Variational two- and three-particle solutions of the relativistic Yukawa model

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The variational method, within the Hamiltonian formalism of quantum field theory, is used to derive momentum space wave equations for any number of fermions interacting via a massive scalar field. These coupled equations are shown to be exactly solvable in the limit of fixed fermions. Approximate solutions are given in the two- and three-particle bound state case for various mass combinations and various strengths of the coupling of the fermion and scalar fields.

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I. INTRODUCTION

The *ab initio* description of relativistic few-particle bound states in quantum field theory has a history that dates back to the 1930s [1], and a literature that is very extensive. The traditional approach to this problem is by means of the Bethe-Salpeter formalism [2], which is described in many standard textbooks on quantum field theory [3]. There are, however, many practical difficulties with the Bethe-Salpeter approach: For one thing it is, in practice, perturbative, and so not straightforwardly applicable to strongly coupled systems. In addition, it leads to four-dimensional wave equations for the relative motion of a two-body system, and the presence of the relative time coordinate leads to ambiguities in interpreting the Bethe-Salpeter amplitude as a traditional wave function. Lastly, the implementation of the Bethe-Salpeter formalism in the case of three- or more-particle system seems to be quite formidable [4].

Recently, the variational approach, within the Hamiltonian formalism of quantum field theory, has been shown to be a viable alternative method for describing relativistic few-particle bound-state systems in quantum electrodynamics [5,6]. Thus, it is of interest to apply this approach to the Yukawa model, as a prototype of a strongly coupled theory, such as arises in quantum field-theoretic models of nuclei [7], and of the standard model.

In this paper we present a study of relativistic two- and three-particle bound states in the Yukawa model, using the variational Hamiltonian (VH) approach. The Yukawa model is too naive to be considered as a realistic field-theoretic model of nuclei. This is because it contains only scalar (or pseudoscalar) meson exchange interactions, and this is insufficient (as is well known) to describe internucleon interactions properly. Nevertheless, the Yukawa model does contain the principal features of a field-theoretic nuclear model and so is a useful one for testing the VH approach. In Sec. II we specify the Lagrangian of the model, the notational conventions, and we introduce the variational ansatz for a system of an arbitrary number of fermions ("nucleons") interacting via a scalar ("meson") field. This ansatz is used to derive the coupled integral wave equations that describe the stationary states of this few-nucleon system. In Sec. III we discuss the exact solution of these equations in the limit of fixed nucleons. Section IV is a discussion of the dynamical system in a spinless approximation. Solutions of the two-particle equations are presented in Sec. V, while three-particle solutions in a separable approximation, are given in Sec. VI. Concluding remarks are given in Sec. VII.

II. LAGRANGIAN, HAMILTONIAN, VARIATIONAL ANSATZ, AND MANY-BODY EQUATIONS

For a system of two distinct types of massive fermions (which we will refer to as "protons" and "neutrons"), described by Dirac fields ψ_1 and ψ_2, interacting via a massive or massless boson (meson) field, ϕ, the Lagrangian density, with $\hbar = c = 1$, is

$$\mathcal{L} = \sum_{k=1}^{2} \bar{\psi}_k (i\gamma^\mu \partial_\mu - M_{0k}) \psi_k + \tfrac{1}{2}(\partial_\mu \phi \partial^\mu \phi - m^2 \phi^2) + \sum_{k=1}^{2} g_k \bar{\psi}_k \Gamma \psi_k \phi \,, \tag{1}$$

where M_{0k}, m, g_k $(k=1,2)$ are the bare masses and coupling constants of the theory, and $\Gamma = I, \gamma^5$ for scalar and pseudoscalar meson fields, respectively. Evidently, the model specified in (1) can be readily generalized to include vector, pseudovector, and other types of meson fields. However, in the present work, we will limit our discussion to scalar or pseudoscalar coupling.

To make the transition to the quantum theory, we employ standard canonical quantization and Fourier transformation to momentum space. Thus, for the meson field, we write

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$$\phi(x)=\int d^3p[(2\pi)^3 2\omega(p,m)]^{-1/2} \times[e^{-ip\cdot x}a(\mathbf{p})+e^{ip\cdot x}a^\dagger(\mathbf{p})] , \quad (2)$$

where $\omega(p,m)=(\mathbf{p}^2+m^2)^{1/2}$ and

$$[a(\mathbf{p}),a^\dagger(\mathbf{q})]=\delta^3(\mathbf{p}-\mathbf{q}) , \quad (3)$$

while all other commutators vanish.

Similarly, for the nucleon fields, we write

$$\psi_k(x)=\sum_s \int d^3p(2\pi)^{-3/2}[U_k(\mathbf{p}s)e^{-ip\cdot x}b_k(\mathbf{p}s) + V_k(\mathbf{p}s)e^{ip\cdot x}d_k^\dagger(\mathbf{p}s)] , \quad (4)$$

where

$$[b_k(\mathbf{p}s),b_k^\dagger(\mathbf{q}\sigma)]_+=[d_k(\mathbf{p}s),d_k^\dagger(\mathbf{q}\sigma)]_+ =\delta_{s\sigma}\delta^3(\mathbf{p}-\mathbf{q}) , \quad (5)$$

and all other anticommutators vanish.

In Eq. (4) the free-nucleon spinors, with masses M_k, are normalized such that

$$\bar{U}_k(\mathbf{p}s)U_k(\mathbf{p}s)=-\bar{V}_k(\mathbf{p}s)V_k(\mathbf{p}s) =M_k/\omega(p,M_k) ; \quad (6)$$

that is, they differ from the standard spinors according to

$$U_k(\mathbf{p}s)=[M_k/\omega(p,M_k)]^{1/2}u_k(\mathbf{p}s) , \quad \text{etc.} \quad (7)$$

For systems consisting of nucleons only (no antinucleons), the effective normal-ordered Hamiltonian, in the Schrödinger representation (with $t=0$), is of the form

$$:H:=\sum_{k,s}\int d^3p\,\Omega_k(\mathbf{p})b_k^\dagger(\mathbf{p}s)b_k(\mathbf{p}s)+\int d^3q\,\omega(q,m)a^\dagger(\mathbf{p})a(\mathbf{q}) - \sum_k \lambda_k \sum_{s\sigma}\int d^3p\,d^3q[\omega(\mathbf{p}-\mathbf{q},m)]^{1/2}b_k^\dagger(\mathbf{p}s)b_k(\mathbf{q}\sigma)[a(\mathbf{p}-\mathbf{q})+a^\dagger(\mathbf{q}-\mathbf{p})]\bar{U}_k(\mathbf{p}s)\Gamma U_k(\mathbf{q}\sigma) , \quad (8)$$

where $\lambda_k=[2(2\pi)^3]^{-1/2}g_k$, and where

$$\Omega_k(\mathbf{p})=\omega(\mathbf{p},M_k)+\frac{(M_{0k}-M_k)M_k}{\omega(\mathbf{p},M_k)} . \quad (9)$$

The expression for $\Omega_k(\mathbf{p})$ reflects the fact that the masses of the physical nucleons are not identical to the bare masses M_{0k} of the Lagrangian. Note that Eq. (8) is not the complete Hamiltonian as we have suppressed all terms containing antinucleon operators, d_k and $d_k^\dagger$.

For a system of N_1 protons and N_2 neutrons and an arbitrary number of quanta of the interaction field, we consider the ansatz

$$|\psi\rangle\equiv|N_1N_2\rangle=\sum_{s\sigma}\int(d^3p)^{N_1}(d^3n)^{N_2}\Big\{\phi^0(p_1s_1,p_2s_2,\ldots,p_{N_1}s_{N_1};n_1\sigma_1,n_2\sigma_2,\ldots,n_{N_2}\sigma_{N_2}) + \int d^3q[\omega(qm)]^{-3/2}\phi^{(1)}(p_1s_1,p_2s_2,\ldots;n_1\sigma_1,n_2\sigma_2,\ldots;q)a^\dagger(q) + \cdots+\frac{1}{k!}\int d^3q_1\cdots d^3q_k[\omega(q_1m)\cdots\omega(q_km)]^{-3/2} \times\phi^{(k)}(p_1s_1,\ldots;n_1\sigma_1,\ldots;q_1q_2,\ldots,q_k) \times\prod_{l=1}^{k}a^\dagger(q_l)+\cdots\Big\}\prod_{i=1}^{N_1}b_1^\dagger(p_is_i)\prod_{j=1}^{N_2}b_2^\dagger(p_js_j)|0\rangle , \quad (10)$$

where $|0\rangle$ is the trial vacuum state annihilated by a, b_1, and b_2. For ease of notation we have suppressed the vector nature of the various three-momenta. In addition, we shall at times use the notation

