+iW\ , \tag{1.1}$$

where ΔM is called the dynamical polarization potential. In the case of an electron plus an atom, or a nucleon plus a nucleus, V_0 could be the Hartree-Fock potential. The correction term ΔM would then arise, as just described, because the target atom or nucleus can be excited, or polarized, by the incident particle.

It is well known that the mean field is frequently found to be nonlocal in the spatial coordinates, and that this nonlocality is equivalent to a dependence on the momentum of the particle [2,4]. The Hartree-Fock potential provides an example of such nonlocality. Furthermore, it is well known that the polarization component ΔM of the mean field depends upon the energy of the particle. Less familiar is the concept that an energy dependence is equivalent to a nonlocality in time. Although the existence of this relationship has been remarked upon previously [5–7], only recently [8,9] has it been explored in any depth.

The recent studies have focused upon the general properties that can be expected for the dynamical polarization potential when the condition of causality is imposed [6]. This condition can be most easily visualized, in a time-dependent description, as requiring that a wave packet $\phi(t)$ at time t should not be influenced by $\phi(t')$ at later times $t'>t$. A popular way of expressing this is that a "scattered wave should not appear before the incident wave has reached the target." This simple condition has as a consequence that the real and imaginary parts of the mean field M are related by dispersion relations. Again, these are close analogs of the Kramers-Kronig relations for the propagation of electromagnetic waves which also result from requiring causality [3].

Current treatments and presentations of the properties of the mean field have been very general, so that the connection with the physical aspects of any particular case is not always evident. It can be very helpful to examine a simple and tractable model from which a dispersive potential can be derived directly from the Schrödinger equation. This enables us to illustrate explicitly the emergence of these and other properties. This we do here,

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paying careful attention to the analytic properties and the conditions under which the results hold.

In the next section, we review briefly the general features expected for a mean field. Section III describes our model and the resulting polarization potential. Section IV establishes the dispersion relations that are satisfied, while the following section examines the time-dependent form and the associated temporal nonlocality. A discussion and summary are given in Sec. VI. Appendix A outlines the construction of the coupled equations and the extraction of the equivalent mean field and its properties, while Appendix B summarizes the properties of the Fourier transforms of the real and imaginary parts of the field and their relationship.

II. GENERAL FEATURES OF THE MEAN FIELD

In general, the mean field is nonlocal in the spatial coordinates. However, the nature of our model ensures that it remains local in our case (Appendix A); thus, since it is spherically symmetric, the mean field depends only on the radial coordinate, $M=M(r;E)$, in addition to the energy E. As will be seen in Sec. III and Appendix A, the spatial locality is not an assumption but a consequence of our model. Our principal results will demonstrate that the existence of dispersion and retardation effects do not depend on spatial locality or nonlocality. The corresponding one-body Schrödinger equation is then

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(r;E)+M(r;E)\psi(r;E)=E\psi(r;E)\ , \qquad (2.1)$$

where μ is the reduced mass. The time-dependent representation is obtained by introducing the Fourier transforms

$$\psi(r;E)=\int_{-\infty}^{\infty} dt\ \phi(r;t)e^{iEt/\hbar} \qquad (2.2)$$

and

$$M(r;E)=\int_{-\infty}^{\infty} d\tau\,\mathcal{M}(r;\tau)e^{iE\tau/\hbar}\ , \qquad (2.3)$$

where $\tau=t-t'$. (Invariance of the Hamiltonian under translation in time ensures that the transform depends only upon the difference $t-t'$.) Then Eq. (2.1) becomes

$$-\frac{\hbar^2}{2\mu}\nabla^2\phi(r;t)+\int_{-\infty}^{\infty} d\tau\,\mathcal{M}(r;\tau)\phi(r;t-\tau)=i\hbar\frac{\partial\phi(r;t)}{\partial t}\ , \qquad (2.4)$$

thus demonstrating the equivalence between energy dependence and temporal nonlocality.

The condition of causality is expressed by requiring that

$$\mathcal{M}(r;t-t')=0\ , \quad \text{if } t'>t\ , \qquad (2.5a)$$

or

$$\mathcal{M}(r;\tau)=0 \quad \text{if } \tau<0\ . \qquad (2.5b)$$

As discussed above, in general we expect the mean field to be comprised of a (real) piece that is independent of energy (and thus instantaneous in a time representation), and the polarization part that is energy dependent (thus nonlocal in time),

$$\mathcal{M}(r;\tau)=V_0\delta(\tau)+\Delta\mathcal{M}(r;\tau)\ . \qquad (2.6)$$

In our specific model we have set V_0 as zero, so our mean field has no instantaneous piece. Since the retardation and dispersive effects come only from the reaction terms, this assumption does not alter the essential physics in any significant way. It is made to allow an explicit closed-form calculation of the time-dependent nonlocal potential. According to a theorem of Titchmarsh [10,11], the causality condition (2.5b) is equivalent to asserting that the real and imaginary parts of

$$\Delta M(r;E)=\Delta V(r;E)+iW(r;E) \qquad (2.7)$$

form a Hilbert transform pair; i.e., they satisfy the dispersion relations

$$\Delta V(r;E)=\frac{P}{\pi}\int_{-\infty}^{\infty} dE'\frac{W(r;E')}{E'-E}\ , \qquad (2.8a)$$

$$W(r;E)=-\frac{P}{\pi}\int_{-\infty}^{\infty} dE'\frac{\Delta V(E')}{E'-E}\ , \qquad (2.8b)$$

where P denotes a principal value. The relations (2.8) are the analogs of the Kramers-Kronig relations [3]. Further properties of the Fourier transforms are implied [8]. For example, relations (2.8) imply that the mean field involves only one function of E. Then it can be shown [8] that the Fourier transform of $\Delta M(r;E)$ can be expressed in terms of the Fourier transform of its imaginary part alone,

$$\Delta\mathcal{M}(r;\tau)=2i\Theta(\tau)\mathcal{W}(r;\tau)\ , \qquad (2.9)$$

where $\Theta(\tau)$ is the Heaviside step function,

$$\Theta(\tau)=\begin{cases}1 & \text{if } \tau>0\\ 0 & \text{if } \tau<0\ .\end{cases} \qquad (2.10)$$

These properties, and how they arise in our model, will be examined explicitly below.

III. THE MODEL

The problem of a projectile incident upon a target which possesses a number of excited states can be reduced to a set of coupled equations ("coupled channels"), one for each state of the target (see [2,4] and Appendix A). Then by projection upon the elastic channel (target in its ground state) [2], one obtains an expression for the equivalent (complex, nonlocal, and energy dependent) one-body potential which gives the same elastic wave function and elastic scattering as the original set of coupled equations. This is our definition of the "mean-field."

For our simple model, we take a spinless projectile incident in an S wave upon a target whose ground state is labeled 0 and has zero spin, and which has a single excited state labeled 1, with excitation energy ε, also with zero spin. The diagonal interaction $V_{00}=V_{11}=V_0$ between the projectile and target is taken to be zero, while it is assumed that there is a spherical shell of interaction at the target surface, $r=a$, which may excite it. The coupling interaction then has the form

$$V_{01}=V_{10}=-(\hbar^2/2\mu)\zeta_{01}\delta(r-a)\ , \quad (3.1)$$

where ζ_{01} is a strength parameter (with the dimension of an inverse length). This model is a very simplified version of a system in which a projectile excites surface oscillations of the target, a system frequently used to describe the excitation of nuclei by hadronic probes [2,4]. The extreme assumption of a delta-function inelastic interaction forces the resulting dispersive potential, Eq. (3.2) below, to be local, eliminating the nonessential complication of a spatial nonlocality.

