Algebraic and coordinate space potentials from heavy ion scattering

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An inversion scheme is presented to derive the potentials of algebraic scattering theory from the corresponding S functions. Representative heavy ion scattering data of ^{12}C, ^{14}N, and ^{16}O ions on ^{208}Pb, accurately fitted by McIntyre strong absorption type S functions, are employed to obtain exact algebraic potentials and to generalize the analytical shapes proposed previously by Alhassid *et al.* The coordinate space potentials corresponding to a number of S functions are also obtained via semiclassical inversion.
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I. INTRODUCTION

Group theoretical methods, which have been applied very succesfully to bound systems for the analyses of their spectra, recently have been extended to investigate scattering systems [1]. This was encouraged by the success of the interacting boson model in the description of nuclear spectra, and by the fact that the algebraic approach gives an exact description of Coulomb scattering. It is an obvious application of the method to study Coulomb modified scattering and, in particular then, heavy ion reactions in which the nuclear interaction in the sensitive radial region is relatively weak compared to the Coulomb interaction. In the algebraic approach the S functions are expressed in terms of the so-called algebraic potentials, which describe the dynamics of the systems. The name is somewhat of a misnomer insofar that the algebraic functions are more akin to modifications of the Coulomb (Sommerfeld) parameter. Until now the algebraic potential due to the strong interaction has been defined phenomenologically [2, 3]. Simple Woods-Saxon (WS) models of the algebraic potentials for heavy ions derived from the SO(3,2) and SO(3,1) group theoretical approaches have been proposed and employed in data analyses. However, such has not been done to the degree of accuracy required by presently available data [2, 3]. For this reason we employ the McIntyre forms of the S function, which has been widely and successfully applied to heavy ion scattering data of good quality, and compare them to the SO(3,1) and SO(3,2) S functions proposed in the literature. We used the same approach in a previous paper [4] and in which the algebraic potentials $v_l(k)$ and $w_l(k)$ of the SO(3,1) and SO(3,2) groups were obtained by matching the algebraic to the McIntyre S functions for ^{12}C+^{12}C scattering over a wide range of energies. Similar investigations at low energies of the $v_l(k)$ for ^{12}C+^{24}Mg were recently made by Lichtenthaler-Filho *et al.* [5]. In this work we obtain explicit analytical expressions in terms of the McIntyre S functions for the $v_l(k)$ and $w_l(k)$ in the asymptotic region of large l values. On the basis of these results it is shown that the original proposals of shapes for $w_l(k)$ and $v_l(k)$ [2, 3] have to be generalized to obtain higher quality fits to the data.

The algebraic scattering formalism is formulated in the angular momentum (l) and wave number (k) space, with corresponding Hamilton operator functions playing the role of Jost functions and the algebraic states corresponding to coordinate space wave functions, etc. We can therefore regard this problem of determining the algebraic potentials as the analog in the space of l and k to the usual inverse scattering formalism relating $S_l(k)$ to $V(r)$ in coordinate space. This point of view is in agreement with the fact that, in principle, a determination of the algebraic potentials via a microscopic theory appears to be possible without recourse to coordinate space [1].

In Sec. II we formulate a general inverse scattering procedure to determine the algebraic potentials from the measured S functions in terms of nonlinear first-order differential equations when the S functions can be regarded as continuous functions of the angular momentum l. We must use difference equations otherwise.

We apply our methods to the cases of ^{12}C, ^{14}N, and ^{16}O+^{208}Pb scattering over a wide spread of energies and obtain McIntyre and algebraic S functions fitted to these data. Therefore we note that generalizations of the proposed algebraic potential shapes [2, 3] are required to resemble the algebraic S functions that fit the data well. That can be inferred from a comparison of the asymptotic values with the corresponding McIntyre S-function ones.

The specific algebraic potentials are given in Sec. III. In addition, for the case ^{12}C+^{208}Pb at 1449 and 2400

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MeV, we obtained the corresponding local potentials via semiclassical inversion [6] and the results are presented in Sec. IV. The physical significance of these results is discussed in the relevant sections (III and IV), respectively, but conclusions that can be drawn are made in Sec. V.

II. THE INVERSE SCATTERING PROBLEM FOR ALGEBRAIC POTENTIALS

The inverse scattering problem to determine the potential $V(r)$ in the Schrödinger equation from the S function at a given fixed energy, is well known and has been the subject of many investigations. Its complexity arises from the need to extract the potential $V(r,k)$ in coordinate space at fixed k from the asymptotic wave function in the form of the S function $S_k(l)$ in the space of l and k.

However, algebraic techniques, which have proved so useful in a number of bound-state problems, particularly nuclear physics problems, also have been applied recently to analyze scattering problems in l-k space only. Algebraic reaction theory starts by assuming that a dynamical group G exists for the problem considered. This approach is similar in spirit to the S-matrix theories of the late sixties except that the dynamic group G restricts the form of the S functions, which can be used. A brief review is given for completeness.

