Crossing-symmetric two-particle reduction of four-point vertex

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We develop a nonperturbative reduction scheme for the four-point vertex T of a generic field theory with interactions among bosons, fermions, and antifermions. We exhibit integral equations which express, in a manifestly crossing-symmetric way, the full vertex in terms of its two-particle irreducible part, together with the dressed three-point vertices and two-point propagators. This scheme generalizes the usual summation of ladder or bubble diagrams, thus providing for the consistent summation of a larger subset of all diagrams.

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The four-point vertex T describes some of the most interesting results of a field theory, since it contains the amplitudes for two-body scattering and for annihilation to two-body final states. The one- and two-body intermediate states play a special role in these processes. Formally, these states are the starting point for bootstrap-type dispersion relations [1] and for Green's function methods in solid-state physics [2] as well as methods introducing parametrized effective interactions [3–8]. Physically, they are important because they dominate the low-energy scattering, since states with more particles typically have higher energy [9]. As a result, it is often important to consider these states explicitly by distinguishing their contributions to the four-point vertex. A partial reduction of this sort is often accomplished by Lippmann-Schwinger or Bethe-Salpeter type integral equations which sum ladders or bubbles, respectively. We present equations which accomplish a complete reduction to a set of fully one- and two-body irreducible kernels.

Here we consider the four-point vertex of a generic field theory with fermions and bosons. Our analysis is independent of the form of the bare interaction, which plays no role in the discussion. Crossing-symmetric equations have been constructed for the pion-nucleon system in Ref. [10], but with the unitary summation of only boson rescattering. We generalize that approach to include the scattering of fermions in both the initial and final states and also in intermediate states, thus allowing for the description of fermion-fermion scattering (and related crossed-channel processes) as well as fermion-boson scattering, and including a much larger class of diagrams for boson-boson scattering. We use the Bethe-Salpeter type equations exploited also in Ref. [11]. While the general form of the equations we develop is quite similar to Refs. [11,12] (the idea goes back to Bethe and Salpeter), there are considerable differences which allow a much broader application. In particular, our treatment of pole contributions to the scattering kernel solves an overcounting difficulty which in Ref. [11] prevented a consistent treatment of one-particle intermediate states. Reference [12] treats the counting problems correctly but considers only systems of either bosons or fermions. The nonrelativistic analog of this problem (potential interaction) has also been considered by many authors [12–15]. The equations for the four-fermion interaction matrix are of the familiar Bethe-Salpeter type, but the bosonic channels are lacking. Our work represents a generalization of Ref. [12] to the case when both bosons and fermions are present, and treats them on an equal footing.

We need to express the four-fermion vertex in terms of the three-point vertex and full propagators. Our aim is to treat all the channels equally, as required by crossing symmetry of scattering amplitudes. In the language of perturbation theory this also means a resummation of a much larger class of diagrams than when treating one channel preferentially, i.e., summing ladder or bubble diagrams only. The price to pay is the proliferation of unknown vertex functions including those corresponding to boson-fermion and boson-boson scattering (and related processes in crossed channels).

The general idea for constructing crossing-symmetric equations is quite simple [10–15]. First, one has to introduce the one-particle irreducible (proper) four-fermion vertex [16]. It is obtained from the four-point connected Green's function by separating the one-particle reducible contributions and then truncating the fermion propagators on the legs (Fig. 1). Next, one separates the contributions to the four-point proper vertex with respect to two-particle reducibility in different channels. Then one writes Bethe-Salpeter type equations for the two-particle reducible contributions in a definite channel, in terms of the two-particle irreducible (2PI) diagrams in that channel. For the four-fermion vertex T we use the following decomposition:

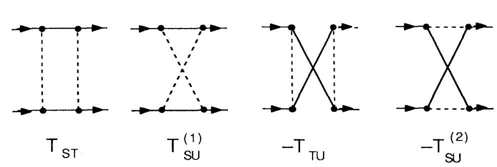

FIG. 1. The four-fermion vertex contributions which are two-particle reducible in more than one channel.

