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# Anatomy of the soft photon approximation in hadron-hadron bremsstrahlung

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A modified Low procedure for constructing soft-photon amplitudes has been used to derive two general soft-photon amplitudes, a two-$s$–two-$t$ special amplitude $M_\mu^{TsTts}$ and a two-$u$–two-$t$ special amplitude $M_\mu^{TuTts}$, where $s$, $t$, and $u$ are the Mandelstam variables. $M_\mu^{TsTts}$ depends only on the elastic $T$ matrix evaluated at four sets of $(s,t)$ fixed by the requirement that the amplitude be free of derivatives ($\partial T/\partial s$ and/or $\partial T/\partial t$). Likewise $M_\mu^{TuTts}$ depends only on the elastic $T$ matrix evaluated at four sets of $(u,t)$ also fixed by the requirement that the amplitude $M_\mu^{TuTts}$ be free of derivatives ($\partial T/\partial u$ and/or $\partial T/\partial t$). In deriving these two amplitudes, we imposed the condition that $M_\mu^{TsTts}$ and $M_\mu^{TuTts}$ reduce to $\overline{M}_\mu^{TsTts}$ and $\overline{M}_\mu^{TuTts}$, respectively, their tree-level approximations. The amplitude $\overline{M}_\mu^{TsTts}$ represents photon emission from a sum of one-particle $t$-channel exchange diagrams and one-particle $s$-channel exchange diagrams, while the amplitude $\overline{M}_\mu^{TuTts}$ represents photon emission from a sum of one-particle $t$-channel exchange diagrams and one-particle $u$-channel exchange diagrams. The precise expressions for $\overline{M}_\mu^{TsTts}$ and $\overline{M}_\mu^{TuTts}$ are determined by using the radiation decomposition identities of Brodsky and Brown. We also demonstrate that two Low amplitudes $M_\mu^{\mathrm{Low}(st)}$ and $M_\mu^{\mathrm{Low}(ut)}$, derived using Low's standard procedure, can be obtained from $M_\mu^{TsTts}$ and $M_\mu^{TuTts}$, respectively, as an expansion in powers of $K$ (photon energy) when terms of order $K$ and higher are neglected. We point out that it is theoretically impossible to describe all nuclear bremsstrahlung processes by using only a single class of soft-photon amplitudes. At least two different classes are required: the amplutides (such as $M_\mu^{TsTts}$, $M_\mu^{\mathrm{Low}(st)}$, and $\overline{M}_\mu^{TsTts}$), which depend on $s$ and $t$, and the amplitudes (such as $M_\mu^{TuTts}$, $M_\mu^{\mathrm{Low}(ut)}$, and $\overline{M}_\mu^{TuTts}$), which depend on $u$ and $t$. When resonance effects are important, the amplitude $M_\mu^{TsTts}$, not $M_\mu^{\mathrm{Low}(st)}$, should be used. For processes with strong $u$-channel exchange effects, the amplitude $M_\mu^{TuTts}$ should be the first choice. As for those processes which exhibit neither resonance effects nor $u$-channel exchange effects, all amplitudes converge essentially to the same description. Finally, we discuss the relationship between the two classes.

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## I. INTRODUCTION

Hadron-hadron bremsstrahlung processes have attracted much attention during the last three decades. Processes, such as nucleon-nucleon bremsstrahlung ($pp\gamma$ and $np\gamma$) [1–3], proton-deuteron bremsstrahlung ($pd\gamma$) [4,5], proton-helium bremsstrahlung ($p\alpha\gamma$) [4,6], proton-carbon bremsstrahlung ($p\,^{12}\mathrm{C}\gamma$) [7], proton-oxygen bremsstrahlung ($p\,^{16}\mathrm{O}\gamma$) [8], and pion-proton bremsstrahlung ($\pi^\pm p\gamma$) [9], are the best-known examples, because they have been studied both experimentally and theoretically. There exist a variety of reasons for investigating these processes. (i) One of the important goals is the investigation of off-shell effects in the scattering amplitude. For instance, the $pp\gamma$ and $np\gamma$ processes have been extensively studied since 1963 to investigate the off-shell behavior of two-nucleon interactions. Most theoretical studies have focused on nonrelativistic potential model calculations using various phenomenological potentials as input, with the goal that the best potential could be selected from comparison with $pp\gamma$ and/or $np\gamma$ data [1–3]. Recently, the observation of energetic photons from heavy-ion collisions has created a growing interest in understanding the basic production mechanism of these high-energy photons [10]. The $np\gamma$ process has received new attention because it appears to be the most likely source of these energetic photons. Moreover, $np\gamma$ is probably an ideal process for studying meson exchange effects [11,12]. (ii) Bremsstrahlung processes have been used as a tool to investigate electromagnetic properties of resonances. The most successful example is the determination of the magnetic moments of the $\Delta^{++}(\Delta^0)$ from the $\pi^+p\gamma$ ($\pi^-p\gamma$) data in the energy region of the $\Delta(1232)$ resonance [9,13]. (iii) The study of nucleon-nucleus and nucleus-nucleus bremsstrahlung processes in the vicinity of resonances, such as the $p\,^{12}\mathrm{C}\gamma$ process near the 1.7- and 0.5-MeV resonances [7] or the $p\,^{16}\mathrm{O}\gamma$ process near the 2.66-MeV resonance [8], was originally suggested for investigating details of nuclear reactions. Such bremsstrahlung measurements can be used to extract the nuclear time delay, and the time delay can be used to distinguish between a direct nuclear reaction and a compound nuclear reaction. That bremsstrahlung emission can be used as a tool to measure time delay has been confirmed experimentally: Three separate experimental groups have measured the $p\,^{12}\mathrm{C}\gamma$ cross sections and then used these cross sections to extract nuclear time delays [7]. (iv) Testing theoretical models and calculational approximations has been another important aspect of studying hadron-hadron bremsstrahlung processes, especially those processes containing significant resonance or exchange effects. The combined experimental and theoretical investigations of the $\pi^{\pm}p\gamma$ and $p\,^{12}\mathrm{C}\gamma$ processes led to a surprising conclusion [14,15]: These cross sections cannot be described by the conventional soft-photon amplitudes (evaluated at a single energy and scattering angle), which had been the standard since 1958 when Low first derived them. They fail completely to fit the experimental data. These observations indicate why the study of bremsstrahlung processes with significant resonance effects or meson exchange effects can provide a sensitive test of theoretical models and approximations.

Among the various models and approximations proposed during the past three decades for bremsstrahlung calculations, the best-known approach is the soft-photon approximation. This approximation is based upon a fundamental theorem: the soft-photon theorem or the low-energy theorem for photons. The theorem was first derived by Low [16]; it was extended and generalized later by many other authors [17,14,9]. Various soft-photon amplitudes, which are consistent with the theorem, have been constructed by using the standard Low procedure [16]. This involves the following steps: (a) Obtain the external amplitude $M_\mu^{(E)}$ from the four external emission diagrams and expand $M_\mu^{(E)}$ in powers of the photon energy $K$. (b) Impose the gauge invariant condition $M_\mu^{(I)}K^\mu=-M_\mu^{(E)}K^\mu$, to obtain the leading term (order $K^0$) of the internal emission amplitude $M_\mu^{(I)}$. (c) Combine $M_\mu^{(E)}$ and $M_\mu^{(I)}$ to obtain the total bremsstrahlung amplitude $M_\mu^{(T)}$. Low's soft-photon amplitude $M_\mu^{\mathrm{Low}}$, which is independent of off-shell effects, is defined by the first two terms of the expansion of $M_\mu^{(T)}$. A universal feature of all soft-photon amplitudes is that they depend only on the corresponding elastic amplitude and electromagnetic constants of the participating particles. Therefore, the soft-photon approximation is referred to as the on-shell approximation, and calculations based on the soft-photon approximation are classified as model independent.

The reader will note that the standard procedure cannot be used to obtain an internal contribution which is separately gauge invariant [9,18]. Therefore, it is difficult to obtain a general form for the internal amplitude by using the standard procedure. In order to derive the general soft-photon amplitude, a modified Low procedure was proposed recently [9,18]. The modified procedure includes four steps. Because the determination of the general amplitude $M_\mu$ is guided by the derivation for the corresponding special amplitude $\overline{M}_\mu$, which can be rigorously derived at the tree level, we first apply the modified procedure to find $\overline{M}_\mu$. (1) Obtain the external amplitude $\overline{M}_\mu^E$ from a set of tree-level external diagrams. (2) Construct the internal contribution $\overline{M}_\mu^I$, which represents photon emission from a dominant internal line (or lines), and split $\overline{M}_\mu^I$ into four quasiexternal amplitudes by using the radiation decomposition identities of Brodsky and Brown [19]. (3) Obtain an additional gauge invariant term $\overline{M}_\mu^G$ by imposing the gauge invariant condition $\overline{M}_\mu^G K^\mu=-\overline{M}_\mu^{EI}K^\mu$. Here $\overline{M}_\mu^{EI}$ is the sum of $\overline{M}_\mu^E$ and $\overline{M}_\mu^I$: $\overline{M}_\mu^{EI}=\overline{M}_\mu^E+\overline{M}_\mu^I$. (4) Combine $\overline{M}_\mu^{EI}$ and $\overline{M}_\mu^G$ to obtain the total amplitude, $\overline{M}_\mu=\overline{M}_\mu^{EI}+\overline{M}_\mu^G$. The amplitude $\overline{M}_\mu$, especially the expression for $\overline{M}_\mu^I$, can then be used to determine the general amplitude $M_\mu$ by applying the modified procedure again. (1) Obtain the external amplitude $M_\mu^E$ from four general external diagrams. (This step is identical to the first step of the standard procedure.) (2) Construct an internal contribution $M_\mu^I$, which reduces to $\overline{M}_\mu^I$ at the tree-level approximation. (3) Obtain an additional gauge invariant term $M_\mu^G$ by imposing the gauge invariant condition, $M_\mu^G K^\mu=-M_\mu^{EI}K^\mu$. Here, $M_\mu^{EI}=M_\mu^E+M_\mu^I$. (4) Combine $M_\mu^{EI}$ with $M_\mu^G$ to obtain the total amplitude $M_\mu=M_\mu^{EI}+M_\mu^G$, which should reduce to $\overline{M}_\mu$ at the tree-level approximation. The first two terms of the expansion of $M_\mu$, which can be written in terms of the complete elastic $T$ matrix and electromagnetic constants of the participating particles, define a general soft-photon amplitude. We emphasize that the expansion of $M_\mu$ is performed in such a way that the expanded $M_\mu$ will depend on the elastic $T$ matrix, evaluated for two Mandelstam variables, but it will be free of any derivative of the $T$ matrix with respect to those two specified Mandelstam variables.

The purpose of this work is to study the soft-photon approximation systematically. We apply both the standard procedure and the modified procedure to derive various soft-photon amplitudes, which fall naturally into two classes delineated by the choice of Mandelstam variables. We find that one of these two classes is completely new; it has been totally ignored in the literature. We show that these two classes are independent and they are equally important for describing bremsstrahlung processes. In order to make this point more precisely, let us consider photon emission accompanying the scattering of two particles $A$ and $B$ ($s$-channel reaction):

$$A(q_i^\mu)+B(p_i^\mu)\rightarrow A(q_f^\mu)+B(p_f^\mu)+\gamma(K^\mu)\ . \tag{1}$$

Here, $q_i^\mu$ ($q_f^\mu$) and $p_i^\mu$ ($p_f^\mu$) are the initial (final) four-momenta for particles $A$ and $B$, respectively, and $K^\mu$ is the four-momentum for the emitted photon with polarization $\varepsilon^\mu$. We assume that the particle $A$ has mass $m_A$ and charge $Q_A$, while the particle $B$ has mass $m_B$ and charge $Q_B$. For simplicity, we also assume that both $A$ and $B$ are spinless particles, since our problem does not depend on the spin of the participating particles. From the process (1), we can define the following Mandelstam variables: $s_i=(q_i+p_i)^2$, $s_f=(q_f+p_f)^2$, $t_p=(p_f-p_i)^2$, $t_q=(q_f-q_i)^2$, $u_1=(p_f-q_i)^2$, and $u_2=(q_f-p_i)^2$. Since a soft-photon amplitude depends only upon two independent variables, chosen from the above possible Mandelstam variables, we can express the two independent classes of soft-photon amplitude as $M_\mu^{(1)}(s,t)$ and $M_\mu^{(2)}(u,t)$. Here, $M_\mu^{(1)}(s,t)$ includes all amplitudes that depend upon $s$ [choosing from $s_i$, $s_f$, and other linear combinations $s_{\alpha\beta}=(\alpha s_i+\beta s_f)/(\alpha+\beta)$, $\alpha\neq 0$ and $\beta\neq 0$] and $t$ [choosing from $t_p$, $t_q$, and other linear combinations $t_{\alpha'\beta'}=(\alpha' t_p+\beta' t_q)/(\alpha'+\beta')$, $\alpha'\neq 0$ and $\beta'\neq 0$], while $M_\mu^{(2)}(u,t)$ involves all those amplitudes that depend upon

$u$ [choosing from $u_1$, $u_2$, and other combinations $u_{\bar{\alpha}\bar{\beta}}=(\bar{\alpha}u_1+\bar{\beta}u_2)/(\bar{\alpha}+\bar{\beta})$, $\bar{\alpha}\neq 0$ and $\bar{\beta}\neq 0$] and $t$.