It is shown in Appendix A that a formal elimination of the inelastic channel, which is equivalent to a Feshbach-type projection onto the ground-state channel, results in a one-channel problem of a particle moving under the influence of an effective, complex, and energy-dependent mean field

$$M(r;E)=-m(E)|\zeta_{01}|^2\delta(r-a)\ , \quad (3.2a)$$

where

$$m(E)=(E-\varepsilon)^{-1/2}\sin[(E-\varepsilon)^{1/2}a]\exp[ia(E-\varepsilon)^{1/2}]\ , \quad (3.2b)$$

and where, to simplify the notation, we have expressed the energy variables E and ε, as well as M itself, in units of $\hbar^2/2\mu$, where μ is the reduced mass. Then $m(E)$ has the dimension of length. It is also to be understood in Eq. (3.2b) that we take the positive root, $k=(E-\varepsilon)^{1/2}>0$ when $E>\varepsilon$, and the positive imaginary root

$$k=(E-\varepsilon)^{1/2}=i\kappa>i0$$

when $E<\varepsilon$. This convention specifies that $M(r;E)$ is on the first Riemann sheet of the double-valued function $m(E)$, with a (finite) branch point at $E=\varepsilon$.

The function (3.2) is complex when $E>\varepsilon$ (above the inelastic threshold), and real when $E<\varepsilon$ (including $E<0$). The imaginary part gives rise to absorption, physically due to the opening of the inelastic channel when $E>\varepsilon$. When $E<\varepsilon$, this channel is closed and there is no absorption, but virtual excitations can still contribute a real part to the field. As described in Appendix A, the field (3.2) exhibits the phenomenon known as a "threshold anomaly" [12] in the vicinity of $E=\varepsilon$. The real and imaginary parts of the energy-dependent factor $m(E)$ are displayed in Fig. 1.

The double valuedness of Eq. (3.2b) comes from the square root. If we write the energy in the complex E plane as

$$E=\varepsilon+\rho e^{i\theta}\ , \quad (3.3)$$

then we have for Eq. (3.2b)

$$m(E)=[\sin(xa)/x]e^{ixa}\ , \quad (3.4a)$$

where

$$x=\rho^{1/2}e^{i\theta/2}\ . \quad (3.4b)$$

When $\theta=0$ or 2π, this reduces to

$$m(E)=[\sin(\rho^{1/2}a)/\rho^{1/2}]e^{\pm i\rho^{1/2}a}\ , \quad (3.4c)$$

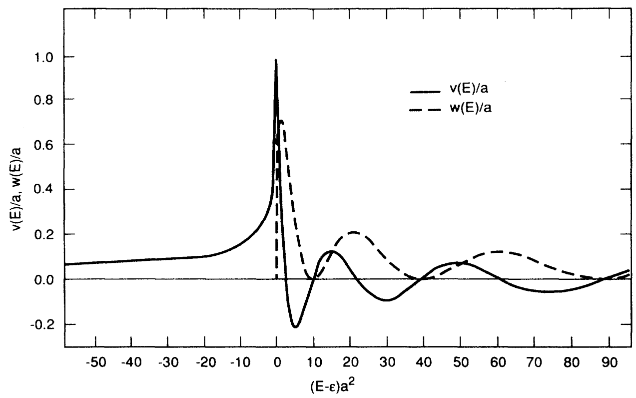

FIG. 1. The real and imaginary parts of the energy-dependent factor $m(E)$ of the polarization potential, as expressed in Eqs. (3.2), (A13), and (A14).

with the upper sign of the exponent for $\theta=0$, and the lower for $\theta=2\pi$. To define a definite function on the first Riemann sheet, $0\le\theta\le 2\pi$, we take a cut along the real axis from $E=\varepsilon$ to ∞. Further, by reexpressing Eqs. (3.4) as

$$m(E)=-\tfrac{1}{2}i\rho^{-1/2}e^{-i\theta/2}\times\exp\left[2\rho^{1/2}a\left[i\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right]-1\right]\ , \quad (3.5)$$

we see that $m(E)$ goes to zero as $\rho\rightarrow\infty$ for all angles $0\le\theta\le 2\pi$. Then $m(E)$ is analytic at all finite values of E except on the cut, where it is double valued, and is zero at infinity. The properties of $m(E)$, including its relation to the effective mass, are discussed further in Appendix A.

IV. DISPERSION RELATIONS

Here we examine the existence of dispersion relations for our model mean field (3.2). We begin by specifying a contour parallel to and d units above the real axis in the E plane. The function $m(E)$ is analytic in this region, so Cauchy's theorem applies. Then we have for E, in the upper half plane,

$$m(E)=\frac{1}{2\pi i}\oint_C\frac{m(E')dE'}{E'-E}\ , \quad (4.1)$$

where the contour C is indicated in Fig. 2(a). Now Eq. (3.5) shows $m(E)$ to vanish as $\rho\rightarrow\infty$ for all angles $0<\theta<2\pi$. Thus we have

$$m(E)=\frac{1}{2\pi i}\int_{-\infty+id}^{\infty+id}dE'\frac{m(E')}{E'-E}\ . \quad (4.2)$$

Next we examine Eq. (4.2) for a value of E just above the lower segment of the contour,

$$E=E_0+id+i\eta\ , \quad (4.3)$$

for example, as shown in Fig. 2(b). Letting $E'=x+id$, we may rewrite Eq. (4.2) as

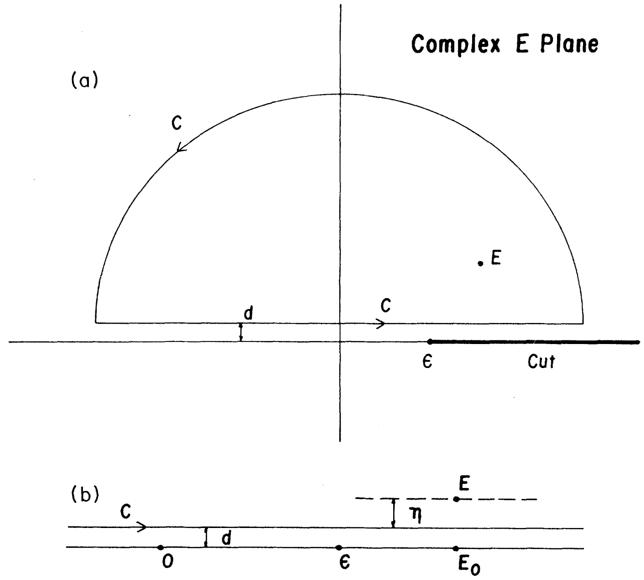

FIG. 2. (a) Contour C for the integral in Eq. (4.1). (b) Illustrating Eq. (4.3).

$$m(E_0)=\frac{1}{2\pi i}\int_{-\infty}^{\infty} dx\,\frac{m(x+id)}{x-E_0-i\eta}$$
$$=\frac{1}{2\pi i}\int_{-\infty}^{\infty} dx\, m(x+id)\left[\frac{x-E_0}{(x-E_0)^2+\eta^2}+\frac{i\eta}{(x-E_0)^2+\eta^2}\right]. \tag{4.4}$$

