Effective interaction of three resonantly interacting particles and the force range

Vitaly Efimov
Nuclear Theory Group, Department of Physics, FM-15, and Nuclear Physics Laboratory, GL-10,
University of Washington, Seattle, Washington 98195
(Received 14 December 1992)

The effect of the force range r_0 on an effective long-range interaction in a system of three resonantly interacting particles is considered. By employing a boundary-condition approach, it is explicitly shown that the account of the force range gives rise to a correction r_0/R^3 in the effective long-range interaction. The correction is universal (i.e., independent of detail structure of the force), and is attractive for symmetric 0^+ three-particle states. The effect of the correction on the well-known correlation between the three-particle binding energy and the particle-pair scattering length is calculated.

PACS number(s): 21.45.+v, 25.10.+s, 03.65.Nk

I. INTRODUCTION

As of today, a great variety of models for the nuclear forces have been used in calculations of low-energy properties of the three-nucleon systems. Despite the differences in the models, the calculated properties display a universal pattern [1–3]. For example, if two models give rise to close values for the triton binding energy, the values of other observables turn out to be close.

It is natural to think that the universality, i.e., the independence of results on details of forces, is due to the fact that at low energies, with the wavelengths greatly exceeding the force range, the details cannot be discerned. Under these conditions only a small number of force parameters are significant, and these parameters determine all totality of low-energy properties of a three-particle system.

An approach can be developed that enables one to obtain the universal properties of a three-particle system directly, without any recourse to specific models of forces. Although this approach, the so-called three-particle zero-range theory, was introduced long ago [4,5], it was not used often until recently, because its physical meaning was not sufficiently clear [3]. Subsequent studies [6,7] made the physics rather clear and prompted a revival of interest in the subject. A number of interesting universal properties of the three-nucleon systems have been revealed and explained [2,3,8,9].

The zero-range theory is only a first approximation to the description of the universal properties. It can be shown that a correction, linear in the force range r_0, is also universal [3]: deviations from the universality start only in the quadratic approximation. Therefore, a next natural step, as far as the universal properties of three-particle systems are concerned, is to calculate the linear correction. The present paper addresses this problem.

The linear correction was first considered in Ref. [10], with a result formulated as an integral equation for the correction. Due to the revival of interest in the subject, we recently revised the problem and showed that the correction can be found explicitly [11]. By this we mean that it can be expressed in terms of wave functions of the zero-range theory. We also suggested that physics of the correction involves a modification of an effective long-range three-nucleon interaction $1/R^2$ existing in the three-particle systems under conditions of the zero-range theory, by a singular perturbation r_0/R^3.

In the present paper we give another independent derivation of the linear correction. We perform our calculations in the coordinate space, which enables us to reveal the origin of the perturbation r_0/R^3 in detail. This procedure has a clear advantage over that used in Ref. [11] where the analysis was performed in the momentum space and where only the answer was indicative of the r_0/R^3 form of the correction.

Since the aim of the present paper is to elucidate the physics of the correction and because the spin and isospin degrees of freedom seem not to bring new features [11], we restrict our consideration to the case of spinless particles. The particle masses are assumed to be unity.

II. EQUATIONS AND BOUNDARY CONDITIONS

We consider the three-body problem subject to the following conditions:

$$r_0\sqrt{|E|} \ll 1, \quad r_0\alpha \ll 1 , \tag{1}$$

where r_0 is the force range, E is the energy of the system, and α is a typical momentum of a loosely bound (or virtual) s state of two particles. The α is equal to $\sqrt{\epsilon}$ (or $-\sqrt{\epsilon}$ for the virtual state), where $-\epsilon$ is the energy of the state.