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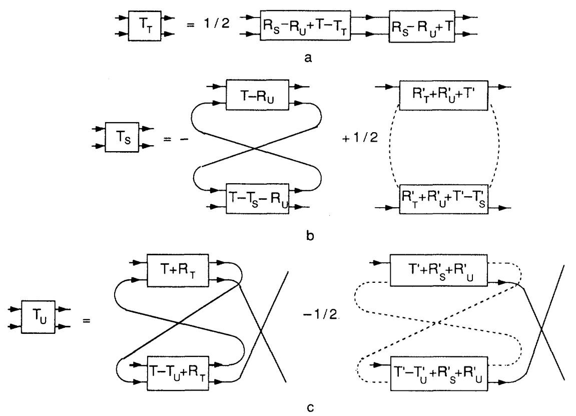

FIG. 2. Bethe-Salpeter type equations for the contributions T_T, T_S, and T_U to the four-fermion vertex T.

$$T=T_I+T_S+T_T+T_U-T_{ST}-T_{SU}-T_{TU} \ , \quad (1)$$

where T_I is 2PI in all channels; T_S, T_T, and T_U are two-particle reducible in s, t, and u channels, respectively; and T_{ST}, T_{SU}, and T_{TU} are two-particle reducible in both of the two indicated channels. The subtraction of the last three terms is necessary to avoid double counting, since these diagrams will appear twice in the sum $T_S+T_T+T_U$. The existence of diagrams reducible in two channels is a consequence of our not distinguishing between boson and fermion reducibility. However, by inspection it is easy to see that the number of diagrams reducible in more than one channel is very limited. They are all given in Fig. 1 and we do not need any additional equations for them. Note that the propagators and three-point vertices are the full ones, i.e., these are skeleton graphs.

We start with the equation for T_T, the vertex reducible in the t channel, which we take to correspond to the fermion-fermion or antifermion-antifermion scattering; see Fig. 2(a). In a somewhat condensed notation we write the equation in the following form:

$$T_T=\tfrac{1}{2}(R_S-R_U+T-T_T)G_T(R_S-R_U+T) \ . \quad (2)$$

G_T is the product of two full fermion propagators in the t channel, and R_S and R_U are the one-particle reducible contributions to the four-fermion connected Green's function defined in Fig. 3. It is necessary to include them, since in the t channel they generate 1PI contributions. On the other hand, R_T has to be excluded since it would then generate one-particle reducible terms. Note

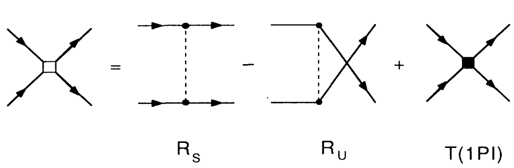

FIG. 3. Decomposition of the truncated four-fermion Green's function into one-particle reducible and one-particle irreducible terms.

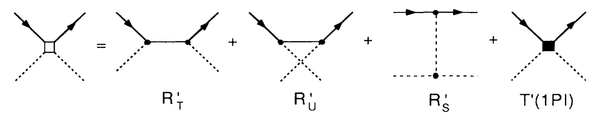

FIG. 4. Decomposition of the truncated two-fermion–two-boson Green's function into one-particle reducible and 1PI terms.

that this result differs from Ref. [11] by treating all the pole terms equally, and solves the related difficulty discussed there. The term $R_S-R_U+T-T_T$ includes all diagrams which are not two-particle reducible in the t channel, while the rest of the diagram, connected by two fermion lines, can be any connected 1PI (looking in the t channel) diagram. The factor $\frac{1}{2}$ compensates for generating each diagram twice, since R_S and R_U are related by the Fermi statistics, and T and T_T possess the correct (anti)symmetry with respect to the exchange of the incoming (1,2) or outgoing (3,4) legs.

