# Pion electroproduction at threshold

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By using the partially conserved axial-vector current hypothesis and the electromagnetic current conservation law, we calculate transverse and longitudinal multipoles for threshold pion electroproduction on the nucleon. We include off-shell form factors for the axial-vector current, the electromagnetic current, and the pion-nucleon vertex. We confirm that if the electromagnetic off-shell form factors are assumed to be finite in the chiral limit of the vanishing pion mass, the low-energy theorems are unaffected by them. However, we show that if one of these form factors has simple-pole singularity in the chiral limit, it shows up in the threshold amplitudes at next-to-leading order predicted by the low-energy theorems.

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## I. INTRODUCTION

It is believed that chiral symmetry of the massless quantum chromodynamics (QCD) is spontaneously broken by the vacuum and that the pion emerges as a massless Nambu-Goldstone boson. The small pion mass $m_\pi$ is attributed to an explicit chiral symmetry breaking by the small quark mass term in the QCD Lagrangian that leads to the partial conservation of the axial-vector current (PCAC) [1]. The PCAC hypothesis has been examined in various aspects. Low-energy theorems have been derived using the PCAC relation and the chiral algebra of currents [2–10]. Recent accurate measurements [11,12] of pion photoproduction close to threshold are of compelling interest in this respect and indeed elicit much discussion about the validity of the predictions of the low-energy theorems.

These low-energy theorems are written in terms of an expansion in powers of the pion to nucleon mass ratio, $\mu=m_\pi/m$. For $\pi^0$ production, the low-energy theorems predict terms up to order $\mu^2$ (next-to-leading order for the reaction $\gamma p\to p\pi^0$ and leading order for the reaction $\gamma n\to n\pi^0$). The coefficients are given by global properties of the nucleon, namely, mass, charge, magnetic moments, and the pion-nucleon coupling constant. Recently, however, Bernard, Kaiser, and Meissner [13] have calculated pion photoproduction amplitudes at threshold within the framework of chiral perturbation theory and found that the triangle diagram that contributes to the half off-shell photon-nucleon vertex produces effects of order $\mu^2$, namely, of the same order as predicted by the low-energy theorems. They argue that one-loop diagrams develop singularities in the limit of the vanishing pion mass and yield a large contribution to the threshold amplitudes. They also calculated $\pi^0$ electroproduction amplitudes [14] and found contributions of order $\nu_2=-k^2/m^2$, with $k$ being the four-momentum of the virtual photon, while Scherer and Koch [10] predicted that model-dependent terms start from order $\mu\nu_2$.

Since derivation of low-energy theorems relies only on general principles, namely, PCAC, current algebra, gauge invariance, Lorentz invariance, and crossing symmetry, it is quite unlikely that these theorems are violated in any event [15]. It is desired that the finding of Bernard *et al.* [13,14] should be understood within the same framework as the low-energy theorems are derived. Since the singularities Bernard *et al.* [13,14] found are related to off-shell form factors of the nucleon, it is necessary to reexamine their role in low-energy theorems. Although it is believed that low-energy theorems are not affected by inclusion of off-shell form factors, this conclusion was deduced from the premises that these form factors do not depend on the pion mass. Furthermore, as was pointed out in the previous paper [16], off-shell form factors of the axial-vector current were not considered for the calculation of its four-divergence so that the Ward-Takahashi identity [17] was not satisfied and therefore off-shell effects of pion electroproduction were not treated consistently in the literature. The purpose of this paper is to rederive low-energy theorems paying particular attention to the effects of off-shell form factors and to show that model-dependent terms appear at order $\mu^2$ and $\nu_2$ for neutral pion production and at order $\mu$ and $\nu_2$ for charged production.

The present paper is organized as follows. In Sec. II we define off-shell form factors for the axial-vector current, the electromagnetic current, and the pion-nucleon vertex function. Consequences of PCAC and gauge invariance are examined for these form factors and for the radiative pion decay vertex. In Sec. III we derive the PCAC constraint condition on the electroproduction amplitude. In Sec. IV we calculate electroproduction multipoles at threshold. A brief summary is given in Sec. V.

## II. VERTEX FUNCTIONS

We assume that the isovector axial-vector current $j_\mu^{Ai}(x)$ at space-time position $x$ and the physical pion field $\phi_\pi^i(x)$ satisfy the operator PCAC relation

$$\partial_\mu j_\mu^{Ai}(x)=f_\pi m_\pi^2\phi_\pi^i(x)\ , \tag{2.1}$$

 

where $f_\pi$ is the pion decay constant defined by the matrix element between the vacuum and the one physical pion state,

$$\langle 0|j_\mu^{Ai}(0)|\pi^j(q)\rangle = i\delta^{ij} f_\pi q_\mu \ . \tag{2.2}$$

The superscript is an isospin index. The nucleon matrix elements of the current operator and the pion source function are

$$\langle N(p')|j_\mu^{Ai}(0)|N(p)\rangle = \bar{u}(p') j_\mu^{Ai}(p',p) u(p) \ , \tag{2.3}$$

$$\langle N(p')|J_\pi^i(0)|N(p)\rangle = \bar{u}(p')\Gamma^i(p',p) u(p) \ , \tag{2.4}$$

where $u(p)$ and $\bar{u}(p')$ are Dirac bispinors of the nucleon in the initial and final states, respectively. The source function is defined by

$$(-\partial_\mu^2 + m_\pi^2)\phi_\pi^i(x) = J_\pi^i(x) \ . \tag{2.5}$$

The nucleon matrix element of the isovector current operator has the factor $\frac{1}{2}\tau^i$ while that of the pion source has $\tau^i$ so that there appears the factor $\frac{1}{2}$ repeatedly. We use a convention in which the isospin dependence can be factored as

$$j_\mu^{Ai}(p',p) = \tau^i j_\mu^A(p',p) \ , \tag{2.6}$$

$$\Gamma^i(p',p) = \tau^i \Gamma(p',p) \ . \tag{2.7}$$

Namely, $\frac{1}{2} j_\mu^{Ai}(x)$ is defined to be the current operator and correspondingly $f_\pi$ is defined to be twice as large as the standard definition. Therefore, the $\pi^\pm$ decay constant is $f_\pi/\sqrt{2}$. The PCAC relation (2.1) in momentum space becomes

$$(p-p')_\mu j_\mu^A(p',p) = -(i\gamma\cdot p' + m)\gamma_5 - \gamma_5(i\gamma\cdot p + m) - i f_\pi m_\pi^2 \frac{1}{(p-p')^2 + m_\pi^2}\Gamma(p',p) \ , \tag{2.8}$$

where (2.5) is used. To be as transparent as possible, we avoid the complication caused by the use of the dressed propagators for nucleons and pions, and employ the free propagators for them. In discussing low-energy phenomena, it is advantageous to separate the pion-pole term from the axial-vector current,

$$j_\mu^A(p',p) = \bar{j}_\mu^A(p',p) + i f_\pi (p-p')_\mu \frac{1}{(p-p')^2 + m_\pi^2}\Gamma(p',p) \ . \tag{2.9}$$

The nonpole part $\bar{j}_\mu^A(p',p)$ must then satisfy

$$(p-p')_\mu \bar{j}_\mu^A(p',p) = -(i\gamma\cdot p' + m)\gamma_5 - \gamma_5(i\gamma\cdot p + m) - i f_\pi \Gamma(p',p) \ . \tag{2.10}$$

In deriving low-energy theorems we want to investigate the effect of off-shell current matrix elements but the literature is not available concerning the off-shell form factors of the axial-vector current. We examine the structure of the axial-vector current from the PCAC equation (2.10). Let us consider the process in which an on-shell nucleon with four-momentum $p$ interacts with the axial-vector current carrying momentum $q$ and makes a transition to the state with momentum $p-q$ that is not necessarily on the mass shell. The most general form of the half off-shell matrix element is characterized by six independent form factors. We take the form

$$\bar{j}_\mu^A(p-q,p)u(p) = \{ iF_1^A \gamma_\mu\gamma_5 - \bar{F}_2^A q_\mu \gamma_5 + iF_3^A \sigma_{\mu\nu} q_\nu \gamma_5 + [i\gamma\cdot(p-q) + m](iF_4^A \gamma_\mu\gamma_5 - \bar{F}_5^A q_\mu\gamma_5 + iF_6^A \sigma_{\mu\nu} q_\nu \gamma_5)\} u(p) \ . \tag{2.11}$$

Similarly, the half off-shell pion-nucleon vertex function has two independent form factors, and we write it as

$$\Gamma(p-q,p)u(p) = i\{ g_1 + [i\gamma\cdot(p-q) + m] g_2 \}\gamma_5 u(p) \ . \tag{2.12}$$

