Approximate seniority-dictated boson-quasifermion mapping and derivation of the interacting boson fermion model

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We derive an approximate boson-quasifermion mapping of a single-j shell-model algebra, applicable to the seniority classification scheme, as an expansion in the ratio of the seniority quantum number to the level degeneracy, with selective summation of higher order terms. The procedure used is an extension of the algebraic technique first applied to the purely bosonic part of the mapping by Bonatsos, Klein, and Li. Only the maps of the generators are needed to derive a Hamiltonian for the interacting boson fermion model (IBFM), and these can be obtained in Hermitian form without having to go through the intermediary of a mapping of the fermion operators. Application of the results is illustrated by a fresh discussion of the so-called exchange interaction of the IBFM. A consistent mapping of the single fermion in terms of bosons and quasifermions is also derived that includes a consideration of the unusual properties of the quasifermions.

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I. INTRODUCTION

Two general methods, each based on a notion of core-particle coupling, remain in wide use for the theoretical study of collective motion in odd nuclei. The older core-particle coupling model (CPCM) [1], based on the geometrical model for the core collective motion, has been improved [2] by the incorporation of microscopic features suggested by the equation of motion method (for recent reviews, see [3, 4]). One can defend the view that the resulting models are the most sophisticated ones we possess at the moment for the study of complex odd nuclei.

At the same time, the interacting boson fermion model (IBFM) [5, 6], which we view as a semimicroscopic extension to odd nuclei of the phenomenological IBM, has proved a very popular alternative to the CPCM. Indeed, the distinction between the two has become somewhat blurred in recent years. For example, it is now common to use the results of IBM analyses to describe core properties in CPCM calculations [4, 7]. On the other hand, the basic microscopic justification of the IBFM resides in the observation that it contains the same physics, mapped to a boson-fermion basis [8], as a shell-model Hamiltonian in which the residual interaction is a sum of pairing, quadrupole-quadrupole, and quadrupole-pairing interactions. This conclusion did not appear full-blown when the IBFM was introduced. In particular, the importance of the quadrupole-pairing interaction as the origin of the main part of the so-called exchange interaction was only gradually affirmed [9–12].

The present work was stimulated by the belief that though the physical conclusions reached by the above-cited work are unlikely to be modified, except in detail, there remains room for technical improvement in the derivations. The original discussions of the IBFM [13] were based on the application of the mapping technique of Otsuka, Arima, and Iachello (OAI) [5] to the single-fermion operator, resulting in an approximate boson-quasifermion mapping of the latter. In this approach the maps of generators of the shell-model algebra, needed for the study of the Hamiltonian, are obtained by coupling the products of fermion maps. The main deficiency of this method as applied thus far is in the accuracy of the single-fermion maps.

A second approach [12] circumvents this difficulty by constructing directly a boson-quasifermion mapping of the generators. This is done in two steps. First one records the exact but non-Hermitian Dyson mapping, without doubt the one that of all mappings is the simplest to obtain, but one that is not appropriate for use with the physics of seniority conservation. The second step is to carry out a special similarity transformation [14] to a seniority basis. This can be done only in successive approximations, and it is apparent from the results that the basis is not orthonormal. Nevertheless, it was possible to give a convincing discussion of the exchange interaction. Separate treatments of the single-fermion operator [15] and of the quadrupole operator [16] followed, the discussion of the exchange interaction having required only the quadrupole-pairing interaction.

In this paper, we extend the method used first by Bonatsos *et al.* [17] to obtain the seniority dictated boson mapping of the single-j shell-model algebra. (For discussion of and references to later work of this type consult Ref. [8].) By this means we derive the corresponding boson-quasifermion mapping of the generators to the first nontrivial order. The method consists of expanding the generators that break seniority in a series of successively more complex tensor operators in the boson-quasifermion space, each multiplied by a scalar operator, often referred to as a Pauli reduction factor. The number of terms is restricted by established selection rules on angular momentum, number conservation, and seniority change. Also a definite rule can be stated for the order of magnitude of each term in reciprocal powers of Ω, the level degeneracy. Equations for the unknown Pauli reduction factors are obtained by demanding that the Lie algebra commutators be satisfied to a consistent order in Ω^{-1}. From the technical point of view, the main task is to evaluate successively more complicated commutators of products of angular-momentum coupled operators with each other. For this purpose a general recursive formula has been developed and applied [18]. This program is carried out in Sec. II for the pair and multipole operators for angular momentum, $J = 2$, though some attention must be also given to multipole operators for $J = 1, 3, 4$.

In Sec. III, we revisit the problem of the exchange interaction. In addition to obtaining the result in the standard form, a simple angular-momentum recoupling suggests an alternative interpretation that the contribution of the original quadrupole pairing interaction to the boson-fermion coupling can be viewed predominantly as the sum of two direct interactions, of quadrupole-quadrupole and hexadecapole-hexadecapole type, respectively, together with smaller contributions of dipole and octupole character.

In Sec. IV, we return to the purely theoretical problem of obtaining the mapping of the single-fermion operators. As was the case for the generators we obtain a result that is slightly more complete than that found in previous work, but not without noticing certain peculiarities associated with our approximations that limit the accuracy with which the product of maps is equal the map of products. In Sec. V, we summarize our findings and suggest avenues for additional investigation. A brief appendix contains the main algebraic formula used for the rearrangement of multiple coupled operators.

II. MAPPING OF GENERATORS

We confine our attention in this paper to the single-j shell model. The associated algebra of bifermion operators, which is SO$(2(2j+1))$, is generated by the multipole and pair operators,

$$\mathcal{A}^\dagger_{JM} = \sqrt{\frac{1}{2}}(\alpha^\dagger \times \alpha^\dagger)_{JM}, \tag{2.1}$$

$$\mathcal{B}_{JM} = \frac{1}{\hat{j}}(\alpha^\dagger \times \tilde{\alpha})_{JM}, \tag{2.2}$$

where we have used a standard notation for the coupling of two spherical tensors to a resultant angular momentum, with $\alpha^\dagger_{jm} \equiv \alpha^\dagger_m$ a shell-model creation operator for the mode jm, α_m the corresponding annihilation operator, $\hat{J} = \sqrt{2J+1}$, and

$$\tilde{\alpha}_m = (-1)^{j-m}\alpha_{-m}. \tag{2.3}$$

We also utilize the definitions

$$\mathcal{A}_{JM} = [\mathcal{A}^\dagger_{JM}]^\dagger, \tag{2.4}$$

$$\tilde{\mathcal{A}}_{JM} = (-1)^{J-M}\mathcal{A}_{J\,-M}, \tag{2.5}$$

and recognize the condition

$$\mathcal{B}^\dagger_{JM} = (-1)^M \mathcal{B}_{J\,-M}. \tag{2.6}$$

The pair and multipole operators given above satisfy the following nonvanishing commutation relations, presented in angular-momentum coupled form,