$$p\equiv\{p_1s_1,\ldots,p_{N_1}s_{N_1}\} , \quad n\equiv\{n_1\sigma_1,\ldots,n_{N_2}\sigma_{N_2}\} , \quad q\equiv\{q_1,\ldots,q_k\} , \quad (11)$$

and we use $\tilde{q}_k$ to denote the set of all the q's excluding q_k. The equations for the coefficient functions $\phi^{(k)}$ follow from the variational principle

$$\langle\delta\psi|:H-E:|\psi\rangle=0 , \quad (12)$$

and are, explicitly,

$$\left[\sum_{i=1}^{N_1}\Omega_1(p_i)+\sum_{j=1}^{N_2}\Omega_2(n_j)+\sum_{k=1}^{\beta}\omega(q_k m)-E\right]\phi^{(\beta)}(p;n;q)$$
$$=\lambda_1\sum_{k=1}^{\beta}\sum_{i=1}^{N_1}\sum_{s}\omega(q_k m)\bar{U}_1(p_i s_i)\Gamma U_1(p_i+q_k,s)\phi^{(\beta-1)}(p_1 s_1,\ldots,p_i+q_k s,\ldots;n;\tilde{q}_k)$$
$$+\lambda_2\sum_{k=1}^{\beta}\sum_{j=1}^{N_2}\sum_{\sigma}\omega(q_k m)\bar{U}_2(n_j\sigma_j)\Gamma U_2(n_j+q_k,\sigma)\phi^{(\beta-1)}(p;n_1,\sigma_1,\ldots,n_j+q_k,\sigma,\ldots;\tilde{q}_k)$$
$$+\lambda_1\sum_{i=1}^{N_1}\sum_{s}\int\frac{d^3Q}{\omega^2(p_i-Q,m)}\bar{U}_1(p_i s_i)\Gamma U_1(Qs)\phi^{(\beta+1)}(p_1 s_1,\ldots,p_i=Q,s_i=\sigma,\ldots;n;q,p_i-Q)$$
$$+\lambda_2\sum_{j=1}^{N_2}\sum_{\sigma}\int\frac{d^3Q}{\omega^2(n_j-Q,m)}\bar{U}_2(n_j\sigma_j)\Gamma U_2(Q\sigma)\phi^{(\beta+1)}(p;n_1\sigma_1,\ldots,n_j=Q,\sigma_j=\sigma,\ldots;q,n_j-Q)\,, \tag{13}$$

where $\beta=0,1,2,\ldots$. We work, of course, in the frame in which the total momentum of the system is zero; hence $\sum_{i=1}^{N_1}\mathbf{p}_i+\sum_{j=1}^{N_2}\mathbf{n}_j+\sum_{k=1}^{\beta}\mathbf{q}_k=0$, etc., in all terms of Eqs. (13), and the stationary state eigenvalue E is just the rest mass of the interacting (N_1+N_2) nucleon system.

In this paper we shall consider in some detail two-nucleon ($N_1=N_2=1$ or $N_1=0$, $N_2=2$) and three-nucleon ($N_1=1$, $N_2=2$) systems. For the two-nucleon case ($N_1=N_2=1$) the first few of the infinite set of coupled integral equations (13) are, in detail,

$$\left[\omega(pM_1)+\frac{(M_{01}-M_1)}{\omega(pM_1)}M_1+\omega(nM_2)+\frac{(M_{02}-M_2)}{\omega(nM_2)}M_2-E_2\right]\phi^{(0)}(ps_1;ns_2)$$
$$=\lambda_1\sum_{\sigma}\int\frac{d^3Q}{\omega^2(p-Q,m)}\bar{U}_1(ps_1)\Gamma U_1(Q\sigma)\phi^{(1)}(Q\sigma;ns_2;p-Q)$$
$$+\lambda_2\sum_{\sigma}\int\frac{d^3Q}{\omega^2(n-Q,m)}\bar{U}_2(ns_2)\Gamma U_2(Q\sigma)\phi^{(1)}(ps_1;Q\sigma;n-Q)\,, \tag{14}$$

$$\left[\omega(pM_1)+\frac{(M_{01}-M_1)}{\omega(pM_1)}M_1+\omega(nM_2)+\frac{(M_{02}-M_2)}{\omega(nM_2)}M_2+\omega(qm)-E_2\right]\phi^{(1)}(ps_1;ns_2;q)$$
$$=\lambda_1\omega(qm)\sum_{\sigma}\bar{U}_1(ps_1)\Gamma U_1(p+q,\sigma)\phi^{(0)}(p+q,\sigma;ns_2)+\lambda_2\omega(qm)\sum_{\sigma}\bar{U}_2(ns_2)\Gamma U_2(n+q,\sigma)\phi^{(0)}(ps_1;n+q,\sigma)$$
$$+\lambda_1\sum_{\sigma}\int\frac{d^3Q}{\omega^2(p-Q,m)}\bar{U}_1(ps_1)\Gamma U_1(Q\sigma)\phi^{(2)}(Q\sigma;ns_2;q,p-Q)$$
$$+\lambda_2\sum_{\sigma}\int\frac{d^3Q}{\omega^2(n-Q,m)}\bar{U}_2(ns_2)\Gamma U_2(Q\sigma)\phi^{(2)}(ps_1;Q\sigma;q,n-Q)\,, \tag{15}$$

$$\left[\omega(pM_1)+\frac{(M_{01}-M_1)}{\omega(pM_1)}M_1+\omega(nM_2)+\frac{(M_{02}-M_2)M_2}{\omega(nM_2)}+\omega(q_1m)+\omega(q_2m)-E_2\right]\phi^{(2)}(ps_1;ns_2;q_1,q_2)$$
$$=\lambda_1\sum_{\sigma}[\omega(q_1m)\bar{U}_1(ps_1)\Gamma U_1(p+q_1,\sigma)\phi^{(1)}(p+q_1,\sigma;ns_2;q_2)$$
$$+\omega(q_2m)\bar{U}_1(ps_1)\Gamma U_1(p+q_2,\sigma)\phi^{(1)}(p+q_2,\sigma;ns_2;q_1)]$$
$$+\lambda_2\sum_{\sigma}[\omega(q_1m)\bar{U}_2(ns_2)\Gamma U_2(n+q_1,\sigma)\phi^{(1)}(ps_1;n+q_1,\sigma;q_2)$$
$$+\omega(q_2m)\bar{U}_2(ns_2)\Gamma U_2(n+q_2,\sigma)\phi^{(1)}(ps_1;n+q_2,\sigma;q_1)]$$
$$+\lambda_1\sum_{\sigma}\int\frac{d^3Q}{\omega^2(p-Q,m)}\bar{U}_1(ps_1)\Gamma U_1(Q\sigma)\phi^{(3)}(Q\sigma;ns_2;q_1,q_2,p-Q)$$
$$+\lambda_2\sum_{\sigma}\int\frac{d^3Q}{\omega^2(n-Q,m)}\bar{U}_2(ns_2)\Gamma U_2(Q\sigma)\phi^{(3)}(ps_1;Q\sigma;q_1,q_2,n-Q)\,, \tag{16}$$

etc.

The equations for the one-nucleon case, which we have not written out explicitly, can be obtained from Eqs. (14)–(16) by setting $\lambda_2=0$ and by eliminating all dependence on (say) n. In this case the energy eigenvalue E_1 (i.e., the one-nucleon mass) depends on λ_1^2, as is evident from the substitution $\phi^{(\beta)}\rightarrow\lambda_1^{\beta}\phi^{(\beta)}$. In the same manner, it follows that the eigenenergy E_N (rest mass of an arbitrary N-neutron or N-proton system) depends only on λ^2; that is, it is independent of the sign of g. More generally for a neutron-proton system, Eq. (16) implies that E_2 will depend on even powers of λ_1 and λ_2.

III. EXACT SOLUTION OF EQUATIONS IN THE FIXED-NUCLEON LIMIT

The Yukawa model for a single-fermion field is exactly solvable in the fixed-nucleon limit [8]. In this paper we consider the more general case of two different kinds of fermion fields and any number of particles. The fixed-nucleon limit corresponds to the case where $\omega(p,M_i)\rightarrow M_i$ or, equivalently, $M_i\rightarrow\infty$, where M_i is the physical (renormalized) mass given by

$$M_i=M_{0i}-\lambda_i^2\int\frac{d^3Q\,f(Q^2)}{\omega^2(Qm)}\ , \tag{17}$$

and $f(Q^2)$ is a cutoff factor used to control the otherwise divergent (when $f=1$) integral in (17). In the fixed-nucleon limit, Eq. (13) reduces to the form

$$\begin{aligned}\left[N_1M_{01}+N_2M_{02}+\sum_{k=1}^{\beta}\omega(q_km)-E\right]\phi^{(\beta)}(p;n;q)&\\ =\sum_{k=1}^{\beta}\omega(q_km)\left[\lambda_1\sum_{i=1}^{N_1}\phi^{(\beta-1)}(p_1,\ldots,p_i+q_k,\ldots;n;\tilde{q}_k)+\lambda_2\sum_{j=1}^{N_2}\phi^{(\beta-1)}(p;\ldots,n_j+q_k,\ldots;\tilde{q}_k)\right]&\\ +\int\frac{d^3Q\,f(Q^2)}{\omega^2(Qm)}\left[\lambda_1\sum_{i=1}^{N_1}\phi^{(\beta+1)}(\ldots,p_i-Q,\ldots;n;qQ)+\lambda_2\sum_{j=1}^{N_2}\phi^{(\beta+1)}(p;\ldots,n_j-Q,\ldots;qQ)\right]\ ,&\end{aligned} \tag{18}$$

since in that limit the fermion nature of the nucleons becomes suppressed [thus, $\bar{U}(ps)\Gamma U(q\sigma)\rightarrow\delta_{s\sigma}$ for $\Gamma=I$, that is for the scalar coupling case].