The PCAC equation (2.10) leads to the two constraints among form factors

$$F_1^A + q^2 \bar{F}_5^A = 1 - f_\pi g_2 \ , \tag{2.13}$$

$$2mF_1^A - q^2 \bar{F}_2^A + [(p-q)^2 + m^2] F_4^A = f_\pi g_1 \ . \tag{2.14}$$

The form factors $F_3^A$ and $F_6^A$ are unconstrained by the PCAC relation. All form factors are functions of $q^2$ as well as the invariant mass squared of the off-shell nucleon $(p-q)^2$, but we restrict ourselves within the form factors that depend on $q^2$ alone. (This is assumed only for simplicity. The most general case is discussed in the previous paper [16].) This assumption leads to $F_3^A = 0$ from the symmetry under charge conjugation. Furthermore, we must have $F_4^A = 0$ for (2.14) to be valid at arbitrary $(p-q)^2$. The Goldberger-Treiman relation [18]

$$2mF_1^A(0) = f_\pi g_1(0) \tag{2.15}$$

follows from (2.14) in the soft limit $q=0$. Eliminating $F_1^A$ from (2.13) and (2.14), one finds for the real pion with $q^2 = -m_\pi^2$,

$$\frac{1}{f_\pi} - g_2 = \frac{g_1}{2m}(1 - \alpha\mu^2) \ , \tag{2.16}$$

where $\mu = m_\pi / m$ and $\alpha$ is defined by

$$\alpha = \frac{m^2}{f_\pi g_1}(\bar{F}_2^A + 2m\bar{F}_5^A) . \tag{2.17}$$

Equation (2.16) will turn out to be of crucial importance in deriving low-energy theorems.

Since $\bar{F}_2^A$ and $\bar{F}_5^A$ are known from $F_1^A$, $g_1$, and $g_2$, the half off-shell current operator becomes

$$\bar{j}_\mu^A(p-q,p) = iF_1^A\left[\gamma_\mu - \frac{q_\mu}{q^2}\gamma\cdot q\right]\gamma_5 + [i\gamma\cdot(p-q)+m]\left[-\frac{q_\mu}{q^2} + iF_6^A\sigma_{\mu\nu}q_\nu\right]\gamma_5 - if_\pi\frac{q_\mu}{q^2}\Gamma(p-q,p) . \tag{2.18}$$

In exactly the same manner, when the nucleon with momentum $p'$ is on shell, we get the current operator in the form

$$\bar{j}_\mu^A(p',p'+q) = iF_1^A\left[\gamma_\mu - \frac{q_\mu}{q^2}\gamma\cdot q\right]\gamma_5 + \left[-\frac{q_\mu}{q^2} - iF_6^A\sigma_{\mu\nu}q_\nu\right]\gamma_5[i\gamma\cdot(p'+q)+m] - if_\pi\frac{q_\mu}{q^2}\Gamma(p',p'+q) . \tag{2.19}$$

Note that the sign of the $F_6^A$ term is reversed because of the charge conjugation symmetry. When the nucleon stays on the mass shell in the initial and final states, (2.18) and (2.19) are reduced to the standard expression

$$\begin{aligned}\bar{j}_\mu^A(p',p) &= iF_1^A\left[\gamma_\mu - \frac{q_\mu}{q^2}\gamma\cdot q\right]\gamma_5 + f_\pi g_1\frac{q_\mu}{q^2}\gamma_5 \\ &= iF_1^A\gamma_\mu\gamma_5 - \bar{F}_2^A q_\mu\gamma_5 ,\end{aligned} \tag{2.20}$$

where the Dirac equations are used on both sides of the current.

As for the off-shell structure of the electromagnetic current there exists the literature [19-24,9]. The current conservation $\partial_\nu j_\nu(x)=0$ implies, in momentum space,

$$\langle N(p')|j_\nu(0)|N(p)\rangle = \bar{u}(p')j_\nu(p',p)u(p) , \tag{2.21}$$

the Ward-Takahashi identity

$$(p'-p)_\nu j_\nu(p',p) = ie_N\gamma\cdot(p'-p) , \tag{2.22}$$

where

$$e_N = \tfrac{1}{2}e(1+\tau^3) \tag{2.23}$$

is the nucleon charge operator and $e$ is the proton charge. We again use the free propagator for the nucleon. We parametrize the half off-shell current operator using six independent form factors as

$$j_\nu(p+k,p)u(p) = \{iF_1\gamma_\nu - iF_2\sigma_{\nu\lambda}k_\lambda + F_3k_\nu + [i\gamma\cdot(p+k)+m](iF_4\gamma_\nu - iF_5\sigma_{\nu\lambda}k_\lambda + F_6k_\nu)\}u(p) . \tag{2.24}$$

Each form factor has isoscalar and isovector components,

$$F_{1,\ldots,6} = \tfrac{1}{2}e(F_{1,\ldots,6}^S + \tau^3F_{1,\ldots,6}^V) . \tag{2.25}$$

The identity (2.22) imposes constraints

$$F_1 + 2mF_4 + k^2F_6 = e_N , \tag{2.26}$$

$$k^2F_3 - [(p+k)^2+m^2]F_4 = 0 . \tag{2.27}$$

We again assume that all form factors are functions of $k^2$ alone. The charge conjugation symmetry then requires $F_3=0$. Therefore we must have $F_4=0$ from (2.27). As a result $F_6$ is not independent any more but is completely determined by $F_1$,

$$F_6 = \frac{e_N - F_1}{k^2} . \tag{2.28}$$

Consequently, the half off-shell electromagnetic current operator becomes

$$\begin{aligned}j_\nu(p+k,p) = ie_N\frac{k_\nu}{k^2}\gamma\cdot k + iF_1\left[\gamma_\nu - \frac{k_\nu}{k^2}\gamma\cdot k\right] \\ - i\{F_2 + [i\gamma\cdot(p+k)+m]F_5\}\sigma_{\nu\lambda}k_\lambda ,\end{aligned} \tag{2.29}$$

and

$$\begin{aligned}j_\nu(p',p'-k) = ie_N\frac{k_\nu}{k^2}\gamma\cdot k + iF_1\left[\gamma_\nu - \frac{k_\nu}{k^2}\gamma\cdot k\right] \\ - i\sigma_{\nu\lambda}k_\lambda\{F_2 + [i\gamma\cdot(p'-k)+m]F_5\} .\end{aligned} \tag{2.30}$$

The photon-pion interaction vertex function is also derived in a similar way. In momentum space

$$\langle \pi^i(q)|j_\nu(0)|\pi^j(q-k)\rangle = j_\nu^{\pi ij}(q,q-k) , \tag{2.31}$$

with

$$j_\nu^{\pi ij}(q,q-k) = -ie\epsilon^{3ij}j_\nu^\pi(q,q-k) , \tag{2.32}$$

the Ward-Takahashi identity

$$k_\nu j_\nu^\pi(q,q-k) = q^2 - (q-k)^2 \tag{2.33}$$

constrains the form of the current operator. Using the electromagnetic form factor $F_\pi$ of pions, which is assumed to be a function of $k^2$, we find

$$j_\nu^\pi(q,q-k)=(F_\pi-1)\left[\delta_{\nu\lambda}-\frac{k_\nu k_\lambda}{k^2}\right](2q-k)_\lambda +(2q-k)_\nu \,. \qquad (2.34)$$

To conclude this section we examine the photon–pion–axial-vector vertex function. We are concerned with the matrix element

$$\langle 0|j_\mu^{Ai}(0)|\pi^j(q-k),\gamma(k)\rangle = S_{\mu\nu}^{ij}\epsilon_\nu(k) \,, \qquad (2.35)$$

where $\epsilon_\nu(k)$ is the polarization vector of the virtual photon with four-momentum $k$. The structure function $S_{\mu\nu}^{ij}$ has the trivial isospin dependence

$$S_{\mu\nu}^{ij}=-ie\epsilon^{3ij}S_{\mu\nu} \,. \qquad (2.36)$$

Since the axial-vector current and the electromagnetic current are involved, the isospin independent function $S_{\mu\nu}$ must satisfy two Ward-Takahashi identities. In the Appendix their derivation is given. From the PCAC,

$$q_\mu S_{\mu\nu}=if_\pi(q-k)_\nu-if_\pi m_\pi^2\frac{1}{q^2+m_\pi^2}j_\nu^\pi(q,q-k) \,, \qquad (2.37)$$

and from the electromagnetic current conservation,

$$k_\nu S_{\mu\nu}=if_\pi(q-k)_\mu-if_\pi q_\mu\frac{(q-k)^2+m_\pi^2}{q^2+m_\pi^2} \,. \qquad (2.38)$$

We can split $S_{\mu\nu}$ into known parts and the rest

$$S_{\mu\nu}=-if_\pi\delta_{\mu\nu}+if_\pi q_\mu\frac{1}{q^2+m_\pi^2}j_\nu^\pi(q,q-k)+\bar{S}_{\mu\nu} \,, \qquad (2.39)$$

where the first part comes from the minimal coupling to (2.2) and the second term from the tree diagram with the pion pole (the pion with momentum $q-k$ adsorbs a photon and propagates with momentum $q$ before being annihilated). From (2.37) we get

$$q_\mu\bar{S}_{\mu\nu}=if_\pi\{(2q-k)_\nu-j_\nu^\pi(q,q-k)\} \,, \qquad (2.40)$$

and from (2.38)

$$k_\nu\bar{S}_{\mu\nu}=0 \,. \qquad (2.41)$$

Consequently we find that $\bar{S}_{\mu\nu}$ has the form

$$\bar{S}_{\mu\nu}=-2if_\pi(F_\pi-1)\left[\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right] -if_\pi(k_\mu q_\nu-q\cdot k\delta_{\mu\nu})S \,, \qquad (2.42)$$

where (2.34) is used and $S$ is the structure function that cannot be determined by the conservation law or the PCAC [25].