$$[\tilde{\mathcal{A}}_J, \mathcal{A}^\dagger_{J'}]_{J''M''} = \hat{J}\delta_{JJ'}\delta_{J''0} + \hat{J}\hat{J}'\hat{J}'' \begin{Bmatrix} J & J' & J'' \\ j & j & j \end{Bmatrix} \mathcal{B}_{J''M''}, \tag{2.7}$$

$$[\mathcal{B}_J, \mathcal{A}^\dagger_{J'}]_{J''M''} = 2\hat{J}' \begin{Bmatrix} J & J' & J'' \\ j & j & j \end{Bmatrix} \mathcal{A}^\dagger_{J''M''}, \tag{2.8}$$

$$[\mathcal{B}_J, \mathcal{B}_{J'}]_{J''M''} = (-1)^{J''}[1-(-1)^{J+J'+J''}]\hat{J}'' \begin{Bmatrix} J & J' & J'' \\ j & j & j \end{Bmatrix} \mathcal{B}_{J''M''}, \tag{2.9}$$

where the curly brackets are the usual 6-j symbols.

We seek an *approximate* realization or mapping of these relations in terms of a restricted set of objects, comprising bosons and quasifermions. Collective effects are to be described in terms of the customary s and d bosons of IBM, satisfying the standard commutation relations, whereas single-particle effects are to be described in terms of quasifermions, designated as $a^\dagger_m, a_m$, whose algebraic properties will be assumed or derived as needed. (The set of bosons and quasifermions chosen will be referred to as mapping operators.) In the present section, we shall assume that the quasifermions are kinematically independent of the bosons (cf. Sec. IV, however), a property that they do not share with the shell-model fermions. For the mapping in question, we shall have need for the multipole operators formed from the quasifermions according to the definition that parallels (2.2),

$$B_{JM} = \frac{1}{\hat{j}}(a^\dagger \times \tilde{a})_{JM}. \tag{2.10}$$

We assume that these operators obey Eq. (2.9). We remark, furthermore, that this property is guaranteed provided the commutation relations

$$[\tilde{a}, B_J]_{j'm'} = -\frac{1}{\hat{j}}\tilde{a}_{jm'}\delta_{jj'} \tag{2.11}$$

are satisfied, all this despite the fact that the quasifermions will turn out to obey anomalous anticommutation relations.

For all cases where exact mappings are known [8], there is no contradiction between these statements. In fact, the stimulus for these assumptions comes from such examples. Two extreme cases are germane to the single-j shell model. At one end, we have the example of the so-called quantized Bogoliubov-Valatin transformation, where one introduces only an s boson, the quasifermion being required to describe all other degrees of freedom [19]. This division of labor is expressed by a subsidiary condition on the quasifermions, forbidding them to form a pair with angular-momentum zero, namely,

$$A^\dagger_0 \equiv (a^\dagger \times a^\dagger)_0 = 0. \tag{2.12}$$

At the other extreme, one bosonizes all fermion pair degrees of freedom. In this case there can be only a single quasifermion of any kind [20], a condition that is expressed by the set of subsidiary conditions, valid for all J and M,

$$A^{\dagger}_{JM} \equiv (a^{\dagger} \times a^{\dagger})_{JM} = 0. \tag{2.13}$$

Both of these examples are associated with exact subalgebras that are decisive in determining the structure of the solutions. In the first case it is the angular-momentum zero or quasispin subalgebra,

$$[\mathcal{A}_0, \mathcal{A}^{\dagger}_0] = 1 - (\mathcal{N}/\Omega), \tag{2.14}$$

$$[\mathcal{N}, \mathcal{A}^{\dagger}_0] = 2\mathcal{A}^{\dagger}_0, \tag{2.15}$$

where $\mathcal{N} = -\sqrt{2\Omega}\mathcal{B}_0$ is the operator for the number of shell-model particles and $2\Omega = 2j+1$. In the second case, it is the full unitary subalgebra of multipole operators described by Eq. (2.9).

It is a complication that for the choice of mapping operators made in the present work there is no exact subalgebra. In the work of Bonatsos and co-workers [17, 8], it is shown that to increase the accuracy of the approximate boson mapping, it is necessary, successively, to add bosons carrying higher values of the angular momentum. Since the properties of the quasifermions depend on the number and character of the associated bosons, as we have already illustrated above in the disparate cases of (2.12) and (2.13), this means that the character of the quasifermion is not fixed, but must successively be modified as we change the level of approximation. It may be helpful to picture this process as one of successive interpolation between the two exact limiting cases that we have described above.

We turn now to the main task of this section. It is to obtain approximate mappings for the bifermion shell-model operators of angular-momentum zero and two. We begin with an appropriate exact solution of the quasispin subalgebra (2.14) and (2.15) that identifies our mapping as a seniority-dictated one, namely,

$$\mathcal{A}^{\dagger}_0 = s^{\dagger}\sqrt{r}, \tag{2.16}$$

$$r = 1 - \frac{\hat{n}_0 + 2\hat{n}_2 + \hat{n}}{\Omega}, \tag{2.17}$$

$$\hat{n}_0 = s^{\dagger}s, \tag{2.18}$$

$$\hat{n}_2 = \sum_{\mu} d^{\dagger}_{\mu}d_{\mu}, \tag{2.19}$$

$$\hat{n} = \sum_{m} a^{\dagger}_{m}a_{m}, \tag{2.20}$$

$$\mathcal{N} = 2\hat{n}_0 + 2\hat{n}_2 + \hat{n}. \tag{2.21}$$

These formulas establish the pattern, to be used throughout this work, of utilizing the same symbol for a shell-model operator and for its boson-fermion map.

We consider next the quadrupole pair operator, for which we assume the approximate form

$$\mathcal{A}^{\dagger}_{2\mu} = d^{\dagger}_{\mu}f_1 + (s^{\dagger})^2\tilde{d}_{\mu}\tilde{f}_1 + s^{\dagger}(d^{\dagger} \times \tilde{d})_{2\mu}f_2 + s^{\dagger}B_{2\mu}f_3 + \sum_{J=1}^{4}(d^{\dagger} \times B_J)_{2\mu}f_{4J} + \sum_{J=1}^{4}(\tilde{d} \times B_J)_{2\mu}\tilde{f}_{4J}, \tag{2.22}$$

where each scalar operator f is allowed to be a function $f(n_0, n_2, n)$ of the occupation numbers alone. The expression as a whole is seen to be a sum of independent tensors that each carry angular-momentum two, increase the fermion number by two, and have a definite seniority selection rule in accordance with the fact that the pair operator may either increase the seniority by two units, decrease it by two units, or leave it unchanged. Each such tensor is multiplied by a scalar operator, f, the so-called Pauli reduction factor, whose value remains to be determined. These are indicated to be functions of the scalar number operators, whose hats have been suppressed. In principle, we should allow the Pauli factors to depend on other number-conserving, seniority-conserving scalars, of which the simplest would appear to be

$$S_1 \equiv (d^{\dagger} \times d^{\dagger})_0(\tilde{d} \times \tilde{d})_0. \tag{2.23}$$