The infinite chain of coupled equations (18) describe the N_1+N_2 nucleon system along with any number of (virtual) mesons which mediate the internucleon interactions and provide the "links" in the chain of equations. Of course, such an infinite set of coupled equations is impossible to solve without approximation (such as, for example, truncation), unless an ansatz can be found that decouples the equations, so that they can each be solved separately. Fortunately it is possible, in this case, to write down an ansatz, which decouples the infinite chain of Eq. (18):

$$\phi^{(\beta)}(p;n;q)=\lambda_1\sum_{k=1}^{N_1}\phi^{(\beta-1)}(\ldots,p_k+q_i,\ldots,n;\tilde{q}_i)+\lambda_2\sum_{l=1}^{N_2}\phi^{(\beta-1)}(p;\ldots,n_l+q_i,\ldots;\tilde{q}_i)\ . \tag{19}$$

We were guided to the particular form (19) by considering the (simpler) two-nucleon case ($N_1=N_2=1$) and small values of β, that is, equations that correspond to a small number of (virtual) mesons. Note that the result for $\phi^{(\beta)}$ is the same irrespective of which q_i is used on the right-hand side of Eq. (19). Thus, substituting Eq. (19) into (18) and using the identification (17) we obtain, without any approximations, the result

$$\begin{aligned}(N_1M_1+N_2M_2-E)\phi^{(\beta)}(p;n;q)=\int\frac{d^3Q}{\omega^2(Qm)}f(Q^2)\Bigg\{&\lambda_1^2\sum_{i\neq j}^{N_1}\phi^{(\beta)}(\ldots,p_i-Q,\ldots,p_j+Q,\ldots;n;q)\\ &+\lambda_2^2\sum_{i\neq j}^{N_2}\phi^{(\beta)}(p;\ldots,n_i-Q,\ldots,n_j+Q,\ldots;q)\\ &+2\lambda_1\lambda_2\sum_{i,j}\phi^{(\beta)}(\ldots,p_i-Q,\ldots;\ldots,n_j+Q,\ldots;q)\Bigg\}\ .\end{aligned} \tag{20}$$

Equation (20) is just the momentum-space Schrödinger equation for a system of N_1+N_2 fixed nucleons interacting via the meson field. Note that the mesonic momenta q appear only as a parameter in the "wave function" $\phi^{(\beta)}$, and so are ignorable coordinates. Indeed, in this fixed nucleons limit the q dependence can be factored out of $\phi^{(\beta)}(p;n;q)=\phi^{(0)}(p;n)G(q)$ and so Eq. (20) can be written in terms of $\phi^{(0)}$. Note, also, that the mass renormalization conditions (17) are just the solutions of Eq. (20) for the one-particle case, $N_1=N_2=1$.

The coordinate-space representation of Eq. (20) is [with $f(Q^2)=1$] given by the expression

$$\left[\sum_{i=1}^{N_1}M_1+\sum_{j=1}^{N_2}M_2+\frac{1}{2}\sum_{i\neq j}V_{ij}(|\mathbf{r}_i-\mathbf{r}_j|)\right]\psi=E\psi\ , \tag{21}$$

where

$$V_{ij}(|\mathbf{r}_i-\mathbf{r}_j|)=-4\pi\lambda_i\lambda_j\frac{e^{-m|\mathbf{r}_i-\mathbf{r}_j|}}{|\mathbf{r}_i-\mathbf{r}_j|} ,$$

that is, a Schrödinger equation for fixed nucleons with attractive Yukawa pairwise interactions. We see, therefore, that the ansatz (19), which decouples the infinite chain of Eq. (18), has the effect of convoluting the field-theoretic meson-exchange interaction into the Yukawa-like internucleon potentials V_{ij} so that the (virtual) mesonic coordinates no longer appear explicitly in the decoupled Eq. (21).

IV. A SPINLESS MODEL

We consider, in the first instance, a simplified spinless version of the present model, in which the spinor products $\bar{U}(ps)U(q\sigma)$ for the scalar meson coupling case ($\Gamma=1$) are replaced, in Eq. (13) [or, equivalently, in the Hamiltonian (8)], by the expression [9]

$$\bar{U}(ps)U(q\sigma)\rightarrow\frac{M}{\omega(P,M)}\delta_{s\sigma} , \tag{22}$$

where $P=(\mathbf{p}+\mathbf{q})/2$. This replacement means that we can suppress the spinor indices in Eq. (13), and so obtain a simplified chain of equations for the spinless model. Specifically, for the case $N_1=N$, $\lambda_1=\lambda$, and $N_2=\lambda_2=0$ (i.e., "protons" only), if we use the replacement $\phi^{(\beta)}\rightarrow\lambda^\beta\phi^{(\beta)}$, the spinless model equations are given by

$$\left\{\sum_{i=1}^{N}\left[\omega(p_i,M)+\frac{(M_0-M)M}{\omega(p_i,M)}\right]+\sum_{k=1}^{\beta}\omega(q_k,m)-E_N\right\}\phi^{(\beta)}(p_1,p_2,\ldots,p_N;q_1,q_2,\ldots,q_\beta)$$
$$=\sum_{k=1}^{\beta}\sum_{i=1}^{N}\omega(q_k,m)\frac{M}{\omega(p_i+\frac{1}{2}q_k,M)}\phi^{(\beta-1)}(p_1,\ldots,p_i+q_k,\ldots,p_N;q_1,q_2,\ldots,\tilde{q}_k,\ldots,q_\beta)$$
$$+\lambda^2\sum_{i=1}^{N}\int\frac{d^3Q}{\omega^2(p_i-Q,m)}\frac{M}{\omega((p_i+Q)/2,M)}\phi^{(\beta+1)}(p_1,\ldots,p_i=Q,\ldots,p_N;q,p_i-Q) . \tag{23}$$

Alternatively, rather than viewing Eqs. (23) as representing a spinless model, one can view them as approximations to Eqs. (13), since the form (22) is valid as an approximation to $\bar{U}(ps)U(q\sigma)$ in the limit when $|\mathbf{p}-\mathbf{q}|\ll\frac{1}{2}|\mathbf{p}+\mathbf{q}|$ (the replacement is exact for the case $\mathbf{p}=\mathbf{q}$), that is small momentum transfer between the nucleons and mesons.

Unfortunately, even for the spinless model, we have not been able to find an ansatz, which uncouples the infinite chain of Eq. (23) exactly. Therefore, we use an approximate decoupling scheme of the form which uncoupled the equations in the fixed nucleon limit [Eq. (19)], namely, we take

$$\phi^{(\beta)}(p_1,p_2,\ldots,p_N;q_1,q_2,\ldots,q_\beta)=\sum_{k=1}^{N}\phi^{(\beta-1)}(p_1,\ldots,p_k+q_i,\ldots,p_N;\tilde{q}_i) . \tag{24}$$

Substituting (24) into Eq. (23) we obtain

$$\left[\sum_{i=1}^{N}\omega(p_i,M)-E_N\right]\phi(p_1,\ldots,p_N)=\lambda^2\sum_{i\neq j}^{N}\int d^3Q\frac{M^2}{\omega^2(Q,m)\omega(p_i-\frac{1}{2}Q,M)\omega(p_i+\frac{1}{2}Q,M)}\phi(\ldots,p_i-Q,\ldots,p_j+Q,\ldots) , \tag{25}$$

where we have written $\phi(p)$ for $\phi^{(0)}(p)$, and where the physical mass M for each particle is related to the bare mass M_0 according to the renormalization condition

$$M=M_0-\lambda^2\int_0^{\Lambda(p_i)}d^3Q\frac{M\omega(p_i,M)}{\omega^2(Q,m)\omega(p_i-\frac{1}{2}Q,M)\omega(p_i+\frac{1}{2}Q,M)} , \tag{26}$$

which is here written in the boosted frame in which the particle has momentum $\mathbf{p}_i$. Note that the cutoff parameter $\Lambda(p_i)$ is chosen such that M is a constant independent of the frame of reference.