The first class, $M_\mu^{(1)}(s,t)$, contains three interesting amplitudes: (i) the conventional Low amplitude $M_\mu^{\mathrm{Low}(st)}(\bar{s},\bar{t})$, (ii) the Feshbach-Yennie amplitude $M_\mu^{\mathrm{FY}}(s_i,s_f;t)$ [20,14], and (iii) the two-$s$–two-$t$ special amplitude $M_\mu^{TsTts}(s_i,s_f;t_p,t_q)$ [or the special two-energy–two-angle amplitude $M_\mu^{\mathrm{TETAS}}(s_i,s_f;t_p,t_q)$] [9,15]. The $M_\mu^{\mathrm{Low}(st)}$ amplitude can be derived using the standard Low procedure. Since this latter amplitude depends on $\bar{s}=s_{11}=(s_i+s_f)/2$ and $\bar{t}=t_{11}=(t_p+t_q)/2$, it is a one-energy–one-angle (OEOA) amplitude [14]. $M_\mu^{\mathrm{Low}(st)}$ has been widely used to calculate cross sections for many bremsstrahlung processes for more than 30 years. However, recent investigations have shown that it fails to describe those bremsstrahlung processes which are dominated by resonance effects. The Feshbach-Yennie amplitude is a special two-energy–one-angle amplitude [14]. It is interesting primarily because it was the first soft-photon amplitude that was used to describe some bremsstrahlung processes with scattering resonances and to extract the nuclear time delay from bremsstrahlung cross sections. The amplitude $M_\mu^{TsTts}$, as we will see later in this work, is the most general amplitude in the $M_\mu^{(1)}(s,t)$ class, since all other amplitudes, such as $M_\mu^{\mathrm{Low}(st)}$ and $M_\mu^{\mathrm{FY}}$, can be reproduced from it. Because the modified procedure is used to derive $M_\mu^{TsTts}$, the amplitude will be shown to be independent of the derivatives of the elastic $T$ matrix with respect to $s$ or $t$. The amplitude has been tested experimentally. The amplitude $M_\mu^{\mathrm{TETAS}}$, which is a practical version of $M_\mu^{TsTts}$, has been used to describe almost all available $\pi^\pm p\gamma$ and $p\,^{12}\mathrm{C}\gamma$ data. It has been used to determine the magnetic moments of the $\Delta^{++}$ and the $\Delta^0$ from $\pi^+p\gamma$ and $\pi^-p\gamma$ data, respectively. Although the $M_\mu^{\mathrm{TETAS}}$ amplitude should be used, it has never actually been applied to extract the nuclear time delay from the bremsstrahlung data.

The second class $M_\mu^{(2)}(u,t)$ is completely new. It has not been previously studied or discussed in the literature. Here, we show (i) how the standard procedure can be used to derive another Low's amplitude, $M_\mu^{\mathrm{Low}(ut)}(u_{11},t_{11})$ where $u_{11}=(u_1+u_2)/2$ and (ii) how the modified procedure can be used to obtain the general amplitude for the second class, the two-$u$–two-$t$ special amplitude $M_\mu^{TuTts}(u_1,u_2;t_p,t_q)$. We explain why we expect $M_\mu^{TuTts}$ to play a major role in predicting and describing those processes that are dominated by exchange current effects.

We also discuss the relationship between $M_\mu^{(1)}(s,t)$ and $M_\mu^{(2)}(u,t)$. In particular, we show that the two classes can be interchanged, $M_\mu^{(1)}(s,t)\leftrightarrow M_\mu^{(2)}(u,t)$, if $Q_B$ is replaced by $-Q_B$, $Q_B\rightarrow -Q_B$, and $p_i^\mu$ is interchanged with $-p_f^\mu$, $p_i^\mu\leftrightarrow -p_f^\mu$.

## II. ELASTIC SCATTERING $T$ MATRIX

The bremsstrahlung process, which we wish to study, is given by Eq. (1). The five four-momenta in Eq. (1) satisfy energy-momentum conservation:

$$q_i^\mu+p_i^\mu=q_f^\mu+p_f^\mu+K^\mu \ . \tag{2}$$

In the limit when $K$ approaches zero, the bremsstrahlung process (1) reduces to the corresponding $A$-$B$ elastic scattering process,

$$A(q_i^\mu)+B(p_i^\mu)\rightarrow A(\bar{q}_f^{\,\mu})+B(\bar{p}_f^{\,\mu}) \ , \tag{3}$$

where

$$\bar{q}_f^{\,\mu}=\lim_{K\rightarrow 0} q_f^\mu$$

and (4)

$$\bar{p}_f^{\,\mu}=\lim_{K\rightarrow 0} p_f^\mu \ .$$

The energy-momentum conservation relation defined in Eq. (2) becomes

$$q_i^\mu+p_i^\mu=\bar{q}_f^{\,\mu}+\bar{p}_f^{\,\mu} \ . \tag{5}$$

A diagram which represents the $A$-$B$ elastic scattering process is shown in Fig. 1(a). In this diagram, $\bar{T}$ represents the $A$-$B$ elastic scattering $T$ matrix. $\bar{T}$ is an on-shell $T$ matrix because all four external lines (legs) are on their mass shells. For the bremsstrahlung process, which will be discussed in the next section, the exact bremsstrahlung amplitude (without the soft-photon approximation) involves half-off-shell $T$ matrices. Each of these $T$ matrices, on-shell or half-off-shell, can be written in terms of six Lorentz invariants, chosen from $s$ ($s_i$ or $s_f$), $t$ ($t_p$ or $t_q$), $u$ ($u_1$ or $u_2$), $q_f'^2$ [$q_f^2$ or $\Delta_a=(q_f+K)^2$], $q_i'^2$ [$q_i^2$ or $\Delta_b=(q_i-K)^2$], $p_f'^2$ [$p_f^2$ or $\Delta_c=(p_f+K)^2$], and $p_i'^2$ [$p_i^2$ or $\Delta_d=(p_i-K)^2$]. Thus, any $T$ matrix can be written as

$$T(s,t,q_i'^2,p_i'^2,q_f'^2,p_f'^2)$$

or (6)

$$T(u,t,q_i'^2,p_i'^2,q_f'^2,p_f'^2) \ .$$

As in the examples, let us define the following $T$ matrices, which will be used later.

(i) The elastic (on-shell) $T$ matrix can be written as a function of two independent variables, e.g.,

$$T(s,t)\equiv T(s,t,m_A^2,m_B^2,m_A^2,m_B^2)$$

or (7)

$$T(u,t)\equiv T(u,t,m_A^2,m_B^2,m_A^2,m_B^2) \ .$$

This is because $q_i^2$, $p_i^2$, $q_f^2$, and $p_f^2$ satisfy the on-mass shell conditions,

$$\begin{aligned} q_i^2&=m_A^2 \ , \\ p_i^2&=m_B^2 \ , \\ q_f^2&\equiv\bar{q}_f^{\,2}=m_A^2 \ , \\ p_f^2&\equiv\bar{p}_f^{\,2}=m_B^2 \ , \end{aligned} \tag{8a}$$

and only two of the three Mandelstam variables are independent, since they satisfy the following condition:

$$s+t+u=2m_A^2+2m_B^2 \;, \tag{8b}$$

where

$$\begin{aligned} s&=(q_i+p_i)^2=(\bar{q}_f+\bar{p}_f)^2 \;, \\ t&=(\bar{p}_f-p_i)^2=(\bar{q}_f-q_i)^2 \;, \\ u&=(\bar{q}_f-p_i)^2=(\bar{p}_f-q_i)^2 \;. \end{aligned} \tag{9}$$

(ii) Five diagrams that represent the bremsstrahlung process (1) are shown in Fig. 2. A half-off-shell $T$ matrix can be defined if a photon of momentum $K^{\mu}$ is emitted from the outgoing $A$ particle [see Fig. 2(a)]. This $T$ matrix can be written as a function of three independent variables,

$$T(s_i,t_p,\Delta_a)\equiv T(s_i,t_p,m_A^2,m_B^2,\Delta_a,m_B^2)$$

or (10)

$$T(u_1,t_p,\Delta_a)\equiv T(u_1,t_p,m_A^2,m_B^2,\Delta_a,m_B^2) \;,$$

where

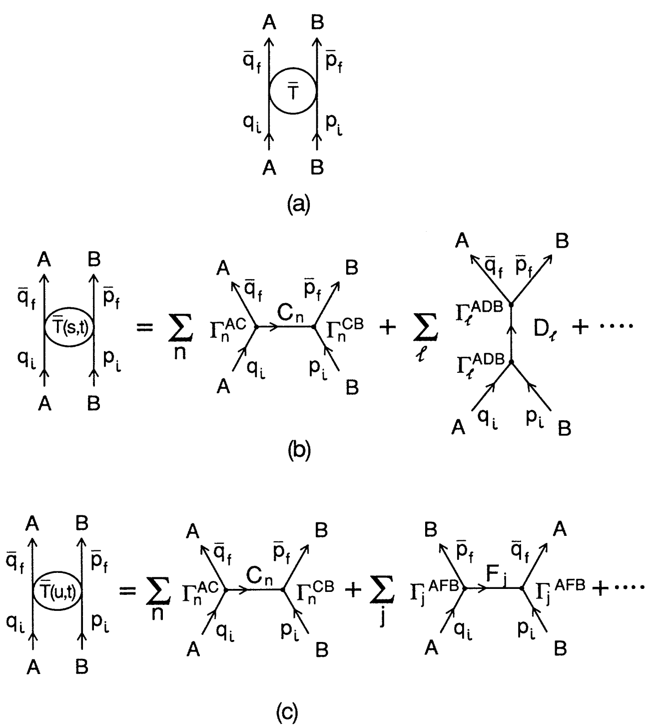


FIG. 1. (a) Graphic representation of the $A$-$B$ elastic scattering process. (b) Feynman diagrams for the $A$-$B$ elastic process at the tree level. The amplitude is approximated by a sum of one-particle $t$-channel exchange diagrams (exchange of $C_n$ particles, $n=1,2,\ldots$) and one-particle $s$-channel exchange diagrams (exchange of $D_l$ particles, $l=1,2,\ldots$). (c) Feynman diagrams for the $A$-$B$ elastic process at the tree level. The amplitude is approximated by a sum of one-particle $t$-channel exchange diagrams (exchange of $C_n$ particles, $n=1,2,\ldots$) and one-particle $u$-channel exchange diagrams (exchange of $F_j$ particles, $j=1,2,\ldots$).

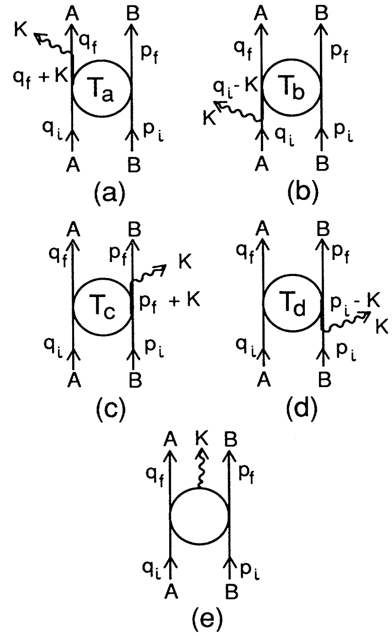


FIG. 2. Feynman diagrams for bremsstrahlung: (a)–(d) are the external emission diagrams; (e) is the internal emission diagram. These diagrams are generated from the source graph, Fig. 1(a).

$$\begin{aligned} s_i&=(q_i+p_i)^2 \;, \\ t_p&=(p_f-p_i)^2 \;, \\ u_1&=(p_f-q_i)^2 \;, \\ \Delta_a&\equiv(q_f+K)^2=m_A^2+2q_f\cdot K \;. \end{aligned} \tag{11}$$

It is easy to show that

$$s_i+t_p+u_1=\Delta_a+m_A^2+2m_B^2 \;. \tag{12}$$

(iii) A half-off-shell $T$ matrix can be defined if a photon of momentum $K^{\mu}$ is emitted from the incoming $A$ particle [see Fig. 2(b)]. This $T$ matrix is a function of three independent variables,

$$T(s_f,t_p,\Delta_b)\equiv T(s_f,t_p,\Delta_b,m_B^2,m_A^2,m_B^2)$$

or (13)

$$T(u_2,t_p,\Delta_b)\equiv T(u_2,t_p,\Delta_b,m_B^2,m_A^2,m_B^2) \;,$$

where

$$\begin{aligned} s_f&=(q_f+p_f)^2 \;, \\ u_2&=(q_f-p_i)^2 \;, \\ \Delta_b&\equiv(q_i-K)^2=m_A^2-2q_i\cdot K \;. \end{aligned} \tag{14}$$

We can show that

$$s_f+t_p+u_2=\Delta_b+m_A^2+2m_B^2 \;. \tag{15}$$

(iv) A half-off-shell $T$ matrix can be defined if a photon of momentum $K^\mu$ is emitted from the outgoing $B$ particle [see Fig. 2(c)]. This $T$ matrix is a function of three independent variables,

$$T(s_i,t_q,\Delta_c)\equiv T(s_i,t_q,m_A^2,m_B^2,m_A^2,\Delta_c)$$

or (16)

$$T(u_2,t_q,\Delta_c)\equiv T(u_2,t_q,m_A^2,m_B^2,m_A^2,\Delta_c)\ ,$$

where

$$t_q=(q_f-q_i)^2$$

and (17)

$$\Delta_c\equiv(p_f+K)^2=m_B^2+2p_f\cdot K\ .$$

The following relation can be easily proved:

$$s_i+t_q+u_2=\Delta_c+m_B^2+2m_A^2\ . \tag{18}$$

(v) A half-off-shell $T$ matrix can be defined if a photon of momentum $K^\mu$ is emitted from the incoming $B$ particle [see Fig. 2(d)]. This $T$ matrix is a function of three independent variables,

$$T(s_f,t_q,\Delta_d)\equiv T(s_f,t_q,m_A^2,\Delta_d,m_A^2,m_B^2)$$

or (19)

$$T(u_1,t_q,\Delta_d)\equiv T(u_1,t_q,m_A^2,\Delta_d,m_A^2,m_B^2)\ ,$$

where

$$\Delta_d\equiv(p_i-K)^2=m_B^2-2p_i\cdot K\ . \tag{20}$$

It is not difficult to prove that

$$s_f+t_q+u_1=\Delta_d+m_B^2+2m_A^2\ . \tag{21}$$

The above discussion illustrates clearly that there are at least two different ways of choosing independent variables for each $T$ matrix. The first choice involves $s$ and $t$, while the second choice involves $u$ and $t$. In the case that one is dealing with the exact amplitude for bremsstrahlung (in contrast to the soft-photon approximation, which is the subject of this paper), these two choices must be equivalent. *However, we shall see below that if one soft-photon amplitude is parametrized in terms of s and t and another soft-photon amplitude is parametrized in terms of u and t, then the two amplitudes are no longer equivalent.* The soft-photon approximation makes the two resulting amplitudes different. Which independent variables to select and how to parametrize $T$ matrices in terms of them is an important question, which must be carefully considered in order to establish the optimal soft-photon amplitude for specific bremsstrahlung processes. Since the elastic scattering diagrams serve as the ultimate source graphs from which all bremsstrahlung diagrams are generated, the independent variables in a soft-photon amplitude are specified by the choice of independent variables made in expressing the elastic $T$ matrix.

