Nonunitary nature of the Dyson boson mapping revisited

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The nonunitary character of the Dyson boson mapping is explored in a model case with the monopole pairing interaction. It is shown that because of the very special form of this interaction, the calculations in the intrinsic state formalism with the particle-particle representation of the Hamiltonian reproduce the exact results of the orginal fermion space. In contrast, we show that with the particle-hole representation of the Hamiltonian the intrinsic state approach produces erroneous results. Of course, upon employing the physical boson states we recover the exact results in the particle-hole representation as well.

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Recently, the Dyson boson mapping (DBM) (Ref. [1]) has been employed quite extensively [2–9] for mapping the bifermion operators in the shell model configuration space onto a physical subspace of the boson space. The advantage in such a mapping procedure is that it has a finite character and correspondingly the mapping of any bifermion operator leads to a finite number of terms in the boson space. On the other hand, the mapping has the disadvantage that it is nonunitary in character because for the bifermion operators: $[A^\dagger_{JM}(ab)]^\dagger_{\rm DBM} \neq [A_{JM}(ab)]_{\rm DBM}$. With this nonunitary nature of the mapping procedure, the mapping of any fermion Hamiltonian in the particle-particle representation [in the form $(A^\dagger A)$] will lead to a non-Hermitian matrix in the boson space. The other possibility for mapping the fermion Hamiltonian is to use the particle-hole representation of the mapping procedure [in the form $(a^\dagger a)$] which leads to a Hermitian matrix in the boson space. A question of fundamental importance to be addressed in the present context is the following: which form of the Hamiltonian (i.e., particle-particle or particle-hole) will result in a better representation [9] of the exact calculations for a given approximate scheme, for example, the incorporation of a basis truncation [7] such as only s and d bosons in the intrinsic state? Clearly, when no approximations are made the two approaches should lead to identical results since the mapping procedure is exact.

The purpose of the present paper is to explore the feasibility of the aforementioned two approaches. We present here a case study of the model Hamiltonian with the monopole pairing interaction. The advantage in such an interaction is that most of the results can be worked out analytically. The monopole pairing Hamiltonian in the particle-particle representation within a single-j shell is given by

$$H^f_{pp} = -\frac{1}{4}G\hat{j}^2[A^\dagger_0(jj)\otimes\tilde{A}_0(jj)]_{00}, \tag{1}$$

where G is the strength of the pairing interaction $\hat{X} = \sqrt{2X+1}$, and the bifermion operator is represented by

$$\begin{aligned} A^\dagger_{JM}(ab) &= (C^\dagger_a C^\dagger_b)_{JM} \\ &= \sum_{m_\alpha m_\beta} \begin{bmatrix} j_\alpha & j_\beta & J \\ m_\alpha & m_\beta & M \end{bmatrix} C^\dagger_\alpha C^\dagger_\beta \end{aligned} \tag{2}$$

and

$$\begin{aligned} \tilde{A}_{JM}(ab) &= (-1)^{J-M}[A^\dagger_{J-M}(ab)]^\dagger \\ &= (-1)^{J-M}A_{J-M}(ab), \end{aligned} \tag{3}$$

where $\alpha, \beta, \ldots,$ denote the $(nljm)$ quantum numbers of the single-particle shell model states and the symbols $a, b, \ldots,$ denote the same quantum numbers but with no projections. The pairing Hamiltonian can be rewritten in the particle-hole representation as

$$\begin{aligned} H^f_{ph} = &\frac{G}{4}\hat{j}(C^\dagger_j\tilde{C}_j)_{00} \\ &-\frac{G}{4}\sum_{J_1}\hat{J}_1[(C^\dagger_j\tilde{C}_j)_{J_1}\otimes(C^\dagger_j\tilde{C}_j)_{J_1}]_{00}. \end{aligned} \tag{4}$$

The ground-state energy of this simple pairing interaction as a function of the femion particle number (N) in a single-j shell has the following simple form [8]:

$$E(N) = -\frac{G}{4}N(2j+3-N). \tag{5}$$

We now employ the DBM for mapping the bifermion operators onto the physical subspace of the boson space. This mapping procedure is given by [7]

$$A^\dagger_{JM} \to b^\dagger_{JM} - 2\sum_{J_1J_2J_3J_4}\begin{Bmatrix} j & j & J_1 \\ j & j & J_2 \\ J_3 & J & J_4 \end{Bmatrix}\hat{J}_1\hat{J}_2\hat{J}_3\hat{J}_4[(b^\dagger_{J_1}b^\dagger_{J_2})_{J_4}\otimes\tilde{b}_{J_3}]_{JM}, \tag{6}$$