Equation (25) is a relativistic momentum-space N-particle equation, which, in the nonrelativistic limit reduces to the form

$$\left[\sum_{i=1}^{N}\left[M-\frac{1}{2M}\nabla_i^2-E_N\right]-\sum_{i\neq j}2\pi\lambda^2\frac{e^{-m|\mathbf{r}_i-\mathbf{r}_j|}}{|\mathbf{r}_i-\mathbf{r}_j|}\right]\times\psi(\mathbf{r}_1,\ldots,\mathbf{r}_N)=0 , \tag{27}$$

i.e., the N-body Schrödinger equation with pairwise Yukawa interactions.

V. SPINLESS TWO-PARTICLE EQUATION AND ITS SOLUTION

For the two-particle ($N=2$) case, Eq. (25) takes on the simple form

$$[2\sqrt{p^2+M^2}-E_2]\phi(\mathbf{p}) = \frac{f}{2\pi^2}\int d^3q \frac{\phi(\mathbf{q})}{(\mathbf{p}-\mathbf{q})^2+m^2}\frac{M^2}{M^2+\frac{1}{4}(\mathbf{p}+\mathbf{q})^2}, \quad (28)$$

where $\mathbf{p}=\mathbf{p}_1=-\mathbf{p}_2$ and $f=(2\pi)^2\lambda^2=g^2/4\pi$. In the coordinate representation this equation has the form

$$[2\sqrt{\mathbf{p}^2+M^2}-E_2]\psi(\mathbf{r})=4\pi^2M^2f \times\int d^2x\ \psi(\mathbf{x})\frac{e^{-m|\mathbf{r}+\mathbf{x}|-M|\mathbf{r}-\mathbf{x}|}}{|\mathbf{r}+\mathbf{x}||\mathbf{r}-\mathbf{x}|}; \quad (29)$$

that is, an equation with nonlocal interactions at small distances ($r\lesssim 1/M$), which, in the nonrelativistic limit ($M\to\infty$), reduces to the usual Yukawa potential.

Equation (28) is of the Fredholm type with a finite norm kernel. It has ground-state solutions for all values of the coupling constant $f\geq f_{\min}$, where

$$f_{\min}=\frac{m}{M}g_0\left[1+c_1\left(\frac{m}{M}\right)^2+O\left(\left(\frac{m}{M}\right)^3\right)\right] \quad (30)$$

for small values of $\mu\equiv m/M$ with $g_0=1.6798$ being the critical (minimum) coupling at which the Yukawa potential $-ge^{-r}/r$ supports a bound state in the nonrelativistic limit, while the constant c_1 is, approximately, $c_1=0.7$.

The eigenenergies E_2 of Eq. (28) decrease monotonically with increasing f from the value $E_2=2M$ at $f=f_{\min}$, as is typical of relativistic two-particle bound-state equation [5,6,10]. The value $E_2=0$ is reached at $f=f_{\max}$, which varies with m/M for each state.

Writing $\phi(\mathbf{p})=\phi_l(p)Y_{lm}(\hat{p})$, Eq. (28) is reduced to the radial form

$$[2\sqrt{p^2+M^2}-E_2]\phi_l(p)=\frac{f}{\pi}\int_0^\infty dq\frac{q}{p}k_l(p,q)\phi_l(q), \quad (31)$$

where

$$k_l(p,q)=\frac{4}{m^2+4M^2+2(p^2+q^2)} \times\left\{Q_l\left[\frac{p^2+q^2+m^2}{2pq}\right]+(-1)^lQ_l\left[\frac{p^2+q^2+4M^2}{2pq}\right]\right\}, \quad (32)$$

and where $Q_l(x)$ are Legendre functions of the second kind. Note that the kernel $k_l(p,q)$ is positive definite for even l, whereas for odd l it is such only if $m<2M$, beyond which it becomes negative definite. It follows, therefore, that there are no odd-l bound states if $m\geq 2M$.

The small-f perturbative expression for the two-particle energy in the Coulombic, $\mu=m/M=0$, case has the general form

$$\frac{E_2^{(n,l)}}{M}=2-\left[\frac{f}{2n}\right]^2+\left[\frac{f}{2n}\right]^4\left[\frac{6n}{2l+1}-\frac{5}{4}-2n\delta_{l0}\right]+O(f^5), \quad (33)$$

where $n=1,2,3,\ldots$ is the principal and $l=0,1,\ldots,n-1$ is the orbital angular-momentum quantum number.

For the case when $\mu\neq 0$, that is, for short-range interactions, bound states are possible only if $f\geq f_{\min}^{(n,l)}$. For s states ($l=0$), the binding energy, $\epsilon_2=2M-E_2$, has a quadratic dependence on $f-f_{\min}^{(n,0)}$ for the small $f-f_{\min}$,

$$\epsilon_2^{(n,0)}\simeq B^{(n,0)}(f-f_{\min}^{(n,0)})^2, \quad (34)$$

whereas for $l>0$ this dependence is linear:

$$\epsilon_2^{(n,l)}\simeq B^{(n,l)}|f-f_{\min}^{(n,l)}|. \quad (35)$$

These results follow simply from Eq. (31). Thus if ϕ_l^0 is the solution of Eq. (31) for $f=f_{\min}^{(n,l)}=f_0$, and so $E_2=2M$, then multiplying Eq. (31) by ϕ_l^0 and similarly multiplying the ϕ_l^0 equation by ϕ_l, integrating both and subtracting we obtain the relation

$$\int d^3p\left[\frac{f-f_0}{ff_0}\frac{p^2}{\sqrt{p^2+M^2}+M}+\frac{\epsilon_2}{2f}\right]\phi_l(p)\phi_l^0(p)=0. \quad (36)$$

Since, for small momenta, the wave function ϕ_l is, explicitly,

$$\phi_l(p)\underset{p\to 0}{\sim}c_1\frac{p^l}{p^2+M\epsilon_2}, \quad (37)$$

it follows that

$$\int d^3p\ \phi_l(p)\phi_l^0(p)\underset{\epsilon_2\to 0}{\sim}c_2+c_3/\sqrt{\epsilon_2}\delta_{l0}, \quad (38)$$

and

$$\int d^3p\frac{p^2}{\sqrt{p^2+M^2}+M}\phi_l(p)\phi_l^0(p)\underset{\epsilon_2\to 0}{\sim}c_4, \quad (39)$$

where $c_1,\ldots,c_4$ are constants. Thereupon, the results (34) and (35) follow from (36) in an obvious manner.

The minimal coupling strength at which binding occurs has the behavior $f_{\min}^{(n,l)}\simeq g_0^{(n,l)}(m/M)$ for small m/M, where $g_0^{(n,l)}$ is the critical coupling at which binding sets in for the nonrelativistic problem with the Yukawa potential, $-g(e^{-r}/r)$. Recall that $g_0^{(n,0)}=1.6798,6.46,14.2,25.1,\ldots$ and $g_0^{(n,0)}\simeq(\pi/2)n^2$ for large n. In contrast, when the coupling is strong (i.e., for $f\gg 1$), the binding energy has a linear dependence on f, $\epsilon_2\simeq\text{const}\times f$. It should be mentioned that when $\mu\gg 1$ and $f\sim\mu^2$, Eq. (28) is no longer of the Fredholm type, and so this case must be considered separately.

Equation (28) generalizes in a straightforward manner for the case of particles of different masses, $M_1\neq M_2$:

$$[\sqrt{p^2+M_1^2}+\sqrt{p^2+M_2^2}-E_2]\phi(\mathbf{p})=\frac{f}{2\pi^2}\int d^3q\frac{\phi(\mathbf{q})}{(\mathbf{p}-\mathbf{q})^2+m^2}\frac{M_1M_2}{[M_1^2+\frac{1}{4}(\mathbf{p}+\mathbf{q})^2]^{1/2}[M_2^2+\frac{1}{4}(\mathbf{p}+\mathbf{q})^2]^{1/2}}\ . \tag{40}$$

In particular, in the fixed force-center limit, $M_2\to\infty$, and for $m=0$, the one-particle eigenenergies of Eq. (39) have the perturbative expansion

$$\frac{E_1^{(n,l)}}{M_1}=1-\frac{1}{2}\left[\frac{f}{n}\right]^2+\frac{1}{2}\left[\frac{f}{n}\right]^4\left[\frac{2n}{2l+1}-\frac{1}{4}-n\delta_{l,0}\right]+O(f^5)\ , \tag{41}$$

in contrast to the expression for the Coulombic Klein-Gordan case,

$$\frac{E_1^{\mathrm{KG}}}{M_1}=1-\frac{1}{2}\left[\frac{f}{n}\right]^2-\frac{1}{2}\left[\frac{f}{n}\right]^4\left[\frac{2n}{2l+1}-\frac{3}{4}\right]+O(f^6)\ , \tag{42}$$

which contains only even powers of f, and for which the lowest-order relativistic correction leads to stronger binding than does the present model, Eq. (41).