In order to illustrate this point, let us consider two special elastic scattering cases. In each case, we assume that the elastic scattering process is determined by a set of one-particle exchange diagrams. The first case is depicted in Fig. 1(b) and the second case in Fig. 1(c).

In the case shown in Fig. 1(b), the elastic $A$-$B$ scattering process is determined by a sum of one-particle $t$-channel exchange diagrams and one-particle $s$-channel exchange diagrams. In other words, we assume that the $A$-$B$ system involves the $t$-channel exchange of particles and the $s$-channel exchange of particles (an intermediate state or scattering resonance). The one-particle $s$-channel exchange diagrams are the dominant elastic diagrams in the resonance regions. [Two well-known examples are $\pi N$ scattering in the $\Delta(1232)$ resonance region and $p\,^{12}\mathrm{C}$ scattering near either the 1.7-MeV resonance or the 0.5-MeV resonance.] The elastic scattering $T$ matrix corresponding to Fig. 1(b) has the form

$$\overline{T}(s,t)=\overline{T}_C(t)+\overline{T}_D(s)\ , \tag{22}$$

where

$$\overline{T}_C(t)=\sum_n \Gamma_n^{AC}\frac{i}{t-(m_n^C)^2+i\varepsilon}\Gamma_n^{CB} \tag{23a}$$

and

$$\overline{T}_D(s)=\sum_l \Gamma_l^{ADB}\frac{i}{s-(m_l^D)^2+i\varepsilon}\Gamma_l^{ADB}\ . \tag{23b}$$

In Eq. (23), $m_n^C$ ($n=1,2,\dots$) are the masses of the $t$-channel exchange particles $C_n$, $\Gamma_n^{AC}$ are the $A$-$C_n$-A vertices, $\Gamma_n^{CB}$ are the $B$-$C_n$-B vertices, $m_l^D$ ($l=1,2,\dots$) are the masses of the intermediate particles ($s$-channel exchange particles) $D_l$, $\Gamma_l^{ADB}$ are the $A$-$D_l$-$B$ vertices, and $s$ and $t$ are defined by Eq. (9). Conservation of charge requires that all $t$-channel exchange particles $C_n$ be neutral and the charge of every $s$-channel exchange particle $D_l$ must be $Q_A+Q_B$. If these diagrams are used as source graphs to describe internal emission, $t$-channel exchange particles make no contribution to internal emission because they have no charge. Therefore, photon emission from the $s$-channel exchange determines the entire internal amplitude in this case.

In the second case, as shown in Fig. 1(c), the elastic $A$-$B$ scattering process is determined by a sum of one-particle $t$-channel exchange diagrams and one-particle $u$-channel exchange diagrams. In other words, we assume that the $A$-$B$ system involves the $t$-channel exchange particles and the $u$-channel exchange particles $F_j$ ($j=1,2,\dots$). The elastic scattering $T$ matrix corresponding to Fig. 1(c) has the form

$$\overline{T}(u,t)=\overline{T}_C(t)+\overline{T}_F(u)\ , \tag{24}$$

where $\overline{T}_C(t)$ is given by Eq. (23a) and

$$\overline{T}_F(u)=\sum_j \Gamma_j^{AFB}\frac{i}{u-(m_j^F)^2+i\varepsilon}\Gamma_j^{AFB}\ . \tag{25}$$

In Eq. (25), $m_j^F$ ($j=1,2,\dots$) are the masses of the $u$-channel exchange particles $F_j$, $\Gamma_j^{AFB}$ are the $A$-$F_j$-$B$ vertices, and $u$ is defined by Eq. (9). The charge of every $u$-channel exchange particle is $Q_A-Q_B$. If $Q_A-Q_B\neq 0$, then photon emission from the $u$-channel exchange particles determines the entire internal amplitude in this case.

## III. BREMSSTRAHLUNG AMPLITUDE AT THE TREE LEVEL

### A. Photon emission from the tree diagrams given by Fig. 1(b)

If the elastic scattering diagrams given by Fig. 1(b) are used as source graphs to generate bremsstrahlung diagrams, then we obtain Fig. 3. Figures 3(a)–3(d) represent the external emission diagrams and Fig. 3(e) represents the internal emission diagram. The external bremsstrahlung amplitude corresponding to Figs. 3(a)–3(d) has the form [21]

$$\overline{M}_\mu^{E(CD)} = Q_A \frac{q_{f\mu}}{q_f\cdot K}\overline{T}(s_i,t_p) - Q_A \frac{q_{i\mu}}{q_i\cdot K}\overline{T}(s_f,t_p) + Q_B \frac{p_{f\mu}}{p_f\cdot K}\overline{T}(s_i,t_q) - Q_B \frac{p_{i\mu}}{p_i\cdot K}\overline{T}(s_f,t_q)\ , \qquad (26)$$

where

$$\overline{T}(s_i,t_p) = \overline{T}_C(t_p) + \overline{T}_D(s_i)\ ,$$
$$\overline{T}(s_f,t_p) = \overline{T}_C(t_p) + \overline{T}_D(s_f)\ ,$$
$$\overline{T}(s_i,t_q) = \overline{T}_C(t_q) + \overline{T}_D(s_i)\ ,$$
$$\overline{T}(s_f,t_q) = \overline{T}_C(t_q) + \overline{T}_D(s_f)\ .$$

$\overline{T}_C(t_p)$ and $\overline{T}_C(t_q)$ are defined by Eq. (23a), and $\overline{T}_D(s_i)$ and $\overline{T}_D(s_f)$ are defined by Eq. (23b). The internal bremsstrahlung amplitude corresponding to Fig. 3(e) can be written as

$$\overline{M}_\mu^{I(D)} = \sum_l (Q_A + Q_B)\Gamma_l^{ADB} \frac{i}{(q_f+p_f)^2 - (m_l^D)^2 + i\varepsilon} \times[-i(q_i+p_i+q_f+p_f+K)_\mu] \times \frac{i}{(q_i+p_i)^2-(m_l^D)^2+i\varepsilon}\Gamma_l^{ADB}\ . \qquad (27)$$

Applying the radiation decomposition identity of Brodsky and Brown to split the amplitude $\overline{M}_\mu^{I(D)}$, we obtain

$$\overline{M}_\mu^{I(D)} = Q_A \overline{T}_D(s_f)\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K} - Q_A \frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\overline{T}_D(s_i) + Q_B \overline{T}_D(s_f)\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K} - Q_B \frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\overline{T}_D(s_i)\ . \qquad (28a)$$

This can be expressed directly in terms of the $T$ matrices defined above plus an exchange term:

$$\overline{M}_\mu^{I(D)} = Q_A[\overline{T}_D(s_f)+\overline{T}_C(t_p)-\overline{T}_C(t_p)]\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K} - Q_A\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}[\overline{T}_D(s_i)+\overline{T}_C(t_p)-\overline{T}_C(t_p)]$$
$$+ Q_B[\overline{T}_D(s_f)+\overline{T}_C(t_q)-\overline{T}_C(t_q)]\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K} - Q_B\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}[\overline{T}_D(s_i)+\overline{T}_C(t_q)-\overline{T}_C(t_q)] \qquad (28b)$$
$$= Q_A\overline{T}(s_f,t_p)\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K} - Q_A\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\overline{T}(s_i,t_p) + Q_B\overline{T}(s_f,t_q)\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K} - Q_B\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\overline{T}(s_i,t_q) + \overline{M}_\mu^x\ , \qquad (28c)$$

where

$$\overline{M}_\mu^x = -Q_A\overline{T}_C(t_p)\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K} + Q_A\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\overline{T}_C(t_p) - Q_B\overline{T}_C(t_q)\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K} + Q_B\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\overline{T}_C(t_q)\ . \qquad (29)$$

Neglecting $\overline{M}_\mu^x$ ($\overline{M}_\mu^x\cdot\varepsilon^\mu=0$ because the $T$ channel contribution $\equiv 0$), the expression for $\overline{M}^{I(D)}$ in terms of the four quasiexternal amplitudes becomes

$$\overline{M}_\mu^{I(D)} = Q_A\overline{T}(s_f,t_p)\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K} - Q_A\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\overline{T}(s_i,t_p) + Q_B\overline{T}(s_f,t_q)\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K} - Q_B\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\overline{T}(s_i,t_q)\ . \qquad (30)$$

We emphasize here that the expression for $\overline{M}_\mu^{I(D)}$ given by Eq. (30) is general. That is, neglecting $\overline{M}_\mu^x$ can be justified on quite general grounds. To see this, consider

$$A(q_i^\mu) + B(p_i^\mu) \rightarrow A'(q_f^\mu) + B'(p_f^\mu) + \gamma(K^\mu)\ .$$

We assume that particles $A$ and $B$ have charges $Q_A$ and $Q_B$, respectively, while particles $A'$ and $B'$ have charges $Q_A'$ and $Q_B'$, respectively. In this case, the amplitude $\overline{M}_\mu^{E(CD)}$ given by Eq. (26) becomes $\tilde{M}_\mu^{E(CD)}$,

$$\tilde{M}_\mu^{E(CD)} = Q_A' \frac{q_{f\mu}}{q_f\cdot K}\overline{T}(s_i,t_p) - Q_A \frac{q_{i\mu}}{q_i\cdot K}\overline{T}(s_f,t_p) + Q_B' \frac{p_{f\mu}}{p_f\cdot K}\overline{T}(s_i,t_q) - Q_B \frac{p_{i\mu}}{p_i\cdot K}\overline{T}(s_f,t_q)\ , \qquad (31)$$

while the amplitude $\overline{M}_\mu^I$ given by Eq. (28c) becomes $\tilde{M}_\mu^{I(D)}$,

$$\widetilde{M}_{\mu}^{I(D)}=Q_A\overline{T}(s_f,t_p)\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K}-Q'_A\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\overline{T}(s_i,t_p)+Q_B\overline{T}(s_f,t_q)\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K}-Q'_B\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\overline{T}(s_i,t_q)+\widetilde{M}_{\mu}^{x}, \quad (32)$$

where

$$\widetilde{M}_{\mu}^{x}=-Q_A\overline{T}_C(t_p)\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K}+Q'_A\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\overline{T}_C(t_p)-Q_B\overline{T}_C(t_q)\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K}+Q'_B\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\overline{T}_C(t_q)\ . \quad (33)$$

Obviously, the amplitude

$$\widetilde{M}_{\mu}^{EI}=\widetilde{M}_{\mu}^{E(CD)}+\widetilde{M}_{\mu}^{I(D)} \quad (34)$$

is not gauge invariant, since

$$\widetilde{M}_{\mu}^{EI}K^\mu=\widetilde{M}_{\mu}^{x}K^\mu=-Q_A\overline{T}_C(t_p)+Q'_A\overline{T}_C(t_p)-Q_B\overline{T}_C(t_q)+Q'_B\overline{T}_C(t_q)\neq 0 \quad (35)$$

if $Q_A\neq Q'_A$ and $Q_B\neq Q'_B$. Therefore, we must construct an additional gauge term by imposing the condition that the total amplitude must be gauge invariant. Let $\widetilde{M}_\mu$ be the total amplitude which is the sum of $\widetilde{M}_{\mu}^{EI}$ and an additional gauge term $\widetilde{M}_{\mu}^{G}$,

$$\widetilde{M}_\mu=\widetilde{M}_{\mu}^{EI}+\widetilde{M}_{\mu}^{G}\ . \quad (36)$$

The gauge invariant condition demands that

$$\widetilde{M}_\mu K^\mu=\widetilde{M}_{\mu}^{EI}K^\mu+\widetilde{M}_{\mu}^{G}K^\mu=\widetilde{M}_{\mu}^{x}K^\mu+\widetilde{M}_{\mu}^{G}K^\mu=0\ . \quad (37)$$

It is clear that we may choose

$$\widetilde{M}_{\mu}^{G}=-\widetilde{M}_{\mu}^{x}\ , \quad (38)$$

so that the term $\widetilde{M}_{\mu}^{x}$ in Eq. (32) is completely canceled by the additional gauge term $\widetilde{M}_{\mu}^{G}$. Hence, we can in general ignore the term $\widetilde{M}_{\mu}^{x}$ in Eq. (32), and therefore in the special case of $Q_A=Q'_A$ and $Q_B=Q'_B$ described by Eq. (28c).