If we first take the limit as $d\to 0$, the bracketed quantity in Eq. (4.4) is unchanged. Then in the limit as $\eta\to 0$, the first term is equal to the principal part integral, and the second term gives a delta function, $i\pi\delta(x-E_0)$. Then we have

$$m(E_0)=\frac{1}{2\pi i}\left[P\int_{-\infty}^{\infty} dx\frac{m(x)}{x-E_0}+i\pi m(E_0)\right]. \tag{4.5}$$

Solving for $m(E_0)=v(E_0)+iw(E_0)$, for example, with v and w being real, gives

$$m(E_0)=\frac{P}{\pi i}\int_{-\infty}^{\infty} dx\,\frac{m(x)}{x-E_0} \tag{4.6a}$$
$$=\frac{P}{\pi i}\int_{-\infty}^{\infty} dx\,\frac{v(x)+iw(x)}{x-E_0}. \tag{4.6b}$$

Equating real and imaginary parts on both sides yields the required dispersion relations

$$v(E_0)=\frac{P}{\pi}\int_{-\infty}^{\infty} dx\,\frac{w(x)}{x-E_0}, \tag{4.7a}$$

$$w(E_0)=-\frac{P}{\pi}\int_{-\infty}^{\infty} dx\,\frac{v(x)}{x-E_0}. \tag{4.7b}$$

Equations (4.7) state that v and w are Hilbert transforms of one another.

We note that the order in which d and η are set to zero does not matter; they both give the same limit (4.5). This is not surprising because nothing special happens at the real axis. Although m is double valued on the cut, we take only the limit as the contour is approached from above.

V. RETARDATION IN THE MEAN FIELD

A. Evaluation of the Fourier transform

We recalled in Sec. II that the mean field in a time-dependent representation was the Fourier transform of $M(r;E)$, namely, the inverse of Eq. (2.3),

$$\mathcal{M}(r;\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dE\, M(r;E)e^{-iE\tau}. \tag{5.1}$$

With our choice of $\hbar^2/2\mu$ as the unit of energy, τ here stands for $(t-t')\hbar/2\mu$. Then introducing our model potential (3.2), the Fourier transform becomes

$$\mathcal{M}(r;\tau)=-\tilde{m}(\tau)|\zeta_{01}|^2\delta(r-a), \tag{5.2a}$$

where in units of $(\hbar^3/4\mu^2)$

$$\tilde{m}(\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dE\, m(E)e^{-iE\tau} \tag{5.2b}$$
$$=\frac{1}{2\pi}\int_{-\infty}^{\infty} dE\,\frac{\sin[(E-\varepsilon)^{1/2}a]}{(E-\varepsilon)^{1/2}}\times\exp[ia(E-\varepsilon)^{1/2}]e^{-iE\tau}. \tag{5.2c}$$

The upper sign in Eq. (3.4c) must be taken for the integration along the cut. [The lower sign would correspond to the complex conjugate of our choice for $m(E)$, giving the emissive potential that would result from choosing an ingoing-wave boundary condition; see Appendix A.]

It is convenient to transform to the wave-number variable $k=(E-\varepsilon)^{1/2}$ in order to evaluate the integral (5.2c), where according to our rules

$$k>0 \quad \text{for } E>\varepsilon \tag{5.3a}$$

and

$$k=i\kappa,\quad \kappa>0 \quad \text{for } E<\varepsilon. \tag{5.3b}$$

This gives us

$$\tilde{m}(\tau)=\frac{1}{\pi}\int_C \sin(ka)e^{ika}\exp[-i(\varepsilon+k^2)\tau]dk, \tag{5.4}$$

where now C is the contour in the complex k plane shown in Fig. 3. Notice that this transformation has simplified the integrand in the sense that it is analytic everywhere in the entire finite plane, whereas the integrand of Eq. (5.2c) has singularities (the branch point at $E=\varepsilon$ and the cut). According to Cauchy's theorem, we may deform the contour C in Eq. (5.4) at our convenience. However, since the contour goes to infinity in two directions, it is necessary to examine the behavior of the integrand in the limit of large $|k|$.

As $|k|\to\infty$, the k^2 factor in the exponent in Eq. (5.4) will dominate. To investigate its behavior at infinity, let $k=\rho e^{i\theta}$. Then

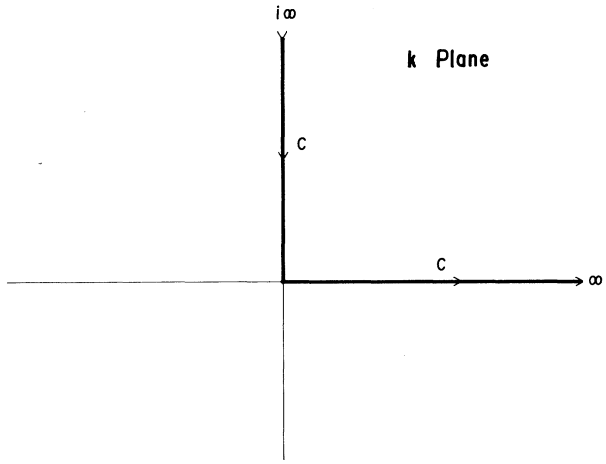

FIG. 3. Contour C for the integral in Eq. (5.4).

$$e^{-ik^2\tau}=\exp[-i\rho^2(\cos2\theta+i\sin2\theta)\tau]$$
$$=\exp[-i\rho^2\tau\cos2\theta]\exp[\rho^2\tau\sin2\theta] \ . \quad (5.5)$$

B. The retarded field, $\tau>0$

First, let us consider $\tau>0$. $\sin2\theta$ must be negative for Eq. (5.5) to remain finite as $\rho\rightarrow\infty$. This will be so if

$$\theta=\left[n+\tfrac{1}{2}\right]\pi+\tfrac{1}{2}\alpha$$

with n an integer and $0<\alpha<\pi$. Then $\frac{1}{2}\pi<\theta<\pi$ if $n=0$, and $\frac{3}{2}\pi<\theta<2\pi$ if $n=1$. Thus the factor $\exp(-ik^2\tau)$ for $\tau>0$ goes to zero as $\rho\rightarrow\infty$ in the second and fourth quadrants, but goes to infinity in the first and third quadrants.

Now we may take advantage of our ability to deform the path C and attempt to change the contour C into one C' consisting of two arcs at infinity plus a straight line passing through the origin, as shown in Fig. 4(a). The two arcs, taken at $|k|\rightarrow\infty$ in the second and fourth quadrants, give zero contributions to the integral along C'. Thus the integral path along C has been unbent into the straight line portion of C', which may be expressed by taking θ to be an arbitrary angle in the fourth quadrant. We may then include the section in the second quadrant by allowing ρ to be the integration variable running from $-\infty$ to ∞.