In scattering theory the asymptotic behavior of the regular solution of the radial Schrödinger equation is given by

$$\phi_l(k,r) \underset{r\to\infty}{\longrightarrow} \frac{1}{2ik}\left[F_l(k)\,e^{ikr} - F_l(-k)\,e^{-ikr}\right] , \tag{1}$$

where the $F_l(\pm k)$ are the Jost functions. The corresponding S function $S_l(k)$ is then given by

$$S_l(k) = \frac{F_l(k)}{F_l(-k)}. \tag{2}$$

If the Hamiltonian of the colliding system is invariant under a transformation group G and the asymptotic behavior is invariant under a group F, the S function connects the representations of these groups [1]. This approach leads to exact solutions for the case of nonrelativistic Coulomb scattering where G≡SO(3,1) is the special orthogonal group and F≡E(3) is the Euclidean group. The equation corresponding to Eq. (1) in the algebraic approach can be written as

$$|\omega l m\rangle = A_l(k)\,|klm\rangle + \,A_l(-k)\,|-klm\rangle, \tag{3}$$

where the quantum numbers are related to the relevant eigenvalues of the different Casimir operators and the $A_l(\pm k)$ play the role of the Jost functions. A Hamiltonian H can also be written down in terms of the Casimir invariants C of the group G. The relation between the generators of the groups SO(3,1) and E(3) yields recursion relations of the form

$$\frac{A_{l+1}(k)}{A_{l+1}(-k)} = -\left(\frac{l+1+iv}{l+1-iv}\right)\frac{A_l(k)}{A_l(-k)} . \tag{4}$$

For pure Coulomb scattering the Hamiltonian is given by

$$H = \frac{\alpha^2 Z_1^2 Z_2^2 \mu c^2}{2(C-1)}, \tag{5}$$

where C is the scalar quadratic invariant of SO(3,1) and v is independent of l and given by $v=\eta\equiv\mu Z_1 Z_2 e^2/(\hbar^2 k)$ when Z_1 and Z_2 are the charges and μ is the reduced mass of the colliding particles. The S function can be defined in terms of the algebraic Jost functions by

$$S_l(k) = (-1)^{l+1}\frac{A_l(k)}{A_l(-k)} . \tag{6}$$

For v independent of l the recursion relation Eq. (4) can be solved exactly in the form

$$S_l(k) = \frac{\Gamma(l+1+iv)}{\Gamma(l+1-iv)}, \tag{7}$$

where $\Gamma(z)$ represents the gamma function. In the case of heavy ion scattering the Coulomb interaction is often dominant compared to the nuclear interaction leading to the suggestions [2, 3] that one should make models of the algebraic potential function $v_l^{\rm tot}(k) = \eta + v_l(k)$, where $v_l(k)$ represents the nuclear part of the interaction. They suggested that the algebraic potentials should be of the following form

$$v_l^{\rm tot}(k) = \eta + \{v_R + iv_I\}\left[\,1 + e^{\frac{(l-l_g)}{\Delta}}\right]^{-1} \tag{8}$$

in the S function of Eq. (7). One therefore goes beyond the pure Coulomb interaction and assumes $v = v_l(k)$; i.e., that it is a function of both l and k. Schematic examples have been given to indicate how useful such a parametrization could be in heavy ion scattering [2, 3].

Another possibility to treat nuclear modified Coulomb potentials for heavy ions is via the dynamical group SO(3,2) expanded into E(2)⊗E(3), which results in an S function of the form

$$S_l = \frac{\Gamma[\frac{1}{2}(l+2+w_l+i\eta)]\;\Gamma[\frac{1}{2}(l+1-w_l+i\eta)]}{\Gamma[\frac{1}{2}(l+2+w_l-i\eta)]\;\Gamma[\frac{1}{2}(l+1-w_l-i\eta)]}e^{2i\,\ln(2\eta)}. \tag{9}$$

Representations of the SO(3,2) algebraic potentials $w_l(k)$ by Fermi distribution forms [Eq. (8)] were suggested earlier than those for $v_l(k)$ and some preliminary investigations for their practical usefulness [3] have been made.

However, the fits to the data previously obtained were not of the accuracy required by present-day heavy ion scattering measurements. Only recently, by matching the algebraic SO(3,2) and SO(3,1) potentials to the fitted McIntyre strong absorption S functions has the algebraic approach accurately described ^{12}C-^{12}C elastic scattering data from 360 to 2400 MeV [4, 5].

In the next section we report on similar investigations for ^{12}C, ^{14}N, and ^{16}O on ^{208}Pb collisions. Those scatterings are more suitable for analysis with algebraic scattering theory than are the light ion collisions of ^{12}C on ^{12}C. With them we shall explore the connection between the McIntyre S function and the algebraic potentials and find a generalization of the proposal of Eq. (8) for $v_l(k)$ and $w_l(k)$.

Others who have investigated algebraic scattering theory have taken the point of view [7] that, ultimately, it would be possible to derive the algebraic potentials from basic microscopic theories as has been done for bound states in the interacting boson model. It is appropriate, therefore, to consider a new type of inverse scattering problem, namely, the determination of the algebraic potentials $v_l(k)$ and $w_l(k)$ from the S function $S_l(k)$ that gives an accurate fit to the scattering data. Solving such an inverse scattering problem also would allow us to determine general shapes of $v_l(k)$ and $w_l(k)$ from the data. Such an attempt was made previously [4], but that attempt can be improved considerably as we shall show in the next section.

A priori it is clear that this new type of inversion problem is much simpler than the inversion from $S_l(k)$ to $V(r)$ at fixed k. Both $S_l(k)$ and the algebraic potentials operate in l and k space and a closed form expression for $S_l(k)$ can usually be found in terms of the algebraic potentials. However, the problem still has some features that are typical of inversion problems in general.