However, we can omit such contributions from the present discussion because they will enter with higher powers of $(\Omega)^{-1}$ than we shall allow. The rule for ascertaining the truth of this assertion that applies to the structure of (2.22) is the following: The factor f_1 associated with the d-boson creation operator turns out to be of leading order unity. For every additional boson operator or every additional quasifermion multipole operator appearing as a factor in one of the elementary tensors, there will be a reduction in the value of the corresponding scalar factor by at least $(1/\sqrt{\Omega})$. Thus in (2.22) we have written down all possible terms, up to relative order $(1/\sqrt{\Omega})$ compared to the leading term, that depend on s bosons, d bosons, and quasifermions. Here we assume that s is of order $\sqrt{\Omega}$, whereas d is of order unity. In ascertaining the correctness of the given form, we must remember that quasifermion pair degrees of freedom carrying zero and two units of angular momentum are required to vanish, according to Eq. (2.12) and to the corresponding equation for angular-momentum two,

$$A^{\dagger}_{2M} = 0. \tag{2.24}$$

We turn our attention to the problem of determining the Pauli factors. The exact dependence on the number of s bosons is given by the Wigner-Eckart theorem, that in our case is expressed by the equation

$$[\mathcal{A}^{\dagger}_0, \mathcal{A}^{\dagger}_{2\mu}] = 0. \tag{2.25}$$

A calculation that involves only the commutation of elementary operators with number operators yields the results

$$f_1 = \sqrt{r\left(r - \frac{1}{\Omega}\right)}\phi_1(n_2, n), \tag{2.26}$$

$$\tilde{f}_1 = \tilde{\phi}_1(n_2, n), \tag{2.27}$$

$$f_2 = \sqrt{r}\phi_2(n_2, n), \tag{2.28}$$
$$f_3 = \sqrt{r}\phi_3(n_2, n), \tag{2.29}$$
$$f_{4J} = \sqrt{r\left(r - \frac{1}{\Omega}\right)}\phi_{4J}(n_2, n), \tag{2.30}$$
$$\tilde{f}_{4J} = \tilde{\phi}_{4J}(n_2, n). \tag{2.31}$$

The next step is to determine the quadrupole operator $\mathcal{B}_{2\mu}$ from the commutation relation

$$[\mathcal{A}_0, \mathcal{A}_{2\mu}] = \sqrt{\frac{10}{\Omega}}\mathcal{B}_{2\mu}, \tag{2.32}$$

where we have inserted the value of a simple 6-j symbol.

After obtaining a preliminary value for the right-hand side of the above equation, by then imposing Eq. (2.6), we obtain the further restrictions on the Pauli factors,

$$\tilde{\phi}_1 = -\frac{1}{\Omega}\phi_1(n_2 - 1), \tag{2.33}$$
$$\tilde{\phi}_{4J} = -\frac{1}{\Omega}\phi_{4J}(n_2 - 1). \tag{2.34}$$

(With these equations we initiate our subsequent practice of suppressing the arguments of the Pauli factors, except when we wish to indicate that they take on other than a reference value.)

We thus find

$$-\sqrt{\frac{10}{\Omega}}\mathcal{B}_{2\mu} = \frac{2}{\Omega}d^\dagger_\mu s\sqrt{r}\phi_1 + \frac{2}{\Omega}s^\dagger\tilde{d}_\mu\sqrt{r + \frac{1}{\Omega}}\phi_1(n_2 - 1) - (d^\dagger \times \tilde{d})_{2\mu}\left(r - \frac{n_0}{\Omega}\right)\phi_2 - B_{2\mu}\left(r - \frac{n_0}{\Omega}\right)\phi_3$$
$$+\frac{2}{\Omega}\sum_{J=1}^{4}(d^\dagger \times B_J)_{2\mu}s\sqrt{r}\phi_{4J} + \frac{2}{\Omega}s^\dagger\sum_{J=1}^{4}(\tilde{d} \times B_J)_{2\mu}\sqrt{r + \frac{1}{\Omega}}\phi_{4J}(n_2 - 1). \tag{2.35}$$

The final sequence of steps in the evaluation of the Pauli factors, ϕ, starts with the calculation of the commutator $[\mathcal{A}_{2\mu}, \mathcal{A}^\dagger_{2\nu}]$ directly from the expression Eq. (2.22) for $\mathcal{A}^\dagger$ and its Hermitian conjugate. Subsequently we equate this result to the right-hand side of the corresponding member of Eq. (2.7), in terms of the required multipole operators. As a consequence we shall also need to have some information concerning the structure of the multipole operators in the range $J = 1, ..., 4$, a question considered below. We also need the updated form of the pair destruction operator,

$$\tilde{\mathcal{A}}_{2\mu} = \sqrt{r\left(r - \frac{1}{\Omega}\right)}\phi_1\tilde{d}_\mu - \frac{1}{\Omega}\phi_1(n_2 - 1)d^\dagger_\mu s^2 + \sqrt{r}\phi_2(d^\dagger \times \tilde{d})_{2\mu}s + \sqrt{r}\phi_3 sB_{2\mu}$$
$$+\sum_{J=1}^{4}\sqrt{r\left(r - \frac{1}{\Omega}\right)}\phi_{4J}(-1)^J(\tilde{d} \times B_J)_{2\mu} - \frac{1}{\Omega}\sum_{J=1}^{4}\phi_{4J}(n_2 - 1)(-1)^J(d^\dagger \times B_J)_{2\mu}s^2. \tag{2.36}$$