The integral equation (31) can be solved by standard quadrature methods, and some results of such numerical solutions are shown in Fig. 1, where we plot the two-particle binding energy as a function of the coupling constant, for vari-

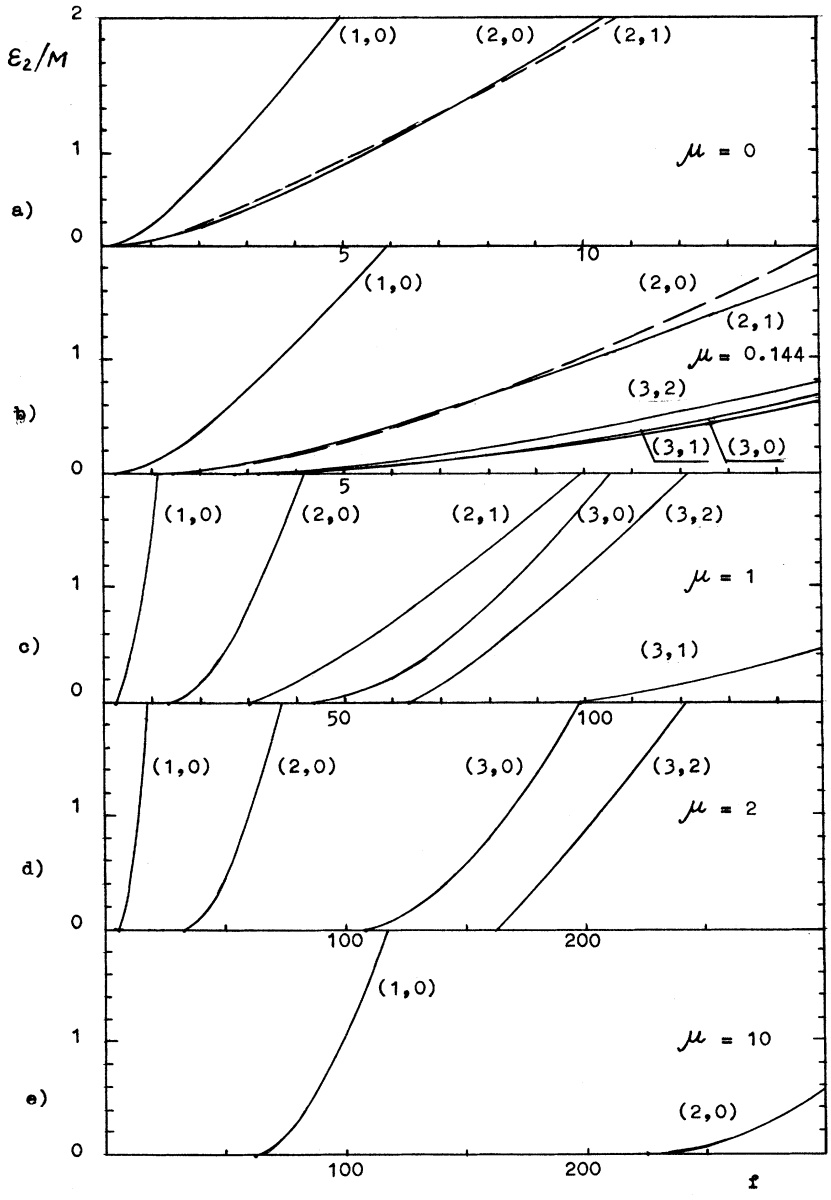

FIG. 1. Two-particle binding energies ϵ_2/M [Eq. (31)] as functions of the coupling constant f, for various values of the mass ratio $\mu=m/M$ and quantum numbers (n,l). Note the different scales of the coupling constant f for various values of μ.

TABLE I. Critical values of the coupling constant, $f_{\min}^{(n,l)}$ for various values of m/M and (n,l).

$\mu=m/M$	(1,0)	(2,0)	(2,1)	(3,0)	(3,1)	(3,2)	(4,0)
0.144	0.245	1.05	1.42	2.71	3.22	3.62	5.45
1.0	2.078	13.709	30.21	43.18	99.14	63.18	
2.0	5.14	33.32	∞	107.76	∞	162.0	
10.0	61.98	224.9	∞	545.6	∞	784.2	

ous values of $\mu=m/M$. (Note that the f scales in Fig. 1 are different for various μ values.) For $m=0, f_{\min}^{(n,l)}=0$ and our numerical results agree at low f ($f\lesssim 0.3$) with the perturbative expression (33). The numerical integration for Eq. (31) at low momenta must be treated with some care for the case $m=0$, $l=0$ because of the singular nature of the integrals in that case. In practice the calculations were done using a small but finite μ, typically $\mu<10^{-3}$. The Coulombic l degeneracy of the energy levels for fixed n is seen in Fig. 1 to be only slightly broken by the relativistic effects when μ is small [cf. Eq. (33)], but the degeneracy disappears completely with increasing μ.

The order of the energy levels, for given μ and f, can vary considerably. Thus, for $\mu=0.144$ and $f=6$ the order of the (n,l) levels is as follows: (1,0), (2,1), (2,0), (3,2), (3,1), (3,0), etc. Yet only a small change in the value of f can induce a reordering of the l sublevels, and there is even complete l degeneracy for particular values of f. The situation in this respect is similar to that which occurs in the nonrelativistic Yukawa problem.

The values of the minimal coupling strength $f_{\min}^{(n,l)}$, at which binding sets in for various states, are listed in Table I. As mentioned previously, these values depend strongly on the ratio m/M [cf. Eq. (30) for the ground state], and are related to the corresponding critical constants for the nonrelativistic Yukawa potential.

Figure 2 is a plot of the minimal coupling strength $f_{\min}^{(1,0)}$ at which binding sets in for the ground state. This critical coupling grows monotonically with the mass ratio $\mu=m/M$, and the numerical results are seen to agree closely with the formula (30) for $\mu\lesssim 0.3$.

VI. SPINLESS THREE-PARTICLE EQUATION

The solution of Eq. (25) for an arbitrary number of particles is no less complicated than for the nonrelativistic N-particle problem, and similar methods, such as variational methods, can be used to obtain approximate solutions. For the case of three equal mass particles Eq. (25) can be written in the form

$$[\sqrt{p_1^2+M^2}+\sqrt{p_2^2+M^2}+\sqrt{p_3^2+M^2}-E_3]\phi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)=\frac{f}{2\pi^2}\int\frac{d^3q}{q^2+m^2}[k(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q})\phi(\mathbf{p}_1-\mathbf{q},\mathbf{p}_2+\mathbf{q},\mathbf{p}_3)$$
$$+k(\mathbf{p}_2,\mathbf{p}_3,\mathbf{q})\phi(\mathbf{p}_1,\mathbf{p}_2-\mathbf{q},\mathbf{p}_3+\mathbf{q})$$
$$+k(\mathbf{p}_1,\mathbf{p}_3,\mathbf{q})\phi(\mathbf{p}_1-\mathbf{q},\mathbf{p}_2,\mathbf{p}_3+\mathbf{q})]\ , \tag{43}$$

where

$$k(\mathbf{p}_1,\mathbf{p}_2,\mathbf{q})=M^2[(\mathbf{p}_1-\tfrac{1}{2}\mathbf{q})^2+M^2]^{-1/2}[(\mathbf{p}_2+\tfrac{1}{2}\mathbf{q})^2+M^2]^{-1/2}\ , \tag{44}$$

and $\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3=\mathbf{P}=0$ in the rest frame.