In the heavy ion context we may assume the S function $S_l(k)$ can be treated as a continuous function of l. By considering the hadronic deflection function

$$\Theta_l(k) = 2\frac{d}{dl}\delta_l(k) \ , \tag{10}$$

we find from the total $S_l(k)$ given by Eq. (7) for the SO(3,1) case, that, with

$$S_l^{\mathrm{had}}(k) = \frac{S_l^{\mathrm{tot}}(k)}{S_l^{\mathrm{Coul}}(k)}, \tag{11}$$

where

$$S_l^{\mathrm{Coul}}(k) = \frac{\Gamma(l+1+i\eta)}{\Gamma(l+1-i\eta)} \ , \tag{12}$$

$$i\Theta_l(k) = \frac{d}{dl}\ln S_l^{\mathrm{had}}(k) = D_1(v_l)\frac{dv_l}{dl} + N_1(v_l) \ . \tag{13}$$

This is a first-order nonlinear differential equation for $v_l(k)$, namely,

$$\frac{dv_l(k)}{dl} = \frac{i\Theta_l(k) - N_1(v_l(k))}{D_1(v_l(k))} \ , \tag{14}$$

where

$$\begin{aligned} N_1(v_l) \equiv\ & \psi(l+1+v_l+iv) - \psi(l+1+iv) \\ & -\psi(l+1-v_l-iv) + \psi(l+1-iv) \end{aligned} \tag{15}$$

and

$$D_1(v_l) = \psi(l+1+v_l+iv) + \psi(l+1-v_l-iv) \ , \tag{16}$$

with $\psi(z)$ being the derivative of the logarithm of the gamma function, $(d/dz)\ln\Gamma(z)$. A similar nonlinear first order differential equation can be obtained for the SO(3,2) algebraic potential in terms of the deflection function. Specifically, we find

$$\frac{dw_l(k)}{dl} = \frac{i\Theta(l) - N_2(w_l)}{D_2(w_l)} \tag{17}$$

in which

$$\begin{aligned} N_2(w_l) \equiv\ & \psi(\tfrac{1}{2}[l+2+w_l+i\eta]) + \psi(\tfrac{1}{2}[l+1-w_l+i\eta]) \\ & -\psi(\tfrac{1}{2}[l+2+w_l-i\eta]) - \psi(\tfrac{1}{2}[l+1-w_l-i\eta]) \\ & -\psi(\tfrac{1}{2}[l+2+\eta]) - \psi(\tfrac{1}{2}[l+1+i\eta]) \\ & +\psi(\tfrac{1}{2}[l+2-i\eta]) + \psi(\tfrac{1}{2}[l+1-i\eta]) \end{aligned} \tag{18}$$

and

$$\begin{aligned} D_2(w_l) =\ & \psi(\tfrac{1}{2}[l+2+w_l+i\eta]) - \psi(\tfrac{1}{2}[l+1-w_l+i\eta]) \\ & -\psi(\tfrac{1}{2}[l+2+w_l-i\eta]) \\ & -\psi(\tfrac{1}{2}[l+1-w_l-i\eta]) \ . \end{aligned} \tag{19}$$

The formalism of algebraic scattering implies that it will result generally in S functions, which are themselves functions of the algebraic potentials $v_l(k)$, i.e., $S_l(k) \equiv S_l(v_l(k), l)$. Consequently, the general form of the differential equation associated with the inversion problem is given by

$$\frac{dS_l}{dl} = i\Theta_l = \frac{\partial S_l}{\partial l} + \frac{\partial S_l}{\partial v_l}\frac{dv_l}{dl} \ , \tag{20}$$

where Θ_l is obtained directly from the data and the functional form $S_l(v_l, l)$ from theory. Thus we have

$$\frac{dv_l}{dl} = \left(i\Theta(l) - \frac{\partial S_l}{\partial l}\right) \Big/ \frac{\partial S_l}{\partial v_l}. \tag{21}$$

If further conditions on the behavior of $v_l(k)$ as a function of l can be obtained from theory, these can be included in this approach as well. They could then serve to impose theoretical constraints on the behavior of the S function; a matter of great importance for more transparent colliding systems where there are many ambiguities in the S function.

An interesting feature of Eq. (18) is that it is a first-order nonlinear differential equation. Such equations also occur in the more conventional $S_l(k) \to V(r,k)$ inverse scattering at fixed energy, e.g., in the method of [8] based on the Bargman-type rational S function in λ^2 ($\lambda = l + \frac{1}{2}$) and its nonrational generalization [9]. They are a general feature of inverse scattering formalisms, and it is interesting that it is retained in this context. We are also in a position to infer that there is an unambiguous relationship between $S_l(k)$ and the algebraic potentials via the differential equation Eq. (21), as long as the functional dependence of $S_l(v_l, l)$ on l and $v_l(k)$ is known from group theory.

Finally we note that the differential equations given here reduce to difference equations when the angular momentum l has to be discretized, e.g., for low-energy light ion scattering.

III. HEAVY ION S FUNCTIONS AND ALGEBRAIC POTENTIALS

In this section we first present the McIntyre S functions required to fit representative data for heavy ion scattering of ^{12}C, ^{14}N, and ^{16}O from ^{208}Pb over a range of energy from 170 MeV to 2400 MeV. Their asymptotic behavior for large angular momentum l is compared to algebraic S functions of the SO(3,1) and SO(3,2) types that describe heavy ion scattering. A generalization of the proposed [1, 2] shapes of the algebraic potentials $v_l(k)$ and $w_l(k)$ is presented as they are necessary to obtain agreement between the algebraic and McIntyre S functions. We show that such generalized analytical shapes provide fits to measured data that are comparable in quality to those found using the McIntyre S function. We also present the algebraic potentials, which correspond exactly to the McIntyre S functions.