Then $\tilde{m}(\tau)$ can be evaluated most conveniently by using a new parameter $\beta=\theta+\pi/4$, in terms of which Eq. (5.4) becomes

$$\tilde{m}(\tau)=\frac{e^{-i\varepsilon\tau}}{2i\pi}\int_C[\exp(2i\rho a e^{i(\beta-\pi/4)})-1]$$
$$\times\exp(-\rho^2\tau e^{2i\beta})\exp[i(\beta-\pi/4)]d\rho \ . \quad (5.6)$$

The conditions on θ ensure that $\cos2\beta$ is positive in the second and fourth quadrants, so the integral converges at infinity. The integral is of Gaussian character and may

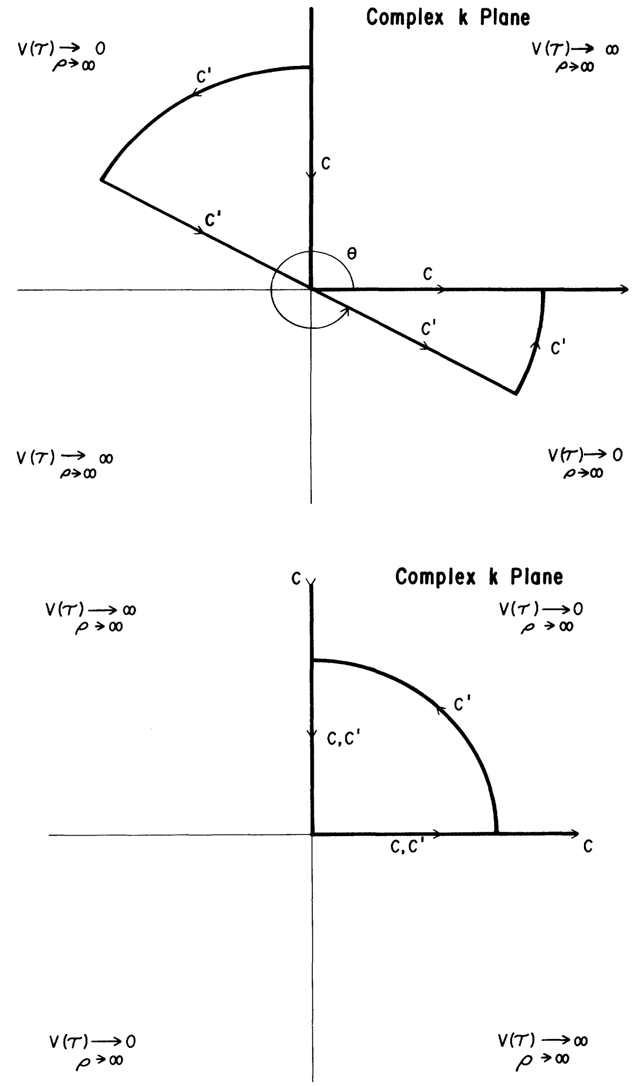

FIG. 4. (a) The deformed contour C' used to evaluate Eq. (5.4) when $\tau>0$. (b) The contour C' used when $\tau<0$.

be done by completing the square on the exponent of the first term,

$$-\rho^2\tau e^{2i\beta}+2i\rho a\exp[i(\beta-\pi/4)]$$
$$=-(\rho\tau^{1/2}e^{i\beta}-ia\tau^{-1/2}e^{-i\pi/4})^2-a^2\tau^{-1}e^{-i\pi/2} \ , \quad (5.7a)$$

and for the second term,

$$-\rho^2\tau e^{2i\beta}=-(\rho\tau^{1/2}e^{i\beta})^2 \ . \quad (5.7b)$$

The two terms (5.7a) and (5.7b) give the same result in the integral (5.6), except for the additional phase factor arising from the last term on the right of Eq. (5.7a), so we have the simple result for positive τ that

$$\tilde{m}(\tau)=\frac{e^{-i\varepsilon\tau}}{2i(i\pi\tau)^{1/2}}[\exp(ia^2/\tau)-1] \ , \quad \tau>0 \ . \quad (5.8)$$

Notice that Eq. (5.8) is independent of our choice for the angle β (except for the restriction that θ be confined to the second and fourth quadrants). This must be so in order to satisfy Cauchy's theorem. The convergence would be most rapid along the $\theta=7\pi/4$ line, and slowest along the axes.

C. The advanced field, $\tau<0$

The result (5.8) applies when $\tau>0$. The causality condition (2.5) requires that $\tilde{m}(\tau)=0$ when $\tau<0$. We now demonstrate how this requirement is satisfied by our model. We must again consider the dominant behavior of Eq. (5.5) as $\rho\to\infty$, but now for negative τ. Rewrite this equation as

$$e^{-ik^2\tau}=\exp(i\rho^2|\tau|\cos2\theta)\exp(-\rho^2|\tau|\sin2\theta)\ . \tag{5.9}$$

In this case we must have $\sin2\theta>0$ for good behavior as $\rho\to\infty$. Thus we need $\theta=n\pi+\alpha/2$ with $0<\alpha<\pi$. Then $0<\theta<\pi/2$ for $n=0$ and $\pi<\theta<3\pi/2$ for $n=1$. These regions, now the first and third quadrants, are indicated in Fig. 4(b); then the contour may be deformed within the first quadrant. The contour C cannot be straightened in this case, but the vertical and horizontal portions can be bent so as to join at infinity, as shown in Fig. 4(b). The joining portion contributes nothing because $\exp(-ik^2\tau)$ vanishes as $k\to\infty$ in the first quadrant. Consequently, the entire integral (5.4) is zero by Cauchy's theorem, or

$$\tilde{m}(\tau)=0 \quad \text{for } \tau<0\ , \tag{5.10}$$

demonstrating that the potential (3.2) is causal.

D. Properties of the model $\tilde{m}(\tau)$ and a new theorem

It can be shown on general grounds [8] that the Fourier transform of a causal function can be expressed in terms of the Fourier transform of its imaginary part alone. This implies that if we write

$$\tilde{m}(\tau)=\tilde{v}(\tau)+i\tilde{w}(\tau)\ , \tag{5.11}$$

where $\tilde{v}(\tau)$ and $\tilde{w}(\tau)$ are the Fourier transforms of $v(E)$ and $w(E)$, respectively, we should have the relation (2.9),

$$\tilde{m}(\tau)=2i\Theta(\tau)\tilde{w}(\tau)\ . \tag{5.12}$$

(We hasten to stress that $\tilde{v}$ and $\tilde{w}$ are *not* the real and imaginary parts of $\tilde{m}$; both are complex quantities.) The implication here is that

$$\begin{aligned}\tilde{v}(\tau)&=i\tilde{w}(\tau)\ ,\quad \tau>0\ ,\\ \tilde{v}(\tau)&=-i\tilde{w}(\tau)\ ,\quad \tau<0\ .\end{aligned} \tag{5.13}$$

Indeed, this result is shown explicitly for our model in Appendix B. The relation (5.13) also implies that $\tilde{v}(\tau)$ is discontinuous at $\tau=0$, unless $\tilde{w}(\tau)$ is zero there, which, in general, it is not. In our case, $\tilde{w}(\tau)$ diverges as $\tau^{-1/2}$ at small τ.

The phase factor $\exp(-i\varepsilon\tau)$ in the expression (5.8) for $\tilde{m}(\tau)$ is trivial in the sense that it arises simply from our choice of zero for the energy scale at $E=0$ [8], i.e., at the point where the incident particle has zero kinetic energy. If, for example, we had chosen $E'=0$ as zero, where $E'=E-\varepsilon$ is the kinetic energy in the inelastic channel, the Fourier transforms would have been taken with respect to E' as the variable and the phase factor would not have appeared. Consequently, we investigate the properties of

$$\hat{m}(\tau)=e^{i\varepsilon\tau}\tilde{m}(\tau)\ , \tag{5.14}$$

rather than $\tilde{m}(\tau)$ itself.