## III. PCAC CONSTRAINT ON ELECTROPRODUCTION

The low-energy theorem follows from the PCAC relation and the current conservation law applied to the radiative axial-vector vertex

$$\langle N(p')|j_\mu^{Ai}(0)|N(p),\gamma(k)\rangle=\bar{u}(p')M_{\mu\nu}^i u(p)\epsilon_\nu(k) \,. \qquad (3.1)$$

The PCAC relation implies [25]

$$q_\mu M_{\mu\nu}^i=-ie\epsilon^{3ij}j_\nu^{Aj}(p',p)-if_\pi m_\pi^2\frac{1}{q^2+m_\pi^2}M_\nu^i \qquad (3.2)$$

and the current conservation law

$$k_\nu M_{\mu\nu}^i=-ie\epsilon^{3ij}j_\mu^{Aj}(p',p) \,. \qquad (3.3)$$

In Eq. (3.2), $M_\nu^i$ is the virtual-pion photoproduction operator defined by

$$\langle N(p')|J_\pi^i(0)|N(p),\gamma(k)\rangle=\bar{u}(p')M_\nu^i u(p)\epsilon_\nu(k) \qquad (3.4)$$

which satisfies the gauge-invariance requirement

$$k_\nu M_\nu^i=-ie\epsilon^{3ij}\frac{q^2+m_\pi^2}{(q-k)^2+m_\pi^2}i\tau^j g_1((q-k)^2)\gamma_5 \,. \qquad (3.5)$$

In taking the low-energy limit we have to separate the infrared divergent term. To this end we split $M_{\mu\nu}^i$ into three pieces

$$M_{\mu\nu}^i=B_{\mu\nu}^i+G_{\mu\nu}^i+R_{\mu\nu}^i \,, \qquad (3.6)$$

where $B_{\mu\nu}^i$ is the isolated-pole terms, namely, the terms that contain nucleon and pion poles, $G_{\mu\nu}^i$ is the term that is required to make $B_{\mu\nu}^i$ gauge invariant, and $R_{\mu\nu}^i$ is the remainder term. We also split $M_\nu^i$ into three pieces

$$M_\nu^i=B_\nu^i+G_\nu^i+R_\nu^i \,. \qquad (3.7)$$

From their definitions, we must have

$$k_\nu R_{\mu\nu}^i=0 \,, \quad k_\nu R_\nu^i=0 \,. \qquad (3.8)$$

It is noted that $B_{\mu\nu}^i$ and $B_\nu^i$, and correspondingly $G_{\mu\nu}^i$ and $G_\nu^i$ are not uniquely defined but the low-energy theorem follows independently of how we define them as long as $B_{\mu\nu}^i$ and $B_\nu^i$ have the correct poles and $G_{\mu\nu}^i$ and $G_\nu^i$ make them gauge invariant. Here we take as $B_{\mu\nu}^i$ the generalized Born term,

$$B_{\mu\nu}^i=j_\mu^{Ai}(p',p+k)\frac{1}{i\gamma\cdot(p+k)+m}j_\nu(p+k,p) +j_\nu(p',p'-k)\frac{1}{i\gamma\cdot(p'-k)+m}j_\mu^{Ai}(p'-k,p) +S_{\mu\nu}^{ij}\frac{1}{(q-k)^2+m_\pi^2}ig_1((q-k)^2)\tau^j\gamma_5 \qquad (3.9)$$

in which all off-shell effects are included. All the necessary vertex functions are defined in the preceding section. One sees that the off-shell current matrix elements produce nonpole contributions to $B_{\mu\nu}^i$.

By taking the four-divergence with the help of (2.22) and (2.38),

$$k_\nu B_{\mu\nu}^i=j_\mu^{Ai}(p',p+k)e_N-e_N j_\mu^{Ai}(p'-k,p) +ie\epsilon^{3ij}if_\pi\left[\frac{q_\mu}{q^2+m_\pi^2}-\frac{(q-k)_\mu}{(q-k)^2+m_\pi^2}\right] \times ig_1((q-k)^2)\tau^j\gamma_5 \,, \qquad (3.10)$$

we confirm that $B_{\mu\nu}^i$ is not gauge invariant by itself. However, we can make $B_{\mu\nu}^i$ gauge invariant by adding to it

$$G^i_{\mu\nu}=\bar{G}^i_{\mu\nu}+if_\pi q_\mu \frac{1}{q^2+m_\pi^2}G^i_\nu \ . \tag{3.11}$$

The first term is obtained from $\bar{j}^A_\mu$ by using the minimal substitution, and the second term describes the decay process of the virtual pion produced by the gauge term

$$G^i_\nu=ie\epsilon^{3ij}\frac{(2q-k)_\nu}{q^2-(q-k)^2}i\tau^j\gamma_5[g_1(q^2)-g_1((q-k)^2)]$$
$$+ie\epsilon^{3ij}\tau^j\gamma_\nu\gamma_5 g_2(q^2) \ , \tag{3.12}$$

which is generated from the pion-nucleon vertex function [24]. The explicit form of $\bar{G}^i_{\mu\nu}$ is [16]

$$\bar{G}^i_{\mu\nu}=ie\epsilon^{3ij}\tau^j\frac{(2q-k)_\nu}{q^2-(q-k)^2}\{i[F^A_1(q^2)-F^A_1((q-k)^2)]\gamma_\mu\gamma_5-[\bar{F}^A_2(q^2)-\bar{F}^A_2((q-k)^2)]q_\mu\gamma_5\}$$
$$-ie\epsilon^{3ij}\tau^j\delta_{\mu\nu}\bar{F}^A_2((q-k)^2)\gamma_5+ie\epsilon^{3ij}\tau^j i\bar{F}^A_5(q^2)q_\mu\gamma_\nu\gamma_5$$
$$+\tfrac{1}{2}e(\delta^{3i}+\tau^i)F^A_6(q^2)\{\sigma_{\mu\lambda}q_\lambda,\gamma_\nu\}\gamma_5-\tfrac{1}{2}ie\epsilon^{3ij}\tau^jF^A_6(q^2)[\sigma_{\mu\lambda}q_\lambda,\gamma_\nu]\gamma_5 \ . \tag{3.13}$$

The first term terms come from the momentum dependence of the form factors $F^A_1$ and $\bar{F}^A_2$, and the last three terms from the momentum dependence of the vertex operators. We can check that the sum of $B^i_{\mu\nu}$ and $G^i_{\mu\nu}$ satisfies the gauge-invariance requirement

$$k_\nu(B^i_{\mu\nu}+G^i_{\mu\nu})=-ie\epsilon^{3ij}j^{Aj}_\mu(p',p) \ . \tag{3.14}$$

As remarked above, the gauge term is not unique. Our choice (3.13) is sufficient for our purpose because it satisfies the condition (3.14).

The validity of the expression (3.11) can be examined in the following way. The full axial-vector current $j^{Ai}_\mu$ can be made gauge invariant if we construct the gauge term from the current operator in which the full pseudoscalar form factors

$$F_2=\bar{F}^A_2+f_\pi\frac{1}{q^2+m_\pi^2}g_1 \ , \tag{3.15}$$

$$F_5=\bar{F}^A_5+f_\pi\frac{1}{q^2+m_\pi^2}g_2 \tag{3.16}$$

are retained. Therefore the gauge term is obtained by substituting $F_2$ and $F_5$ in place of $\bar{F}^A_2$ and $\bar{F}^A_5$ in (3.13). The result is the sum of (3.11) and

$$-ie\epsilon^{3ij}if_\pi\left[-\delta_{\mu\nu}+q_\mu\frac{1}{q^2+m_\pi^2}(2q-k)_\nu\right]$$
$$\times\frac{1}{(q-k)^2+m_\pi^2}ig_1((q-k)^2)\tau^j\gamma_5 \ . \tag{3.17}$$

This additional term is contained in the last term of $B^i_{\mu\nu}$ [see Eq. (2.39)] and should be dropped.