In calculating the commutator of (2.36) with $\mathcal{A}^\dagger_{2\nu}$, it is important to observe that the only reliable terms are those proportional to $\delta_{\mu\nu}$, to $d^\dagger_{\nu-\mu}s$ and its Hermitian conjugate, and to terms linear in $B_{J\mu-\nu}$. The coefficients of other tensors, such as those that can be constructed from $(d^\dagger \times \tilde{d})$ or normal-ordered products of two factors of B_J, that occur as a result of the algebra, cannot be used because of the absence of contributions of the same size that would arise from commutators involving terms not included in the approximate mapping. It is the nature of our approach that the reliable terms will prove sufficient to determine the unknown operators, as will be seen. To identify independent tensors, the technique is to arrange all tensors into normal form and to discard higher order terms. It is simplest to present the result in angular-momentum coupled form, in which the magnetic quantum number has been suppressed:

$$[\tilde{\mathcal{A}}_2, \mathcal{A}^\dagger_2]_J = \left[r\left(r - \frac{1}{\Omega}\right) - \frac{n_0(n_0 - 1)}{\Omega^2}\right]\phi_1^2\sqrt{5}\delta_{J0} + s^\dagger\tilde{d}\left(r + \frac{n_0}{\Omega}\right)\sqrt{r + \frac{1}{\Omega}}\phi_1(n_2 - 1)\phi_2\delta_{J2}$$
$$+d^\dagger s\left(r + \frac{n_0}{\Omega} - \frac{2}{\Omega}\right)\sqrt{r}\phi_1\phi_2(n_2 + 1)\delta_{J2} + 2B_J(-1)^J\frac{\sqrt{5}}{\hat{J}}\left[r\left(r - \frac{1}{\Omega}\right) - \frac{n_0(n_0 - 1)}{\Omega^2}\right]\phi_1\phi_{4J}$$
$$+\hat{J}B_J\begin{Bmatrix} 2 & j & j \\ j & 2 & J \end{Bmatrix}\left[(-1)^J(n_0 + 1)r - n_0\left(r + \frac{1}{\Omega}\right)\right]\phi_3^2$$
$$+5\hat{J}B_J\sum_{J',J''=1}^{4}(-1)^{J'+J''}\begin{Bmatrix} J' & J'' & J \\ 2 & 2 & 2 \end{Bmatrix}\begin{Bmatrix} J' & J'' & J \\ j & j & j \end{Bmatrix} \times \left[r\left(r - \frac{1}{\Omega}\right) - (-1)^J\frac{n_0(n_0 - 1)}{\Omega^2}\right]\phi_{4J'}\phi_{4J''}. \tag{2.37}$$

In the derivation of this result in the most expeditious manner, we have made use of a recursive formula for computing the commutator of two products of operators, each coupled to a good angular momentum [18], which is described in a brief appendix.

In order to utilize the results just presented, we need representations for the corresponding terms in the multipole operators for $J = 1,...,4$. The quadrupole operator has already been given in Eq. (2.36) more accurately than actually required in the current calculation. We need only the purely quasifermion parts of the odd multipoles. These are known exactly for the odd multipoles, which necessarily have the form

$$J \text{ odd}: \quad \mathcal{B}_{JM} = B_{JM} + \text{ (boson terms)}, \tag{2.38}$$

and the boson terms can be determined from the fact that the odd multipoles satisfy a closed subalgebra. Since they involve tensors not utilized in the present calculation, no further details will be given here. Finally, we note that the hexadecapole will have the form

$$\mathcal{B}_{4\mu} = \sqrt{\frac{\Omega}{10}} B_{4\mu}\left(r - \frac{n_0}{\Omega}\right)\phi_3 + \text{ (boson terms)}. \tag{2.39}$$

The specific term shown here is the analogue of the corresponding term of the quadrupole operator, Eq. (2.35). That such a term must be present can be concluded by concentrating on the quasifermion contribution to the commutator $[(-1)^{\mu}\mathcal{B}_{2\,-\mu}, \mathcal{B}_{3\nu}]$ that contains the term $\mathcal{B}_{4\,\nu-\mu}$.

We now have all the ingredients necessary to compare the two ways of computing the commutator Eq. (2.8). For the purely bosonic terms, we are simply reproducing the results of Bonatsos *et al.* By comparing the coefficients of $\delta_{\mu\nu}$, we find

$$\phi_1 = \frac{1}{\sqrt{r_2 - \frac{1}{\Omega}}}, \quad r_2 = 1 - \frac{2n_2 + n}{\Omega}. \tag{2.40}$$

From the coefficient of the operator $d^\dagger s$, we can derive

$$\phi_2 = \frac{10}{r_2}\sqrt{\frac{2}{\Omega}}\begin{Bmatrix} 2 & 2 & 2 \\ j & j & j \end{Bmatrix}. \tag{2.41}$$

We turn then to the equations for the new coefficients ϕ_3 and ϕ_{4J}. We obtain two sets of equations, the first valid for $J = 1, 3$,

$$2(2J+1)\sqrt{5}\begin{Bmatrix} 2 & 2 & J \\ j & j & j \end{Bmatrix} = 2\left[r_2\left(r_2 - \frac{1}{\Omega}\right) - 2\left(\frac{n_0}{\Omega}\right)\left(r_2 - \frac{1}{\Omega}\right)\right]\phi_1\phi_{4J} + \frac{(2J+1)}{\sqrt{5}}\left(r_2 + 2n_0 r_2 - \frac{2n_0^2}{\Omega}\right)\begin{Bmatrix} 2 & 2 & J \\ j & j & j \end{Bmatrix}\phi_3^2 - M_J, \tag{2.42}$$

$$M_J = (2J+1)\sqrt{5}\sum_{J'J''}(-1)^{J'+J''}\left[r_2\left(r_2 - \frac{1}{\Omega}\right) - \frac{2n_0 r_2}{\Omega} - (-1)^J\frac{2n_0^2}{\Omega^2}\right]\begin{Bmatrix} J & J' & J'' \\ j & j & j \end{Bmatrix}\begin{Bmatrix} J & J' & J'' \\ 2 & 2 & 2 \end{Bmatrix}\phi_{4J'}\phi_{4J''}. \tag{2.43}$$

For $J = 2, 4$, on the other hand, we find

$$-2(2J+1)\sqrt{5}\begin{Bmatrix} 2 & 2 & J \\ j & j & j \end{Bmatrix}\sqrt{\frac{\Omega}{10}}\left(r_2 - \frac{2n_0}{\Omega}\right)\phi_3 = 2\left[r_2\left(r_2 - \frac{1}{\Omega}\right) - 2\left(\frac{n_0}{\Omega}\right)\left(r_2 - \frac{1}{\Omega}\right)\right]\phi_1\phi_{4J} + \frac{(2J+1)}{\sqrt{5}}\left(r_2 - \frac{2n_0}{\Omega}\right)\begin{Bmatrix} 2 & 2 & J \\ j & j & j \end{Bmatrix}\phi_3^2 + M_J. \tag{2.44}$$

In addition to the equations that have been recorded above, Eq. (2.37) contains a contribution proportional to $n\delta_{\mu\nu}$. We can deal with this term in one of two equivalent ways. Either we can consider it to modify the value of ϕ_1 or else we can introduce a term ϕ_{40}, chosen so as to cancel these extra terms. It can be verified *a posteriori* that in any event this term is smaller by one order in (n/Ω) than the accuracy that we shall maintain for our result, and therefore we do not pursue this question to the end.