Using Eq. (5.8), we may break $\hat{m}(\tau)$ into its real and imaginary parts,

$$\hat{m}(\tau)=\hat{m}_R(\tau)+i\hat{m}_I(\tau)\ , \tag{5.15a}$$

where for $\tau>0$

$$\hat{m}_R(\tau)=(8\pi\tau)^{-1/2}[1-\cos(a^2/\tau)+\sin(a^2/\tau)]\ , \tag{5.15b}$$

$$\hat{m}_I(\tau)=(8\pi\tau)^{-1/2}[1-\cos(a^2/\tau)-\sin(a^2/\tau)]\ . \tag{5.15c}$$

These quantities are displayed in Fig. 5.

As has been found in other cases [8], it is difficult to assign a "range" to the temporal nonlocality expressed in Eqs. (5.15) because the dependence on τ is not monotonic. The simple $\tau^{-1/2}$ behavior, which diverges as $\tau\to+0$, is modulated by factors which oscillate between $(1-\sqrt{2})$ and 2 with a period of $2\pi\tau^2/a^2$. Thus, these oscillations become infinitely fast as τ approaches zero.

The divergence as $\tau\to+0$ may be compared with the result for an energy-independent potential, which also has an infinite Fourier transform at $\tau=0$, namely, a Dirac delta function. Furthermore, this also has an infinitely rapidly oscillating representation, $\lim_{\alpha\to\infty}\sin(\alpha\tau)/\tau$. However, in contrast, $\tilde{m}(\tau)$ has nonzero values for integrals over finite domains of τ for $\tau>0$, and does not have a finite integral over an infinitesimal interval about $\tau=0$.

Asymptotically, we have for $\tau\gg a^2$

$$\hat{m}(\tau)\to(1-i)(8\pi)^{-1/2}a^2\tau^{-3/2}\ , \tag{5.16}$$

which is in curious contrast to the τ^{-n}, with n integer,

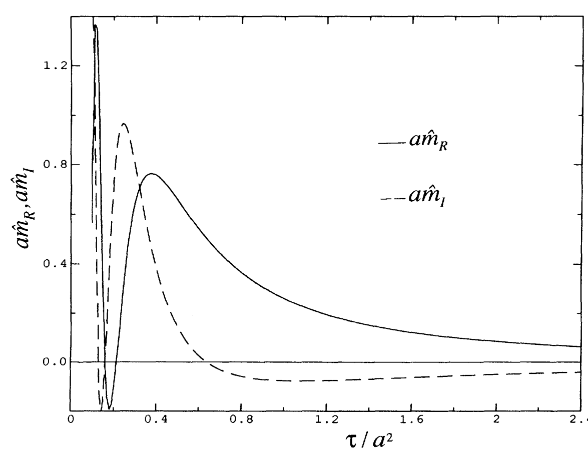

FIG. 5. The real and imaginary parts of the temporally nonlocal factor $\hat{m}(\tau)$, where $\tau=t-t'$, of the polarization potential $\mathcal{M}(\tau)$, as expressed in Eqs. (5.2), (5.14), and (5.15). Both parts oscillate very rapidly as τ approaches zero, and diverge like $\tau^{-1/2}$.

behavior found for several examples by Mahaux, Davies, and Satchler [8]. Those examples are in agreement with a theorem quoted by Bracewell [13], which states that if the nth derivative of a function $f(E)$ is impulsive (i.e., a delta function in energy), then its Fourier transform $\tilde{f}(\tau)$ falls off as τ^{-n} at larger τ. However, our model $m(E)$ has no derivative with such an impulsive behavior, so its transform $\tilde{m}(\tau)$ is not bound by the theorem. Rather, we derive in Appendix C a similar theorem which states that if the nth derivative of $f(E)$ with respect to E has an $(E-\varepsilon)^{1/2}$ singularity, the Fourier transform $\tilde{f}(\tau)$ behaves as $\tau^{-(n+1/2)}$ asymptotically. Both theorems require that $f(E)$ and its derivatives go to zero as $E\rightarrow\pm\infty$. In our case, $m(E)$ has no infinities on the real axis but the first derivative ($n=1$) has an $(E-\varepsilon)^{-1/2}$ singularity. Consequently, the new theorem requires that $\tilde{m}(\tau)$ behaves as $\tau^{-3/2}$ as $\tau\rightarrow\infty$, in agreement with Eq. (5.16).

VI. POTENTIAL AVERAGED OVER ENERGY

Realistic systems generally have many more degrees of freedom than our simple model, and tend to give rise to mean fields that show a rapidly fluctuating behavior with energy [2]. In such cases, empirical optical potentials are identified with some suitable average over energy of the underlying mean field $M(E)$. A Lorentzian with a full width at half maximum (FWHM) of $2I$ provides a suitable weight factor, and has the consequence that the average of $M(E)$ is simply $\overline{M}(E)=M(E+iI)$ [14].

The effect of this averaging on the Fourier transform of our model potential is easy to see. The calculation of $\tilde{m}(\tau)$ is again done by transforming to the k plane, the origin of which changes from $(E-\varepsilon)^{1/2}$ to $(E+iI-\varepsilon)^{1/2}$. The right-angle path shown in Fig. 2 then changes to a curve in the first quadrant resembling an hyperbola which approaches the axes at infinity. The k-space integrand is still analytic and has the same behavior at infinity as before. Again the path may be deformed to a straight line, giving a Gaussian integral. The only change from the result (5.8) is an additional factor $\exp(-I\tau)$, as has been proved, in general, by Lipperheide [7]. For $\tau<0$, the path can again be closed in the first quadrant by a quarter circle at infinity, proving that $\tilde{m}(\tau)=0$ for $\tau<0$, so that the energy-averaged field $\overline{M}(E)$ remains causal.

The appearance of the factor $\exp(-I\tau)$ means that the degree ("range") of nonlocality in time has been reduced. This is consistent with the slower variation with energy of the averaged field.

VII. SUMMARY AND PHYSICAL INTERPRETATION

In order to demonstrate explicitly the various general properties expected for a dynamical polarization potential, a simple, but physically meaningful, two-channel model was introduced, and an effective energy-dependent mean field, or optical potential, derived by elimination of the inelastic channel. The resulting polarization potential was expressed in Eqs. (3.2), and its properties were explored in Sec. III and Appendix A. In Sec. IV we demonstrated that its real and imaginary parts form a Hilbert transform pair satisfying the dispersion relations (4.7). The temporal nonlocality of its Fourier transform (5.2) was derived and examined in Sec. V. The dependence upon $\tau=t-t'$ is contained in $\tilde{m}(\tau)$ of Eq. (5.8) for $\tau>0$, while the potential vanishes for $\tau<0$, confirming that the model potential is causal. This causal nature of the polarization potential follows from our choice (A7) of the outgoing scattered wave solution of the original coupled channels equations; the two conditions are equivalent.

The model time-dependent potential was found to decrease as $\tau^{-3/2}$ for large τ, as in Eq. (5.16). This result led to the formulation of a new theorem (Appendix C) which states that if the nth derivative with respect to E of a function $f(E)$ has an $(E-\varepsilon)^{-1/2}$ singularity, its Fourier transform $\tilde{f}(\tau)$ decreases as $\tau^{-(n+1/2)}$ at large τ. Our model potential has $n=1$.