The generalized Born approximation $B^i_{\mu\nu}$ corrected by $G^i_{\mu\nu}$, however, does not satisfy the PCAC requirement (3.2). To see this, we first calculate the divergence of $B^i_{\mu\nu}$ using the Ward-Takahashi identities for the axial-vector vertices, (2.8) and (2.37),

$$q_\mu B^i_{\mu\nu}=-ie\epsilon^{3ij}if_\pi(q-k)_\nu\frac{1}{(q-k)^2+m_\pi^2}i\tau^jg_1((q-k)^2)\gamma_5-if_\pi m_\pi^2\frac{1}{q^2+m_\pi^2}B^i_\nu-\tau^i\gamma_5 j_\nu(p+k,p)-j_\nu(p',p'-k)\tau^i\gamma_5 \ , \tag{3.18}$$

where the generalized Born approximation for the virtual-pion photoproduction, $B^i_\nu$, is constructed as

$$B^i_\nu=\Gamma^i(p',p+k)\frac{1}{i\gamma\cdot(p+k)+m}j_\nu(p+k,p)+j_\nu(p',p'-k)\frac{1}{i\gamma\cdot(p'-k)+m}\Gamma^i(p'-k,p)$$
$$+j^{\pi ij}_\nu(q,q-k)\frac{1}{(q-k)^2+m_\pi^2}i\tau^jg_1((q-k)^2)\gamma_5 \ . \tag{3.19}$$

The gauge term $G^i_\nu$ (3.12) makes $B^i_\nu$ gauge invariant, namely,

$$k_\nu(B^i_\nu+G^i_\nu)=-ie\epsilon^{3ij}\frac{q^2+m_\pi^2}{(q-k)^2+m_\pi^2}i\tau^jg_1((q-k)^2)\gamma_5 \ , \tag{3.20}$$

where (2.22) and (2.38) have been used. On the other hand, $q_\mu G^i_{\mu\nu}$ becomes

$$q_\mu G^i_{\mu\nu} = -ie\epsilon^{3ij}\bar{j}^{Aj}_\nu(p',p) - if_\pi m_\pi^2 \frac{1}{q^2+m_\pi^2} G^i_\nu$$
$$-ie\epsilon^{3ij}\tau^j i[F_1^A(q^2) - F_1((q-k)^2)]\left[\gamma_\nu - \frac{(2q-k)_\nu}{q^2-(q-k)^2}\gamma\cdot k\right]\gamma_5 + ie\epsilon^{3ij}\tau^j i\gamma_\nu\gamma_5 , \quad (3.21)$$

where we have used the relations (2.13) and (2.14) to eliminate the form factors $\bar{F}_2^A$ and $\bar{F}_5^A$, except for $\bar{F}_2^A$ in the first term of (3.21). Summing up (3.18) and (3.21), and using (2.9), we find

$$q_\mu(B^i_{\mu\nu} + G^i_{\mu\nu}) = -ie\epsilon^{3ij} j^{Aj}_\nu(p',p) - if_\pi m_\pi^2 \frac{1}{q^2+m_\pi^2}(B^i_\nu + G^i_\nu)$$
$$-ie\epsilon^{3ij}\tau^j i[F_1^A(q^2) - F_1^A((q-k)^2)]\left[\gamma_\nu - \frac{(2q-k)_\nu}{q^2-(q-k)^2}\gamma\cdot k\right]\gamma_5$$
$$+ie\epsilon^{3ij}\tau^j i\gamma_\nu\gamma_5 - \tau^i\gamma_5 j_\nu(p+k,p) - j_\nu(p',p'-k)\tau^i\gamma_5 . \quad (3.22)$$

The form factor $F_6^A$ drops out in this procedure since it appears in divergence-free forms both in $B^i_{\mu\nu}$ and $G^i_{\mu\nu}$.

By subtracting (3.22) from $q_\mu M^i_{\mu\nu}$ (3.2) we find the relation between $R^i_{\mu\nu}$ and $R^i_\nu$. The residual term $R^i_{\mu\nu}$ still contains the pion-pole term which should be written in terms of $R^i_\nu$,

$$R^i_{\mu\nu} = \bar{R}^i_{\mu\nu} + if_\pi q_\mu \frac{1}{q^2+m_\pi^2} R^i_\nu . \quad (3.23)$$

Consequently $\bar{R}^i_{\mu\nu}$ and $R^i_\nu$ must satisfy the PCAC relation

$$q_\mu \bar{R}^i_{\mu\nu} = -if_\pi R^i_\nu + ie\epsilon^{3ij}\tau^j i[F_1^A(q^2) - F_1^A((q-k)^2)]\left[\gamma_\nu - \frac{(2q-k)_\nu}{q^2-(q-k)^2}\gamma\cdot k\right]\gamma_5$$
$$-ie\epsilon^{3ij}\tau^j i\gamma_\nu\gamma_5 + \tau^i\gamma_5 j_\nu(p+k,p) + j_\nu(p',p'-k)\tau^i\gamma_5 . \quad (3.24)$$

We can solve this constraint equation in the following way,

$$R^i_\nu = -i\frac{1}{f_\pi}\Bigg[ie\epsilon^{3ij}\tau^j i[F_1^A(q^2) - F_1^A((q-k)^2)\left[\gamma_\nu - \frac{(2q-k)_\nu}{q^2-(q-k)^2}\gamma\cdot k\right]\gamma_5$$
$$+ie\epsilon^{3ij}\tau^j i(F_1^V - 1)\left[\gamma_\nu - \frac{k_\nu}{k^2}\gamma\cdot k\right]\gamma_5 - ie(\tau^i F_2^S + \delta^{i3}F_2^V)\sigma_{\nu\lambda}k_\lambda\gamma_5$$
$$-ie(\tau^i F_5^S + \delta^{i3}F_5^V)[(p+p')\cdot k\gamma_\nu - (p+p')_\nu\gamma\cdot k]\gamma_5 + ie^{3ij}\tau^j i F_5^V(q\cdot k\gamma_\nu - q_\nu\gamma\cdot k)\gamma_5\Bigg] + \bar{R}^i_\nu . \quad (3.25)$$

The residual amplitude $\bar{R}^i_\nu$ must obey the conditions

$$q_\mu \bar{R}^i_{\mu\nu} = -if_\pi \bar{R}^i_\nu \quad (3.26)$$

and

$$k_\nu \bar{R}^i_\nu = 0 . \quad (3.27)$$

There is no further restriction on $\bar{R}^i_\nu$. The electromagnetic form factors of the nucleon in (3.25) are functions of $k^2$.

## IV. THRESHOLD ELECTROPRODUCTION AMPLITUDES

We are now in the position to derive the multipoles for the pion electroproduction. For this purpose we follow the standard procedure and decompose $M^i_\nu$ according to the isospin dependence

$$M^i_\nu = \delta^{i3}M^{(+)}_\nu + \tfrac{1}{2}[\tau^i,\tau^3]M^{(-)}_\nu + \tau^i M^{(0)}_\nu . \quad (4.1)$$

Each of the three amplitudes is further decomposed into six invariants

$$M^{(\pm,0)}_\nu = e(A^{(\pm,0)}M_A + B^{(\pm,0)}M_B + \cdots + F^{(\pm,0)}M_F) , \quad (4.2)$$

where

$$\begin{aligned} M_A &= -\gamma_5\sigma_{\nu\lambda}k_\lambda , \\ M_B &= i\gamma_5[P_\nu(4m\nu_B - k^2) + (2q_\nu - k_\nu)m\nu] , \\ M_C &= \gamma_5(2m\nu_B\gamma_\nu - \gamma\cdot k q_\nu) , \\ M_D &= 2\gamma_5(-m\nu\gamma_\nu - P_\nu\gamma\cdot k) - 2mM_A , \\ M_E &= i\gamma_5(2m\nu_B k_\nu - k^2 q_\nu) , \\ M_F &= \gamma_5(\gamma\cdot k k_\nu - k^2\gamma_\nu) , \end{aligned} \quad (4.3)$$

with $P = \frac{1}{2}(p+p')$ and

$$\nu = -\frac{P\cdot k}{2m} , \quad \nu_B = \frac{q\cdot k}{2m} . \quad (4.4)$$

We follow the definition of Dennery [26] instead of the original one by Fubini, Nambu, and Wataghin [27] and choose $M_B$ so that the invariant amplitudes $B$ and $E$ are free from kinematical singularities as $\nu_B \to 0$.