We proceed to solve Eqs. (2.42)–(2.44) to leading order in $(1/\Omega)$. In order to do this we must recognize (by examining a table of elementary 6-j coefficients) that for large j,

$$\begin{Bmatrix} 2 & 2 & J \\ j & j & j \end{Bmatrix} = \begin{cases} O\left(\frac{1}{\sqrt{\Omega}}\right), & J \text{ even}, \\ O\left(\frac{1}{\sqrt{\Omega^3}}\right), & J \text{ odd}. \end{cases} \tag{2.45}$$

For $J = 1$ and from the terms independent of n_0, the dominant terms of (2.42) are

$$6\sqrt{5}\begin{Bmatrix} 2 & 2 & 1 \\ j & j & j \end{Bmatrix} = 2r_2^2\phi_1\phi_{41}. \tag{2.46}$$

To leading order in $(1/\Omega)$ this has the solution

$$\phi_{41} = 3\sqrt{5}\begin{Bmatrix} 2 & 2 & 1 \\ j & j & j \end{Bmatrix}\frac{1}{r_2^{3/2}}, \tag{2.47}$$

which shows that ϕ_{41} is $O(\Omega^{-3/2})$.

From the terms proportional to n_0, again for $J = 1$, we obtain in leading order an equation to determine ϕ_3, namely,

$$0 = -4r_2\phi_1\phi_{41} + \frac{3}{\sqrt{5}}\begin{Bmatrix} 2 & 2 & 1 \\ j & j & j \end{Bmatrix}2\Omega r_2\phi_3^2, \tag{2.48}$$

of which a solution to leading order, with a definite and consistent choice of phase, is

$$\phi_3 = \sqrt{\frac{10}{\Omega}\frac{1}{r_2}}. \qquad (2.49)$$

A similar analysis may be applied for the other values of J, using Eqs. (2.42)–(2.44). One thereby verifies the consistency of Eq. (2.49) and also obtains the values of the remaining factors ϕ_{4J}. The results may be summarized neatly by the formula

$$\phi_{4J} = (-1)^{J+1}(2J+1)\sqrt{5}\begin{Bmatrix} 2 & 2 & J \\ j & j & j \end{Bmatrix}\frac{1}{r_2^{3/2}}. \qquad (2.50)$$

We have thus obtained mappings at the indicated level of accuracy for the bifermion operators of angular-momentum zero and two, as well as more limited information about the multipole operators up to $J = 4$. In the next section we shall apply these results to the study of the boson-fermion exchange interaction.

III. BOSON-QUASIFERMION INTERACTION FROM A QUADRUPOLE-PAIRING HAMILTONIAN

In this section we apply the results of the previous section by deriving the boson-quasifermion interaction that follows from the quadrupole pairing interaction

$$H_{2P} = \sum_\mu \mathcal{A}_{2\mu}^\dagger \mathcal{A}_{2\mu}, \qquad (3.1)$$

where the coupling constant has been omitted. If we enumerate the terms of the ingredient operators as follows

$$\mathcal{A}_{2\mu}^\dagger = 1' + 2' + 3' + 4' + 5' + 6', \qquad (3.2)$$

$$\mathcal{A}_{2\mu} = 1 + 2 + 3 + 4 + 5 + 6, \qquad (3.3)$$

where the various terms can be read off from Eqs. (2.22) and (2.36), respectively, inspection shows that the leading order contributions to the exchange interaction involve the eight products $(1'4), (2'4), (4'1), (4'2), (1'5), (2'6), (5'1)$, and $(6'2)$. These decompose into sets that are Hermitian conjugate to one another, a property that we shall use in the combinations

$$(4'1) + (2'4) = [(1'4) + (4'2)]^\dagger, \qquad (3.4)$$

$$(5'1) + (2'6) = [(1'5) + (6'2)]^\dagger. \qquad (3.5)$$

The evaluation of the terms above is elementary. In the following presentation we shall discard $(1/\Omega)$ compared to unity, but retain quantities of order (n_2/Ω) and (n/Ω), at the same time remembering that (n_0/Ω) is itself of order unity. We thus find

$$(1'4) + (4'2) = [(d^\dagger s) \times (a^\dagger \times \tilde{a})_2]_0 \left(r_2 - \frac{2n_0}{\Omega}\right)\sqrt{r_2 - \frac{n_0}{\Omega}}\,\phi_1\phi_3, \qquad (3.6)$$

$$(1'5) + (6'2) = \sum_{J=1}^{4}[(d^\dagger \times \tilde{d})_J \times B_J]_0(-1)^J\sqrt{5}\left[r^2 + (-1)^J\frac{n_0^2}{\Omega^2}\right]\phi_1\phi_{4J}. \qquad (3.7)$$

In order to reach (3.7), we have utilized the recoupling statement

$$\{d^\dagger \times [(a^\dagger \times \tilde{a})_J \times \tilde{d}]_2\}_0 = (-1)^J\{(a^\dagger \times \tilde{a})_J \times (d^\dagger \times \tilde{d})_J\}_0. \qquad (3.8)$$

The quantities ϕ_1, ϕ_3, and ϕ_{4J} are given, respectively, in Eqs. (2.41), (2.49), and (2.50).

The contributions, Eq. (3.6), plus Hermitian conjugate were pointed out by Bijker and Scholten [9], but are claimed to be small by Otsuka *et al.* [10] (by an argument that bears reexamination). The interaction that is usually identified as the exchange interaction is that contained in Eq. (3.7), but is normally presented in a different form, requiring a further recoupling that is described below, following the current discussion. The form we have given above does not seem to have interested previous authors, but remembering that the coefficients ϕ_{4J}, for $J = 1, 3$ are smaller by order $(1/\Omega)$ compared to those for $J = 2, 4$, we reach the conclusion that the sum of (3.6) and (3.7) is equivalent to the sum of *direct* interactions of quadrupole-quadrupole and of hexadecapole-hexadecapole types.

We turn then to the further recoupling necessary to exhibit (3.7) in the customary form of exchange interaction. The first step involves the statement

$$\{d^\dagger \times [(a^\dagger \times \tilde{a})_J \times \tilde{d}]_2\}_0 = \sum_{J'}(-1)^{j+J'}\sqrt{2J'+1}\begin{Bmatrix} j & j & J \\ 2 & 2 & 2 \end{Bmatrix} : \{(a^\dagger \times \tilde{d})_{J'} \times (\tilde{a} \times d^\dagger)_{J'}\}_0 :, \qquad (3.9)$$

where the colons, as usual, indicate normal ordering. As the consequence (3.7) becomes

$$(1'5) + (6'2) = \sum_{J,J'}(-1)^{J+j+J'}\sqrt{5(2J'+1)}\begin{Bmatrix} j & j & J \\ 2 & 2 & 2 \end{Bmatrix} : \{(a^\dagger \times \tilde{d})_{J'} \times (\tilde{a} \times d^\dagger)_{J'}\}_0 : \left[r^2 + (-1)^J\frac{n_0^2}{\Omega^2}\right]\phi_1\phi_{4J}. \qquad (3.10)$$