The equation for T_S is analogous to expression (2). Since only 1PI diagrams contribute, R_S should not be included in the equation, but R_U still gives a 1PI contribution. The total fermion number flowing through this channel is zero and this means that the two-particle reducibility can also be of the "bosonic" type, i.e., we can separate the diagram into two disconnected pieces by cutting two boson lines. This forces us to consider four-point vertices with two fermion and two boson legs (distinguished by a prime from four-fermion vertices). First, we decompose the truncated Green's function into one-particle reducible and 1PI terms (Fig. 4). Then, similar to expression (1) for the four-fermion vertex, we perform the decomposition with respect to two-particle reducibility in different channels:

$$T'=T'_I+T'_S+T'_T+T'_U-T'_{ST}-T'_{SU}-T'_{TU} \ . \quad (3)$$

The last two terms on the right-hand side of the above expression are the terms reducible in more than one channel and they are shown in Fig. 5. The equations for the

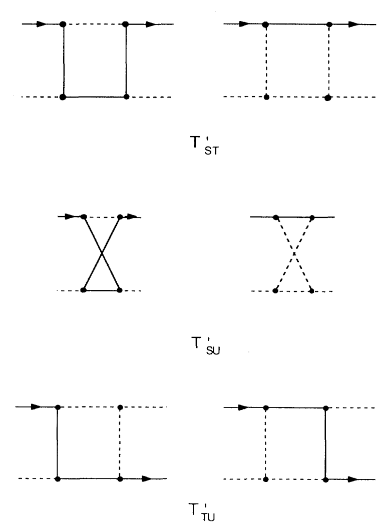

FIG. 5. The two-fermion–two-boson vertex contributions reducible in more than one channel.

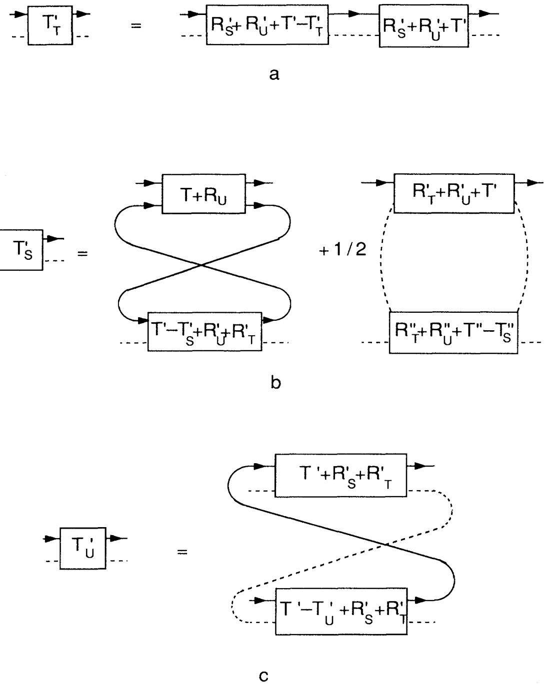

FIG. 6. Bethe-Salpeter type equations for the contributions T'_T, T'_S, and T'_U to the two-fermion–two-boson vertex T'.

other terms are given below. Now we can write the equation for T_S:

$$T_S=-(-R_U+T-T_S)G_S^{(F)}(-R_U+T)+\tfrac{1}{2}(T'-T'_S+R'_S+R'_U)G_S^{(B)}(T'+R'_S+R'_U)\ , \tag{4}$$

where $G_S^{(F)}$ denotes the product of two fermion propagators in the s channel, while $G_S^{(B)}$ corresponds to two boson propagators [for the graphical representation of the equation see Fig. 2(b)]. We note that there is no overlap between these two sets of diagrams [those with $G_S^{(F)}$ and $G_S^{(B)}$ in expression (4)], since the first two-particle reducibility (encountered when going from the bottom to the top) of the diagram is in one case of fermion and in the other of boson type.