At threshold we only have the $E_{0+}$ and $L_{0+}$ amplitudes,

$$E_{0+}=\frac{e}{4\pi(1+\mu)}\mathcal{F}_1\ ,$$
$$L_{0+}=\frac{e}{4\pi(1+\mu)}(\mathcal{F}_1+\mathcal{F}_5)\ , \tag{4.5}$$

where $\mathcal{F}_1$ and $\mathcal{F}_5$ are the coefficients of the amplitude

$$\mathcal{F}=i\sigma\cdot\mathbf{a}\mathcal{F}_1+i\frac{\sigma\cdot\mathbf{kk}\cdot\mathbf{a}}{\mathbf{k}^2}\mathcal{F}_5 \tag{4.6}$$

with

$$\mathbf{a}=\epsilon-\frac{\mathbf{k}\cdot\epsilon}{k_0^2}\mathbf{k}\ . \tag{4.7}$$

The amplitude $\mathcal{F}$ is obtained by reducing the four-component Dirac matrices to two-component ones in the center-of-mass system. The center-of-mass amplitudes $\mathcal{F}_1$ and $\mathcal{F}_5$ can be calculated from the invariant amplitudes as

$$\mathcal{F}_1=\frac{1}{2}m\frac{[(2+\mu)^2-\nu_2]^{1/2}}{(1+\mu)^{1/2}}\left[\mu A+m\mu\frac{2\mu+\mu^2+\nu_2}{2(1+\mu)}C+m\mu\frac{\mu^2-\nu_2}{2(1+\mu)}D-m\nu_2F\right]\ ,$$
$$\mathcal{F}_5=\frac{1}{4}m\frac{[(2+\mu)^2-\nu_2]^{1/2}}{(1+\mu)^{3/2}}(\mu^2-\nu_2)\left[-A-m^2\frac{(2-\mu)(2\mu+\mu^2+\nu_2)}{4(1+\mu)}B-m\mu D-m^2\mu\frac{2\mu+\mu^2+\nu_2}{2(1+\mu)}E-m(2+\mu)F\right]\ . \tag{4.8}$$

It is straightforward to decompose the amplitudes obtained in the preceding section into these invariants. The Born approximation supplemented by the gauge term satisfies $k_\nu(B_\nu^i+G_\nu^i)=0$ for the on-shell pion [see Eq. (3.20)] so that it can be decomposed into invariants

$$A^{(+,0)}=-\frac{g_1}{4m}F_1^{V,S}\left[\frac{1}{\nu_B-\nu}+\frac{1}{\nu_B+\nu}\right]-g_1F_5^{V,S}-g_2(F_2^{V,S}+2mF_5^{V,S})\ ,$$
$$A^{(-)}=-\frac{g_1}{4m}F_1^V\left[\frac{1}{\nu_B-\nu}-\frac{1}{\nu_B+\nu}\right]\ ,$$
$$B^{(+,0)}=-\frac{g_1}{2m}\frac{F_1^{V,S}}{\Delta^2+m_\pi^2}\left[\frac{1}{\nu_B-\nu}+\frac{1}{\nu_B+\nu}\right]\ ,$$
$$B^{(-)}=-\frac{g_1}{2m}\frac{F_1^V}{\Delta^2+m_\pi^2}\left[\frac{1}{\nu_B-\nu}-\frac{1}{\nu_B+\nu}\right]\ ,$$
$$C^{(+,0)}=\frac{g_1}{4m}F_2^{V,S}\left[\frac{1}{\nu_B-\nu}-\frac{1}{\nu_B+\nu}\right]\ ,$$
$$C^{(-)}=\frac{g_1}{4m}F_2^V\left[\frac{1}{\nu_B-\nu}+\frac{1}{\nu_B+\nu}\right]-g_2F_5^V\ , \tag{4.9}$$
$$D^{(+,0)}=\frac{g_1}{4m}F_2^{V,S}\left[\frac{1}{\nu_B-\nu}+\frac{1}{\nu_B+\nu}\right]-g_2F_5^{V,S}\ ,$$
$$D^{(-)}=\frac{g_1}{4m}F_2^V\left[\frac{1}{\nu_B-\nu}-\frac{1}{\nu_B+\nu}\right]\ ,$$
$$E^{(+,0)}=-\frac{g_1}{4m}\frac{F_1^{V,S}}{\Delta^2+m_\pi^2}\left[\frac{1}{\nu_B-\nu}-\frac{1}{\nu_B+\nu}\right]\ ,$$
$$E^{(-)}=-\frac{g_1}{4m}\frac{F_1^V}{\Delta^2+m_\pi^2}\left[\frac{1}{\nu_B-\nu}+\frac{1}{\nu_B+\nu}\right]-\frac{2}{k^2}\frac{1}{\Delta^2+m_\pi^2}[(F_\pi-1)g_1(\Delta^2)-(F_1^V-1)g_1]\ ,$$
$$F^{(+,0)}=0\ ,\quad F^{(-)}=\frac{1}{k^2}(F_1^V-1)g_2\ .$$

The nucleon pole contributions completely agree with the results using the pseudoscalar pion-nucleon coupling without off-shell effects. The off-shell effects produced by $g_2$ and $F_5$ appear as nonpole contributions to (4.9). In (4.9) all strong form factors are evaluated at $q^2=-m_\pi^2$ except $g_1(\Delta^2)$, which is evaluated at

$$\Delta^2=(q-k)^2=k^2-4m\nu_B-m_\pi^2\ . \tag{4.10}$$

In the invariant amplitude $E^{(-)}$, $F_\pi-1$, and $F_1^V-1$ appear instead of $F_\pi$ and $F_1^V$ because we included off-shell effects in the photon-nucleon and photon-pion vertices. However, these effects do not influence low-energy amplitudes because $E^{(-)}$ will be multiplied by the quantity of $O(\mu^4,\mu^2\nu_2)$ [see Eq. (4.8)] and we can safely make an expansion

$$g_1(\Delta^2)=g_1+(\Delta^2+m_\pi^2)\times O(1)$$

to get the standard result

$$-\frac{2}{k^2}\frac{1}{\Delta^2+m_\pi^2}(F_\pi-F_1^V)g_1 \tag{4.11}$$

for the pion pole term. The pion poles in $B$ and $E$ in (4.9) in conjunction with the nucleon poles arise from the definition of the invariant amplitudes devoid of kinematical singularities and do not remain in the final expressions.

The correction $R_\nu^i$ induced by the PCAC constraint is also decomposed into the invariant amplitudes

$$A^{(+,0)}=\frac{1}{f_\pi}(F_2^{V,S}+2mF_5^{V,S})\ ,$$
$$C^{(-)}=\frac{2}{f_\pi}\frac{F_1^A(\Delta^2)-F_1^A(-m_\pi^2)}{\Delta^2+m_\pi^2}+\frac{1}{f_\pi}F_5^V\ ,$$
$$D^{(+,0)}=\frac{1}{f_\pi}F_5^{V,S}\ ,$$
$$F^{(-)}=\frac{1}{f_\pi}\frac{F_1^A(\Delta^2)-F_1^A(-m_\pi^2)}{\Delta^2+m_\pi^2}-\frac{1}{f_\pi}\frac{F_1^V-1}{k^2}\ . \tag{4.12}$$

The residual amplitude $\bar{R}_\nu^i$ in Eq. (3.25) will be treated separately at the end of this section. We now add the PCAC correction (4.12) to the Born approximation corrected by the gauge term (4.9). Apart from the nucleon and pion pole terms we get

$$\begin{aligned}
A^{(+,0)}&=\left[\frac{1}{f_\pi}-g_2\right]F_2^{V,S}+\left[\frac{1}{f_\pi}-g_2-\frac{g_1}{2m}\right]2mF_5^{V,S}\ ,\\
C^{(-)}&=\frac{2}{f_\pi}\frac{F_1^A(\Delta^2)-F_1^A(-m_\pi^2)}{\Delta^2+m_\pi^2}+\left[\frac{1}{f_\pi}-g_2\right]F_5^V\ ,\\
D^{(+,0)}&=\left[\frac{1}{f_\pi}-g_2\right]F_5^{V,S}\ ,\\
F^{(-)}&=\frac{1}{f_\pi}\frac{F_1^A(\Delta^2)-F_1^A(-m_\pi^2)}{\Delta^2+m_\pi^2}-\left[\frac{1}{f_\pi}-g_2\right]\frac{F_1^V-1}{k^2}\ .
\end{aligned}\tag{4.13}$$

One notices that the off-shell pion-nucleon form factor $g_2$ appears only in the combination $1/f_\pi - g_2$.