A. Strong absorption model S functions

Heavy ion scattering is absorptive in character, and very strongly so, for most cases of heavy targets and for kinetic energies of 20 MeV/nucleon and greater [10]. To describe such scattering, the algebraic potentials must be complex and the S functions that fit cross-section data have the form

$$S_l = |S_l|\ e^{2i\mathrm{Re}(\delta_l)}\ S_l^{\mathrm{Coul}}(k) = e^{2i\delta_l(k)}\ S_l^{\mathrm{Coul}}(k) \tag{22}$$

with δ_l being complex hadronic scattering phase shifts and S_l^{Coul} an approximate Coulomb S function. For many heavy ion scattering cross sections [11, 12], the empirical hadronic S functions can be well represented in the strong absorption model (SAM) by the McIntyre form [11], wherein

$$|\ S_l^M\ | = \left[1 + e^{\left(\frac{l_g - l}{\Delta}\right)}\right]^{-1} \tag{23}$$

and, for the real part of the phase shifts,

$$\mathrm{Re}(\delta_l^M) = \mu\left[1 + e^{\left(\frac{l - l_g'}{\Delta'}\right)}\right]^{-1} . \tag{24}$$

For large l values, the phase shift has the form

$$2i\delta_l^{(M)} \underset{l\to\infty}{\longrightarrow} l\ e^{\left(\frac{l_g - l}{\Delta}\right)} + 2i\mu\ e^{\left(\frac{l_g' - l}{\Delta'}\right)} . \tag{25}$$

The McIntyre parameter values that fit the measured cross sections for the examples to be considered in this paper are those labeled SAM in Table I.

B. Properties of the SO(3,1) algebraic potentials

The hadronic scattering phase shifts, $\delta_l(k)$, are related to the algebraic potentials $v_l(k)$ by

$$\begin{aligned} 2i\delta_l(k) = {} & \ln\Gamma(l + 1 + i\eta + v_l) + \ln\Gamma(l + 1 - i\eta) \\ & - \ln\Gamma(l + 1 - i\eta - v_l) - \ln\Gamma(l + 1 + i\eta) , \end{aligned} \tag{26}$$

which, by using Stirling's formula to first order, viz.,

$$\ln\Gamma(z + a) - \ln\Gamma(z) \underset{z\to\infty}{\longrightarrow} a\left[\ln(z) - \frac{1}{2z}\right] , \tag{27}$$

expands to give

$$\begin{aligned} 2i\delta_l(k) \underset{l\to\infty}{\longrightarrow} {} & v_l(k)\left[\ln(l + 1 + i\eta) + \ln(l + 1 - i\eta) - \frac{1}{2}\frac{1}{(l + 1 + i\eta)} - \frac{1}{2}\frac{1}{(l + 1 - i\eta)}\right] \\ = {} & v_l(k)\left\{\ln[(l + 1)^2 + \eta^2\ ] - \frac{l + 1}{(l + 1)^2 + \eta^2}\right\} \\ \underset{l\gg\eta}{\longrightarrow} {} & 2v_l(k)\left[\ln(l) - \frac{1}{2l}\right] . \end{aligned} \tag{28}$$

Equating the McIntyre asymptotic form to the asymptotic algebraic potential result gives the result

$$v_l(k) \underset{l\to\infty}{\longrightarrow} A_l\ e^{\left(-\frac{l}{\Delta}\right)} + iB_l\ e^{\left(-\frac{l}{\Delta'}\right)} \tag{29}$$

with coefficients

$$A_l = -\left[\ln[(l + 1)^2 + \eta^2] - \frac{l + 1}{(l + 1)^2 + \eta^2}\right]^{-1} e^{\left(\frac{l_g}{\Delta}\right)} \tag{30}$$

TABLE I. Scattering function parameter values.

Case			l_g	Δ	l_g'	Δ'	μ	v_0	χ^2/F
125 MeV	^{12}C-^{208}Pb	SAM	66.32	4.20	66.32	4.20	38.96		1.13
147 MeV	^{14}N-^{208}Pb	SAM	79.00	3.05	79.00	3.05	33.80		1.11
170 MeV	^{16}O-^{208}Pb	SAM	90.95	4.75	90.95	4.75	37.82		1.13
1449 MeV	^{12}C-^{208}Pb	SAM	266.18	23.89	207.92	20.25	129.98		0.96
1449 MeV	^{12}C-^{208}Pb	WS	266.54	21.90	122.79	34.90	158.90	0.38	0.77
1449 MeV	^{12}C-^{208}Pb	ASM	239.65	32.33	160.23	57.78	67.47		1.10
2400 MeV	^{12}C-^{208}Pb	SAM	289.91	40.56	175.11	28.06	224.17		1.36
2400 MeV	^{12}C-^{208}Pb	WS	196.08	46.24	63.11	40.00	224.17	0.50	1.77
2400 MeV	^{12}C-^{208}Pb	ASM	250.52	58.49	79.02	47.29	393.04		2.13

and

$$B_l = 2\,\mu \left[\ln[(l+1)^2+\eta^2] - \frac{l+1}{(l+1)^2+\eta^2} \right]^{-1} e^{\left(\frac{l_g'}{\Delta'}\right)} \tag{31}$$

that are weakly dependent upon l. This result can be further approximated to

$$v_l(k) \underset{l\to\infty}{\longrightarrow} \frac{1}{(\ln l - \frac{1}{2l})} \left[-\frac{1}{2} e^{\left(\frac{l_g - l}{\Delta}\right)} + i\mu\, e^{\left(\frac{l_g' - l}{\Delta'}\right)} \right] . \tag{32}$$

Consequently, the proposal of Alhassid and co-workers [1, 2] of a Woods-Saxon (WS) form for the strong (hadronic) part of the algebraic potential [note that their $v_l(k)$ corresponds to $-iv_l(k)$ in our notation], i.e.,

$$v_l(k) = [\, v_R + iv_I \,] \left\{ 1 + e^{\left(\frac{l-l_0}{\Delta}\right)} \right\}^{-1} , \tag{33}$$

has to be generalized to allow for different grazing values l_g and l_g' and surface thicknesses Δ and Δ' for the real and imaginary parts, respectively. The algebraic strength parameters A_l and B_l then become slightly l dependent. We ignore this dependence and use the following parametrization for v_l

$$v_l(k) = v_0 \left[-\left(1 + e^{\left(\frac{l-l_g}{\Delta}\right)}\right)^{-1} + 2i\mu \left(1 + e^{\left(\frac{l-l_g'}{\Delta'}\right)}\right)^{-1} \right] . \tag{34}$$

This corresponds in the large l limit with Eq. (28), provided the identification

$$(2v_0)^{-1} \approx \ln l_{\rm av} - \frac{1}{2l_{\rm av}} \approx \ln l - \frac{1}{2l} \tag{35}$$

is made, where $l_{\rm av} = \frac{1}{2}(l_g + l_g')$. However, accurate fits to data lead to modified values of v_0.