Combining the external amplitude of Eq. (26) and the quasiexternal amplitudes of Eq. (30), we obtain the total amplitude $\overline{M}_{\mu}^{TsTts}$,

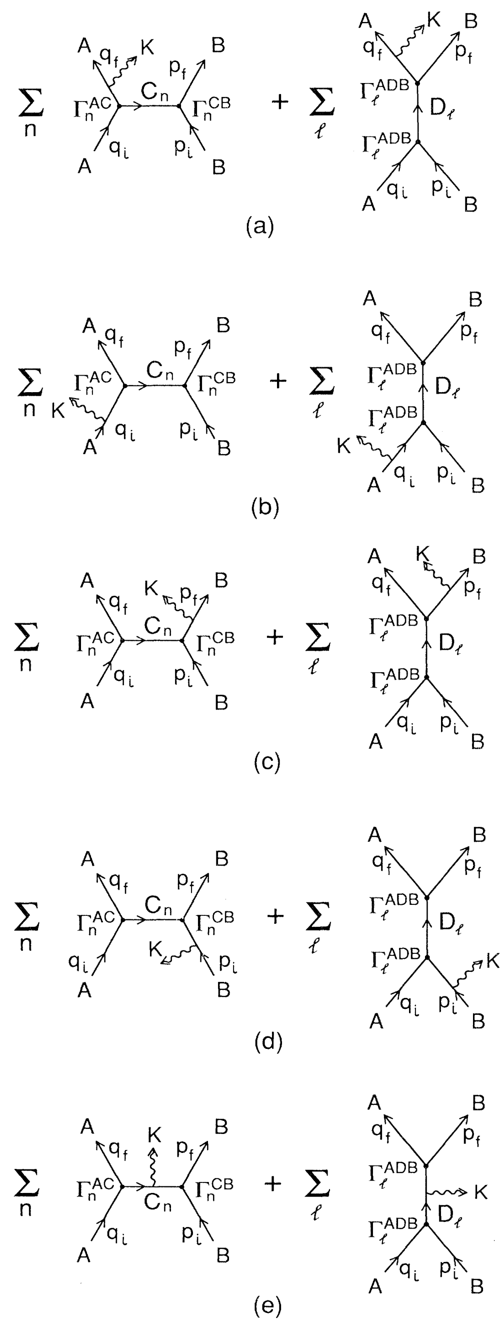

FIG. 3. Feynman diagrams for bremsstrahlung at the tree level: (a)–(d) are the external emission diagrams; (e) is the internal emission diagram. These diagrams are generated from the source graphs, Fig. 1(b).

$$\overline{M}_\mu^{TsTts}=\overline{M}_\mu^{E(CD)}+\overline{M}_\mu^{I(D)}$$
$$=Q_A\left[\frac{q_{f\mu}}{q_f\cdot K}-\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\right]\overline{T}(s_i,t_p)-Q_A\overline{T}(s_f,t_p)\left[\frac{q_{i\mu}}{q_i\cdot K}-\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K}\right]$$
$$+Q_B\left[\frac{p_{f\mu}}{p_f\cdot K}-\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\right]\overline{T}(s_i,t_q)-Q_B\overline{T}(s_f,t_q)\left[\frac{p_{i\mu}}{p_i\cdot K}-\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K}\right] . \quad (39)$$

It is easy to show that $\overline{M}_\mu^{TsTts}$ is gauge invariant; that is,

$$\overline{M}_\mu^{TsTts}K^\mu=0 . \quad (40)$$

Here, we have used "*TsTts*" to identify the amplitude given by Eq. (39), because the amplitude can be classified as the two-*s*–two-*t* special (*TsTts*) amplitude [22].

### B. Photon emission from the tree diagrams given by Fig. 1(c)

Using the elastic scattering diagrams given by Fig. 1(c) as source graphs to generate bremsstrahlung diagrams, we obtain Fig. 4. Figures 4(a)–4(d) represent the external emission diagrams and Fig. 4(e) represents the internal emission diagrams. The external bremsstrahlung amplitude corresponding to Figs. 4(a)–4(d) has the form [23]

$$\overline{M}_\mu^{E(CF)}=Q_A\frac{q_{f\mu}}{q_f\cdot K}\overline{T}(u_1,t_p)-Q_A\frac{q_{i\mu}}{q_i\cdot K}\overline{T}(u_2,t_p)$$
$$+Q_B\frac{p_{f\mu}}{p_f\cdot K}\overline{T}(u_2,t_q)-Q_B\frac{p_{i\mu}}{p_i\cdot K}\overline{T}(u_1,t_q) , \quad (41)$$

where

$$\overline{T}(u_1,t_p)=\overline{T}_C(t_p)+\overline{T}_F(u_1) ,$$
$$\overline{T}(u_2,t_p)=\overline{T}_C(t_p)+\overline{T}_F(u_2) ,$$
$$\overline{T}(u_2,t_q)=\overline{T}_C(t_q)+\overline{T}_F(u_2) ,$$
$$\overline{T}(u_1,t_q)=\overline{T}_C(t_q)+\overline{T}_F(u_1) .$$

$\overline{T}_C(t_p)$ and $\overline{T}_C(t_q)$ are defined by Eq. (23a), $\overline{T}_F(u_1)$ and $\overline{T}_F(u_2)$ are defined by Eq. (25), and $u_1$ and $u_2$ are defined by Eqs. (11) and (14), respectively. The internal bremsstrahlung amplitude corresponding to Fig. 4(e) can be written as

$$\overline{M}_\mu^{I(F)}=\sum_j(Q_A-Q_B)\Gamma_j^{AFB}\frac{i}{(q_i-p_f-K)^2-(m_j^F)^2+i\varepsilon}$$
$$\times[-i(q_i-p_f+q_i-p_f-K)_\mu]$$
$$\times\frac{i}{(q_i-p_f)^2-(m_j^F)^2+i\varepsilon}\Gamma_j^{AFB} \quad (42)$$

which can be decomposed by using the Brodsky-Brown identity as was done with Eq. (27). The decomposed amplitude

$$\overline{M}_\mu^{I(F)}=Q_A\overline{T}_F(u_2)\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}-Q_A\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}\overline{T}_F(u_1)$$
$$+Q_B\overline{T}_F(u_1)\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}-Q_B\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}\overline{T}_F(u_2) \quad (43a)$$

can be written as

$$\overline{M}_\mu^{I(F)}=Q_A[\overline{T}_F(u_2)+\overline{T}_C(t_p)-\overline{T}_C(t_p)]\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}-Q_A\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}[\overline{T}_F(u_1)+\overline{T}_C(t_p)-\overline{T}_C(t_p)]$$
$$+Q_B[\overline{T}_F(u_1)+\overline{T}_C(t_q)-\overline{T}_C(t_q)]\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}-Q_B\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}[\overline{T}_F(u_2)+\overline{T}_C(t_q)-\overline{T}_C(t_q)] \quad (43b)$$
$$=Q_A\overline{T}(u_2,t_p)\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}-Q_A\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}\overline{T}(u_1,t_p)$$
$$+Q_B\overline{T}(u_1,t_q)\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}-Q_B\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}\overline{T}(u_2,t_q)+\overline{M}_\mu^Y , \quad (43c)$$

where

$$\overline{M}_\mu^Y=-Q_A\overline{T}_C(t_p)\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}+Q_A\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}\overline{T}_C(t_p)-Q_B\overline{T}_C(t_q)\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}+Q_B\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}\overline{T}_C(t_q) . \quad (44)$$

Again, we can apply the same reasoning given in the last section, Sec. III A, to neglect the term $\overline{M}_\mu^Y$ ($\equiv 0$ in this case). Hence, we obtain the four quasiexternal amplitudes

$$\overline{M}_\mu^{I(F)}=Q_A\overline{T}(u_2,t_p)\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}-Q_A\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}\overline{T}(u_1,t_p)+Q_B\overline{T}(u_1,t_q)\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}-Q_B\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}\overline{T}(u_2,t_q) . \quad (45)$$

The total amplitude $\overline{M}_\mu^{TuTts}$ is therefore the sum of $\overline{M}_\mu^{E(CF)}$ and $\overline{M}_\mu^{I(F)}$ [given by Eq. (45)]:

$$\overline{M}_{\mu}^{TuTts}=\overline{M}_{\mu}^{E(CF)}+\overline{M}_{\mu}^{I(F)}$$

$$=Q_A\left[\frac{q_{f\mu}}{q_f\cdot K}-\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}\right]\overline{T}(u_1,t_p)-Q_A\overline{T}(u_2,t_p)\left[\frac{q_{i\mu}}{q_i\cdot K}-\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}\right]$$

$$+Q_B\left[\frac{p_{f\mu}}{p_f\cdot K}-\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}\right]\overline{T}(u_2,t_q)-Q_B\overline{T}(u_1,t_q)\left[\frac{p_{i\mu}}{p_i\cdot K}-\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}\right] . \tag{46}$$

Obviously, the amplitude $\overline{M}_{\mu}^{TuTts}$ is gauge invariant; that is,

$$\overline{M}_{\mu}^{TuTts}K^{\mu}=0 . \tag{47}$$

We have classified this amplitude as the two-$u$–two-$t$ special ($TuTts$) amplitude [24].

It should be pointed out that if we change $p_i^\mu$ to $-p_f^\mu$, $p_f^\mu$ to $-p_i^\mu$, and $Q_B$ to $-Q_B$, then the amplitude $\overline{M}_{\mu}^{TuTts}$ becomes the amplitude $\overline{M}_{\mu}^{TsTts}$:

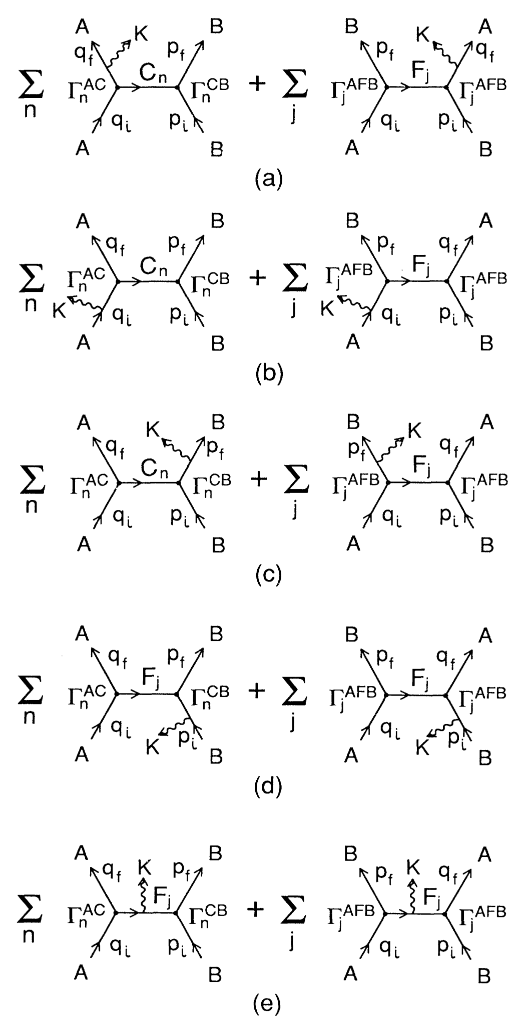


FIG. 4. Same as Fig. 3, but the diagrams are generated from the source graphs, Fig. 1(c).

$$\overline{M}_{\mu}^{TuTts}\xrightarrow[p_i^\mu\leftrightarrow -p_f^\mu]{Q_B\to -Q_B}\overline{M}_{\mu}^{TsTts} . \tag{48}$$

The reverse is also true. This interchange equivalence is expected from a close examination of Figs. 1(c) and 1(b).

## IV. SOFT-PHOTON AMPLITUDE

If the elastic scattering diagram given by Fig. 1(a) is used as the source graph to generate a set of bremsstrahlung diagrams, we obtain Fig. 2. Figures 2(a)–2(d) are the external emission diagrams and Fig. 2(e) is the internal emission diagram. $T_a$, $T_b$, $T_c$, and $T_d$ in these diagrams represent the half-off-shell $T$ matrices. It is well known that there is no general method which can be used to determine the exact internal amplitude without introducing dynamical models. It is also true that it is difficult to calculate all internal terms derived from a given model without introducing some approximations. This is why various soft-photon amplitudes, approximate amplitudes consistent with the soft-photon theorem, have been constructed and applied to describe many different nuclear bremsstrahlung processes. In the past, the utility of these amplitudes was determined only by comparison with experimental measurements. Recently, however, there has been some effort to determine the range of validity of various soft-photon amplitudes theoretically without comparing with experimental data. Here, we investigate methods for selecting optimal independent Lorentz invariants to parametrize the $T$ matrices ($T_a,T_b,T_c,T_d$) in the soft-photon amplitudes. We show that the question of validity of a given soft-photon approximation is directly related to the choice of independent Lorentz invariants. Four different soft-photon amplitudes are derived using two different procedures: the standard Low procedure and our modified Low procedure. The first two amplitudes are derived in Secs. IV A and IV B and the last two amplitudes, which are more general, are derived in Secs. IV C and IV D.

### A. Low's original soft-photon amplitude $M_{\mu}^{\mathrm{Low}(st)}$

Below, we review the procedure for deriving the first of two Low's soft-photon amplitudes. The independent Lorentz invariants are $s_x$ ($x=i,f$), $t_y$ ($y=p,q$), and $\Delta_z$ ($z=a,b,c,d$). (These invariants were defined in Sec. II). In other words, the four half-off-shell $T$ matrices are chosen to be

$$\begin{aligned} T_a &= T(s_i,t_p,\Delta_a)\ ,\\ T_b &= T(s_f,t_p,\Delta_b)\ ,\\ T_c &= T(s_i,t_q,\Delta_c)\ ,\\ T_d &= T(s_f,t_q,\Delta_d)\ . \end{aligned} \tag{49}$$

In terms of the above $T$ matrices, the external amplitude can be written in the familiar form

$$\begin{aligned} M_\mu^E(s,t,\Delta) = {} & Q_A \frac{q_{f\mu}}{q_f\cdot K} T(s_i,t_p,\Delta_a) \\ & - Q_A T(s_f,t_p,\Delta_b)\frac{q_{i\mu}}{q_i\cdot K} \\ & + Q_B \frac{p_{f\mu}}{p_f\cdot K} T(s_i,t_q,\Delta_c) \\ & - Q_B T(s_f,t_q,\Delta_d)\frac{p_{i\mu}}{p_i\cdot K}\ . \end{aligned} \tag{50}$$

Following Low, we introduce the average values of $s$ and $t$:

$$\begin{aligned} \bar{s} &= \tfrac{1}{2}(s_i+s_f)\ ,\\ \bar{t} &= \tfrac{1}{2}(t_p+t_q)\ . \end{aligned} \tag{51}$$

It is easily demonstrated that

$$\begin{aligned} s_i &= \bar{s}+(q_i+p_i)\cdot K = \bar{s}+(q_f+p_f)\cdot K\ ,\\ s_f &= \bar{s}-(q_i+p_i)\cdot K = \bar{s}-(q_f+p_f)\cdot K\ ,\\ t_p &= \bar{t}-(q_i-q_f)\cdot K = \bar{t}+(p_i-p_f)\cdot K\ ,\\ t_q &= \bar{t}+(q_i-q_f)\cdot K = \bar{t}-(p_i-p_f)\cdot K\ . \end{aligned} \tag{52}$$