Let us make a systematic expansion of (4.13) in terms of $\mu$. From the relation (2.16), we find

$$\frac{1}{f_\pi}-g_2=\frac{g_1}{2m}+O(\mu^2)\ .\tag{4.14}$$

Furthermore, we assume that $F_1^A$ can be expanded as

$$F_1^A(-m_\pi^2)=F_1^A(0)+O(\mu^2)\ .\tag{4.15}$$

At threshold we have

$$\Delta^2=m^2\frac{\mu^2-\nu_2}{1+\mu}\tag{4.16}$$

so that

$$\Delta^2=k^2+O(\mu^2,\mu\nu_2)\tag{4.17}$$

and

$$F_1^A(\Delta^2)=F_1^A(k^2)+O(\mu^2,\mu\nu_2)\ .\tag{4.18}$$

To leading order, we get

$$\begin{aligned}
A^{(+,0)}&=\frac{g_1}{2m}F_2^{V,S}\ ,\\
C^{(-)}&=\frac{2}{f_\pi}\frac{F_1^A(k^2)-F_1^A(0)}{k^2}+\frac{g_1}{2m}F_5^V\ ,\\
D^{(+,0)}&=\frac{g_1}{2m}F_5^{V,S}\ ,\\
F^{(-)}&=\frac{1}{f_\pi}\frac{F_1^A(k^2)-F_1^A(0)}{k^2}-\frac{g_1}{2m}\frac{F_1^V-1}{k^2}\ .
\end{aligned}\tag{4.19}$$

In calculating the threshold amplitudes, $C^{(-)}$ will be multiplied by the factor of $O(\mu^2,\mu\nu_2)$ and $D^{(+,0)}$ by the factor of $O(\mu^3,\mu\nu_2)$ [see Eq. (4.8)], so that they will not survive in low-energy theorems if $F_5$ is considered to be $O(\mu^0)$. Moreover, if we assume the difference between $g_1$ and $g_1(0)$ to be order $\mu^2$,

$$g_1=g_1(0)+O(\mu^2)\ ,\tag{4.20}$$

we can replace all $g_1$ by $g_1(0)$ in (4.19). We now recover the soft-pion limit [2–5,8]

$$\begin{aligned}
A^{(+,0)}&=\frac{g_1(0)}{2m}F_2^{V,S}(k^2)\ ,\\
F^{(-)}&=\frac{g_1(0)}{2m}\left[\frac{F_1^A(k^2)}{F_1^A(0)}-F_1^V(k^2)\right]\frac{1}{k^2}\ ,
\end{aligned}\tag{4.21}$$

where the Goldberger-Treiman relation (2.15) has been used in deriving $F^{(-)}$. Consequently, apart from the $F_5$ terms, all the amplitudes are in complete agreement with the conventional results. The off-shell form factors of the axial-vector current, $F_3^A$ and $F_4^A$, and those of the electromagnetic current, $F_3$ and $F_4$ were dropped to be consistent with the assumption that all form factors do not depend on virtual nucleon masses. The form factors $\bar F_2^A$, $\bar F_5^A$, and $F_6$ are eliminated using the relations (2.14), (2.13), and (2.28), respectively. The form factor $F_6^A$ drops out in taking the four-divergence of $M_{\mu\nu}^i$. Finally the off-shell form factor $g_2$ disappears because of the relation (4.14).

Since our amplitudes coincide with the usual results except for $F_5$, there is no need to repeat the derivation of the remaining terms in detail. Nonetheless, we would like to make a remark concerning the derivation of $\mathcal{F}_5^{(-)}$ as there is subtlety in $\mu$ and $\nu_2$ expansions. Firstly, in $\mathcal{F}_5^{(-)}$, we can replace $2+\mu$ in the coefficient of $F^{(-)}$ by 2 since the ignored piece contributes the amplitude of $O(\mu^3,\mu\nu_2)$ [note that $F^{(-)}$ is finite at $k^2=0$ as is seen from (4.21)]. Secondly, we replace $g_1(0)$ by $g_1$ in (4.21) using (4.20) again. Then we can prove that all terms containing the electromagnetic form factor $F_1^V$ cancel out exactly by the use of

$$\nu_B=-m\mu\frac{2\mu+\mu^2+\nu_2}{4(1+\mu)}\ ,\quad \nu=m\mu\frac{(2+\mu)^2-\nu_2}{4(1+\mu)}\ .\tag{4.22}$$

The term containing $F_2^V$ becomes

$$\frac{g_1}{4m}2mF_2^V\left[-\frac{\nu_2}{2-\nu_2}+\frac{1}{4}\nu_2\cdot\frac{\nu_2}{2-\nu_2}\right]+O(\mu^2,\mu\nu_2)\tag{4.23}$$

up to a normalization factor. In the curly brackets of $\mathcal{F}_5^{(-)}$, the term containing $F_\pi$ [see Eq. (4.11)] and the term containing $F_1^A(k^2)$ [see Eq. (4.19) with $g_1(0)$ replaced by $g_1$] are combined to

$$-\frac{1}{\Delta^2+m_\pi^2}g_1F_\pi+g_1\left[F_\pi-\frac{F_1^A(k^2)}{F_1^A(0)}\right]\frac{1}{k^2}\ ,\tag{4.24}$$

where (4.16) is used to split the $F_\pi$ term into two pieces. By multiplying $\frac{1}{2}m(\mu^2-\nu_2)$, we obtain

$$\frac{g_1}{2m}\frac{\mu(\mu+\nu_2)}{\mu^2(2+\mu)-\nu_2}F_\pi-\frac{g_1}{2m}\frac{F_1^A(k^2)}{F_1^A(0)}\ ,\tag{4.25}$$

where we have dropped terms of $\mu^2\times$(a function of $\nu_2$). Note that it is not necessary to assume that $F_1^A$ and $g_1$ have identical $q^2$ dependence, in contrast to Ref. [10].

We now turn to the evaluation of the $F_5$ contribution to the multipoles for threshold electroproduction. The $F_5$ terms in (4.13) bring about the amplitudes

$$\mathcal{F}_1^{(+,0)}=\frac{1}{4}\frac{[(2+\mu)^2-\nu_2]^{1/2}}{(1+\mu)^{3/2}} \times m^2\mu\left[4(1+\mu)\left[\frac{1}{f_\pi}-g_2-\frac{g_1}{2m}\right]+(\mu^2-\nu_2)\left[\frac{1}{f_\pi}-g_2\right]\right]F_5^{V,S} ,$$

$$\mathcal{F}_1^{(-)}=\frac{1}{4}\frac{[(2+\mu)^2-\nu_2]^{1/2}}{(1+\mu)^{3/2}} \times m^2\mu(2\mu+\mu^2+\nu_2)\left[\frac{1}{f_\pi}-g_2\right]F_5^{V} ,$$

$$\mathcal{F}_5^{(+,0)}=\frac{1}{4}\frac{[(2+\mu)^2-\nu_2]^{1/2}}{(1+\mu)^{3/2}} \times m^2(\mu^2-\nu_2)\left[-2\left[\frac{1}{f_\pi}-g_2-\frac{g_1}{2m}\right]-\mu\left[\frac{1}{f_\pi}-g_2\right]\right]F_5^{V,S} ,$$

$$\mathcal{F}_5^{(-)}=0 . \tag{4.26}$$

One immediately notices that the off-shell form factor $F_5$ produces corrections of order $\mu^3F_5$ and $\mu\nu_2F_5$ for the (+) and (0) amplitudes, and those of order $\mu^2F_5$ and $\mu\nu_2F_5$ for the (−) amplitude. We confirm that the off-shell effects do not influence low-energy theorems as long as we assume that $F_5$ is independent of $m_\pi$.

If we calculate off-shell form factors in explicit models, however, they depend on the pion mass and we cannot take them to be $O(\mu^0)$. In fact, Bernard, Kaiser, and Meissner [13,14] calculated pion photo- and electroproduction amplitudes using chiral perturbation theory and found that the triangle diagrams produces terms of $O(\mu^{-1})$. Many years ago, using dispersion relations in the nucleon mass and assuming the threshold dominance, Nyman [22] predicted that

$$F_2^- \approx \frac{2m+m_\pi}{m_\pi}F_2^+ . \tag{4.27}$$

In our notation, $F_2^+=2mF_2$ and $F_2^-=2m(F_2+2mF_5)$, we find

$$F_5\approx\frac{1}{m_\pi}F_2 . \tag{4.28}$$

In light of these facts, we have to derive the corrections to the multipoles $E_{0+}$ and $L_{0+}$ by taking $\mu F_5$ to be $O(\mu^0)$. We give the results in units,

$$E_{0+}=\frac{eg_1}{2m}\frac{1}{8\pi(1+\mu)}\frac{[(2+\mu)^2-\nu_2]^{1/2}}{(1+\mu)^{1/2}}\mathcal{E}_{0+} ,$$

$$L_{0+}=\frac{eg_1}{2m}\frac{1}{8\pi(1+\mu)}\frac{[(2+\mu)^2-\nu_2]^{1/2}}{(1+\mu)^{1/2}}\mathcal{L}_{0+} . \tag{4.29}$$

Our results are

$$\mathcal{E}_{0+}^{(+,0)}=-\frac{\mu}{2+\mu}(1+\kappa^{V,S})+\tfrac{1}{2}\mu\kappa^{V,S}-\tfrac{1}{2}G_M^{V,S}\frac{\nu_2}{2-\nu_2}+\tfrac{1}{2}\mu^2(\delta_1^{V,S}+\delta_2^{V,S})-\tfrac{1}{2}\nu_2\delta_2^{V,S}+O(\mu^3,\mu\nu_2) ,$$

$$\mathcal{L}_{0+}^{(+,0)}=-\frac{\mu}{2+\mu}-\tfrac{1}{2}G_E^{V,S}\frac{\nu_2}{2-\nu_2}+\tfrac{1}{2}\mu^2\delta_1^{V,S}+O(\mu^3,\mu\nu_2) ,$$

$$\mathcal{E}_{0+}^{(-)}=\frac{F_1^A}{F_1^A(0)}+\tfrac{1}{2}G_M^V\frac{\nu_2}{2-\nu_2}+(\mu+\tfrac{1}{2}\nu_2)\delta_2^V+O(\mu^2,\mu\nu_2) ,$$

$$\mathcal{L}_{0+}^{(-)}=\frac{\mu(\mu+\nu_2)}{\mu^2(2+\mu)-\nu_2}F_\pi+\tfrac{1}{2}G_E^V\frac{\nu_2}{2-\nu_2}+(\mu+\tfrac{1}{2}\nu_2)\delta_2^V+O(\mu^2,\mu\nu_2) . \tag{4.30}$$