C. Properties of SO(3,2) algebraic potentials

To first order in the SO(3,2) algebraic potential $w_l(k)$ and to all orders in η, we have [3]

$$2i\delta_l(k) \approx -\frac{1}{2} w_l(k)\, \ln \left[\frac{(l+1-i\eta)\,(l+i\eta)}{(l+1+i\eta)\,(l-i\eta)} \right] , \tag{36}$$

which, for $l \gg 1$, reduces to

$$w_l(k) \approx -\frac{2}{\eta}(l^2+\eta^2)\,\delta_l(k). \tag{37}$$

Equating $\delta_l(k)$ to the corresponding McIntyre result given by Eq. (25) we find

$$w_l(k) \approx C_l e^{\left(-\frac{l}{\Delta'}\right)} + iD_l e^{\left(-\frac{l}{\Delta}\right)} , \tag{38}$$

where

$$C_l = -\frac{2\mu}{\eta}(l^2+\eta^2)\, e^{\left(\frac{l_g'}{\Delta'}\right)} \tag{39}$$

and

$$D_l = -\frac{1}{\eta}(l^2+\eta^2)\, e^{\left(\frac{l_g}{\Delta}\right)} . \tag{40}$$

The coefficients C_l and D_l do not vary as slowly with l as do A_l and B_l for $v_l(k)$, and consequently the asymptotic behavior of the SO(3,2) algebraic potentials cannot be as directly related to the McIntyre (and other SAM type) S functions. They belong to a different class of S functions. However, this does not imply that the SO(3,2) algebraic potentials are not suitable for use in analyses of heavy ion scattering. As discussed in Sec. II we can invert any given S function to obtain the corresponding $w_l(k)$. But that may not be a simple function of l. Examples will be given in the next section.

The simple forms of the type suggested by Alhassid and co-workers [1, 2] [see Eq. (33)] for $w_l(k)$ have been employed by Amado and Sparrow [3] for schematic examples of heavy ion scattering at low energies. This form for $w_l(k)$ should also be generalized to one having different ranges for the real and imaginary parts and in the next section we consider their application to realistic heavy ion scattering.

D. Examples of algebraic potentials

The first applications of the algebraic scattering approach [1–3] considered "low"-energy data (specifically the elastic scattering of 20 MeV ^{16}O ions) with the assumption that the SO(3,2) algebraic potentials were complex and had a Woods-Saxon variation with l. But at such low energies, while Coulomb dominance is assured, the "sensitive radial region" of the nuclear interaction is well outside the summed half density radius [13]. Nuclear effects are therefore very peripheral so that the data are insensitive to the specific values of the low-l scattering amplitudes. Almost any low l values of the algebraic potentials would suffice in analyses. We first consider data from ^{12}C, ^{14}N, and ^{16}O scattering from the heavy mass target ^{208}Pb and with energies in the range 125–2400 MeV; data whose analyses require several hundred partial waves. Even so, cross sections are all very well fitted by using a five parameter McIntyre function for the S matrices. For the data in question, Baker and McIntyre [11] have determined those parameter values and we give them for completeness in Table I. From those S functions, we obtained the relevant algebraic potentials, $v_l(k)$ and $w_l(k)$. We stress that using the McIntyre parametrizations of S functions is but a convenience as any $S_l(k)$ determined from fits to data by any sensible means can be used equally in this procedure to specify algebraic potentials. But the McIntyre form is particularly convenient as it permits asymptotic identifications as discussed in Sec. III B and has been used widely in heavy ion analyses [12].

The exact algebraic potentials corresponding to the McIntyre S functions are displayed in Figs. 1 and 2 for the elastic scattering of 125 MeV ^{12}C ions, of 147 MeV ^{14}N ions, and of 170 MeV ^{16}O ions off of ^{208}Pb. In Fig. 1 the real and imaginary parts of the $v_l(k)$ are given, while

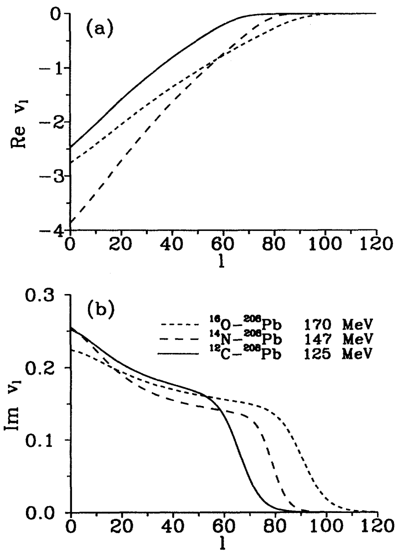

FIG. 1. Algebraic potentials $v_l(k)$ corresponding to the SAM for 125 Mev ^{12}C ions, 147 MeV ^{14}N ions, and 170 MeV ^{16}O ions scattered from ^{208}Pb.