If all half-off-shell $T$ matrices are expanded about [$\bar{s}$, $\bar{t}$, $\Delta=(\text{mass})^2$], then we obtain

$$\begin{aligned} M_\mu^E(s,t,\Delta) = {} & Q_A \frac{q_{f\mu}}{q_f\cdot K}\left[T(\bar{s},\bar{t}) + \frac{\partial T(\bar{s},\bar{t})}{\partial \bar{s}}(q_i+p_i)\cdot K + \frac{\partial T(\bar{s},\bar{t})}{\partial \bar{t}}(p_i-p_f)\cdot K + \frac{\partial T_a}{\partial \Delta_a}2q_f\cdot K\right] \\ & - Q_A \left[T(\bar{s},\bar{t}) - \frac{\partial T(\bar{s},\bar{t})}{\partial \bar{s}}(q_i+p_i)\cdot K + \frac{\partial T(\bar{s},\bar{t})}{\partial \bar{t}}(p_i-p_f)\cdot K - \frac{\partial T_b}{\partial \Delta_b}2q_i\cdot K\right]\frac{q_{i\mu}}{q_i\cdot K} \\ & + Q_B \frac{p_{f\mu}}{p_f\cdot K}\left[T(\bar{s},\bar{t}) + \frac{\partial T(\bar{s},\bar{t})}{\partial \bar{s}}(q_i+p_i)\cdot K - \frac{\partial T(\bar{s},\bar{t})}{\partial \bar{t}}(p_i-p_f)\cdot K + \frac{\partial T_c}{\partial \Delta_c}2p_f\cdot K\right] \\ & - Q_B \left[T(\bar{s},\bar{t}) - \frac{\partial T(\bar{s},\bar{t})}{\partial \bar{s}}(q_i+p_i)\cdot K - \frac{\partial T(\bar{s},\bar{t})}{\partial \bar{t}}(p_i-p_f)\cdot K - \frac{\partial T_d}{\partial \Delta_d}2p_i\cdot K\right]\frac{p_{i\mu}}{p_i\cdot K} + O(K)\ , \end{aligned} \tag{53}$$

where $T(\bar{s},\bar{t}) \equiv T(\bar{s},\bar{t},m_A^2,m_B^2,m_A^2,m_B^2)$ is the elastic scattering (on-shell) $T$ matrix evaluated at $\bar{s}$ and $\bar{t}$. Now, we impose the gauge invariant condition

$$[M_\mu^E(s,t,\Delta)+M_\mu^I(s,t,\Delta)]K^\mu = 0\ ,$$

which gives

$$M_\mu^I K^\mu = -2(Q_A+Q_B)\frac{\partial T(\bar{s},\bar{t})}{\partial \bar{s}}(q_i+p_i)\cdot K - 2Q_A\frac{\partial T_a}{\partial \Delta_a}q_f\cdot K - 2Q_A\frac{\partial T_b}{\partial \Delta_b}q_i\cdot K - 2Q_B\frac{\partial T_c}{\partial \Delta_c}p_f\cdot K - 2Q_B\frac{\partial T_d}{\partial \Delta_d}p_i\cdot K\ . \tag{54}$$

Hence, the leading term of $M_\mu^I(s,t,\Delta)$ has the form

$$M_\mu^I(s,t,\Delta) = -2(Q_A+Q_B)\frac{\partial T(\bar{s},\bar{t})}{\partial \bar{s}}(q_i+p_i)_\mu - 2Q_A\frac{\partial T_a}{\partial \Delta_a}q_{f\mu} - 2Q_A\frac{\partial T_b}{\partial \Delta_b}q_{i\mu} - 2Q_B\frac{\partial T_c}{\partial \Delta_c}p_{f\mu} - 2Q_B\frac{\partial T_d}{\partial \Delta_d}p_{i\mu}\ . \tag{55}$$

Combining Eqs. (53) and (55), we obtain the total bremsstrahlung amplitude $M_\mu^{\mathrm{Low}(st)}$,

$$M_\mu^{\mathrm{Low}(st)} = M_\mu^{E(st)} + M_\mu^{I(st)}\ , \tag{56}$$

where $M_\mu^{E(st)}$ is the on-shell part of the external amplitude which depends on $\bar{s}$ and $\bar{t}$,

$$M_\mu^{E(st)}=\left[Q_A\left[\frac{q_{f\mu}}{q_f\cdot K}-\frac{q_{i\mu}}{q_i\cdot K}\right]+Q_B\left[\frac{p_{f\mu}}{p_f\cdot K}-\frac{p_{i\mu}}{p_i\cdot K}\right]\right]T(\bar{s},\bar{t})$$
$$+\left[Q_A\left[\frac{q_{f\mu}}{q_f\cdot K}+\frac{q_{i\mu}}{q_i\cdot K}\right]+Q_B\left[\frac{p_{f\mu}}{p_f\cdot K}+\frac{p_{i\mu}}{p_i\cdot K}\right]\right](q_i+p_i)\cdot K\frac{\partial T(\bar{s},\bar{t})}{\partial\bar{s}}$$
$$+\left[Q_A\left[\frac{q_{f\mu}}{q_f\cdot K}-\frac{q_{i\mu}}{q_i\cdot K}\right]-Q_B\left[\frac{p_{f\mu}}{p_f\cdot K}-\frac{p_{i\mu}}{p_i\cdot K}\right]\right](p_i-p_f)\cdot K\frac{\partial T(\bar{s},\bar{t})}{\partial\bar{t}}\ , \tag{57}$$

and $M_\mu^{I(st)}$ is the on-shell part of the internal amplitude which depends on $\bar{s}$ and $\bar{t}$,

$$M_\mu^{I(st)}=-2(Q_A+Q_B)(q_i+p_i)_\mu\frac{\partial T(\bar{s},\bar{t})}{\partial\bar{s}}\ . \tag{58}$$

It is clear that $M_\mu^{I(st)}\varepsilon^\mu$ contributes nothing to the bremsstrahlung cross section, since $(q_i+p_i)_\mu\varepsilon^\mu$ vanishes in the c.m. system and in the Coulomb gauge.

### B. A new Low amplitude $M_\mu^{\mathrm{Low}(ut)}$

A second Low soft-phonon amplitude can be derived if the independent Lorentz invariants are chosen to be $u_j$ $(j=1,2)$, $t_y$ $(y=p,q)$, and $\Delta_z$ $(=a,b,c,d)$. Here, $u_1$ and $u_2$ are defined by Eqs. (11) and (14), respectively. With this choice, the four half-off-shell $T$ matrices are parametrized in terms of $u_j$, $t_y$, and $\Delta_z$ as

$$\begin{aligned}T_a&=T(u_1,t_p,\Delta_a)\ ,\\ T_b&=T(u_2,t_p,\Delta_b)\ ,\\ T_c&=T(u_2,t_q,\Delta_c)\ ,\\ T_d&=T(u_1,t_q,\Delta_d)\ .\end{aligned} \tag{59}$$

The external amplitude has the form

$$\begin{aligned}M_\mu^E(u,t,\Delta)=&Q_A\frac{q_{f\mu}}{q_f\cdot K}T(u_1,t_p,\Delta_a)\\ &-Q_AT(u_2,t_p,\Delta_b)\frac{q_{i\mu}}{q_i\cdot K}\\ &+Q_B\frac{p_{f\mu}}{p_f\cdot K}T(u_2,t_q,\Delta_c)\\ &-Q_BT(u_1,t_q,\Delta_d)\frac{p_{i\mu}}{p_i\cdot K}\ .\end{aligned} \tag{60}$$

Introducing the average $u$,

$$\bar{u}=\tfrac{1}{2}(u_1+u_2)\ , \tag{61}$$

we have

$$u_1=\bar{u}-(p_f-q_i)\cdot K=\bar{u}-(p_i-q_f)\cdot K$$

and

$$u_2=\bar{u}+(p_f-q_i)\cdot K=\bar{u}+(p_i-q_f)\cdot K\ . \tag{62}$$

If we expand all half-off-shell $T$ matrices in Eq. (60) about $[\bar{u},\bar{t},\Delta=(\mathrm{mass})^2]$, we find

$$M_\mu^E(u,t,\Delta)=Q_A\frac{q_{f\mu}}{q_f\cdot K}\left[T(\bar{u},\bar{t})-\frac{\partial T(\bar{u},\bar{t})}{\partial\bar{u}}(p_f-q_i)\cdot K+\frac{\partial T(\bar{u},\bar{t})}{\partial\bar{t}}(p_i-p_f)\cdot K+\frac{\partial T_a}{\partial\Delta_a}2q_f\cdot K\right]$$
$$-Q_A\left[T(\bar{u},\bar{t})+\frac{\partial T(\bar{u},\bar{t})}{\partial\bar{u}}(p_f-q_i)\cdot K+\frac{\partial T(\bar{u},\bar{t})}{\partial\bar{t}}(p_i-p_f)\cdot K-\frac{\partial T_b}{\partial\Delta_b}2q_i\cdot K\right]\frac{q_{i\mu}}{q_i\cdot K}$$
$$+Q_B\frac{p_{f\mu}}{p_f\cdot K}\left[T(\bar{u},\bar{t})+\frac{\partial T(\bar{u},\bar{t})}{\partial\bar{u}}(p_f-q_i)\cdot K-\frac{\partial T(\bar{u},\bar{t})}{\partial\bar{t}}(p_i-p_f)\cdot K+\frac{\partial T_c}{\partial\Delta_c}2p_f\cdot K\right]$$
$$-Q_B\left[T(\bar{u},\bar{t})-\frac{\partial T(\bar{u},\bar{t})}{\partial\bar{u}}(p_f-q_i)\cdot K-\frac{\partial T(\bar{u},\bar{t})}{\partial\bar{t}}(p_i-p_f)\cdot K-\frac{\partial T_d}{\partial\Delta_d}2p_i\cdot K\right]\frac{p_{i\mu}}{p_i\cdot K}+O(K)\ . \tag{63}$$

Here, $T(\bar{u},\bar{t})\equiv T(\bar{u},\bar{t},m_A^2,m_B^2,m_A^2,m_B^2)$ is the elastic (on-shell) $T$ matrix evaluated at $\bar{u}$ and $\bar{t}$. To obtain the leading term of the internal amplitude $M_\mu^I(u,t,\Delta)$, we again impose the gauge invariant condition

$$[M_\mu^E(u,t,\Delta)+M_\mu^I(u,t,\Delta)]K^\mu=0\ , \tag{64}$$

from which we obtain

$$M_\mu^IK^\mu=2(Q_A-Q_B)\frac{\partial T(\bar{u},\bar{t})}{\partial\bar{u}}(p_f-q_i)\cdot K-2Q_A\frac{\partial T_a}{\partial\Delta_a}q_f\cdot K-2Q_A\frac{\partial T_b}{\partial\Delta_b}q_i\cdot K-2Q_B\frac{\partial T_c}{\partial\Delta_c}p_f\cdot K-2Q_B\frac{\partial T_d}{\partial\Delta_d}p_i\cdot K\ . \tag{65}$$

Equation (65) gives

$$M_\mu^I(u,t,\Delta)=2(Q_A-Q_B)\frac{\partial T(\bar{u},\bar{t})}{\partial\bar{u}}(p_f-q_i)_\mu-2Q_A\frac{\partial T_a}{\partial\Delta_a}q_{f\mu}-2Q_A\frac{\partial T_b}{\partial\Delta_b}q_{i\mu}-2Q_B\frac{\partial T_c}{\partial\Delta_c}p_{f\mu}-2Q_B\frac{\partial T_d}{\partial\Delta_d}p_{i\mu}\ . \tag{66}$$

The total bremsstrahlung amplitude $M_\mu^{\mathrm{Low}(ut)}$ is the sum of Eqs. (63) and (66):

$$M_\mu^{\mathrm{Low}(ut)}=M_\mu^{E(ut)}+M_\mu^{I(ut)}\ , \tag{67}$$

where

$$\begin{aligned}M_\mu^{E(ut)}=&\left[Q_A\left[\frac{q_{f\mu}}{q_f\cdot K}-\frac{q_{i\mu}}{q_i\cdot K}\right]+Q_B\left[\frac{p_{f\mu}}{p_f\cdot K}-\frac{p_{i\mu}}{p_i\cdot K}\right]\right]T(\bar{u},\bar{t})\\&+\left[-Q_A\left[\frac{q_{f\mu}}{q_f\cdot K}+\frac{q_{i\mu}}{q_i\cdot K}\right]+Q_B\left[\frac{p_{f\mu}}{p_f\cdot K}+\frac{p_{i\mu}}{p_i\cdot K}\right]\right](p_f-q_i)\cdot K\frac{\partial T(\bar{u},\bar{t})}{\partial\bar{u}}\\&+\left[Q_A\left[\frac{q_{f\mu}}{q_f\cdot K}-\frac{q_{i\mu}}{q_i\cdot K}\right]-Q_B\left[\frac{p_{f\mu}}{p_f\cdot K}-\frac{p_{i\mu}}{p_i\cdot K}\right]\right](p_i-p_f)\cdot K\frac{\partial T(\bar{u},\bar{t})}{\partial\bar{t}}\end{aligned} \tag{68}$$

and

$$M_\mu^{I(ut)}=2(Q_A-Q_B)\frac{\partial T(\bar{u},\bar{t})}{\partial\bar{u}}(p_f-q_i)_\mu\ . \tag{69}$$

Again, $M_\mu^{E(ut)}$ is the on-shell part of the external amplitude which depends on $\bar{u}$ and $\bar{t}$ while $M_\mu^{I(ut)}$ is the on-shell part of the internal amplitude, which depends on $\bar{u}$ and $\bar{t}$. Unlike the internal amplitude $M_\mu^{I(st)}\varepsilon^\mu$ [Eq. (58)], which is identically zero, the internal amplitude $M_\mu^{I(ut)}\varepsilon^\mu$ does not vanish when $Q_A\neq Q_B$. (It should be remembered that we consider specifically an $s$-channel reaction here.) Thus, we see by this simple example that different choices of independent variables (Lorentz invariants) can lead to different soft-photon amplitudes. We shall discuss this further in Sec. V.

### C. The general soft-photon amplitude $M_\mu^{TsTts}$

As we have already mentioned, more general soft-photon amplitudes can be derived by using the modified Low procedure described in Sec. I. In using this new procedure, the construction of the internal amplitude is guided by the elastic scattering and the bremsstrahlung processes at the tree level. For example, if the $A$-$B$ elastic scattering is dominated by the one-particle $s$-channel exchange diagrams, then the internal amplitude will be determined by photon emissions from the $s$-channel exchange particles. (Since a $t$-channel exchange particle should be neutral, there is no internal emission from it.) In this case, we should choose a set of independent Lorentz invariants which includes $s$ and $t$. On the other hand, if the $A$-$B$ elastic scattering is dominated by the one-particle $u$-channel exchange diagrams, then the internal emissions will come from the $u$-channel exchange particles, and we should choose a set of independent Lorentz invariants which includes $u$ and $t$. In this subsection, a general bremsstrahlung amplitude for a process whose elastic scattering is dominated by the diagrams shown in Fig. 1(b)will be derived. [The derivation of another general bremsstrahlung amplitude for a process, whose elastic scattering is dominated by the diagrams shown in Fig. 1(c), will be discussed in IV D.]