$$A_{JM} \to b_{JM}, \tag{7}$$

and

$$(C_j^\dagger \tilde{C}_j)_{JM} \to 2\sum_{J_1 J_2} \hat{J}_1 \hat{J}_2 (-1)^{J+J_2} \left\{ \begin{matrix} j & j & J_1 \\ J_2 & J & j \end{matrix} \right\} (b_{J_1}^\dagger \tilde{b}_{J_2})_{JM}, \tag{8}$$

where $b_{JM}^\dagger$ (b_{JM}) are the boson creation (annihilation) operators. In Eqs. (6)–(8) and hereafter the quantum number j in the operators $A^\dagger$ and $b^\dagger$ is disregarded. It can be seen here that the mapping is non-Hermitian, i.e., $[A_{JM}^\dagger(ab)]_{\rm DBM}^\dagger \neq [A_{JM}(ab)]_{\rm DBM}$. Using this mapping procedure, the boson image of the pairing Hamiltonian in the particle-particle representation is given by

$$H_{pp}^B = -\frac{G}{2}\hat{j}^2 \left[s^\dagger s - 2\sum_{J_1 J_2 J_3} (-1)^{J_2+J_3} \left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j & j & j \end{matrix} \right\} \frac{\hat{J}_1 \hat{J}_2 \hat{J}_3}{\hat{j}} [(b_{J_1}^\dagger b_{J_2}^\dagger)_{J_3} \otimes \tilde{b}_{J_3}]_{00} s \right], \tag{9}$$

where $s = b_{00}$. The corresponding boson image in the particle-hole representation is

$$H_{ph}^B = H_{ph}^B(1) + H_{ph}^B(2), \tag{10}$$

where

$$H_{ph}^B(1) = -\frac{G}{2} \sum_{J_1 M_1} b_{J_1 M_1}^\dagger b_{J_1 M_1} \tag{11}$$

and

$$H_{ph}^B(2) = -G \sum_{J_1 J_2 J_3 J_4 J} (-1)^{J+M} (-1)^{2J+J_2+J_4} \times \hat{J}_1 \hat{J}_2 \hat{J}_3 \hat{J}_4 \hat{J} \times \left\{ \begin{matrix} j & j & J_1 \\ J_2 & J & j \end{matrix} \right\} \left\{ \begin{matrix} j & j & J_3 \\ J_4 & J & j \end{matrix} \right\} \times [(b_{J_1}^\dagger \tilde{b}_{J_2})_J \otimes (b_{J_3}^\dagger \tilde{b}_{J_4})_J]_{00}. \tag{12}$$

For the energy calculation we make use of the intrinsic function defined through

$$|\phi) = \frac{b_0^{\dagger p}}{\sqrt{p!}} |0), \tag{13}$$

where

$$b_0^\dagger = \sum_J \alpha_{J0} b_{J0}^\dagger \tag{14}$$

in which p denotes the number of fermion pairs or bosons. The weight factors (α) in the above equation depend on the dynamics of the system. These are obtained through the variational procedure

$$\delta \frac{(\phi|H^B|\phi)}{(\phi|\phi)} = 0, \tag{15}$$

with the constraint

$$\sum_J \alpha_{J0}^2 = 1, \tag{16}$$

which is incorporated in Eq. (15) through the Lagrangian multiplier. The energy matrix in the particle-particle representation is given by

$$(\phi|H_{pp}^B|\phi) = -\frac{G}{2}\hat{j}^2 p \left(\alpha_0^2 + 2(p-1) \sum_{J_1 J_2 J_3} (-1)^{J_2} \frac{\hat{J}_1 \hat{J}_2}{\hat{j}} \begin{bmatrix} J_1 & J_2 & J_3 \\ 0 & 0 & 0 \end{bmatrix} \left\{ \begin{matrix} J_1 & J_2 & J_3 \\ j & j & j \end{matrix} \right\} \alpha_0 \alpha_{J_3} \alpha_{J_1} \alpha_{J_2} \right) \tag{17}$$

and the corresponding expression in the particle-hole case is

$$(\phi|H_{ph}^B|\phi) = -\frac{G}{2}p - \frac{G}{\hat{j}^2}\frac{p}{2} \sum_{JM} [1 + (-1)^J] - Gp(p-1) \sum_{J_1 J_2 J_3 J_4 J} (-1)^J \hat{J}_1 \hat{J}_2 \hat{J}_3 \hat{J}_4 \left\{ \begin{matrix} j & j & J_1 \\ J_2 & J & j \end{matrix} \right\} \left\{ \begin{matrix} j & j & J_3 \\ J_4 & J & j \end{matrix} \right\} \times \begin{bmatrix} J_1 & J_2 & J \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} J_3 & J_4 & J \\ 0 & 0 & 0 \end{bmatrix} \alpha_{J_1 0} \alpha_{J_2 0} \alpha_{J_3 0} \alpha_{J_4 0}. \tag{18}$$