With the aid of the explicit form of the coefficients ϕ_{4J}, Eq. (2.50), we can carry out the summation over J. This is done most conveniently if we include a $J=0$ term in the summation. In that case, we may use the standard sum rules for 6-j symbols that apply, respectively, to the two terms of Eq. (3.10). In accordance with our resolve to retain only the leading order contributions, we can calculate from a table of elementary 6-j symbols that

$$\begin{Bmatrix} j\,2\,j \\ j\,2\,j \end{Bmatrix} \cong -\frac{1}{2\Omega}. \tag{3.11}$$

This is the largest of the 6-j quantities that results from the application of the sum rules; the symbol where in the upper right-hand corner j is replaced by $J'=j\pm 1$ is smaller by order $(1/\Omega)$, and for $J'=\pm 2$, smaller still by the same factor. To the stated accuracy, we thus obtain an explicit form for the exchange interaction

$$\begin{aligned}(1'5)+(6'2) = {} & \frac{5}{\sqrt{2\Omega}}\left[\frac{r^2}{r_2^2}+\left(\frac{n_0}{\Omega}\right)^2\frac{1}{r_2^2}\right] \\ & \times :[(a^\dagger\times\tilde{d})_j\times(\tilde{a}\times d^\dagger)_j]_0: \\ & +\frac{1}{2\Omega}nn_2\left[\frac{r^2}{r_2^2}+\left(\frac{n_0}{\Omega}\right)^2\frac{1}{r_2^2}\right]. \end{aligned} \tag{3.12}$$

To obtain the total contribution, we must multiply this result by a factor of 2. Of the two terms of (3.12), the first one is of the standard form found in the previous literature [12], except that the dependence on boson and quasifermion occupation numbers does not agree with that provided by previous authors. We ascribe that difference to the more complete and consistent solution that we have given to the mapping problem. The second term, of the form of a monopole interaction between the core and quasifermion, is the compensation for the extra piece that had to be provided to permit the 6-j sum rules to be used. In addition to these terms, there are, as we mentioned, contributions of higher order in $(1/\Omega)$ that we have omitted. Though we have thus substantiated the qualitative results, there would be some quantitative differences in application to the analysis of data.

IV. MAPPING OF FERMIONS

One of the aims of this paper has been to show how to derive an Hermitian boson-fermion mapping without having first to introduce a mapping for the individual shell-model fermion operators. It is nevertheless a fundamental problem to study these operators, not only because of the empirical interest of single-particle properties, but also because of the technical requirements of consistency between the mapping of single-fermion operators and bifermion operators, namely, that the map of a product should equal the product of the maps. In the following, this requirement will be used in part to help determine the mappings studied in this section.

We shall be somewhat briefer than in Sec. II and start by recording the form taken by the approximate fermion mapping that already satisfies the commutation relations

$$[\alpha_m^\dagger, \mathcal{A}_0^\dagger] = 0, \tag{4.1}$$

$$[\alpha_m^\dagger, \mathcal{A}_0] = -\frac{1}{\sqrt{\Omega}}\tilde{\alpha}_m. \tag{4.2}$$

The resulting expressions are

$$\begin{aligned}\alpha_m^\dagger = {} & a_m^\dagger\sqrt{r}E+\frac{1}{\sqrt{\Omega}}s^\dagger\tilde{a}_mE(n-1)+(d^\dagger\times\tilde{a})_{jm}\sqrt{r}F-\frac{1}{\sqrt{\Omega}}s^\dagger(\tilde{d}\times a^\dagger)_{jm}F(n_2-1,n+1) \\ & +\sum_{J=1}^{4}[a^\dagger\times(d^\dagger\times\tilde{d})_J]_{jm}\sqrt{r}G_J+\frac{1}{\sqrt{\Omega}}\sum_{J=1}^{4}s^\dagger[(d^\dagger\times\tilde{d})_J\times\tilde{a}]_{jm}G_J(n-1),\end{aligned} \tag{4.3}$$

$$\begin{aligned}\tilde{\alpha}_m = {} & \sqrt{r}E\tilde{a}_m-\frac{1}{\sqrt{\Omega}}E(n-1)a_m^\dagger s-\sqrt{r}F(\tilde{d}\times a^\dagger)_{jm}-\frac{1}{\sqrt{\Omega}}F(n_2-1,n+1)s(d^\dagger\times\tilde{a})_{jm} \\ & +\sqrt{r}\sum_{J=1}^{4}G_J[(d^\dagger\times\tilde{d})_J\times\tilde{a}]_{jm}-\frac{1}{\sqrt{\Omega}}\sum_{J=1}^{4}G_J(n-1)s[a^\dagger\times(d^\dagger\times\tilde{d})_J]_{jm}.\end{aligned} \tag{4.4}$$

In these equations, the Pauli factors E, f, and G_J are functions of n_2 and n, the full dependence on the s-boson occupation operator n_0 having been separated in the factors r.

The natural next step to help determine the unknown scalar factors in (4.3) and (4.4), that depend on n_2 and n only, is to study the commutation relations

$$[\alpha_m^\dagger, \mathcal{A}_{2\mu}^\dagger] = 0, \tag{4.5}$$

$$[\tilde{\alpha}, \mathcal{A}_2^\dagger]_{j'm} = -\sqrt{\frac{5}{\Omega}}\delta_{jj'}\alpha_m^\dagger. \tag{4.6}$$

By analogy with (4.1) and (4.2) these conditions should determine the dependence of the coefficient functions on n_2. When we sought to carry through this program, however, we found that we could not derive results that were fully consistent with both of the above commutation relations. Later we shall explain how we were able to circumvent this difficulty and determine the required coefficients, but first let us try to explain the source of the trouble. First note that the order of magnitude of the coefficients in Eq. (4.3), e.g., is determined by the same considerations that we have explained in connection with the representation of the bifermion operators. Thus the function E is of order unity, F is of order $\Omega^{-1/2}$, and the G_J no larger than order Ω^{-1}. In fact we know the value of E in the limit $n_2 = 0$ from the exact SU(2) case, namely,

$$E(0,n) = [1-(n/\Omega)]^{1/2}. \tag{4.7}$$

If we now consider the commutator (4.6), for instance, we see from the right-hand side that the leading term is of order $\Omega^{-1/2}$, and successive terms decrease by the same factor. When we examine contributions to both commutators of leading order, we discover that they all give the same relation,

$$F = \sqrt{\frac{5}{\Omega}}\left(r_2 - \frac{1}{\Omega}\right)^{-1/2} E(n-1), \tag{4.8}$$

to terms of order $(1/\Omega)$. At the next order, the commutators yield several difference equations for the coefficient E, which are, unfortunately not consistent with each other. The source of the difficulty can be traced to the assumption, used in the derivation, that the quasifermion operators commute with the d-boson creation and annihilation operators. However in the approximate theory constructed in this paper, this assumption fails precisely at the level of accuracy, $(1/\Omega)$ needed to obtain usable difference equations for E. We shall leave this as an assertion for the moment, and return to it at the end of this section to provide a further discussion that we hope the reader will find convincing. What we shall show now is that there are means of determining the still unknown operators E and G_J, where the difficulties to which we have called attention are bypassed.