Choosing the set of independent Lorentz invariants, which includes $s_x$ $(x=i,f)$, $t_y$ $(y=p,q)$, and $\Delta_z$ $(z=a,b,c,d)$, the external emission amplitude $M_\mu^E(s,t,\Delta)$ is identical to that given by Eq. (50). Because we assume that the elastic scattering depicted in Fig. 1(a) is dominated by the diagrams shown in Fig. 1(b) and, likewise, that the bremsstrahlung processes represented in Fig. 2 are dominated by the diagrams shown in Fig. 3, we can write the internal emission amplitude in the form

$$\begin{aligned}M_\mu^{I(D)}(s,t,\Delta)=&Y_aT(s_i,t_p,\Delta_a)+T(s_f,t_p,\Delta_b)Y_b\\&+Y_cT(s_i,t_q,\Delta_c)+T(s_f,t_q,\Delta_d)Y_d\ ,\end{aligned} \tag{70}$$

where $Y_z$ $(z=a,b,c,d)$ are electromagnetic factors to be specified. To determine $Y_z$, we demand that $M_\mu^{I(D)}$ reduce to the expression for $\overline{M}_\mu^{I(D)}$ given by Eq. (30) when the general diagram in Fig. 2(e) reduces to the tree approximation in Fig. 3(e). Since $T(s_x,t_y,\Delta_z)$ reduces to $\overline{T}(s_x,t_y)$ in the tree approximation in this special case, we find

$$\begin{aligned}Y_a&=-Q_A\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\ ,\\Y_b&=Q_A\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K}\ ,\\Y_c&=-Q_B\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\ ,\\Y_d&=Q_B\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K}\ .\end{aligned} \tag{71}$$

Now, combining $M_\mu^E(s,t,\Delta)$ given by Eq. (50) with $M_\mu^{I(D)}(s,t,\Delta)$ given by Eqs. (70) and (71), we obtain

$$M_\mu^{TsTt}=Q_A\left[\frac{q_{f\mu}}{q_f\cdot K}-\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\right]T(s_i,t_p,\Delta_a)$$
$$-Q_A T(s_f,t_p,\Delta_b)\left[\frac{q_{i\mu}}{q_i\cdot K}-\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K}\right]$$
$$+Q_B\left[\frac{p_{f\mu}}{p_f\cdot K}-\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\right]T(s_i,t_q,\Delta_c)$$
$$-Q_B T(s_f,t_q,\Delta_d)\left[\frac{p_{i\mu}}{p_i\cdot K}-\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K}\right]. \quad (72)$$

Because $M_\mu^{TsTt}$ is already explicitly gauge invariant,

$$M_\mu^{TsTt}K^\mu=0 ,$$

no additional gauge term is needed. The amplitude $M_\mu^{TsTt}$ is an off-shell two-$s$–two-$t$ ($TsTt$) amplitude derived for the $A$-$B$ bremsstrahlung process when internal emission from the $s$-channel exchange particles is dominant. To obtain an on-shell $TsTt$ special amplitude $M_\mu^{TsTts}$, which is free of any derivative of the $T$ matrix with respect to $s$ or $t$, we expand $T(s_x,t_Y,\Delta_z)$ only about the on-shell point (mass)$^2$ in $\Delta_z$:

$$T(s_i,t_p,\Delta_a)=T(s_i,t_p)+\frac{\partial T(s_i,t_p,\Delta_a)}{\partial\Delta_a}2q_f\cdot K ,$$
$$T(s_f,t_p,\Delta_b)=T(s_f,t_p)-\frac{\partial T(s_f,t_p,\Delta_b)}{\partial\Delta_b}2q_i\cdot K ,$$
$$T(s_i,t_q,\Delta_c)=T(s_i,t_q)+\frac{\partial T(s_i,t_q,\Delta_c)}{\partial\Delta_c}2p_f\cdot K ,$$
$$T(s_f,t_q,\Delta_d)=T(s_f,t_q)-\frac{\partial T(s_f,t_q,\Delta_d)}{\partial\Delta_d}2p_i\cdot K , \quad (73)$$

where

$$T(s_i,t_p)\equiv T(s_i,t_p,m_A^2) ,$$
$$T(s_f,t_p)\equiv T(s_f,t_p,m_A^2) ,$$
$$T(s_i,t_q)\equiv T(s_i,t_q,m_B^2) ,$$

and

$$T(s_f,t_q)\equiv T(s_f,t_q,m_B^2) .$$

Inserting Eq. (73) into Eq. (72) gives

$$M_\mu^{TsTt}=M_\mu^{TsTts}+M_\mu^{\text{off}(st)} , \quad (74)$$

where

$$M_\mu^{TsTts}=Q_A\left[\frac{q_{f\mu}}{q_f\cdot K}-\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\right]T(s_i,t_p)$$
$$-Q_A T(s_f,t_p)\left[\frac{q_{i\mu}}{q_i\cdot K}-\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K}\right]$$
$$+Q_B\left[\frac{p_{f\mu}}{p_f\cdot K}-\frac{(q_f+p_f)_\mu}{(q_f+p_f)\cdot K}\right]T(s_i,t_q)$$
$$-Q_B T(s_f,t_q)\left[\frac{p_{i\mu}}{p_i\cdot K}-\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K}\right] \quad (75)$$

and $M_\mu^{\text{off}(st)}$ represents those terms involving off-shell derivatives of the $T$ matrix. The amplitude $M_\mu^{\text{off}(st)}$ is neglected in the soft-photon approximation. The soft-photon amplitude $M_\mu^{TsTts}$ is the on-shell $TsTt$ special amplitude and is more general than the amplitude $M_\mu^{\text{Low}(st)}$ given by Eq. (56), the soft-photon amplitude derived by using Low's standard procedure. To see this point, let us rewrite $M_\mu^{TsTts}$ in two parts, an external term $M_\mu^{E(TsTt)}$ and an internal term $M_\mu^{I(TsTt)}$:

$$M_\mu^{E(TsTt)}=Q_A\frac{q_{f\mu}}{q_f\cdot K}T(s_i,t_p)-Q_A T(s_f,t_p)\frac{q_{i\mu}}{q_i\cdot K}$$
$$+Q_B\frac{p_{f\mu}}{p_f\cdot K}T(s_i,t_q)-Q_B T(s_f,t_q)\frac{p_{i\mu}}{p_i\cdot K} \quad (76)$$

and

$$M_\mu^{I(TsTt)}=-\{Q_A[T(s_i,t_p)-T(s_f,t_p)]$$
$$+Q_B[T(s_i,t_q)-T(s_f,t_q)]\}\frac{(q_i+p_i)_\mu}{(q_i+p_i)\cdot K} \quad (77)$$

in a manner analogous to Eqs. (57) and (58). Here, we have used the fact that $(q_i+p_i)_\mu\varepsilon^\mu=(q_f+p_f)_\mu\varepsilon^\mu$ and $(q_i+p_i)\cdot K=(q_f+p_f)\cdot K$. [Again, the amplitude $M_\mu^{I(TsTt)}$ vanishes in the c.m. system and the Coulomb gauge, since $M_\mu^{I(TsTt)}\varepsilon^\mu$ is proportional to a factor $(q_i+p_i)_\mu\varepsilon^\mu$.] If we use Eq. (52) to expand all $T$ matrices in Eqs. (76) and (77) about $(\bar{s},\bar{t})$, then we can prove that

$$M_\mu^{E(TsTt)}=M_\mu^{E(st)}+O(K) , \quad (78a)$$
$$M_\mu^{I(TsTt)}=M_\mu^{I(st)}+O(K) . \quad (78b)$$

Here, $\bar{s}$ and $\bar{t}$ are defined by Eq. (51), $M^{E(st)}$ is the external term given by Eq. (57), and $M^{I(st)}$ is the internal term given by Eq. (58). Equations (78a) and (78b) show clearly that the amplitude $M_\mu^{\text{Low}(st)}$, derived by using Low's standard procedure, is a special case of the amplitude $M_\mu^{TsTts}$. In other words, $M_\mu^{E(TsTt)}$ reduces to $M_\mu^{E(st)}$ and $M_\mu^{I(TsTt)}$ reduces to $M_\mu^{I(st)}$ when $T$ matrices, $T(s_x,t_y)$, in the expressions for $M_\mu^{E(TsTt)}$ and $M_\mu^{I(TsTt)}$ are expanded about $(\bar{s},\bar{t})$ and the $O(K)$ term is neglected. It should be emphasized that if $T$ matrices $T(s_x,t_y)$ vary rapidly with $s_x$ and/or $t_y$ in the vicinity of a resonance, then the expansion of $T(s_x,t_y)$ about $(\bar{s},\bar{t})$, which is the essential step in the derivation of the amplitude $M_\mu^{\text{Low}(st)}$, is obviously not valid. In that case, the amplitude $M_\mu^{TsTts}$, which is free of $\partial T/\partial s$ and/or $\partial T/\partial t$, is the proper choice. In fact, recent studies reveal that the amplitude $M_\mu^{TsTts}$ (or more precisely the special two-energy–two-angle amplitude $M_\mu^{\rm TETAS}$) can be used to describe almost all the available $p\,^{12}{\rm C}\gamma$ data (near both the 1.7 and 0.5-MeV resonances) and $\pi^{\pm}p\gamma$ data [near the $\Delta(1232)$ resonance]. These studies also show that the amplitude $M_\mu^{{\rm Low}(st)}$ fails to describe both data adequately.

### D. The general soft-photon amplitude $M_\mu^{TuTts}$

In this subsection, we derive a second general bremsstrahlung amplitude, in the soft-photon approximation, for a process whose elastic scattering is dominated by the diagrams shown in Fig. 1(c). Since photon emission from the $u$-channel exchange particles, $F_j$, are involved, we choose the set of independent Lorentz invariants that includes $u_j$ $(j=1,2)$, $t_y$ $(y=p,q)$, and $\Delta_z$ $(=a,b,c,d)$. The external emission amplitude is identical to the amplitude $M_\mu^E(u,t,\Delta)$ given by Eq. (60). Since Fig. 1(a) is now dominated by Fig. 1(c) and Fig. 2 is dominated by Fig. 4, the internal emission amplitude can be written as

$$M_\mu^{I(F)}(u,t,\Delta)=X_aT(u_1,t_p,\Delta_a)+T(u_2,t_p,\Delta_b)X_b + X_cT(u_2,t_q,\Delta_c)+T(u_1,t_q,\Delta_d)X_d\ , \quad (79)$$

where $X_z$ $(z=a,b,c,d)$ are coefficients to be specified. They can be uniquely determined if we demand that $M_\mu^{I(F)}$ reduces to $\overline{M}_\mu^{I(F)}$ [given by Eq. (45)], with $T(u_j,t_y,\Delta_z)$ reduces to $\overline{T}(u_j,t_y)$, when Fig. 2(e) reduces to Fig. 4(e). We find

$$\begin{aligned} X_a&=-Q_A\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}\ ,\\ X_b&=Q_A\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}\ ,\\ X_c&=-Q_B\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}\ ,\\ X_d&=Q_B\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}\ . \end{aligned} \quad (80)$$

Now, combining $M_\mu^E(u,t,\Delta)$ given by Eq. (60) with $M_\mu^{I(F)}(u,t,\Delta)$ given by Eqs. (79) and (80), we obtain

$$\begin{aligned} M_\mu^{TuTt}=&\,Q_A\left[\frac{q_{f\mu}}{q_f\cdot K}-\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}\right]T(u_1,t_p,\Delta_a)\\ &-Q_AT(u_2,t_p,\Delta_b)\left[\frac{q_{i\mu}}{q_i\cdot K}-\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}\right]\\ &+Q_B\left[\frac{p_{f\mu}}{p_f\cdot K}-\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}\right]T(u_2,t_q,\Delta_c)\\ &-Q_BT(u_1,t_q,\Delta_d)\left[\frac{p_{i\mu}}{p_i\cdot K}-\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}\right]\ . \end{aligned} \quad (81)$$