In this paper we shall consider approximate solutions of Eq. (43) based on the Faddeev formalism [11–13]. To this end we express the three-particle wave function in the form

$$\phi(\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3)=\phi(\mathbf{k},\mathbf{p})+\hat{P}(12)\phi(\mathbf{k},\mathbf{p})+\hat{P}(23)\hat{P}(12)\phi(\mathbf{k},\mathbf{p})\ , \tag{45}$$

where $\mathbf{k}$ and $\mathbf{p}$ are the usual Jacobi momenta such that

$$\mathbf{p}_1=\mathbf{p}+\tfrac{1}{3}\mathbf{P},\quad \mathbf{p}_2=\mathbf{k}-\tfrac{1}{2}\mathbf{p}+\tfrac{1}{3}\mathbf{P},\quad \mathbf{p}_3=-\mathbf{k}-\tfrac{1}{2}\mathbf{p}+\tfrac{1}{3}\mathbf{P}\ ,$$

and where $\hat{P}(ij)$ is an operator that interchanges particles i and j. Thus Eq. (43) becomes the following equation for the amplitude $\phi(\mathbf{k},\mathbf{p})$:

$$[\sqrt{(\mathbf{k}-\tfrac{1}{2}\mathbf{p})^2+M^2}+\sqrt{(\mathbf{k}+\tfrac{1}{2}\mathbf{p})^2+M^2}+\sqrt{p^2+M^2}-E_3]\phi(\mathbf{k},\mathbf{p})$$
$$=\frac{f}{2\pi^2}\int d^3qV(\mathbf{k},\mathbf{q})[\phi(\mathbf{q},\mathbf{p})+\phi(\tfrac{1}{2}\mathbf{q}+\tfrac{3}{4}\mathbf{p},\mathbf{q}-\tfrac{1}{2}\mathbf{p})+\phi(-\tfrac{1}{2}\mathbf{q}+\tfrac{3}{4}\mathbf{p},-\mathbf{q}-\tfrac{1}{2}\mathbf{p})]\ , \tag{46}$$

where

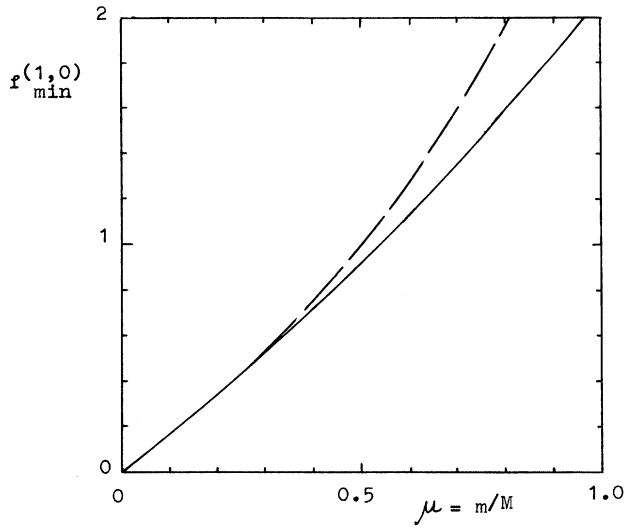

FIG. 2. The ground-state critical (minimal) coupling constant $f^{(1,0)}_{\min}$, as a function of the mass ratio $\mu=m/M$. Solid curve: from the numerical solution of Eq. (31). Broken curve: the asymptotic expansion (30).

$$V(\mathbf{k},\mathbf{q})=\frac{1}{(\mathbf{k}-\mathbf{q})^2+m^2}\frac{M^2}{\frac{1}{4}(\mathbf{k}+\mathbf{q})^2+M^2}\ . \tag{47}$$

In the nonrelativistic limit $p,k\ll M$, Eq. (46) reduces to the usual Faddeev equation with Yukawa interactions, but written in the potential rather than T-matrix form. Equations (43) and (46) are in all respects equivalent, except that whereas the Schrödinger form (43) is more convenient for direct variational solutions, the Faddeev form (46) is more suitable for various separable or approximating potential approaches.

Equation (46), for given values of m, M, and f, has a finite number of bound state eigenvalues, such that $E_3<3M$ and $\epsilon_3=3M-E_3\geq\epsilon_2=2M-E_2$ provided that $\epsilon_2\neq 0$. The ground three-particle bound-state exists if $f>f^{(0)}_{3,\min}$, where $f^{(0)}_{3,\min}<f^{(0)}_{2,\min}$, that is, the three-particle system, with the interaction (47), is more strongly bound than the two-particle system.

We note that, as in the two-particle case, we can determine the threshold behavior of the three-particle binding energy from Eq. (43). Thus the relation (36) generalizes, for the three particles, to the expression

$$\left\langle\psi_0\left|\frac{1}{f}G^{-1}(\epsilon_3)\right|\psi\right\rangle=\left\langle\psi_0\left|\frac{1}{f_0}G^{-1}(0)\right|\psi\right\rangle\ , \tag{48}$$

where $G(\epsilon_3)=[\sum_i\sqrt{p_i^2+M^2}-3M+\epsilon_3]^{-1}$ is the free-particle Green's function, and ψ is the exact solution of Eq. (43) (ψ_0 and f_0 correspond to the case when $\epsilon_3=0$).

From Eq. (46) it follows that

$$\phi(\mathbf{k},\mathbf{p})\underset{\epsilon_3,k,p\to 0}{\sim}\frac{\text{const}}{k^2+\frac{3}{4}p^2+M\epsilon_3} \tag{49}$$

whereupon we obtain from Eq. (48) the threshold law

$$\frac{\epsilon_3}{M}\simeq B|f-f_{3,\min}|\ , \tag{50}$$

for all states. This threshold behavior is essentially the same as occurs in the nonrelativistic case (see, for example, the remarks in Refs. [14,15]). The threshold behavior (50) is immediately generalizable to the N-body case for short-range interparticle interactions, provided that this interaction has no anomalous dependence on the energy.

The case $f\to f^{(1,0)}_{2,\min}$ when $E_2\to 2M$ ($\epsilon_2\to 0$), and $E_3\to 3M$, is of particular interest as this corresponds to the onset of the so-called Efimov effect [14]. The situation is dominated by small momenta $p,q\to 0$, and Eq. (46) becomes essentially nonrelativistic [16], with an infinite number of weakly bound three-particle states of total orbital angular momentum $L=0$. The number $\mathcal{N}$ of such Efimov levels increases as

$$\mathcal{N}\underset{\epsilon_2\to 0}{\sim}\frac{s_0}{\pi}\ln\left[\frac{1}{\epsilon_2}\right]\to\infty\ , \tag{51}$$

and the energy levels maintain the asymptotic relationship

$$\epsilon_3^{(n)}/\epsilon_3^{(n+1)}\underset{n\to\infty}{\sim}e^{2\pi/s_0}\sim 500 \tag{52}$$

as in the nonrelativistic case [14], where $s_0=1.006\,23$ is a characteristic constant.

In this paper we determine three-particle eigenenergies numerically, using a monotonic sequence of approximations to the "potential" (47) in Eq. (46), for the case $m/M=\frac{135}{940}=0.144\ll 1$. In the first place we replace $V(\mathbf{k},\mathbf{q})$ by the weaker s-type potential $V_0(k,q)$, where

$$V_0(k,q)=\tfrac{1}{2}\int_{-1}^{1}V(\mathbf{k},\mathbf{q})dz<V(\mathbf{k},\mathbf{q})\ , \tag{53}$$

and where $z=\mathbf{p}\cdot\mathbf{q}/pq$, that is,

$$V_0(k,q)=\frac{1}{2}\int_{-1}^{1}dz\frac{1}{(k^2+q^2+m^2-2kqz)}\times\frac{M^2}{M^2+\frac{1}{4}(k^2+q^2+2kqz)}\ . \tag{54}$$

Since $m\ll M$ for the case being considered, it is reasonable to simplify the short-range part of the potential further, by replacing it with its simplest separable approximation, $\tilde{V}_0(k,q)$, where

$$\begin{aligned}\tilde{V}_0(k,q)&=\frac{1}{(1+k^2/4M^2)(1+q^2/4M^2)}\times\frac{1}{2}\int_{-1}^{1}dz\frac{1}{k^2+q^2+m^2-2kqz}\\&=\frac{1}{(1+k^2/4M^2)(1+q^2/4M^2)}\frac{1}{4kq}\times\ln\frac{(k+q)^2+m^2}{(k-q)^2+m^2}\ ,\end{aligned} \tag{55}$$

and where we note that $\tilde{V}_0<V_0<V$ (see also Appendix A). We shall further use the Bateman approximation [17,18,12,13], in which the kernel $\tilde{V}_0(k,q)$ is replaced by

a sum of N separable forms $V_B^{(N)}(k,q)$, which, as is shown in Appendix B, form a monotonic sequence,

$$V_B^{(1)} < V_B^{(2)} < \cdots \leq \tilde{V}_0 < V_0 < V \ . \tag{56}$$

This replacement reduces Eq. (46) to a system of one-dimensional integral equations that can be solved by standard quadrature methods.