We may attempt a physical interpretation of the temporal nonlocality that appears in our model potential. For this purpose, we should abandon the special units of time and energy that we adopted in order to simplify the notation. Then from Eqs. (5.2), (5.8), and (5.14), we have

$$\mathcal{M}(r;\tau)=-\exp\left[\frac{-i\varepsilon\tau}{\hbar}\right]\left[\frac{\hbar^3}{4\mu^2 a}\right]\hat{m}\left[\frac{\tau}{\tau_0}\right]|\zeta_{01}|^2 \times\delta(r-a)\,, \tag{7.1a}$$

where

$$\hat{m}\left[\frac{\tau}{\tau_0}\right]=\frac{(1+i)}{[8\pi(\tau/\tau_0)]^{1/2}}[1-\exp(i\tau_0/\tau)] \tag{7.1b}$$

and the characteristic time is

$$\tau_0=2\mu a^2/\hbar\,. \tag{7.1c}$$

We remember that μ is the reduced mass of the system, and a is the radius of the spherical shell of interaction (3.1), with strength ζ_{01}, that couples the two channels. Thus, except for the trivial phase factor $\exp(-i\varepsilon\tau)$, $\mathcal{M}$ is a function only of the ratio τ/τ_0.

We now present an interpretation of the characteristic time τ_0. Consider the projectile colliding with the target and virtually exciting it by crossing the shell of interaction. It is then confined within a sphere of radius a until it again collides with the shell of interaction, when it may deexcite the system and scatter back to the elastic channel. Within the confining sphere, the uncertainty principle implies a momentum of order $\hbar/2a$, corresponding to a classical velocity of $\hbar/2\mu a$. Thus the time from entry into the inelastic channel until its reemergence is the average distance traveled, of order a, divided by the velocity, namely,

$$\tau_0=a/(\hbar/2\mu a)=2\mu a^2/\hbar\,. \tag{7.2}$$

This simple argument suggests that the characteristic time of the nonlocality is connected with some average classical transit time through the nucleus, which is also the average time between inelastic collisions. Thus we relate the nonlocality to a time delay experienced by the system while it propagates in the inelastic channel before suffering another interaction and returning to the elastic channel. It is interesting to note that the extent of the nonlocality depends only upon the reduced mass and the size of the system but is independent of the strength of the interaction. The interaction strength ζ_{01} determines only the magnitude of $M(r;E)$, as shown by Eq. (3.2a).

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APPENDIX A: DERIVATION OF EFFECTIVE POTENTIAL

Consider a spinless projectile incident upon a target with a ground state ($n=0$) and a number of excited states ($n>0$) with excitation energies ε_n and internal state vectors Φ_n. Let each target state also be spinless so that, if the incident state of relative motion is an S wave, there are only S waves in the inelastic channels. The corresponding Schrödinger equation is

$$\left[E-H_0+\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}-V\right]\Psi=0 , \tag{A1}$$

where H_0 is the target Hamiltonian, V is the interaction, and r is the separation between projectile and target. Expanding Ψ in terms of the target states,

$$\Psi=\sum_n \Phi_n u_n(r) , \tag{A2}$$

we obtain as usual a set of coupled equations for the states of relative motion, $u_n(r)$,

$$\left[(E-\varepsilon_n)+\frac{\hbar^2}{2\mu}\frac{d^2}{dr^2}-V_{nn}(r)\right]u_n(r) =\sum_{n'\neq n} V_{nn'}(r)u_{n'}(r) . \tag{A3}$$

We now assume that the interaction V couples only the ground state to each excited state, or each state to itself. Further, we take a square well for the diagonal elements and a spherical shell for the off-diagonal ones. Then we have

$$V_{nn}=\begin{cases}-\dfrac{\hbar^2}{2\mu}\xi_n , & \text{if } r\le a ,\\ 0 , & \text{if } r>a ,\end{cases} \tag{A4a}$$

$$V_{n0}=V_{0n}=-(\hbar^2/2\mu)\xi_{n0}\delta(r-a) , \tag{A4b}$$

and

$$V_{nn'}=0 , \quad \text{if } n\neq n'\neq 0 . \tag{A4c}$$

Substitution into Eqs. (A3) gives

$$\left[k_0^2+\frac{d^2}{dr^2}+\xi_0\Theta(a-r)\right]u_0(r)=\sum_n \zeta_{0n}\delta(r-a)u_n(r) , \tag{A5a}$$

$$\left[k_n^2+\frac{d^2}{dr^2}+\xi_n\Theta(a-r)\right]u_n(r)=\zeta_{0n}\delta(r-a)u_0(r) , \tag{A5b}$$

where $\Theta(x)$ is the Heaviside step function and

$$k_n=[(2\mu/\hbar^2)(E-\varepsilon_n)]^{1/2} \tag{A5c}$$

is the wave number in the nth channel, with the rule that the positive square root is taken if $E>\varepsilon_n$, while $k_n=i\kappa_n$, $\kappa_n>0$, if $E<\varepsilon_n$.

Integrating Eq. (A5b) from $r=a-\eta$ to $r=a+\eta$, where η is a small element of length, and taking the limit $\eta\to 0$, results in the continuity equations at $r=a$,

$$u_n(a_+)=u_n(a_-) , \tag{A6a}$$

$$\frac{du_n(a_+)}{dr}-\frac{du_n(a_-)}{dr}=\zeta_{n0}u_0(a) , \tag{A6b}$$

where $a_\pm=\lim_{\eta\to 0}(a\pm\eta)$.

We adopt outgoing waves for the open inelastic channels, $n>0$, so that

$$u_n(r)=\begin{cases}B_n e^{ik_n r} , & r>a , \quad \text{(A7a)}\\ C_n \sin K_n r , & r<a , \quad \text{(A7b)}\end{cases}$$

where

$$K_n=[(2\mu/\hbar^2)(E-\varepsilon_n)+\xi_n]^{1/2} . \tag{A7c}$$

Then Eqs. (A6) yield

$$B_n e^{ik_n a}-C_n \sin K_n a=0 , \tag{A8a}$$

$$ik_n B_n e^{ik_n a}-C_n K_n \cos K_n a=\zeta_{n0}u_0(a) . \tag{A8b}$$

These can be solved to give

$$u_n(a)=B_n e^{ik_n a}=\frac{\zeta_{n0}u_0(a)}{ik_n-\gamma_n} , \tag{A9a}$$

where

$$\gamma_n=K_n \cot K_n a . \tag{A9b}$$

Substitution into Eq. (A5a) gives an equation for u_0

$$\left[k_0^2+\frac{d^2}{dr^2}+\xi_0\Theta(a-r)-\sum_n \frac{|\zeta_{0n}|^2\delta(r-a)}{ik_n-\gamma_n}\right]u_0(r)=0 . \tag{A10}$$

This equation describes the elastic scattering by an effective potential consisting of the original ground-state interaction plus a complex, energy-dependent "dynamic polarization potential,"

$$M(r;E)=-\xi_0\Theta(a-r)+\sum_n \frac{|\zeta_{0n}|^2\delta(r-a)}{ik_n-\gamma_n} . \quad (A11)$$

The resulting elastic wave and scattering are identical to those obtained by solving the original set (A5) of coupled equations.