Again, no additional gauge term is required, since $M_\mu^{TuTt}$ is manifestly gauge invariant; that is,

$$M_\mu^{TuTt}K^\mu=0\ .$$

The amplitude $M_\mu^{TuTt}$ is an off-shell, two-$u$–two-$t$ $(TuTt)$ amplitude that can be derived by using the modified Low procedure for the $A$-$B$ bremsstrahlung process when internal emission from the $u$-channel exchange particles is dominant. To find an on-shell $TuTt$ special amplitude $M_\mu^{TuTts}$, we first expand $T(u_j,t_y,\Delta_z)$:

$$\begin{aligned} T(u_1,t_p,\Delta_a)&=T(u_1,t_p)+\frac{\partial T(u_1,t_p,\Delta_a)}{\partial\Delta_a}2q_f\cdot K\ ,\\ T(u_2,t_p,\Delta_b)&=T(u_2,t_p)-\frac{\partial T(u_2,t_p,\Delta_b)}{\partial\Delta_b}2q_i\cdot K\ ,\\ T(u_2,t_q,\Delta_c)&=T(u_2,t_q)+\frac{\partial T(u_2,t_q,\Delta_c)}{\partial\Delta_c}2p_f\cdot K\ ,\\ T(u_1,t_q,\Delta_d)&=T(u_1,t_q)-\frac{\partial T(u_1,t_q,\Delta_d)}{\partial\Delta_d}2p_i\cdot K\ , \end{aligned} \quad (82)$$

where

$$T(u_1,t_p)\equiv T(u_1,t_p,m_A^2)\ ,$$
$$T(u_2,t_p)\equiv T(u_2,t_p,m_A^2)\ ,$$
$$T(u_2,t_q)\equiv T(u_2,t_q,m_B^2)\ ,$$

and

$$T(u_1,t_q)\equiv T(u_1,t_q,m_B^2)\ .$$

We then substitute Eq. (82) into Eq. (81) to obtain

$$M_\mu^{TuTt}=M_\mu^{TuTts}+M_\mu^{{\rm off}(ut)}\ , \quad (83)$$

where

$$\begin{aligned} M_\mu^{TuTts}=&\,Q_A\left[\frac{q_{f\mu}}{q_f\cdot K}-\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}\right]T(u_1,t_p)\\ &-Q_AT(u_2,t_p)\left[\frac{q_{i\mu}}{q_i\cdot K}-\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}\right]\\ &+Q_B\left[\frac{p_{f\mu}}{p_f\cdot K}-\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}\right]T(u_2,t_q)\\ &-Q_BT(u_1,t_q)\left[\frac{p_{i\mu}}{p_i\cdot K}-\frac{(p_i-q_f)_\mu}{(p_i-q_f)\cdot K}\right] \end{aligned} \quad (84)$$

and $M_\mu^{{\rm off}(ut)}$ includes those terms that involve off-shell derivatives of the $T$ matrix. Again, the off-shell amplitude $M_\mu^{{\rm off}(ut)}$ is ignored in the soft-photon approximation. The amplitude $M_\mu^{TuTts}$ is the on-shell $TuTt$ special amplitude, which should be used when internal emission from the $u$-channel exchange particles, $F_j$, are important. It is easy to demonstrate that $M_\mu^{TuTts}$ given by Eq. (84) is more general than the amplitude $M_\mu^{{\rm Low}(ut)}$ given by Eq. (67). Again, we divide the amplitude $M_\mu^{TuTts}$ into two parts, an external term $M_\mu^{E(TuTt)}$ and an internal term $M_\mu^{I(TuTt)}$:

$$M_\mu^{E(TuTt)}=Q_A\frac{q_{f\mu}}{q_f\cdot K}T(u_1,t_p)-Q_A T(u_2,t_p)\frac{q_{i\mu}}{q_i\cdot K}+Q_B\frac{p_{f\mu}}{p_f\cdot K}T(u_2,t_q)-Q_B T(u_1,t_q)\frac{p_{i\mu}}{p_i\cdot K} \tag{85}$$

and

$$M_\mu^{I(TuTt)}=-\{Q_A[T(u_1,t_p)-T(u_2,t_p)]+Q_B[T(u_2,t_q)-T(u_1,t_q)]\}\frac{(q_i-p_f)_\mu}{(q_i-p_f)\cdot K}\ . \tag{86}$$

Here, we have used the following relation:

$$\frac{(p_i-q_f)_\mu \varepsilon^\mu}{(p_i-q_f)\cdot K}=\frac{(q_i-p_f)_\mu \varepsilon^\mu}{(q_i-p_f)\cdot K}\ .$$

If we use Eqs. (52) and (62) to expand all $T$ matrices in Eqs. (85) and (86) about $(\bar{u},\bar{t})$, then we obtain

$$M_\mu^{E(TuTt)}=M_\mu^{E(ut)}+O(K) \tag{87a}$$

and

$$M_\mu^{I(TuTt)}=M_\mu^{I(ut)}+O(K)\ . \tag{87b}$$

Here, $M^{E(ut)}$ is the external term given by Eq. (68), and $M^{I(ut)}$ is the internal term given by Eq. (69), and we have used the relation, $(q_i-q_f)\cdot K=-(p_i-p_f)\cdot K$. Eqs. (87a) and (87b) demonstrate that $M_\mu^{E(TsTt)}$ and $M_\mu^{I(TuTt)}$ reduce to the Low amplitudes $M_\mu^{E(ut)}$ and $M_\mu^{I(ut)}$, respectively, if the $T(u_j,t_y)$ in Eqs. (85) and (86) are expanded about $(\bar{u},\bar{t})$ and if $O(K)$ terms are neglected.

To summarize briefly, we have derived four soft-photon amplitudes, $M_\mu^{\text{Low}(st)}(\bar{s},\bar{t})$, $M_\mu^{\text{Low}(ut)}(\bar{u},\bar{t})$, $M_\mu^{TsTts}(s_i,s_f;t_p,t_q)$, and $M_\mu^{TuTts}(u_1,u_2;t_p,t_q)$. $M_\mu^{\text{Low}(st)}(\bar{s},\bar{t})$ and $M_\mu^{\text{Low}(ut)}(\bar{u},\bar{t})$ were derived using Low's standard procedure, while $M_\mu^{TsTts}(s_i,s_f;t_p,t_q)$ and $M_\mu^{TuTts}(u_1,u_2;t_p,t_q)$ were derived using a modified Low procedure. The amplitudes $M_\mu^{\text{Low}(st)}$ and $M_\mu^{TsTts}$ depend on a set of Lorentz invariants that include $s$ and $t$. The amplitudes $M_\mu^{\text{Low}(ut)}$and $M_\mu^{TuTts}$, on the other hand, are parametrized in terms of Lorentz invariants $u$ and $t$. In deriving $M_\mu^{TsTts}(s_i,s_f;t_p,t_q)$, we have imposed a condition that it reduce to the amplitude $\bar{M}_\mu^{TsTts}(s_i,s_f;t_p,t_q)$, which represents photon emission from a sum of one-particle $t$-channel exchange diagrams and one-particle $s$-channel exchange diagrams (the tree approximation). Similarly, in our derivation of the amplitude $M_\mu^{TuTts}(u_1,u_2;t_p,t_q)$, we have imposed the condition that it reduce to the tree approximation amplitude $\bar{M}_\mu^{TuTts}(u_1,u_2;t_p,t_q)$, which represents photon emission from a sum of one-particle $t$-channel exchange diagrams and one-practice $u$-channel exchange diagrams. Note that the expressions for $\bar{M}_\mu^{TsTts}$ and $\bar{M}_\mu^{TuTts}$ were derived in last section by using the radiation decomposition identities of Brodsky and Brown. We have proved that $M_\mu^{\text{Low(st)}}$ and $M_\mu^{\text{Low}(ut)}$ can be reproduced from $M_\mu^{TsTts}$ and $M_\mu^{TuTts}$, respectively; furthermore, the amplitudes $M_\mu^{TsTts}$ and $M_\mu^{TuTts}$ are the most general soft-photon amplitudes for hadron-hadron bremsstrahlung processes which can be constructed by using the modified Low procedure. Finally, it is easy to show that the amplitudes $M_\mu^{\text{Low}(st)}$ and $M_\mu^{\text{Low}(ut)}$ and the amplitudes $M_\mu^{TsTts}$ and $M_\mu^{TuTts}$ can be interchanged when $p_i^\mu$, $p_f^\mu$, and $Q_B$ are replaced by $-p_f^\mu$, $-p_i^\mu$, and $-Q_B$, respectively. The relationships among the amplitudes $\bar{M}_\mu^{TsTts}$, $\bar{M}_\mu^{TuTts}$, $M_\mu^{\text{Low}(st)}$, $M_\mu^{\text{Low}(ut)}$, $M_\mu^{TsTts}$, and $M_\mu^{TuTts}$ are illustrated in Fig. 5.

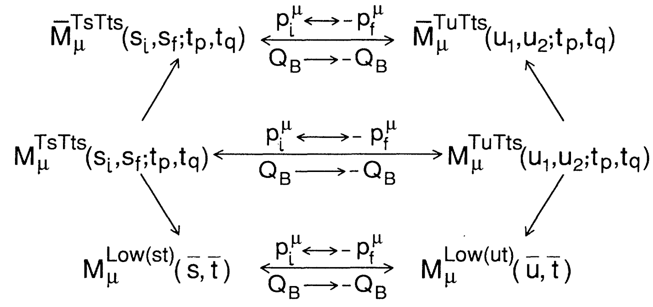


FIG. 5. Schematic representation of the relations among the six soft-photon amplitudes derived in this work. Five important relations are shown here: (i) These six amplitudes can be divided into two independent classes ($M_\mu^{TsTts}$, $\bar{M}_\mu^{TsTts}$, $M_\mu^{\text{Low}(st)}$) as the first class and ($M_\mu^{TuTts}$, $\bar{M}_\mu^{TuTts}$, $M_\mu^{\text{Low}(ut)}$) as the second class. (ii) The general amplitudes for the first and second classes are $M_\mu^{TsTts}$ and $M_\mu^{TuTts}$, respectively. (iii) In the tree-level approximation, $M_\mu^{TsTts}$ reduces to $\bar{M}_\mu^{TsTts}$, while $M_\mu^{TuTts}$ reduces to $\bar{M}_\mu^{TuTts}$. (iv) When all $T$ matrices in $M_\mu^{TsTts}$ are expanded about $(\bar{s},\bar{t})$ and all $T$ matrices in $M_\mu^{TuTts}$ are expanded about $(\bar{u},\bar{t})$, then $M_\mu^{\text{Low}(st)}$ and $M_\mu^{\text{Low}(ut)}$ can be obtained. (v) The two classes of amplitude can be interchanged ($\bar{M}_\mu^{TsTts}\leftrightarrow\bar{M}_\mu^{TuTts}$, $M_\mu^{TsTts}\leftrightarrow M_\mu^{TuTts}$, $M_\mu^{\text{Low}(st)}\leftrightarrow M_\mu^{\text{Low}(ut)}$) when $Q_B$ is replaced by $-Q_B$ ($Q_B\to -Q_B$) and $p_i^\mu$ is interchanged with $-p_f^\mu$ ($p_i^\mu\leftrightarrow -p_f^\mu$).

## V. DISCUSSION

Six soft-photon amplitudes, $\bar{M}_\mu^{TsTts}$ [Eq. (39)], $\bar{M}_\mu^{TuTts}$ [Eq. (46)], $M_\mu^{\text{Low}(st)}$ [Eq. (56)], $M_\mu^{\text{Low}(ut)}$ [Eq. (67)], $M_\mu^{TsTts}$ [Eq. (75)], and $M_\mu^{TuTts}$ [Eq. (84)], have been derived in Secs. III and IV. A primary purpose of this investigation is to explicate their relationships and to explore their ranges of validity. These six amplitudes can be divided into two classes: (i) $\bar{M}^{TsTts}$, $M^{\text{Low}(st)}$, and $M_\mu^{TsTts}$ as the first class $[M_\mu^{(1)}(s,t)]$ and (ii) $\bar{M}_\mu^{TuTts}$, $M_\mu^{\text{Low}(ut)}$, and $M_\mu^{TuTts}$ as the second class $[M_\mu^{(2)}(u,t)]$. As shown in Fig. 5, the following relationships have already been established: (a) $M_\mu^{TsTts}$ and $M_\mu^{TuTts}$ reduce to $\bar{M}_\mu^{TsTts}$ and $\bar{M}_\mu^{TuTts}$, respectively, in the tree level approximation. (b) If $M_\mu^{TsTts}$ is expanded about $(\bar{s},\bar{t})$ and $M_\mu^{TuTts}$ is expanded about $(\bar{u},\bar{t})$, assuming that such expansions are valid, then the first two terms of the expansions for $M_\mu^{TsTts}$ and $M_\mu^{TuTts}$ give $M_\mu^{\text{Low}(st)}$ and $M_\mu^{\text{Low}(ut)}$, respectively. (c) If $p_i^\mu\to -p_f^\mu$, $p_f^\mu\to -p_i^\mu$ and $Q_B\to -Q_B$, then $\bar{M}_\mu^{TsTts}\to\bar{M}_\mu^{TuTts}$, $M_\mu^{\text{Low}(st)}\to M_\mu^{\text{Low}(ut)}$ and $M_\mu^{TsTts}\to M_\mu^{TuTts}$, and vice versa.

Now, let us consider the question about their ranges of validity. Which amplitude, $M_\mu^{TsTts}$ or $M_\mu^{TuTts}$, should be used to describe a particular bremsstrahlung measurement? The answer will depend upon the nature of the bremsstrahlung process. Let us examine three cases.

(a) For a process whose elastic scattering is dominated by the tree diagrams shown in Fig. 1(b) or whose internal emission is dominated by the diagrams shown in Fig. 3(e), we must use the amplitude $M_\mu^{TsTts}$ for bremsstrahlung calculations. That is, when the process is resonance dominated, $M_\mu^{TsTts}$ is the correct choice. Some well-known examples are the $\pi^\pm p\gamma$ processes near the $\Delta(1232)$ resonance [9], the $p\,^{12}\mathrm{C}\gamma$ process near either the 1.7-MeV resonance or the 461-keV resonance [7], and the $p\,^{16}\mathrm{O}\gamma$ process near the 2.66-MeV resonance [8]. These radiative resonant scattering processes have been systematically studied both experimentally and theoretically. The following findings illustrate why the amplitude $M_\mu^{TsTts}$, not $M_\mu^{\mathrm{Low}(st)}$, should be used to describe bremsstrahlung processes involving a resonance: (i) Using a one-energy–two-angle amplitude, which is slightly different from the amplitude $M_\mu^{\mathrm{Low}(st)}$, a UCLA group has calculated the $\pi^\pm p\gamma$ cross sections in order to compare with the cross sections measured by the group [25]. The UCLA calculations have been repeated but using the amplitude $M_\mu^{\mathrm{Low}(st)}$ [14,15]. These two independent calculations yield essentially the same result. Typically, the calculated spectra at 298 MeV, for example, rise steeply with increasing photon energy above $K=80$ MeV in complete disagreement with the experimental data. The amplitude $M_\mu^{\mathrm{Low}(st)}$ has also been used to calculate the $p\,^{12}\mathrm{C}\gamma$ cross sections at 1.88 MeV for a scattering angle of 155° [14,15]. The calculated cross sections show a large resonance peak around $K=270$ keV in stark contrast with the small peak observed experimentally around $K=135$ keV. In short, neither the $\pi^\pm p\gamma$ nor the $p\,^{12}\mathrm{C}\gamma$ data can be described by the amplitude $M_\mu^{\mathrm{Low}(st)}$ or any other one-energy amplitude. These studies also show that the terms which involve $\partial T/\partial\bar{s}$ and/or $\partial T/\partial\bar{t}$ cause the problem. This is because the elastic $T$ matrix, which has been used as an input for bremsstrahlung calculations in the soft-photon approximation, varies rapidly with $s$ and/or $t$ in the vicinity of a resonance. In other words, the problem is directly related to the invalid expansions of the four half-off-shell $T$ matrices about $(\bar{s},\bar{t})$ [or about $(s_{\alpha\beta}, t_{\alpha'\beta'})$, where $s_{\alpha\beta}=(\alpha s_i+\beta s_f)/(\alpha+\beta)$ and $t_{\alpha'\beta'}=(\alpha' t_p+\beta' t_q)/(\alpha'+\beta')$], which are used in Low's standard procedure for the derivation of $M_\mu^{\mathrm{Low}(st)}$ and other one-energy amplitudes. These expansions give rise to those terms which depend upon $\partial T/\partial\bar{s}$ and $\partial T/\partial\bar{t}$ in all one-energy amplitudes. (ii) From the amplitude $M_\mu^{TsTts}$ one may define an amplitude called the special two-energy–two-angle (TETAS) amplitude $M_\mu^{\mathrm{TETAS}}$, which is free of $\partial T/\partial s$ and/or $\partial T/\partial t$. The amplitude $M_\mu^{\mathrm{TETAS}}$ has been thoroughly tested and has been found to describe the data well for bremsstrahlung processes near a scattering resonance. For example, $M_\mu^{\mathrm{TETAS}}$ has been successfully applied to extract the magnetic moments of the $\Delta^{++}(1232)$ [9] and $\Delta^0(1232)$ [13] from the experimental $\pi^+ p\gamma$ data and $\pi^- p\gamma$ data, respectively. It is now well established that this amplitude can be used to describe almost all available $p\,^{12}\mathrm{C}\gamma$ and $\pi^\pm p\gamma$ data. Furthermore, a direct, sensitive experimental test of various soft-photon amplitudes was made recently by the Brooklyn group [7]. This test showed that the amplitude $M_\mu^{\mathrm{TETAS}}$ provides an excellent description of the $p\,^{12}\mathrm{C}\gamma$ data not only in the soft-photon region but also in the hard-photon region.