Fig. 2 contains the components of $w_l(k)$. Clearly there are general forms for these algebraic potentials independent of the exact projectile energy or mass; particularly when only the important l values (in excess of 40) are considered. The potentials $v_l(k)$ monotonically decrease with l and exponentially so for large l. The $w_l(k)$ likewise decrease monotonically for large l but have maximum values in the region 40 to 70 in l. Also, with increasing projectile mass and energy, the potentials spread over a larger range of l values and tend to be stronger. It does seem that for 100–200 MeV light mass (heavy) ion scattering from heavy mass nuclei, algebraic scattering potentials are smoothly varying functions of l. Those functions, for the important angular momenta, may be simply parametrized; albeit that the parameters would themselves not be simple functions of mass and energy. But neither are the parameter values of the McIntyre representations to which they are asymptotically related in the way discussed in Secs. III A–III C. However, it is at higher energies that the sensitive radial region for scattering lies within the strong absorption radius, whence scattering can be more influenced by the non-Coulombic interaction. Data at 1449 and 2400 MeV for ^{12}C ions have been taken from ^{208}Pb and algebraic potentials obtained from those data are shown in Figs. 3 and 4 for $v_l(k)$ and in Fig. 5 for $w_l(k)$, respectively. The potentials are displayed as real (above) and imaginary (below) components. Again the algebraic potentials have a simple variation with l. The $v_l(k)$ show quite simple variations, the real parts especially. The $w_l(k)$ potential variations (Fig. 5) have more structure as expected from our discussion in Sec. III C. Their real and imaginary components are relatively smooth, single peaked functions of l. The peak values occur at values of l that scale as the target mass but the actual peak values show a trend that has additional mass dependence.

Besides deriving algebraic potentials corresponding to the SAM by inversion for ^{12}C-^{208}Pb at 1449 and 2400 MeV, we have fitted the data directly using the WS form of algebraic potential proposed in Eq. (21) and also, by using the asymptotic form given by Eqs. (32) and (38) (for all l). The parameters found in these searches are shown in Table I and are labeled WS and ASM (for asymptotic model). The ASM algebraic potentials are given by Eqs. (32) and (38), where $l = l_{\rm av}$ is used to cal-

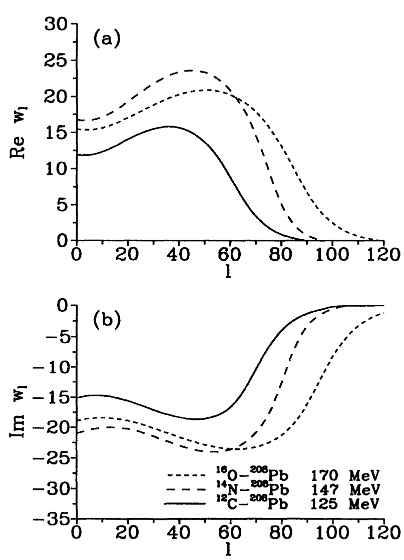

FIG. 2. Algebraic potentials $w_l(k)$ corresponding to the SAM for 125 Mev ^{12}C ions, 147 MeV ^{14}N ions, and 170 MeV ^{16}O ions scattered from ^{208}Pb.

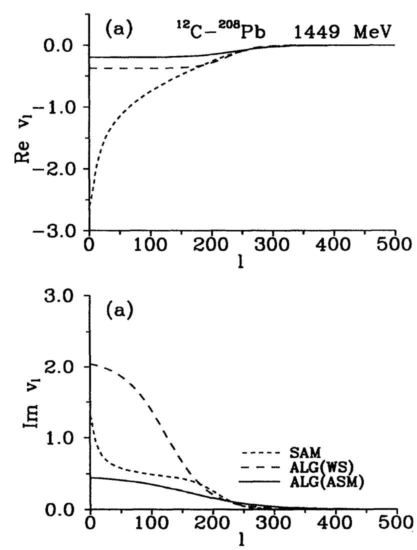

FIG. 3. Algebraic potentials $v_l(k)$ corresponding to the SAM, WS, and ASM S-functions for ^{12}C-^{208}Pb scattering at 1449 MeV.

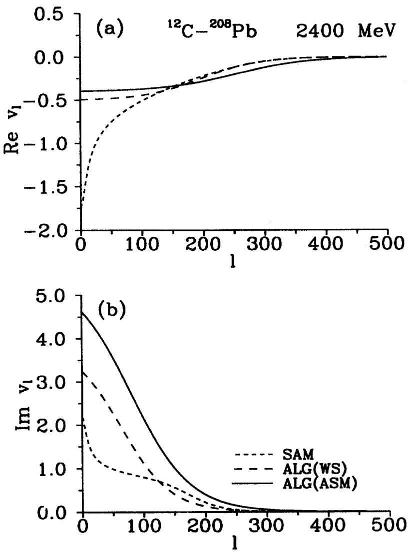

FIG. 4. Algebraic potentials $v_l(k)$ corresponding to the SAM, WS, and ASM S functions for ^{12}C-^{208}Pb scattering at 2400 MeV.

culate the coefficients A_l, B_l, C_l, and D_l. Note, however, that we can no longer attach the same physical meaning to l_g and l'_g as grazing angular momenta, as is the case for the SAM model. As can be seen in Figs. 6 and 8, Eq. (32) leads to a strongly suppressed S functions at low l values, even more strongly suppressed than the SAM, while the WS form Eq. (34) gives a finite value implying more transparency. However, the data are not sensitive to this low-l behavior in the cases we consider here. It is seen from Figs. 3, 4, and 5 that the different algebraic potentials show large variations, particularly for l values less than about 200.