Usually [2,4] this elimination, or projection, procedure gives an effective potential that is nonlocal in the spatial coordinates, but in our case the choice (A4b) of an inelastic interaction localized on a spherical shell forces the potential to be local.

The potential (A11) is further simplified by setting to zero the diagonal interactions (A4a), $\xi_n=0$, so that $K_n=k_n$. If, in addition, we limit ourselves to a system with a single excited state, $n=1$, for example, we have

$$M(r;E)=\frac{|\zeta_{10}|^2\delta(r-a)}{ik_1-k_1\cot(k_1a)}$$
$$=-|\zeta_{10}|^2\delta(r-a)\sin(k_1a)e^{ik_1a}/k_1 . \quad (A12)$$

If we set $\varepsilon_1=\varepsilon$ and $k_1=k$, this becomes the result (3.2) quoted in the main text,

$$M(r;E)=-m(E)|\zeta_{01}|^2\delta(r-a) , \quad (A13a)$$

where

$$m(E)=k^{-1}\sin(ka)e^{ika} , \quad (A13b)$$

with

$$k=[2\mu(E-\varepsilon)/\hbar^2]^{1/2} . \quad (A13c)$$

The energy-dependent factor $m(E)$ may be separated into real and imaginary parts,

$$m(E)=\nu(E)+iw(E) , \quad (A14a)$$

where

$$\nu(E)=\begin{cases}\sin(2ka)/2k , & \text{if } E>\varepsilon ,\\ \sinh(\kappa a)e^{-\kappa a}/\kappa , & \text{if } E<\varepsilon ,\end{cases} \quad (A14b)$$

$$w(E)=\begin{cases}\sin^2(ka)/k , & \text{if } E>\varepsilon ,\\ 0 , & \text{if } E<\varepsilon .\end{cases} \quad (A14c)$$

We note from Eqs. (3.2a) and (A14c) that the imaginary part of $M(r;E)$ is always negative, representing absorption, or zero. This results from our physical choice (A7) of outgoing waves in the inelastic channels. The opposite choice of incoming waves would have resulted in a polarization potential that is the complex conjugate of (A12); that is, it would be emissive.

The potential (A14) provides an example of a "threshold anomaly" [12]; that is, the rise of the imaginary part $w(E)$ from zero at the threshold $E=\varepsilon$ is accompanied by a real component $\nu(E)$ which has its maximum at the threshold. This feature is ordained by the dispersion relations (4.7) that ν and w satisfy.

Note that, because of the sign in Eqs. (3.2a) and (A13a), this corresponds to an increase in the magnitude of the imaginary potential $W(E)$ being accompanied by an *attractive* real potential $\Delta V(E)$.

Below the inelastic threshold, $E<\varepsilon$, the imaginary part $w(E)=0$ and the real part $\nu(E)$ has the dimensionless form

$$\frac{\nu(E)}{a}=\frac{e^{-x}\sinh x}{x} , \quad x=\kappa a ; \quad (A15)$$

it decreases monotonically with decreasing energy. Initially the decrease is rapid, but becomes slow, $\sim(2\kappa a)^{-1}$, far below the threshold.

Above threshold, $w(E)$ has the form

$$\frac{w(E)}{a}=\frac{\sin^2 x}{x} , \quad x=ka , \quad (A16)$$

which is always positive but oscillates, as k increases, with period $\Delta k=\pi/a$ and with an amplitude damped by the envelope $(ka)^{-1}$. The real part,

$$\frac{\nu(E)}{a}=\frac{\sin 2x}{2x} , \quad x=ka \quad (A17)$$

oscillates about zero with period $\Delta k=\pi/a$, but with amplitude also damped as $(ka)^{-1}$. This behavior is illustrated in Fig. 1. The zeros in $w(E)$, which occur when $ka=n\pi$ (with n integer), arise because at these energies the matching condition (A8a) can be satisfied only if $B_1=0$ when $k_1=K_1$. That is, the matching condition prevents any flux escaping in this channel.

The analyticity of $m(E)$ which resulted in the dispersion relations (4.7) has another consequence. We may write $m(E)/a$ as

$$\frac{m(E)}{a}=\frac{e^{2ika}-1}{2ika}$$
$$=\frac{(e^{iz}-1)/i}{z}$$
$$=\frac{\nu'+iw'}{z} . \quad (A18)$$

The oscillatory behavior as a function of E or k arises totally from the numerator, which is analytic. Putting $z=x+iy$, the real and imaginary parts of the numerator,

$$\nu'+iw'=e^{-y}\sin x+i(1-e^{-y}\cos x) , \quad (A19)$$

therefore, satisfy the Cauchy-Riemann condition

$$\frac{\partial w'}{\partial x}=-\frac{\partial \nu'}{\partial y} , \quad (A20)$$

as may be checked explicitly. Furthermore, because of the exponential dependence on y,

$$\frac{\partial w'}{\partial x}=-\frac{\partial \nu'}{\partial y}=\nu' . \quad (A21)$$

Thus, aside from the slowly varying z^{-1} factor in Eq. (A18), the real part of m equals the derivative of the imaginary part. This relation is exact at the minima of $w(E)$, where $ka=n\pi$, n integer, and both $\nu(E)$ and $w(E)$ are zero [see Eqs. (A14)]. The z^{-1} factor slightly displaces the other zeros of $\nu(E)$ from the maxima of $w(E)$. The displacement is largest at the first maximum but rapidly becomes small as n increases.

In general, we see that whenever the mean field

$$M(k)=V(k)+iW(k)$$

is an analytic function of k, the Cauchy-Riemann relations give us the exact relations

$$\frac{\partial V}{\partial x}=\frac{\partial W}{\partial y}\ , \quad \frac{\partial W}{\partial x}=-\frac{\partial V}{\partial y}\ . \tag{A22}$$

Evaluating these on the real axis will give relationships between the real and imaginary parts of M.

The real part of $m(E)$ is continuous at threshold, where $v(0)=1$, but its slope $dv(E)/dE$ is discontinuous. Approached from below, the slope at threshold is positive and infinite, while approached from above it is finite with the value $-2a^3/3$. The energy dependence of the real polarization potential is often discussed in terms of the effective mass, defined as [15]

$$\mu^*/\mu=1-d(\mathrm{Re}M)/dE\ . \tag{A23}$$

(This is the E mass [15] in our case, since our M is local in the spatial coordinates.) From Eqs. (A13a) and (A14a), this becomes

$$\mu^*(E)/\mu=1+|\zeta_{01}|^2\delta(r-a)dv(E)/dE\ , \tag{A24}$$

which differs from unity only on the shell $r=a$. Just below threshold the μ^*/μ becomes infinitely large, while just above it is less than unity. This reflects the general behavior observed for μ^*/μ in more realistic models [8,15], namely, $\mu^*/\mu>1$ below and in the vicinity of the threshold and $\mu^*/\mu<1$ above threshold. The rather singular behavior at threshold observed in our case results from the choice of a singular delta-function interaction.

APPENDIX B: DERIVATION OF FOURIER TRANSFORM RELATIONS

Here we show explicitly that the Fourier transforms $\tilde{v}(\tau)$ and $\tilde{w}(\tau)$ of the real and imaginary parts of the energy-dependent factor $m(E)$ of our model polarization potential (3.2) satisfy the general relations (2.9), (5.12), and (5.13).