Finally, we write the equation for T_U [Fig. 2(c)]:

$$T_U=(R_S+T-T_U)G_U(R_S+T)-\tfrac{1}{2}(T'-T'_U+R'_S+R'_U)G_U^{(B)}(T'+R'_S+R'_U)\ , \tag{5}$$

which is analogous to the contribution for T_S. In the first term R_S replaces R_U, since the latter would give a one-particle reducible contribution, leading to overcounting. The second term is analogous to the one in Eq. (4), but with a sign changed, assuring the antisymmetry under exchange of outgoing fermions.

The four-point vertex with two fermion and two boson legs, T', can be treated in the same way as the four-fermion vertex T. The equations for the terms T'_S, T'_T, and T'_U in the decomposition (3) are graphically shown in Fig. 6. Algebraically they read

$$T'_T=(T'-T'_T+R'_U+R'_S)G_T^{(BF)}(T'+R'_U+R'_S)\ , \tag{6a}$$

$$T'_S=\tfrac{1}{2}(T''-T''_S+R''_T+R''_U)G_S^{(B)}(T'+R'_T+R'_U)+(T'-T'_S+R'_T+R'_U)G_S^{(F)}(T+R_U)\ , \tag{6b}$$

$$T'_U=(T'-T'_U+R'_S+R'_T)G_U^{(BF)}(T'+R'_S+R'_T)\ . \tag{6c}$$

Equation (6b) represents one of the two possibilities of writing down the contributions to the s-channel reducible terms. The other is to divide each diagram into a part not s-channel reducible starting from the fermion lines [in distinction from starting with the boson lines in Eq. (6b)] and the rest of the diagram, leading to an alternative equation:

$$T'_S=\tfrac{1}{2}(T''+R''_T+R''_U)G_S^{(B)}(T'+R'_S+R'_U-T'_S)+(T'+R'_S+R'_U)G_S^{(F)}(T-T_S+R_U)\ . \tag{6b'}$$

The double primed symbols refer to (truncated) four-boson Green's functions; see Fig. 7 for their definition. The labeling of the left-hand side of Eq. (6c) is somewhat misleading, implying the contribution of diagrams reducible in the u channel. What it actually means is shown in Fig. 6(c). These diagrams are necessary and insure the symmetry of T' under exchange of the two external boson legs. We checked that in the sum $T'_S+T'_T+T'_U$ the diagrams T'_{ST}, T'_{SU}, and T'_{TU} are generated correctly in this way, i.e., each of them twice.

Finally, we have to construct the equations satisfied by the four-boson proper vertex T'', defined in Fig. 7. After classifying the contributions with respect to two-particle reducibility

$$T''=T''_I+T''_S+T''_T+T''_U-T''_{ST}-T''_{SU}-T''_{TU}\ , \tag{7}$$

where the doubly reducible contributions are shown in Fig. 8, we can write the equations for the two-particle reducible contributions (Fig. 9):

$$T''_S=\tfrac{1}{2}(T''+R''_T+R''_U)G_S^{(B)}(T''+R''_T+R''_U-T''_S)+(T'+R'_T+R'_U)G_S^{(F)}(T'+R'_T+R'_U-T'_S)\ , \tag{8a}$$

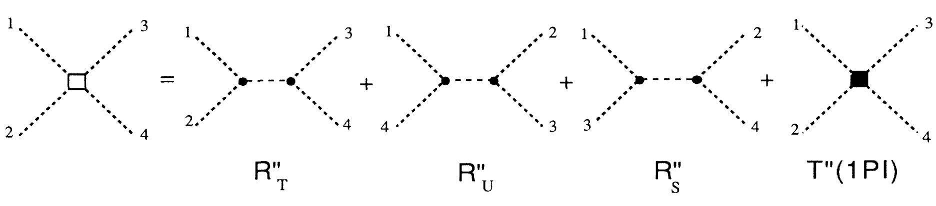

FIG. 7. Decomposition of the truncated four-boson Green's function into one-particle reducible and 1PI terms.