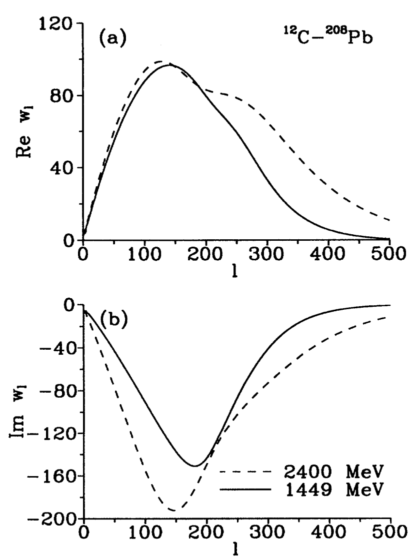

FIG. 5. Algebraic potentials $w_l(k)$ corresponding to the SAM, WS, and ASM S functions of equation for ^{12}C-^{208}Pb scattering at 1449 MeV.

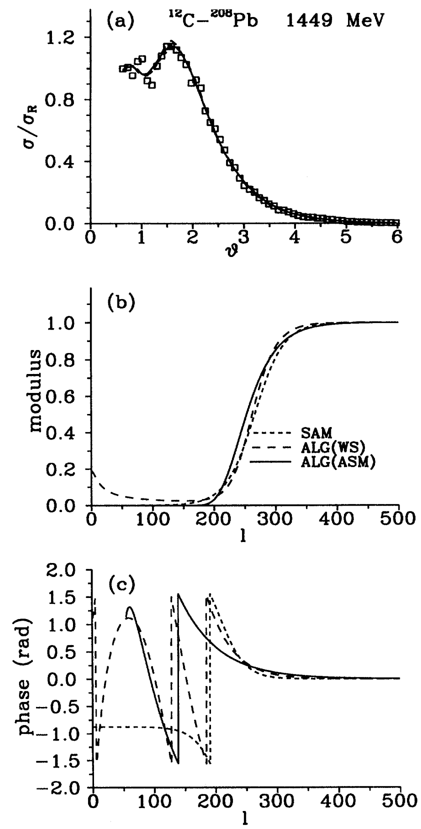

FIG. 6. Cross section and S functions for ^{12}C-^{208}Pb scattering at 1449 MeV.

The algebraic potentials are embedded in a group theoretical formalism, which holds the promise of their being susceptible to microscopic calculation and that they apply to a wider range of scattering problems than heavy ion scattering. These are major advantages compared to the purely phenomenological SAM type S functions.

IV. COORDINATE POTENTIALS FROM ALGEBRAIC POTENTIALS

Hussein, Pato, and Iachello [3] and Hussein and Pato [14] performed a semiclassical analysis of the SO(3,1) S function inverting the algebraic potential $v_l(k)$ to give a physical model in the Schrödinger picture that is reasonable for the peripheral region of coordinate space. A quantal inversion of heavy ion scattering at low energies was performed by Maas and Scheid [15]. Here we employ a semiclassical WKB inversion scheme to obtain potentials from the S functions discussed in the preceding section but it is valid over the whole region of physical relevance of the potential and not only in the periphery.

The coordinate space potential can be derived from the phase shifts using the inverse scattering problem at fixed energy within the WKB approximation. The details of the procedure are discussed elsewhere [6]. The WKB inversion is facilitated if the S function is mapped to the following rational form:

$$S(\lambda) = S^{(0)}(\lambda) \prod_{n=1}^{N} \frac{\lambda^2 - \beta_n^2}{\lambda^2 - \alpha_n^2} , \tag{41}$$

where $\lambda = l + \frac{1}{2}$ is the orbital angular momentum variable and $\{\alpha_n, \beta_n\}$ are a set of (N) poles and zeros of the S matrix. $S^{(0)}(\lambda)$ is a reference S function associated with a reference potential $V^{(0)}$ that is of convenience (an approximate Coulomb potential in this case). Briefly the quasipotential is related to the phase shift $\delta(\lambda)$ by

$$Q(\sigma) = \frac{4E}{\pi}\frac{1}{\sigma}\frac{d}{d\sigma}\int_{\sigma}^{\infty} \frac{\delta(\lambda) d\lambda}{\sqrt{(\lambda^2 - \sigma^2)}} , \tag{42}$$

and the potential $V(\rho = kr)$, is related to $Q(\sigma)$ by

$$V(\rho) = E\left[1 - e^{\left(\frac{Q(\sigma)}{E}\right)}\right] \tag{43}$$

and

$$\rho = \sigma e^{Q(\sigma)/2E}. \tag{44}$$

Using the parametrization implied by Eq. (41) for $\delta(\lambda)$, $Q(\sigma)$ can be found analytically and $V(\sigma)$ is then found by solving the set of transcendental equations given by Eqs. (43) and (44).

The coordinate space potentials, obtained by means of WKB inversion, have been obtained from the McIntyre SAM parametrizations for 125 MeV ^{12}C ions and 170 MeV ^{16}O ions off of ^{208}Pb and the results were considered in detail therein [10]. Here we have obtained potentials by inversion from the S functions corresponding to the two forms of algebraic potentials (WS) and (ASM) and compared these with the potentials corresponding to the SAM parametrization for ^{12}C-^{208}Pb at both 1449 and 2400 MeV. These results are shown in Figs. 6 and 7 for 1449 MeV and in Figs. 8 and 9 for 2400 MeV. The small l-value behavior of the scattering functions that are shown in Figs. 6 and 8 were commented upon in the preceding section.

Furthermore, the local coordinate space potentials corresponding to the various S function parametrizations are shown in Figs. 7 and 9 and they differ considerably at distances less than about 7 to 8 fm, although they all produce comparable values of χ^2/F in fits to the cross-section data. The large degree of ambiguity inherent in the data demonstrates the importance of pursuing the goal of a microscopic derivation of the algebraic potentials to the maximum extent possible.