It is convenient to work in k space, as in Sec. V. Using the expression (A14c), the transform of the imaginary part $w(E)$ may be written

$$\tilde{w}(\tau)=\frac{e^{-i\varepsilon\tau}}{\pi}\int_0^\infty e^{-ik^2\tau}\sin^2(ka)dk\ , \tag{B1a}$$

where we made use of the fact that $w(E)=0$ if $E-\varepsilon$. The transform of the full potential, $m(E)=v(E)+iw(E)$, is

$$\tilde{m}(\tau)=\tilde{v}(\tau)+i\tilde{w}(\tau)$$

$$=\frac{e^{-i\varepsilon\tau}}{\pi}\int_C e^{-ik^2\tau}e^{ika}\sin(ka)dk\ , \tag{B2a}$$

where we introduced the expression (A13b), and C is the contour shown in Fig. 2.

For $\tau>0$, we now deform the contour as was done in Sec. V B, but taking $k=(1-i)\rho/\sqrt{2}$ for convenience so that the straightened path C' is along the $\theta=7\pi/4$ line in the complex k plane. Equations (B1) and (B2) then become

$$\tilde{w}(\tau)=e^{-i\varepsilon\tau}\frac{1-i}{\pi\sqrt{2}}\int_0^\infty e^{-\rho^2\tau}\sin^2\left[\left(\frac{1-i}{\sqrt{2}}\right)\rho a\right]d\rho \tag{B1b}$$

and

$$\tilde{m}(\tau)=e^{i\varepsilon\tau}\frac{1-i}{\pi\sqrt{2}}\int_{-\infty}^\infty e^{-\rho^2\tau}\left\{\sin\left[\left(\frac{1-i}{\sqrt{2}}\right)\rho a\right]\cos\left[\left(\frac{1-i}{\sqrt{2}}\right)\rho a\right]+i\sin^2\left[\left(\frac{1-i}{\sqrt{2}}\right)\rho a\right]\right\}d\rho\ . \tag{B2b}$$

The first term in braces in Eq. (B2b) is an odd function of ρ and integrates to zero, while the second term is an even function and integrates to twice the integral from zero to infinity found in Eq. (B1b). Consequently, we have the desired result

$$\tilde{m}(\tau)=\tilde{v}(\tau)+i\tilde{w}(\tau)=2i\tilde{w}(\tau)\ , \quad \tau>0\ . \tag{B3}$$

We showed in Sec. V C that $\tilde{m}(\tau)$ vanished for $\tau<0$, so

$$\tilde{m}(\tau)=\tilde{v}(\tau)+i\tilde{w}(\tau)=0\ , \quad \tau<0\ . \tag{B4}$$

Equations (B3) and (B4) thus imply

$$\begin{aligned}\tilde{v}(\tau)&=i\tilde{w}(\tau)\ , \quad \tau>0\\ &=-i\tilde{w}(\tau)\ , \quad \tau<0\ ,\end{aligned} \tag{B5}$$

in agreement with the general result deduced from Titchmarsh's theorem [8].

APPENDIX C: A THEOREM FOR THE ASYMPTOTIC DEPENDENCE ON TIME

In this appendix we derive a theorem that specifies the asymptotic dependence on τ of a function $\tilde{m}(\tau)$ that is the Fourier transform of a function $m(E)$, under the conditions that

$$\frac{d^n m(E)}{dE^n}\xrightarrow[E\to\varepsilon]{}\frac{c}{(E-\varepsilon)^{1/2}} \tag{C1a}$$

and

$$\frac{d^p m(E)}{dE^p}\xrightarrow[E\to\infty]{}0\ , \tag{C1b}$$

where $p=0,1,\ldots,n-1$, and n is the lowest order of derivative in which an $(E-\varepsilon)^{1/2}$ infinity occurs. There may be other terms with positive powers of $(E-\varepsilon)^{1/2}$ besides the singularity (C1a).

The Fourier transform $\tilde{m}(\tau)$ is

$$
\begin{aligned}
\tilde{m}(\tau) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} m(E)e^{-iE\tau}dE \\
&= \frac{1}{2\pi}\left[m(E)\left[-\frac{1}{i\tau}\right]e^{-iE\tau}\Bigg|_{-\infty}^{\infty} + \frac{1}{i\tau}\int_{-\infty}^{\infty}\frac{dm(E)}{dE}e^{-iE\tau}dE \right] \\
&= \frac{1}{2\pi}\left[\frac{1}{i\tau}\right]^n \int_{-\infty}^{\infty}\frac{d^n m(E)}{dE^n}e^{-iE\tau}dE \\
&= \frac{1}{2\pi}\left[\frac{1}{i\tau}\right]^n \int_{-\infty}^{\infty}\sum_{q=-1}^{\infty} C_q(E-\varepsilon)^{q/2}e^{-iE\tau}dE \ ,
\end{aligned}
\tag{C2}
$$

where $m(E)$ has been written as a Laurent expansion.

Changing variables to $k=(E-\varepsilon)^{1/2}$ gives the result

$$\tilde{m}(\tau)=\frac{1}{\pi}\left[\frac{1}{i\tau}\right]^n \sum_{q=-1}^{\infty} C_q \int_C k^{q+1}e^{-ik^2\tau}dk \ , \tag{C3}$$

where C is the familiar contour shown in Fig. 2. We deform the contour, as was done in Sec. V A, into a line along the $7\pi/4$ line, so that k becomes

$$k=\rho\exp(i7\pi/4)=\rho\exp(-i\pi/4) \ . \tag{C4}$$

The exponential factor in the integrands of Eq. (C3) then becomes

$$\exp(-ik^2\tau)=\exp(-\rho^2\tau) \ ,$$

which results in Gaussian integrals,

$$\tilde{m}(\tau)=\frac{1}{\pi}\left[\frac{1}{i\tau}\right]^n \sum_{q=-1}^{\infty} C_q \exp[-i\pi(q+2)/4] \times\int_{-\infty}^{\infty}\rho^{q+1}e^{-\rho^2\tau}d\rho \ . \tag{C5}$$

The integrals for odd powers, $q=-1,1,3,\ldots$, are

$$\int_{-\infty}^{\infty}\rho^{q+1}e^{-\rho^2\tau}d\rho=\frac{q!!\pi^{1/2}}{\sqrt{2^{q+1}\tau^{q+2}}} \ , \tag{C6a}$$

while those for even powers, $q=0,2,4,\ldots$, are

$$\int_{-\infty}^{\infty}\rho^{q+1}e^{-\rho^2\tau}d\rho=\frac{(q/2)!}{\sqrt{\tau^{q+2}}} \ . \tag{C6b}$$

The leading term at large τ is the term with the lowest power of q, namely, $q=-1$. Thus, as $\tau\to\infty$,

$$\tilde{m}(\tau)\to\frac{C_{-1}}{\pi^{1/2}(i\tau)^{n+1/2}} \ . \tag{C7}$$

In our case, $m(E)$ has no infinity, but $dm(E)/dE$ does. Thus we have $n=1$, so the use of $C_{-1}=ia^2/2$ gives

$$\tilde{m}(\tau)\to\frac{ia^2}{2\pi^{1/2}}\left[\frac{1}{i\tau}\right]^{3/2} , \tag{C8}$$

in agreement with Eq. (5.16).

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