Here $\kappa=2mF_2(0)$ is the anomalous magnetic moment of the nucleon and $G_M$ and $G_E$ are the Sachs magnetic and electric form factors:

$$G_M=F_1+2mF_2 , \quad G_E=F_1+\tfrac{1}{4}\nu_2 2mF_2 . \tag{4.31}$$

The parameters $\delta_1$ and $\delta_2$ characterize the form factor $F_5$,

$$\delta_1=-4\alpha m^2\mu F_5 , \quad \delta_2=m^2\mu F_5 , \tag{4.32}$$

where $\alpha$ is given by (2.17). In deriving the first term of $\mathcal{E}_{0+}^{(-)}$ in (4.30), the Kroll-Ruderman term [28], we have made use of the expansion (4.20) and the Goldberger-Treiman relation (2.15). In Table I we display the physical channel amplitudes given by

$$M(\gamma p\to n\pi^+)=\sqrt{2}(M^{(0)}+M^{(-)}) ,$$
$$M(\gamma n\to p\pi^-)=\sqrt{2}(M^{(0)}-M^{(-)}) ,$$
$$M(\gamma p\to p\pi^0)=M^{(+)}+M^{(0)} ,$$
$$M(\gamma n\to n\pi^0)=M^{(+)}-M^{(0)} . \tag{4.33}$$

To conclude this section, we comment on the residual amplitude $\bar{R}_\nu^i$ which satisfies

$$-m_\pi\bar{R}_{0\nu}^i=-if_\pi\bar{R}_\nu^i \tag{4.34}$$

at threshold, $q_0=m_\pi$. We expand $\bar{R}_\nu^i$ in terms of the invariants as in (4.2). The crossing symmetry requires that the amplitudes $A^{(+,0)}$, $B^{(+,0)}$, $C^{(-)}$, $D^{(+,0)}$, $E^{(-)}$, and $F^{(-)}$ are even functions of $\nu$, and $A^{(-)}$, $B^{(-)}$, $C^{(+,0)}$, $D^{(-)}$, $E^{(+,0)}$, and $F^{(+,0)}$ are odd functions of $\nu$. At threshold where $\nu$ and $\nu_B$ are small, we can make double power-series expansions of even functions, e.g.,

$$A^{(+,0)}=a_{00}^{(+,0)}+a_{01}^{(+,0)}\nu_B+a_{20}^{(+,0)}\nu^2+\cdots . \tag{4.35}$$

Each of the expansion coefficients is a function of $m_\pi^2$ and $k^2$. The first term $a_{00}^{(+,0)}$ vanishes in the chiral limit as is seen from (4.34). As a result,

$$a_{00}^{(+,0)}=\bar{a}_{00}^{(+,0)}m_\pi^2+\cdots . \tag{4.36}$$

With the aid of the relations (4.22), we can make an expansion in powers of $\mu$ and $\nu_2$,

$$A^{(+,0)}=m(m\bar{a}_{00}^{(+,0)}+ma_{20}^{(+,0)}-\tfrac{1}{2}a_{01}^{(+,0)})\mu^2+\cdots . \tag{4.37}$$

We have retained only terms that contribute to the final results. Similarly odd functions can be expanded as

$$A^{(-)}=a_{10}^{(-)}\nu+\cdots=ma_{10}^{(-)}\mu+\cdots \ , \tag{4.38}$$

and

$$C^{(+,0)}=mc_{10}^{(+,0)}\mu+\cdots \ , \qquad F^{(+,0)}=mf_{10}^{(+,0)}\mu+\cdots \ . \tag{4.39}$$

The leading-order contributions from $\bar{R}_{\nu}^{\,i}$ can be summarized as

$$\begin{aligned} E_{0+}^{(+,0)}&=m^2e(m\bar{a}_{00}^{(+,0)}+ma_{20}^{(+,0)}-\tfrac{1}{2}a_{01}^{(+,0)}+mc_{10}^{(+,0)})\\ &\quad\times\mu^3-m^3ef_{10}^{(+,0)}\mu\nu_2 \ ,\\ L_{0+}^{(+,0)}&=m^2e(m\bar{a}_{00}^{(+,0)}+ma_{20}^{(+,0)}-\tfrac{1}{2}a_{01}^{(+,0)}\\ &\quad+mc_{10}^{(+,0)}-mf_{10}^{(+,0)})\mu^3 \ ,\\ E_{0+}^{(-)}&=m^2ea_{10}^{(-)}\mu^2 \ ,\\ L_{0+}^{(-)}&=m^2ea_{10}^{(-)}\mu(\mu+\tfrac{1}{2}\nu_2) \ . \end{aligned} \tag{4.40}$$

If expansion coefficients possess singularities in the chiral limit, these terms can also contribute to low-energy amplitudes.

## V. SUMMARY

We have investigated the consequences of including off-shell form factors in the derivation of the pion electroproduction amplitudes at low energies. The PCAC relation and gauge invariance are rigorously respected. We have shown that as long as off-shell form factors are taken to be parameters independent of the pion mass, we recover the conventional results. There occurs a cancellation among various off-shell effects and the low-energy theorems remain unchanged. In reality, off-shell form factors also contain the pion mass because loop diagrams involve virtual-pion propagation. In the chiral limit where the pion mass vanishes, the loop diagrams can develop singularities so that the electroproduction amplitudes get modified to a large extent. We have seen that if we assume $F_5$ to behave as $\mu^{-1}$, it gives contributions to the threshold amplitudes at the order where the low-energy theorems are considered to hold true. This does not necessarily mean that the low-energy theorems are violated. If one distinguishes the pion mass that enters the Born approximation from the pion mass that is carried by internal lines in loop diagrams, one can still differentiate model-independent terms and others theoretically [15]. However, it is not possible to distinguish the

TABLE I. Threshold pion electroproduction multipoles $\mathcal{E}_{0+}$ and $\mathcal{L}_{0+}$ in units defined in Eq. (4.29). Terms of $O(\mu^2,\mu\nu_2)$ are dropped for the charged pion production and terms of $O(\mu^3,\mu\nu_2)$ are dropped for the neutral pion production. The quantities with superscripts $p$ and $n$ are defined by $G_M^{p,n}=\frac{1}{2}(G_M^S\pm G_M^V)$, etc. All form factors are functions of $k^2$.

| Channel | $\mathcal{E}_{0+}$ |
|---|---|
| $\pi^+$ | $\sqrt{2}\left[\frac{F_1^A}{F_1^A(0)}-G_M^n\frac{\nu_2}{2-\nu_2}-\frac{1}{2}\mu+\mu\delta_2^V-\nu_2\delta_2^n\right]$ |
| $\pi^-$ | $\sqrt{2}\left[-\frac{F_1^A}{F_1^A(0)}-G_M^p\frac{\nu_2}{2-\nu_2}-\frac{1}{2}\mu-\mu\delta_2^V-\nu_2\delta_2^p\right]$ |
| $p\pi^0$ | $-\mu+\frac{1}{2}\mu^2(1+\kappa^p)-G_M^p\frac{\nu_2}{2-\nu_2}+\mu^2(\delta_1^p+\delta_2^p)-\nu_2\delta_2^p$ |
| $n\pi^0$ | $-\frac{1}{2}\mu^2\kappa^n+G_M^n\frac{\nu_2}{2-\nu_2}-\mu^2(\delta_1^n+\delta_2^n)+\nu_2\delta_2^n$ |
| | $\mathcal{L}_{0+}$ |
| $\pi^+$ | $\sqrt{2}\left[\frac{\mu(\mu+\nu_2)}{\mu^2(2+\mu)-\nu_2}F_\pi-G_E^n\frac{\nu_2}{2-\nu_2}-\frac{1}{2}\mu+\left[\mu+\frac{1}{2}\nu_2\right]\delta_2^V\right]$ |
| $\pi^-$ | $\sqrt{2}\left[-\frac{\mu(\mu+\nu_2)}{\mu^2(2+\mu)-\nu_2}F_\pi-G_E^p\frac{\nu_2}{2-\nu_2}-\frac{1}{2}\mu-\left[\mu+\frac{1}{2}\nu_2\right]\delta_2^V\right]$ |
| $p\pi^0$ | $-\mu+\frac{1}{2}\mu^2-G_E^p\frac{\nu_2}{2-\nu_2}+\mu^2\delta_1^p$ |
| $n\pi^0$ | $G_E^n\frac{\nu_2}{2-\nu_2}-\mu^2\delta_1^n$ |

internal and external pion masses experimentally. In this respect, it is difficult to check the validity of the low-energy theorems by experiments.