V. CONCLUSIONS

In the framework of algebraic scattering theory we have presented a scheme to invert the corresponding S functions fitted to the data in terms of the algebraic potentials of the theory, both in general and in the particular cases of the SO(3,1) and SO(3,2) dynamical groups appropriate to nuclear modified heavy ion scattering. If we treat the angular momentum l as a continuous variable as was applicable for the cases considered herein, the method requires the solutions of a nonlinear first-order differential equation. Otherwise solutions of a first-order

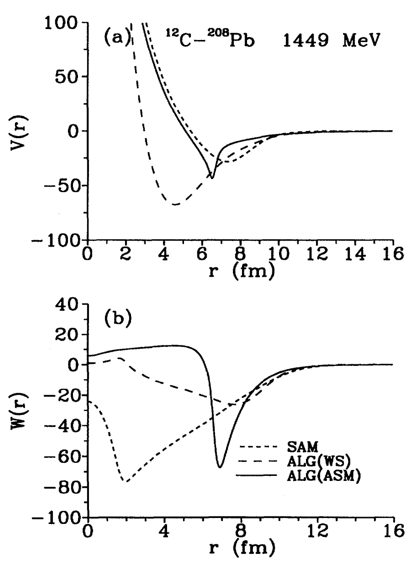

FIG. 7. Coordinate space potentials for ^{12}C-^{208}Pb scattering at 1449 MeV.

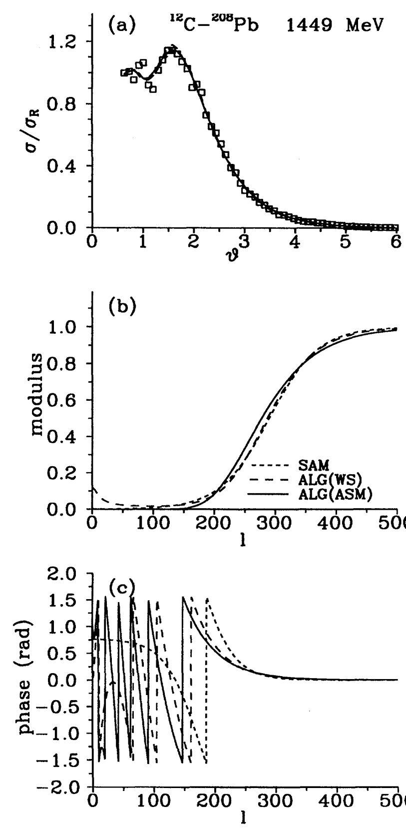

FIG. 8. Cross section and S functions for ^{12}C-^{208}Pb scattering at 2400 MeV.

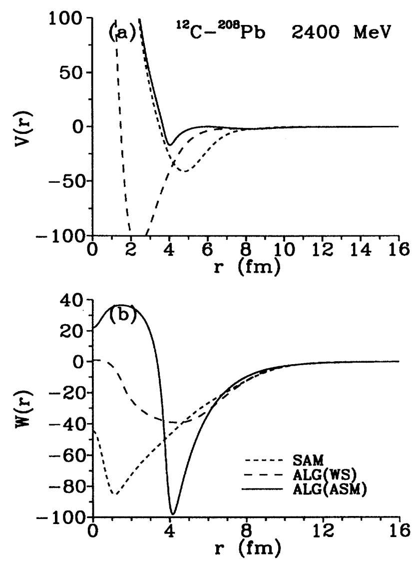

FIG. 9. Coordinate space potentials for ^{12}C-^{208}Pb scattering at 2400 MeV.

difference equation are to be found. The analogy with conventional inverse scattering theory is obvious. This new inversion scheme is of particular interest where the algebraic formalism can largely replace the usual analyses of the scattering problem via the Schrödinger equation in coordinate space.

We have considered representative heavy ion scattering data for ^{12}C, ^{14}N, and ^{16}O on ^{208}Pb from 170 to 2400 MeV and analyzed them by means of the widely used McIntyre parametrization of the S function. In the large l limit a comparison of the McIntyre S function to the algebraic ones leads to a generalization of the Woods-Saxon shape proposed [1, 2] for the algebraic potentials $v_l(k)$ and $w_l(k)$ of the SO(3,1) and SO(3,2) dynamical groups. The data have also been analyzed directly in terms of the corresponding S functions and fits of comparable quality to the McIntyre S functions have been obtained, although the resultant algebraic S functions are not identical. The algebraic potentials $v_l(k)$ and $w_l(k)$ exactly equivalent to the McIntyre S function have also been obtained for comparison purposes for ^{12}C-^{208}Pb at 1449 and 2400 MeV. They differ considerably from the Woods-Saxon and ASM (asymptotic) parametrizations, which also differ among themselves for l less than about 200.

In addition, we calculated the corresponding coordinate space local potentials for ^{12}C-^{208}Pb at 1449 and 2400 MeV via semiclassical inversion. The potentials obtained in this way also display a considerable degree of variation especially at radial distances smaller than the sensitive region, much more so than the corresponding S functions.

It is an interesting feature of our results that both the algebraic potentials and the coordinate space potentials are only reasonably well determined in their large l- and r-space regions, respectively, by the available data. This also applies, though to a lesser extent, to the S functions themselves. It is a general feature of inverse scattering problems that differences in the S functions are magnified in the potentials. The algebraic potentials are no exception to this rule.

Our present results indicate that the analytical generalizations we suggest of the proposed Woods-Saxon algebraic potentials of Alhassid and co-workers [1, 2] for heavy ion scattering, are quite adequate for a good description of the data. They provide a similar level of accuracy as the usual SAM type S functions, which are widely used in the literature. When more constraints on the shapes of algebraic potentials can be derived from a microscopic theory or better data become available the algebraic inversion scheme presented here could be used in their determination to a greater extent than is now possible. The same applies to the corresponding coordinate space potentials. The major advantage of the algebraic potentials is that, at a theoretical level, they are more directly related to the S functions than are coordinate space potentials.

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