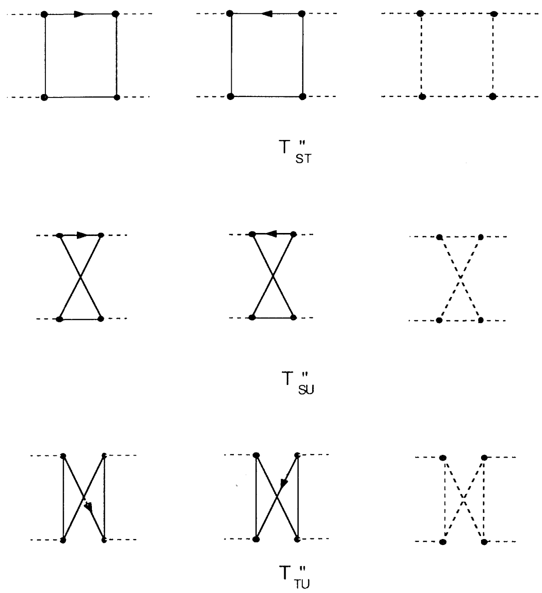

FIG. 8. The four-boson vertex contributions which are two-particle reducible in more than one channel.

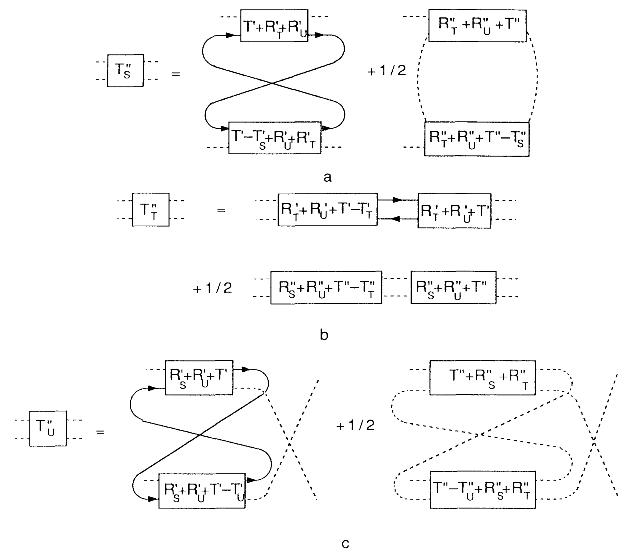

FIG. 9. Bethe-Salpeter type equations for the contributions T''_S, T''_T, and T''_U to the four-boson vertex T''.

$$T''_T=\tfrac{1}{2}(T''+R''_S+R''_U)G^{(B)}_T(T''+R''_S+R''_U-T''_T)+(T'+R'_S+R'_U)G^{(F)}_T(T'+R'_S+R'_U-T'_T)\ , \qquad (8b)$$

$$T''_U=\tfrac{1}{2}(T''+R''_S+R''_T)G^{(B)}_U(T''+R''_S+R''_T-T''_U)+(T'+R'_S+R'_U)G^{(F)}_U(T'+R'_S+R'_U-T'_U)\ . \qquad (8c)$$

The parts of the four-boson vertex reducible in more than one channel, T''_{ST}, T''_{SU}, and T''_{TU}, are skeleton graphs analogous to the contributions of this type to T and T'.

Until now we did not say anything about the two-particle irreducible contributions to the considered four-particle vertices, T_I, T'_I, and T''_I. The reduction is especially useful when the particles involved have masses, or are coupled by gradient couplings. In these cases, the description of low-energy scattering and annihilation may be simplified by our reduction, since then the 2PI parts are used in kinematic regimes where they are not singular. This means that they might be small, or if not they can be parametrized in a simple way [3]. It also means that they may be dominated by a few skeleton graphs, as is the case for nonrelativistic potential scattering [13]. We discuss the applications of the above result in another publication [9].

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