The tool that we shall use to complete the calculation undertaken in this section is the requirement that the product of maps should equal the map of the products. Since we are dealing with approximate realizations, we must again apply this requirement with care. We have, for instance, examined the application of this criterion to the quadrupole pair operator. Not surprisingly, we run into the same problem here that we encountered in trying to apply (4.5) and (4.6). The solution consists in applying the criterion circumspectly only to the angular-momentum zero operators. Here we have carried out two calculations that confirm one another.

We first study the number operator, which must be given by the two equivalent expressions

$$\mathcal{N} = \sum_m \alpha_m^\dagger \alpha_m \cong n + 2n_2 + 2n_0. \tag{4.9}$$

If we write

$$\alpha_m^\dagger = 1+2+3+4+5+6, \tag{4.10}$$

$$\alpha_m = 1'+2'+3'+4'+5'+6', \tag{4.11}$$

where the numbers refer to the terms of Eqs. (4.3) and (4.4), then the requirement is that a product of 36 terms should collapse into the simple sum of occupation number operators shown in (4.9). As indicated in the writing of this equation, this requirement is not to be taken too literally. We can only expect the relationship to hold to order $(1/\Omega)$. This immediately rules out fully one-third of the products. An additional third can be dropped because they involve a product of quasifermion creation or annihilation operators. (Here we are using a bit of licence since strictly only the $J = 0$ and $J = 2$ projections of these products are required to vanish according to our starting assumptions.) We then verified explicitly that four more terms linear in d bosons cancel out. Of the remaining eight terms, four are proportional to n_0. These will provide results consistent with what we report below, but will not be discussed in detail. The remaining terms that will yield the results we seek, the products $(11')$, $(33')$, $(15')$, and $(51')$, will be evaluated for $n_0 = 0$. To put the answer into the required form it is further necessary to utilize the anticommutation relations of the quasifermions. In the present calculation, we assume that these are the same as for fermions, since the error incurred is one order smaller than the accuracy of our evaluation. (The commutation relations of the quasifermions will be presented at the end of this section and the present assertion thereby verified.)

With these preliminaries out of the way, we simply present the results for the indicated terms, evaluated for $n_0 = 0$. We also have dropped $(1/\Omega)$ compared to unity in the following. This also means replacing $n_2 - 1$ and $n - 1$ by n_2 and n, respectively, in the unknown Pauli factors, since these occupation numbers always appear in a ratio to Ω. The calculation finally involves some standard angular-momentum coupling. We thus find

$$(11') = nr_2E^2, \tag{4.12}$$

$$(33') = 2n_2E^2 - \sum_{J=0}^{4} 10\sqrt{2J+1}\begin{Bmatrix} 2 & j & j \\ j & 2 & J \end{Bmatrix} \times[(a^\dagger \times \tilde{a})_J \times (d^\dagger \times \tilde{d})_J]_0 E^2, \tag{4.13}$$

$$(15') + (51') = -\sum_{J=1}^{4} 2\sqrt{2\Omega}(-1)^J r_2 E G_J \times[(a^\dagger \times \tilde{a})_J \times (d^\dagger \times \tilde{d})_J]_0. \tag{4.14}$$

These equations contain a good deal of information that we proceed to extract. For instance, if we set all d-boson factors to zero, then we are back to the SU(2) limit and the coefficient of n in (4.12) must be unity. This verifies Eq. (4.7). Next we rewrite the first term of (4.13) keeping only linear terms in n_2 and treating (n/Ω) as a small quantity,

$$2n_2E^2 \cong 2n_2 + 2n_2\frac{n}{\Omega} \cong 2n_2 + 2n_2E^2\frac{n}{\Omega}. \tag{4.15}$$

In this equation, the first term provides the required coefficient, unity, of $2n_2$. The second term, when considered as a correction to the coefficient of n, provides part of the n_2 dependence of the quantity E. There is another contribution of like form arising from the $J = 0$ part of the second term of (4.13) that cancels half of the second term of (4.15). The requirement that the coefficient of n be unity now easily yields the result we have been seeking,

$$E(n_2,n) = [1-(n+n_2)/\Omega]^{-1/2}. \tag{4.16}$$

There remain the $J \neq 0$ terms of (4.13) that must cancel against the content of (4.14). Having determined the value of the quantity E, this provides a set of equations to fix the G_J. We thus find

$$G_J = (-1)^{J+1} 5 \sqrt{\frac{2J+1}{2j+1}} \begin{Bmatrix} 2 & 2 & J \\ j & j & j \end{Bmatrix} \frac{E}{r_2}$$
$$\cong (-1)^{J+1} 5 \sqrt{\frac{2J+1}{2j+1}} \begin{Bmatrix} 2 & 2 & J \\ j & j & j \end{Bmatrix}. \quad (4.17)$$

We have also studied the related condition

$$\mathcal{A}_0^\dagger = \frac{1}{2\sqrt{\Omega}} \sum_m (-1)^{j-m} \alpha_m^\dagger \alpha_{-m}^\dagger$$
$$\cong s^\dagger \sqrt{r}. \quad (4.18)$$

Requiring the same level of accuracy as in the study of Eq. (4.9) for the number operator, we have verified all the results obtained from a study of the latter, but we shall spare the reader any additional details.

It remains for us only to dispose of several questions that we have raised during the course of the discussion in this section. Let us first consider the anticommutation relations satisfied by the quasifermions. This subject is attacked by requiring that the realizations (4.3) and (4.4) satisfy the fermion anticommutation relations. This somewhat tedious calculation can be simplified by requiring that the fermion anticommutator will have some terms that are independent of the boson operators and others that depend on the boson operators. In an argument, whose analog has been used repeatedly in this work, the relation we obtain when we set all normal-ordered boson operators to zero will determine the anticommutation relations sought. Afterwards we must verify that the boson-dependent terms will vanish, to the required accuracy, when these new results are utilized. Again we shall spare the reader any algebraic details, and only quote the final result:

$$\{\tilde{a}, a^\dagger\}_J = \hat{j}\delta_{J0} - \left(\frac{2j+1}{\Omega - n + 1}\right) \sum_{s=0,2} (2s+1) \begin{Bmatrix} j & j & J \\ j & j & s \end{Bmatrix} (a^\dagger \times \tilde{a})_J. \quad (4.19)$$

Except for the term with $s = 2$, this equation has the same form as for the SU(2) limit and verifies the assertion used in our previous calculations that the usual anticommutation relations were violated only in relative order $(1/\Omega)$.