(b) For a process whose elastic scattering is dominated by the tree diagrams shown in Fig. 1(c) or whose internal emission is dominated by the diagrams shown in Fig. 4(e), $M_\mu^{TuTts}$ should be used for bremsstrahlung calculations. That is, when the process is exchange-current dominated, $M_\mu^{TuTts}$ is optimal. An example of this is neutron-proton bremsstrahlung ($np\gamma$): (i) In the one-boson-exchange model, the $np$ interaction involves the $u$-channel exchange of charged bosons. (ii) The $np\gamma$ cross section is dominated by the internal emission from the exchanged bosons. More precisely, Brown and Franklin have calculated the $np\gamma$ cross sections using a nonrelativistic potential model [11]. The electromagnetic Hamiltonian used by these authors includes the coupling of the electromagnetic field to the nucleon currents $V_{\mathrm{em}}^1$ and the coupling of the electromagnetic field to the exchange currents $V_{\mathrm{em}}^2$. As a result, large exchange effects from $V_{\mathrm{em}}^2$ were predicted. The inclusion of the $V_{\mathrm{em}}^2$ term has been found to increase the $np\gamma$ cross section by about a factor of 2. This finding has been confirmed very recently by Nakayama [12]. (iii) The $np\gamma$ cross sections at 200 MeV have been calculated by Baier, Kuhnelt, and Urban [26] using a one-boson-exchange model and by Nyman [27] using a soft-photon amplitude derived via Low's standard procedure. The amplitude used by Baier, Kuhnelt, and Urban is equivalent to the amplitude $\overline{M}_\mu^{TuTts}$, while the amplitude used by Nyman is equivalent to $M_\mu^{\mathrm{Low}(st)}$. When those two calculations are compared, one can see that the $np\gamma$ cross sections obtained by Baier, Kuhnelt, and Urban are consistently a factor of 1.8–2 times larger than those obtained by Nyman. The obvious explanation of this result is that the amplitude $M_\mu^{\mathrm{Low}(st)}$ does not contain any exchange effect, since we have shown above that its internal contribution is identically zero, while the amplitude $\overline{M}_\mu^{TuTts}$ used by Baier, Kuhnelt, and Urban does include a nonzero internal contribution from all charged bosons. [Note that the internal contribution of the amplitudes $M_\mu^{TsTts}$ and $M_\mu^{\mathrm{Low}(st)}$ involves a factor of the form $(q_i+p_i)_\mu \varepsilon^\mu$, which vanishes in the c.m. system and in the Coulomb gauge.] Thus, the finding of Brown and Franklin that the internal exchange contribution dominates the $np\gamma$ cross section could also have been observed by comparing the relativistic calculations of Nyman and Baier *et al.* The one-boson-exchange calculations of Baier, Kuhnelt, and Urban are in much better agreement with the experimental data of Brady and Young [28] than many other calculations. This illustrates why the amplitude $M_\mu^{TuTts}$, not the amplitude $M_\mu^{TsTts}$ (or $M_\mu^{\mathrm{Low}(st)}$), should be used for $np\gamma$ calculations.

(c) For a process that involves little resonance effect (i.e., it contains no resonant state or is observed in an energy region far from resonance) and has very little contribution from exchange effects (those due to the $u$-channel exchange particles), we expect all six amplitudes $\overline{M}_\mu^{TsTts}$, $M_\mu^{TsTts}$, $M_\mu^{\mathrm{Low}(st)}$, $\overline{M}_\mu^{TuTts}$, $M_\mu^{TuTts}$, and $M_\mu^{\mathrm{Low}(ut)}$ to yield similar results, at least in the soft-photon region. This does not mean that they will give identical results but that the differences should not be large. A typical example is proton-proton bremsstrahlung ($pp\gamma$): (i) As we have already mentioned, there is no internal contribution from the amplitudes $\overline{M}_{\mu}^{TsTts}$, $M_{\mu}^{TsTts}$, and $M_{\mu}^{\text{Low}(st)}$, since it vanishes in the c.m. system and in the Coulomb gauge. If $M_{\mu}^{TsTts}$ is expanded about $(\bar{s},\bar{t})$, we obtain

$$M_{\mu}^{TsTts}=M_{\mu}^{\text{Low}(st)}+O(K)\ ,$$

which is exactly the sum of Eqs. (78a) and (78b). Here, $O(K)$ involves the derivatives of $T$ matrix with respect to $\bar{s}$ and $\bar{t}$. If there is no resonance effect, then derivatives of $T$ with respect to $\bar{s}$ and $\bar{t}$ will not produce significant structure and such an expansion is valid. Hence, the contribution from the $O(K)$ term will be small, and we expect the amplitudes $M_{\mu}^{TsTts}$ and $M_{\mu}^{\text{Low}(st)}$ to give similar results. (ii) For a process that has very little contribution from exchange effects, the amplitude $M_{\mu}^{TuTts}$ may be expanded about $(\bar{u},\bar{t})$. We find

$$M_{\mu}^{TuTts}=M_{\mu}^{\text{Low}(ut)}+O(K)$$

which is identical to the sum of Eqs. (87a) and (87b). Again if the derivatives of $T$ with respect to $\bar{u}$ and $\bar{t}$ are small, we expect that the contribution from the $O(K)$ term will be small. Therefore, the amplitudes $M_{\mu}^{TuTts}$ and $M_{\mu}^{\text{Low}(ut)}$ should predict similar cross sections. (iii) From Eq. (69), we can see that the internal amplitude $M_{\mu}^{I(ut)}$ (of the amplitude $M_{\mu}^{\text{Low}(ut)}$) contributes nothing if $Q_A=Q_B$. Thus, like the amplitude $M_{\mu}^{\text{Low}(st)}$, there is no internal contribution from $M_{\mu}^{\text{Low}(ut)}$ for the $pp\gamma$ process. We therefore do not expect that the $pp\gamma$ cross sections calculated using the external part of the amplitude $M_{\mu}^{\text{Low(st)}}$ to be very different from those calculated using the external part of the amplitude $M_{\mu}^{\text{Low}(ut)}$. (iv) The $pp\gamma$ process has been extensively studied, both experimentally and theoretically, during the last three decades. Many different calculations (based on various models and approximations), including a soft-photon approach, which uses an amplitude equivalent to $M_{\mu}^{\text{Low}(st)}$ and a one-boson-exchange approach, which uses an amplitude equivalent to $\overline{M}_{\mu}^{TuTts}$, have been performed. The results of these calculations do differ, but their differences are indeed not large [29]. (v) Since two-nucleon interactions have been successfully described by the one-boson-exchange model, we expect the difference between $\overline{M}_{\mu}^{TsTts}$ and $M_{\mu}^{TsTts}$ or the difference between $\overline{M}_{\mu}^{TuTts}$ and $M_{\mu}^{TuTts}$ to be small when these amplitudes are applied to predict the $pp\gamma$ cross sections.

## VI. SUMMARY AND CONCLUSIONS

In conclusion, the primary purpose of this work is to point out that there exist at least two independent classes of soft-photon amplitudes, both of which are equally important for describing hadron-hadron bremsstrahlung processes. The two-$s$-two-$t$ special amplitude $M_{\mu}^{TuTts}(s_i,s_f;t_p,t_q)$, Eq. (75), is the general amplitude for the first class, and this amplitude should be used to describe those processes which are resonance dominated. The two-$u$–two-$t$ special amplitude $M_{\mu}^{TuTts}(u_1,u_2;t_p,t_q)$, Eq. (84), is the general amplitude for the second class, and it should be used to describe those processes, which are exchange current dominated. These two amplitudes can be derived using a modified Low procedure, but not the standard (Low's original) procedure. The modified procedure involves one additional step, which allows us to take into account photon emission from the internal line by imposing the condition that $M_{\mu}^{TsTts}$ and $M_{\mu}^{TuTts}$ reduce to $\overline{M}_{\mu}^{TsTts}$ and $\overline{M}_{\mu}^{TuTts}$, respectively, at the tree level approximation. The $\overline{M}_{\mu}^{TsTts}$ and $\overline{M}_{\mu}^{TuTts}$ amplitudes can be rigorously derived from the relevant set of fundamental bremsstrahlung diagrams at the tree level, if we apply the radiation decomposition identities of Brodsky and Brown to decompose the internal amplitude into four quasiexternal amplitudes.

If $M_{\mu}^{TsTts}$ is expanded about $(\bar{s},\bar{t})$ and $M_{\mu}^{TuTts}$ is expanded about $(\bar{u},\bar{t})$, assuming that such expansions are valid, the first two terms of the expansions yield $M_{\mu}^{\text{Low}(st)}(\bar{s},\bar{t})$ and $M_{\mu}^{\text{Low}(ut)}(\bar{u},\bar{t})$, respectively. Here, $M_{\mu}^{\text{Low}(st)}$ is a one-$s$–one-$t$ (or one-energy–one-angle) amplitude, a typical Low amplitude, which can be derived using the standard procedure. This amplitude has been regarded as the sole soft-photon amplitude in the past, and it has been applied to describe all possible bremsstrahlung processes without justification. In addition to exploring why $M_{\mu}^{\text{Low}(st)}$ cannot be used to describe processes containing significant resonance effects, we also demonstrated why it should fail to describe those processes with large exchange effects. The amplitude $M_{\mu}^{\text{Low}(ut)}$, on the other hand, is a one-$u$–one-$t$ amplitude. It is a new Low amplitude, which can also be derived by using the standard procedure. This new amplitude has never before been studied.

We have demonstrated that we can transform the soft-photon amplitudes in the first class ($M_{\mu}^{TsTts}$, $\overline{M}_{\mu}^{TsTts}$, $M_{\mu}^{\text{Low}(st)}$) into the soft-photon amplitudes in the second class ($M_{\mu}^{TuTts}$, $\overline{M}_{\mu}^{TuTts}$, $M_{\mu}^{\text{Low}(ut)}$) by making the following variable transformations: $p_i^{\mu}\leftrightarrow -p_f^{\mu}$ and $Q_B\rightarrow -Q_B$. This establishes the relationship between the two independent classes.

Many amplitudes, especially those in the second class, discussed in this work are new. Their ranges of validity and other properties are not well understood. Further systematic studies are required to understand these amplitudes thoroughly. These studies should include comparison with new experimental work, since the ultimate test of the utility of these soft-photon amplitudes lies in a comparison between the theoretical predictions and the experimental data.


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[21] In general, all $\overline{T}$ matrices in Eq. (26) should be half-off-shell $T$ matrices. That is, we should have $\overline{T}(s_i,t_p,\Delta_a)$, $\overline{T}(s_f,t_p,\Delta_b)$, $\overline{T}(s_i,t_q,\Delta_c)$, and $\overline{T}(s_f,t_q,\Delta_d)$ rather than $\overline{T}(s_i,t_p)$, $\overline{T}(s_f,t_p)$, $\overline{T}(s_i,t_q)$, and $\overline{T}(s_f,t_q)$, respectively, to indicate that one of the four legs is off its mass shell. We have ignored the $\Delta_z$ $(z=a,b,c,d)$ dependence here simply because none of the $\overline{T}$ matrices explicitly depend upon $\Delta_z$ in this special case.

[22] As indicated in Ref. [21] (footnote), in general all $\overline{T}$ matrices in Eq. (39) may depend not only on $s_x$ $(x=i,j)$ and $t_y$ $(y=p,q)$ but also on $\Delta_z$ $(z=a,b,c,d)$. In that case, the amplitude given by Eq. (39) should be classified as the two-$s$-two-$t$ amplitude $M_\mu^{TsTt}$. If $M_\mu^{TsTt}$ is expanded and all off-shell terms are ignored, then the first two terms of the expansion define the amplitude $M_\mu^{TsTts}$. The expression for $M_\mu^{TsTts}$ will be exactly the same as that given by Eq. (39).

[23] In this special case, none of the $\overline{T}$ matrices in Eq. (41) depend on $\Delta_z$ $(z=a,b,c,d)$ even though $\overline{T}$ may depend on $\Delta_z$ in general. See Ref. [21].

[24] If all $\overline{T}$ matrices in Eq. (46) depend on $\Delta_z$ $(z=a,b,c,d)$, then the amplitude given by Eq. (46) should be classified as the two-$u$-two-$t$ amplitude $M_\mu^{TuTt}$. Expanding $M_\mu^{TuTt}$ and neglecting all off-shell terms, the amplitude $M_\mu^{TuTts}$, which has exactly the same expression given by Eq. (46) (i.e., no $\overline{T}$ matrix depends on $\Delta_z$), is defined by the first two terms of the expansion.

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