The Bateman approximation to the kernel $\tilde{V}_0(k,p)$ is given by

$$V_B^{(N)}(k,p) = \sum_{n=1}^{N} u_n(k) u_n(p) \ , \tag{57}$$

and the corresponding expression for the solution of (46) is

$$\phi(\mathbf{k},\mathbf{p}) = \frac{1}{K(\mathbf{k},\mathbf{p}) - E_3} \sum_{n=1}^{N} u_n(k) \phi_n(p) \ , \tag{58}$$

where

$$K(\mathbf{k},\mathbf{p}) = \omega(\mathbf{k} - \tfrac{1}{2}\mathbf{p}) + \omega(\mathbf{k} + \tfrac{1}{2}\mathbf{p}) + \omega(\mathbf{p}) \ .$$

The functions $\phi_n(p)$ satisfy the following system of one-dimensional equations:

$$\sum_n \left\{ d_{nm}(p)\phi_n(p) - \frac{f}{\pi^2} \int u_{nm}(\mathbf{p},\mathbf{q}) \phi_n(q) d^3q \right\} = 0 \ , \tag{59}$$

where

$$u_{nm}(\mathbf{p},\mathbf{q}) = \frac{u_n(|\mathbf{q} + \frac{1}{2}\mathbf{p}|) u_m(|\mathbf{p} + \frac{1}{2}\mathbf{q}|)}{[\omega(p) + \omega(q) + \omega(\mathbf{p}+\mathbf{q}) - E_3]} \tag{60}$$

and

$$d_{kn}(p) = \delta_{kn} - \frac{f}{2\pi^2} \int d^3q \frac{u_k(q) u_n(q)}{K(\mathbf{q},\mathbf{p}) - E_3} \ . \tag{61}$$

The analysis here is very similar to that of the nonrelativistic three-particle problem [13], except that the propagator has the relativistic form defined by $K(\mathbf{k},\mathbf{p})$.

We have applied the Bateman approximation to the two-body case discussed earlier in order to check its accuracy. Thus, for the case when $f = 0.359\,72$, $\epsilon_2/M = 0.0024$ (2.256 MeV) and $m/M = 0.144$ we obtain 1% agreement with the results of numerical quadrature, using a Bateman approximation of order $N = 3$. The rate of convergence of the Bateman approximation improves with increasing μ, whereas for $\mu \to 0$, that is, for a long-range potential, the convergence is very slow, although this has not been investigated in detail to date. Table II is a list of ground-state two- and three-particle eigenenergies, ϵ_2 and ϵ_3, as well as the single $L = 0$ three-particle excited state energy ϵ_3^*, that exists for the case $\mu = m/M = 0.144$ and $f = 0.359\,72$. The results given in Table II indicate that there is reasonable convergence of the Bateman approximation with $n = 3$, at least for the ground states.

TABLE II. Three particle binding energies, in the Bateman approximation of order N, for $\mu = 0.144$, $f = 0.359\,72$.

N	ϵ_3	ϵ_3^*	ϵ_2
1	16.2	1.50	1.435
2	23.52	2.47	2.11
3	25.0	2.83	2.235
"Exact"			2.256

Our numerical results indicate that the relativistic correction to the three-particle energy is not large. Indeed, when the parameters μ and f are adjusted such that the relativistic and nonrelativistic two-particle energies are both 2.256 MeV, the relativistic three-particle energy is about 0.1 MeV below the nonrelativistic value. In general, for values of the parameters in the vicinity of $\mu = 0.144$, $f = 0.36$ we find that ϵ_3 (rel) $\lesssim \epsilon_3$ (nonrel).

Figure 3 is plot of three-particle binding energies for a range of values of the coupling constant f. These plotted results were obtained using an order, $N = 1$ Bateman approximation, since higher order calculations require a lot of computer time. They are meant to illustrate the structure for the bound state spectrum of Eq. (46), which is seen to be similar to that of the nonrelativistic case.

We carried out detailed numerical calculations in the vicinity of $f_{3,\min}$, and our results confirm the linear threshold behavior $\epsilon_3/M \simeq B_3 |f - f_{3,\min}|$ for all three particle states. In particular, we find that $B_3 \simeq 0.006$ with $f_{3,\min} = 0.234\,65$ ($\mu = 0.144$) for the ground state. The constant B_3 varies with μ, and for small values, $\mu \to 0$, $B_3 \simeq c_3 \mu^3$ where $c_3 \simeq 2$, whereas $f_{3,\min}$ varies linearly with μ.

In the parameter domain when $\epsilon_2 \to 0$ our numerical calculations confirm the onset of the Efimov effect. The number of such excited three-particle levels grows as

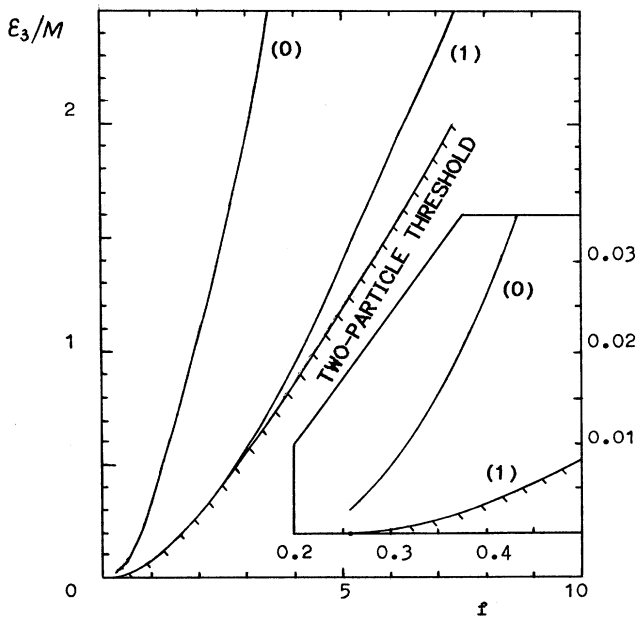

Fig. 3. The ground, (0), and first-excited, (1), three-particle binding energies as functions of the coupling constant f for $\mu = 0.144$ in the Bateman approximation of order 1. Hashed curve: two-particle threshold. Inset: details of the behavior near $f = f_{\min}^{(1,0)}$.

$\epsilon_2 \to 0$. For the first two levels we find that $\epsilon_3/\epsilon_3^* \simeq 660$, whereas for the next pair this ratio is $\epsilon_3^*/\epsilon_3^{**} \simeq 550$, which is in agreement with the asymptotic formula (52).

VII. CONCLUSIONS

We have used the variational method, within the Hamiltonian formalism of quantum field theory to derive momentum-space relativistic integral wave equations for a system of N_1 fermions ("protons") of mass M_1 and N_2 fermions ("neutrons") with mass M_2, which interact via a real scalar meson field of mass m. The variational ansatz is a superposition of N_1+N_2 free fermion states and any number of free boson states. The variational coefficient functions in this expansion are shown to satisfy an infinite chain of coupled integral equations, which are explicitly written out. In the limit of fixed nucleons it is shown that this infinite chain of equations can be uncoupled exactly by means of an ansatz in which the mesonic coordinates appear as explicit factors. The resulting integral equation for the fermionic part of the wave function is a momentum-space Schrödinger-like equation with Yukawa interparticle potentials.

We have, thereafter, considered a spinless model, which is valid for small momentum transfer among the nucleons and mesons. The infinite chain of equations were approximately decoupled for this model, using the same ansatz which decouples the equations exactly in the fixed-nucleon limit. The resulting N-"fermion" equation is shown to reduce to the N-particle Schrödinger equation, in the nonrelativistic limit, with Yukawa interparticle interactions. In general (i.e., relativistically), the interaction has a nonlocal character at small distances, $r < 1/M$, where M is the nucleon mass.

The relativistic two-particle equation was solved numerically to determine the bound-state spectrum for arbitrary quantum numbers n,l. We find that the binding is weaker than in the analogous nonrelativistic case, and this is corroborated by perturbative expansions of the two-particle binding energy in powers of the coupling constant. For massive meson exchange bound states exist only for values of the coupling constant, f, such that $f \geq f_{\min}^{(n,l)}$, where $f_{\min}^{(n,l)} \to 0$, as the mesonic mass, m, approaches zero, i.e, for long-range Coulombic interactions. For s states, the two-particle binding energy ϵ_2 is shown to have a quadratic dependence on $f - f_{\min}^{(n,l)}$, for small $f - f_{\min}$, whereas this dependence is found to be linear for $l \neq 0$. The interaction kernel is such that two-particle bound states are possible only if $m < 2M$ for odd l, whereas such bound states exist for all m/M values for states with even l.

For the scalar Coulombic case ($\mu = 0, f_{\min}^{(n,l)} = 0$) we have obtained asymptotic expansions of the two-particle binding energy in the domain $f \ll 1$, for two particles of equal mass M, and for the case where one of the masses becomes infinite (i.e., a fixed center of force). The binding, in this latter case, is found to be weaker than for the corresponding Klein-Gordon Coulomb equation.

Our numerical solutions of the relativistic two-particle equation, which were carried out for various values of the mass ratio m/M and coupling strength f, are in good agreement with and confirm the analytic approximations in various limits, such as the weak-coupling limit or the asymptotic expansion of $f_{\min}$ in terms of the mass ratio m/M.