We are now in a position to check the validity of Eq. (2.11), the commutation relations of the quasifermion operator with the quasifermion multipole operators. Using (4.19), we find for the monopole

$$[\tilde{a}_{m'}, n] = \tilde{a}_{m'} - \sqrt{2} \frac{1}{\Omega - n + 1} \sum_{J=0,2} \hat{j}\hat{J} (a^\dagger \times A_J)_{jm'}$$
$$= \tilde{a}_{m'}, \quad (4.20)$$

i.e., the exact result. On the other hand, for $J \neq 0$, we calculate

$$[(a^\dagger \times \tilde{a})_J, \tilde{a}]_{jm} = \frac{\hat{J}}{\hat{j}} \tilde{a}_m + \left(\frac{2j+1}{\Omega - n + 1}\right) \sum_{J'} \sum_{s=0,2} \sum_{J''=4,6,\ldots} (-1)^{J+J'} (2s+1)(2J'+1) \hat{J}\hat{J}''$$
$$\times \begin{Bmatrix} j & j & J \\ j & j & J' \end{Bmatrix} \begin{Bmatrix} j & j & J' \\ j & j & s \end{Bmatrix} \begin{Bmatrix} j & j & J' \\ j & j & J'' \end{Bmatrix} \sqrt{2} (a^\dagger \times A_{J''})_{jm}, \quad (4.21)$$

having taken into account the fact that $A_{J''} = 0$ for $J'' = 0, 2$. These relations differ by terms of relative order $(1/\Omega)$ from those that have been assumed and utilized previously in this work. The reader may verify, however, that this failure does not change any of the results derived in this paper.

This brings us to the final gap in our work. Earlier in this section we remarked how we could not use the commutation relations (4.5) and (4.6) to derive the value of the factor E, because of the asserted failure of the quasifermion operators to commute with the d-boson operators. We are now in a position to find values of those nonvanishing commutators of order $(1/\Omega)$ that would bring these equations back into consistency. This exercise would be necessary if we wished to carry the calculation to the next order.

V. SUMMARY AND DISCUSSION

Using the example of the single-j shell model, we have shown how to construct an approximate boson-quasifermion mapping applicable to the physical situation that we are near the limit of good seniority. Such a mapping can provide a basis for the derivation of an IBFM model valid near such a limit. (This is the only limit for which existing derivations have any validity.) The method generalizes a relatively straightforward, if unavoidably tedious, algebraic technique first applied to the purely bosonic part of this problem by Bonatsos *et al.* that produces a systematic expansion in the ratio of the seniority to the level degeneracy with selection summation of higher order terms subsumed in so-called Pauli reduction factors. The results are both more consistent and complete than those produced by methods that have been previously applied to the same problem.

The order of events in the derivation is to first find an approximate realization of the Lie algebra, SO$(2(2j+1))$, generated by all products of fermion shell-model operators. To carry out this program at the level that only s and d bosons are admitted to the mapping, we add to the mix a set of quasifermions that are in one-to-one correspondence with the shell-model fermions, but with different algebraic properties *vis à vis* the bosons. They are assumed to be kinematically independent of the bosons and to be algebraically indistinguishable from fermions except in the anticommutation of two quasifermions. These assumptions are known to be correct for all cases where exact boson-quasifermion mappings are known. It is later discovered that in the approximate mapping studied in this paper, the properties assumed for the quasifermions both with respect to the bosons and with respect to each other cannot be maintained exactly. This failure is also seen later not to have affected the accuracy of the results presented for the generators of the Lie algebra.

The mapping of the generators is sufficient to permit a discussion of the IBFM Hamiltonian and, in particular, of the so-called exchange interaction, that is an established ingredient of the model. We apply our results to a cogent source of this interaction, namely, the quadrupole-pairing interaction in the shell-model space and derive results that parallel those given previously in the literature. Nevertheless there are some, possibly significant, differences of detail, arising presumably from the more complete character of our mapping.

Finally we turn to the problem of constructing a boson-fermion mapping for the shell-model fermion operator. Here, when we study the commutation relations of single fermions with the generators of the Lie algebra, we are only partially successful in determining the Pauli factors for the fermions from the known factors for the generators. We recognize the source of the difficulty to be the failure of the ideal properties that were assumed for the quasifermions. We show nevertheless that a conservative (and justified) application of these commutation relations and of the requirement that the map of a product equal the product of the maps suffices to determine all unknown coefficients. We also describe the derivation of the anomalous anticommutation relations of the quasifermions. Together with previous results, this puts us in a position to study the deviations of the properties of the quasifermions from the ideal ones assumed at the beginning of the work. Since the results of such an investigation are of interest only if one intends to carry out higher order mapping procedures, our enthusiasm for this exercise flags after a few examples.

Problems of remaining interest that are not too tedious are to extend the results of this paper to the multi-j case and to consider an alternative method for cases other than the seniority-dictated limit. One such possibility is the so-called hybrid mapping method suggested by one of us [21].

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APPENDIX: COMMUTATORS OF COUPLED OPERATORS

In this appendix we quote the main formulas used to calculate the nontrivial terms of the commutator Eq. (2.37). These formulas are special cases of results obtained by the authors [18] for evaluating commutators of coupled operator products. Let A, B, C, D be operators carrying angular momentum a, b, c, d. We consider a special case where one of the commutators or anticommutators (both indicated by the bracket here) is elementary,

$$[B, C^\dagger]_J = \delta_{bc}\delta_{J0} \times \text{const.} \tag{A1}$$

We then have the following formula general for rearrangement into normal form:

$$[(A^\dagger \times B)_e \times (C^\dagger \times D)_f]_k = Y + \theta_{bc} \sum_{gh} N_{cd}[(A^\dagger \times C^\dagger)_g \times (B \times D)_h]_k, \tag{A2}$$

$$Y = (-1)^{a+b+c+d+k} \times \delta_{bc}(\hat{e}\hat{f}/\hat{b}) \begin{Bmatrix} e & f & k \\ d & a & b \end{Bmatrix} [B, C^\dagger]_0 \times (A^\dagger \times D)_k, \tag{A3}$$

$$\theta_{ab} = \mp, \text{ if } \begin{cases} \text{both } a \text{ and } b \text{ are half integers,} \\ \text{otherwise,} \end{cases} \tag{A4}$$

$$N_{cd} = \begin{bmatrix} a & b & e \\ c & d & f \\ g & h & k \end{bmatrix}. \tag{A5}$$

For the special case that A and D are fermions and B and C are bosons, the following alternative formula is useful:

$$[(A^\dagger \times B)_e \times (C^\dagger \times D)_f]_k = Y + \theta_{bc}(-1)^{f+b-d-h} \sum_{gh} N_{cd}[(A^\dagger \times D)_g \times (C^\dagger \times B)_h]_k. \tag{A6}$$

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