## APPENDIX

In this appendix we derive the Ward-Takahashi identities for the radiative pion decay vertex function defined by the vacuum expectation value of the time-ordered product,

$$\langle 0|T[\phi_\pi^j(x)j_\mu^{Ai}(y)j_\nu(z)]|0\rangle = \int d^4x' S_{\mu\nu}^{ij}(x'yz)\Delta_F'(x'-x)\ , \quad \text{(A1)}$$

where $\Delta_F'$ is the dressed pion propagator

$$\langle 0|T[\phi_\pi^i(x')\phi_\pi^j(x)]|0\rangle = \delta^{ij}\Delta_F'(x'-x)\ . \quad \text{(A2)}$$

The vertex function $S_{\mu\nu}^{ij}$ possesses the isospin dependence

$$S_{\mu\nu}^{ij}(xyz) = -ie\epsilon^{3ij}S_{\mu\nu}(xyz)\ . \quad \text{(A3)}$$

One of the Ward-Takahasi identities follows from the electromagnetic current conservation $\partial_\nu j_\nu(z)=0$, the equal-time commutator

$$[\phi_\pi^i(x), j_0(z)]\delta(x_0-z_0) = -ie\epsilon^{3ij}\phi_\pi^j(x)\delta(x-z)\ , \quad \text{(A4)}$$

and the chiral SU(2)×SU(2) algebra

$$[j_\mu^{Ai}(y), j_0(z)]\delta(y_0-z_0) = -ie\epsilon^{3ij}j_\mu^{Aj}(y)\delta(y-z)\ . \quad \text{(A5)}$$

By taking the derivative of (A1) with respect to the photon position $z_\nu$, we obtain

$$\frac{\partial}{\partial z_\nu}S_{\mu\nu}(xyz) = -i\delta(z-y)G_\mu(y-x) + i\int d^4x' G_\mu(y-x')\Delta_F'(x'-z) \times \Delta_F'^{-1}(z-x)\ , \quad \text{(A6)}$$

where $G_\mu$ is the nonradiative pion decay vertex

$$\langle 0|T[\phi_\pi^j(x)j_\mu^{Ai}(y)]|0\rangle = i\delta^{ij}\int d^4x' G_\mu(y-x')\Delta_F'(x'-x)\ . \quad \text{(A7)}$$

In momentum space,

$$S_{\mu\nu}(xyz) = \frac{-i}{(2\pi)^8}\int d^4q' d^4q\, e^{iq\cdot(y-z)+iq'\cdot(z-x)}S_{\mu\nu}(q,q')\ ,$$

$$\Delta_F'(x'-x) = \frac{-i}{(2\pi)^4}\int d^4q\, e^{iq\cdot(x'-x)}\Delta_F'(q)\ , \quad \text{(A8)}$$

$$G_\mu(x'-x) = \frac{-i}{(2\pi)^4}\int d^4q\, e^{iq\cdot(x'-x)}G_\mu(q)\ ,$$

we find

$$(q-q')_\mu S_{\mu\nu}(q,q') = G_\mu(q') - G_\mu(q)\Delta_F'(q)\Delta_F'^{-1}(q')\ . \quad \text{(A9)}$$

For the physical pion, we can use the reduction formula to rewrite the pion decay matrix element as

$$\langle 0|j_\mu^{Ai}(0)|\pi^j(q)\rangle = \delta^{ij}\lim_{q^2\to -m_\pi^2}(q^2+m_\pi^2)\Delta_F'(q)G_\mu(q)\ . \quad \text{(A10)}$$

From the definition of the pion decay constant (2.2) we get $G_\mu = if_\pi q_\mu$ and therefore Eq. (2.38) in the text.

Another Ward-Takahashi identity follows by taking the derivative of (A1) with respect to the position of the axial-vector current, $y_\mu$,

$$\frac{\partial}{\partial y_\mu}S_{\mu\nu}(xyz) = i\delta(z-y)G_\nu(y-x) + if_\pi m_\pi^2\int d^4y'\Delta_F'(y-y')j_\nu^\pi(xy'z)\ , \quad \text{(A11)}$$

where we have used the PCAC relation (2.1) and the chiral algebra,

$$[j_\nu(z), j_0^{Ai}(y)]\delta(y_0-z_0) = ie\epsilon^{3ij}j_\nu^{Aj}(y)\delta(y-z)\ . \quad \text{(A12)}$$

The term coming from the equal-time commutator

$$[\phi_\pi^i(x), j_0^{Ai}(y)]\delta(x_0-z_0)$$

does not contribute to (A11). In (A11) $j_\nu^\pi$ is the photon-pion interaction vertex defined by

$$\langle 0|T[\phi_\pi^j(x)\phi_\pi^i(y)j_\nu(z)]|0\rangle = i\int d^4x' d^4y' \Delta_F'(y-y')j_\nu^{\pi ij}(x'y'z)\Delta_F'(x'-x) \quad \text{(A13)}$$

with the isospin dependence

$$j_\mu^{\pi ij}(xyz) = -ie\epsilon^{3ij}j_\nu^\pi(xyz)\ . \quad \text{(A14)}$$

Using the momentum representation,

$$j_\nu^\pi(xyz) = \frac{-i}{(2\pi)^8}\int d^4q' d^4q\, e^{iq\cdot(y-z)+iq'\cdot(z-x)} \times j_\nu^\pi(q,q')\ , \quad \text{(A15)}$$

we obtain

$$q_\mu S_{\mu\nu}(q,q') = G_\nu(q') - if_\pi m_\pi^2\Delta_F{}'(q)j_\nu^\pi(q,q') \quad \text{(A16)}$$

which leads to Eq. (2.37) in the text.


[1] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960); Y. Nambu, Phys. Rev. Lett. **4**, 380 (1960).

[2] Y. Nambu and E. Shrauner, Phys. Rev. **128**, 862 (1962).

[3] S. Fubini, G. Furlan, and C. Rossetti, Nuovo Cimento **40**, 1171 (1965).

[4] Riazuddin and B. W. Lee, Phys. Rev. **146**, B1202 (1966).

[5] S. L. Adler and F. J. Gilman, Phys. Rev. **152**, B1460 (1966).

[6] I. A. Vainshtein and V. I. Zakharov, Yad. Fiz. **12**, 610 (1970) [Sov. J. Nucl. Phys. **12**, 333 (1971)].

[7] P. de Baenst, Nucl. Phys. **B24**, 633 (1970).

[8] N. Dombey and R. J. Read, Nucl. Phys. **B60**, 65 (1973).

[9] H. W. L. Naus, J. H. Koch, and J. L. Friar, Phys. Rev. C **41**, 2852 (1990).

[10] S. Scherer and J. H. Koch, Nucl. Phys. **A534**, 461 (1991).
[11] E. Mazzucato *et al.*, Phys. Rev. Lett. **57**, 3144 (1986).
[12] R. Beck *et al.*, Phys. Rev. Lett. **65**, 1841 (1991).
[13] V. Bernard, N. Kaiser, and U.-G. Meissner, Phys. Lett. B **268**, 291 (1992); Nucl. Phys. **B383**, 442 (1992).
[14] V. Bernard, N. Kaiser, and U.-G. Meissner, Phys. Lett. B **282**, 448 (1992).
[15] S. Scherer, J. H. Koch, and J. L. Friar, Nucl. Phys. A (to be published).
[16] K. Ohta, Phys. Rev. C **46**, 2519 (1992).
[17] J. C. Ward, Phys. Rev. **78**, 182 (1950); Y. Takahashi, Nuovo Cimento **6**, 371 (1957).
[18] M. L. Goldberger and S. B. Treiman, Phys. Rev. **111**, 354 (1958).
[19] E. Kazes, Nuovo Cimento **13**, 1226 (1959).
[20] A. M. Bincer, Phys. Rev. **118**, 854 (1960).
[21] F. A. Berends and G. B. West, Phys. Rev. **188**, 2538 (1969).
[22] E. M. Nyman, Nucl. Phys. **A154**, 97 (1970).
[23] F. Gross and D. O. Riska, Phys. Rev. C **36**, 1928 (1987).
[24] K. Ohta, Phys. Rev. C **40**, 1335 (1989).
[25] S. L. Adler and Y. Dothan, Phys. Rev. **151**, B1267 (1966).
[26] P. Dennery, Phys. Rev. **124**, 2000 (1961).
[27] S. Fubini, Y. Nambu, and V. Wataghin, Phys. Rev. **111**, 329 (1958).
[28] N. M. Kroll and M. A. Ruderman, Phys. Rev. **93**, 233 (1954).