The spinless model three-particle integral equation was recast in the Faddeev form, and was solved approximately using a separable approximation due to Bateman. Detailed numerical calculations were carried out for the case $m/M \simeq m_\pi/m_N = 0.144$ and with the coupling constant chosen such that the two-particle binding energy is close to the deuteron binding energy. For this case the third-order Bateman approximation, which gives a two-particle binding energy with 1% accuracy, yields a three-particle binding energy of about 25 MeV. Relativistic effects are found to be weak in the three particle system for this choice of parameters. Our numerical results confirm the onset of an infinite series of weakly bound excited three-particle states (the Efimov effect) in the domain where the coupling constant f approaches the critical value, $f_{\min}^{(1,0)}$, at which two-particle bound states set in.

This paper demonstrates that the variational Hamiltonian method is potentially a useful approach in a field-theoretic description of few-nucleon systems. Although in this work we have suppressed the nucleon spin degrees of freedom, this was done primarily to simplify the mathematics of the present expository work. The full and explicit retention of the spin coordinates presents no fundamental difficulties (as has been demonstrated in the atomic case [5]), although it does make the reduction of the equations to radial form somewhat more tedious.

It is also a straightforward matter to include various types of mesonic internucleon interactions (scalar, pseudoscalar, vector, pseudovector) by adding suitable terms to the interaction part of the Lagrangian (1). All the techniques of the present calculation can be just as readily carried through in such a more general case, and the resulting internucleon interaction would be modified from the monotonically attractive Yukawa form [cf. Eqs. (28), (29), (40), and (47)] to include a short-range repulsive core (as is needed in a realistic description of the internucleon interaction). It would be of interest to do such a more general calculation, containing various types of meson exchange terms in the Hamiltonian, in order to make a realistic comparison with the observed properties (energy levels, form factors, etc.) of few-nucleon systems.

Finally, we point out that the present VH approach is straightforwardly applicable to quasibound states, such as nuclear excited states, or a nucleon-antinucleon system, which decays into mesons. In the latter case, it is easy to accommodate a final-state two- (or more) meson channel into the trial state [cf. Eq. (19)] of the system. This results in equations that couple the $N\bar{N}$ and (say) $\pi\pi$ channels, and the short-lived $N\bar{N}$ system then appears as a resonance in the $\pi\pi$ scattering cross section [19].

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APPENDIX A

The kernel (32) of the integral Eq. (31), for states with orbital angular momentum l, has the form

$$k_l(p,q)=4M^2\int_{-1}^{1}dzP_l(z)/[(p^2+q^2+m^2-2pqz)(p^2+q^2+4M^2+2pqz)]\ , \tag{A1}$$

where $P_l(z)$ is the usual Legendre polynomial. We note the following identities:

$$\frac{1}{(A-2pqz)(B+2pqz)}=\frac{1}{AB}\left\{1+2pqz\left[\frac{1}{A^2}-\frac{1}{B^2}\right]\Big/\left[\frac{1}{A}+\frac{1}{B}\right]+(2pqz)^2\left[\frac{1}{A^3}+\frac{1}{B^3}\right]\Big/\left[\frac{1}{A}+\frac{1}{B}\right]+\cdots\right.$$
$$\left.+(2pqz)^n\left[\frac{1}{A^{n+1}}+\frac{(-1)^n}{B^{n+1}}\right]\Big/\left[\frac{1}{A}+\frac{1}{B}\right]+\cdots\right\}, \tag{A2}$$

where $A=p^2+q^2+m^2$ and $B=p^2+q^2+4M^2$, and

$$\left[\frac{1}{A^{n+1}}+\frac{(-1)^n}{B^{n+1}}\right]\Big/\left[\frac{1}{A}+\frac{1}{B}\right]=\frac{1}{A^n}-\frac{1}{A^{n-1}B}+\frac{1}{A^{n-2}B^2}-\cdots+\frac{(-1)^n}{B^n}\ . \tag{A3}$$

Therefore the kernel $k_l(p,q)$ can be represented by the series

$$k_l(p,q)\propto\frac{1}{AB}\sum_{n=0}^{\infty}(p^2q^2)^n(d_{n,l})^2\left[\frac{1}{A^{2n+1}}+\frac{1}{B^{2n+1}}\right]\Big/\left[\frac{1}{A}+\frac{1}{B}\right], \tag{A4}$$

for even l, whereas for odd l we have an analogous expression but multiplied by the factor $B-A=4M^2-m^2$.

The positive definite nature of the kernel $k_l(p,q)$ follows from the analogous property of the kernel $1/(p^2+q^2+m^2)$. Indeed, since

$$\frac{1}{p^2+q^2+m^2}=\frac{m^2}{(p^2+m^2)(q^2+m^2)-p^2q^2}=\frac{m^2}{(p^2+m^2)(q^2+m^2)}\sum_{k=0}^{\infty}\left[\frac{p^2}{p^2+m^2}\frac{q^2}{q^2+m^2}\right]^k \tag{A5}$$

is a series of positive terms, then substituting (A5), and the analogous expansion for $1/B$, into (A4), we find that the expression (A4) has the similar positive definite form,

$$k_l(p,q)\propto\sum_n\mathcal{K}_n f_n(p)f_n(q)\ , \tag{A6}$$

where $\mathcal{K}_n\geq 0$. This establishes the positive definite nature of the kernel (A1) for even values of l. All the above arguments can be applied in the case of odd l given that k_l can be written in the form

$$k_l\propto(4M^2-m^2)n_l\ , \tag{A7}$$

where $n_l(p,q)$ is a positive definite kernel.

APPENDIX B

We verify that a symmetric, positive-definite kernel $K(x,y)$ is not smaller than its Bateman approximation $K^B(x,y)$. The Bateman approximation has the form [12,17]

$$K^B(x,y)=\sum_{i=1}^{N}K_i(x,s_i)K_i(s_i,y)/K_i(s_i,s_i)\ , \tag{B1}$$

where

$$K_{i+1}(x,y)=K_i(x,y)-K_i(x,s_i)K_i(s_i,y)/K_i(s_i,s_i) \tag{B2}$$

and $K_1(x,y)=K(x,y)$. For a positive-definite kernel of the Hilbert-Schmidt type,

$$K(x,y)=\sum_i\frac{1}{\lambda_i}\phi_i(x)\phi_i(y)\ , \tag{B3}$$

where all eigenvalues $\lambda_i>0$, we have

$$K_1(s_1,s_1)=K(s_1,s_1)=\sum_i\frac{1}{\lambda_i}[\phi_i(s_i)]^2>0\ . \tag{B4}$$

Thus the first-order Bateman approximation yields a positive-definite kernel.

In a similar fashion,

$$K_2(s_2,s_2)=K(s_2,s_2)-[K(s_1,s_2)]^2/K(s_1,s_1)$$
$$=\frac{1}{K(s_1,s_1)}\sum_{i,j}\frac{1}{\lambda_i\lambda_j}[\phi_i^2(s_1)\phi_j^2(s_2)+\phi_j^2(s_1)\phi_i^2(s_2)-2\phi_i(s_1)\phi_i(s_2)\phi_j(s_1)\phi_j(s_2)]$$
$$=\frac{1}{K(s_1,s_1)}\sum_{i,j}\frac{1}{\lambda_i\lambda_j}[\phi_i(s_1)\phi_j(s_2)-\phi_j(s_1)\phi_i(s_2)]^2>0\ , \tag{B5}$$

which demonstrates the positive-definite nature of the second-order Bateman approximation, since

$$(\phi,K_2\phi)=\int d^3x\,d^3y\,\phi(x)K_2(x,y)\phi(y)=\sum_i\frac{1}{\lambda_i}\alpha_i^2-\frac{1}{K_1(s_1s_1)}\left[\sum_i\frac{1}{\lambda_i}\alpha_i\phi_i(s_1)\right]^2$$
$$=\frac{1}{K(s_1s_1)}\sum_{i,j}\frac{1}{\lambda_i\lambda_j}[\alpha_i\phi_j(s_1)-\alpha_j\phi_i(s_1)]^2>0\ , \tag{B6}$$

where $\alpha_i=\phi_1(s_1)\int d^3x\,\phi(x)$. Thus the positive-definite nature of $K_2(x,y)$ follows from that of $K_1(x,y)$. By induction, using the recurrence relation (B2), it follows that all kernels $K_i(x,y)$ are positive definite. Thus, all $K_i(s_is_i)$ being positive, the bilinear form (B1) is positive and increasing so that increasing the order N of the Bateman approximation increases the kernel K^B monotonically. From this it follows that K^B increases monotonically to K, i.e., $K^B\leq K$.

[1] For early work on this subject, see, for example, H. A. Bethe and F. de Hoffman, *Mesons and Fields, Vol. II: Mesons* (Row, Peterson, Evanston, IL, 1955), and references contained therein.

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