In principle, for the evaluation of the matrix elements in the boson space, the fermion states should be mapped onto the boson space using the explicit mapping procedure. This state referred to as the physical boson state which for the J=0 condensate is given by

$$(S^\dagger)^p |0\rangle \to \left[N_0^p (s^\dagger)^p + \sum_{\nu \neq 0} s_\nu^\dagger s_0^\dagger + \sum_{\nu\rho \neq 0} N_{\nu\rho}^2 s_\nu^\dagger s_\rho^\dagger + \cdots \right] |0)_R, \tag{19}$$

where $S^\dagger = A^\dagger_{00}$. The closed algebraic expressions for the various expansion coeffcients appearing in Eq. (19) are given in Ref. [6]. The bra state corresponding to the ket, Eq. (19), is given by

$$\langle 0|(S)^p \to_L (0|(s)^p. \quad (20)$$

It has been shown [3] in model situations like the Lipkin model that the use of these biorthonormal basis states with the normalization condition

$$_L(i|i)_R = 1 \quad (21)$$

reproduces the exact results. This observation is independent of the representation of the Hamiltonian, both particle-particle as well as the particle-hole representations lead to identical results. However, in a realistic situation with various approximations, it is not obvious whether the two representations will lead to identical results. It can be easily shown that, when using the intrinsic state with only the s boson in the expression of Eq. (14), the particle-particle representation, Eq. (17) leads to the exact result as in the fermion space. In contrast, the matrix element in the particle-hole represenation is given by

$$E(p) = -\frac{G}{2}p - \frac{G}{\hat{j}^2}\frac{p}{2}\sum_{JM}[1+(-1)^J] - \frac{Gp(p-1)}{\hat{j}^2} \quad (22)$$

with all the α coefficients set equal to one in Eq. (18) since we are considering only s bosons. This particle-hole representation considerably underestimates the ground-state energy for $p > 1$. For example, in the case of j=11/2 with only the s boson in the intrinsic state, this expression gives rise to $E(p) = -6G$, $-12.167G$, and $-18.5G$ corresponding to p=1, 2, and 3, in comparison to the exact numbers which are $-6G$, $-10G$, and $-12G$, respectively, for the same p values.

It is interesting to realize that we obtain the exact result in the particle-particle case on account of the presence of the s boson which appears on the right-hand side of Eq. (9). This fact can be clearly demonstrated in the case of four particles in j=7/2 for which the explicit physical boson state, Eq. (19) is given by

$$(S^\dagger)^2|0\rangle \to \left[\frac{1}{2}\frac{(s^\dagger)^2}{\sqrt{2}} - \frac{\sqrt{5}}{6}\frac{(d^\dagger d^\dagger)_{00}}{\sqrt{2}} - \frac{\sqrt{9}}{6}\frac{(g^\dagger g^\dagger)_{00}}{\sqrt{2}} - \frac{\sqrt{13}}{6}\frac{(I^\dagger I^\dagger)_{00}}{\sqrt{2}}\right]|0)_R. \quad (23)$$

It is easily seen that with s operating on this state, only the first term contributes. For all other terms the s operator commutes through, hence there is no contribution from these terms. Therefore, the results with the intrinsic state are exactly the same as with the proper physical boson state. [The intrinsic state happens to be the first term on the right-hand side of Eq. (23).] This is in contrast with the particle-hole representation, Eq. (18) wherein all the terms in the physical boson state contribute, thereby leading to the incorrect results with the intrinsic state. In the present case of four particles in j=7/2, the energy matrix element as obtained from Eq. (22) is $-8.25G$, the exact value is $-6G$. With the use of the physical boson state, Eq. (23), the exact result $-6G$ is reproduced.

In conclusion, we note here that the particle-particle representation is preferable to the particle-hole representation in the case of the pairing interaction. The results of the present analysis are somewhat surprising since normally one would have expected that both the particle-particle and particle-hole representations should lead to similar results. The present investigation has shown that the particle-hole representation can lead to somewhat incorrect results with the employment of the intrinsic states for the monopole pairing interaction. This clearly indicates a need for a proper understanding of the nonunitary character of the Dyson boson mapping in a realistic scenario.

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