For the two-body case the solution of this problem is well known [4,12]. Under conditions (1), only s-wave forces are of importance. The s-state wave function can be found by solving the free Schrödinger equation

$$\left[\frac{d^2}{dr^2}+E\right] r\psi=0 \tag{2}$$

subject to the boundary condition

$$\left[\frac{d}{dr}+\alpha\right]_{r=0} r\psi=\frac{r_0}{2}\left[-\frac{d^2}{dr^2}+\alpha^2\right]_{r=0} r\psi , \tag{3}$$

where the right-hand side is considered as a small perturbation. It can be readily verified that the well-known expression for the scattering amplitude,

$$f(k) = -(\alpha+ik)^{-1} - \frac{r_0}{2}\frac{\alpha-ik}{\alpha+ik} \tag{4}$$

(where $k=\sqrt{E}$), follows from (2) and (3). Employing the standard relation between the scattering amplitude and the phase shift, one can also check that r_0 is the effective range of two-body scattering. At $r_0=0$ Eqs. (2)–(4) are the basic relations of the Bethe-Peierls zero-range theory [12]; the terms proportional to r_0 constitute a linear correction.

A similar approach for the three-body problem was developed in Ref. [6]. The leading approximation in expansion parameters (1) is formulated as follows. The wave function is written as a sum of three terms. For the most interesting case of the symmetric S state

$$\Psi = \chi(r_{12},\rho_3) + \chi(r_{23},\rho_1) + \chi(r_{31},\rho_2) , \tag{5}$$

where the coordinates $\mathbf{r}_{12},\boldsymbol{\rho}_3$ are defined as $\mathbf{r}_{12}=\mathbf{r}_1-\mathbf{r}_2$, $\boldsymbol{\rho}_3=(2/\sqrt{3})[\mathbf{r}_3-\frac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)]$. The other coordinate sets $\mathbf{r}_{23},\boldsymbol{\rho}_1$ and $\mathbf{r}_{31},\boldsymbol{\rho}_2$ are obtained by permutations. The wave function χ is found by solving the free Schrödinger equation [cf. Eq. (2)]

$$\left[\frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial \rho^2} + E\right] r\rho\chi = 0 \tag{6}$$

subject to the boundary condition [cf. Eq. (3) for $r_0=0$]

$$\left[\frac{\partial}{\partial r} + \alpha\right]_{r=0} r\Psi = 0 . \tag{7}$$

The states with other symmetries and angular momenta are considered in a similar way.

The comparison of expressions (2) and (3) with (6) and (7) makes it plausible that the account of a correction linear in expansion parameter (1) can be taken by modification of boundary condition (7) by a perturbation proportional to r_0:

$$\left[\frac{\partial}{\partial r} + \alpha\right]_{r=0} r\Psi = \frac{r_0}{2}\left[-\frac{\partial^2}{\partial r^2} + \alpha^2\right]_{r=0} r\Psi . \tag{8}$$

We show in Appendix A that condition (8) indeed holds. The formulations of the two- and three-body problem turn out to be similar, the latter being more complicated due to the additional variable ρ.

Denoting the function $r\rho\chi$ by χ_0 and substituting decomposition (5) for Ψ into (8), we finally obtain the equation and boundary condition for the three-body case:

$$(\Delta + E)\chi_0 = 0 , \tag{9}$$

$$\left[\frac{\partial}{\partial r} + \alpha\right]_{r=0} \chi_0 + \frac{8}{\sqrt{3}\rho}\chi_0\left[\frac{\sqrt{3}\rho}{2},\frac{\rho}{2}\right] = \frac{r_0}{2}\left[-\frac{\partial^2}{\partial r^2} + \alpha^2\right]_{r=0} \chi_0 , \tag{10}$$

where Δ is the two-dimensional Laplacian. In deriving (10) we used the fact that at $r_{12}=0$ one has $r_{23}=r_{31}=\sqrt{3}\rho_3/2$ and $\rho_1=\rho_2=\rho_3/2$. We also took into account that, being an even function of $\mathbf{r}_{12}$, the sum of the last two terms on the right-hand side of (5) does not contain terms linear in r_{12}. From the definition of χ_0 we have another boundary condition,

$$\chi_0(r,0) = 0 . \tag{11}$$

In addition to (10) and (11), there is one more boundary condition, involving small distances $r_{ik}\sim r_0$ between all three particles [6]. To impose it, a radius R is introduced according to $R^2=\frac{2}{3}(r_{12}^2+r_{23}^2+r_{31}^2)=r^2+\rho^2$. The close approach of three particles corresponds to $R\sim r_0$. The boundary condition is imposed at $R_0=Cr_0$, where C is a sufficiently large arbitrary number such that the wave function at $R\approx R_0$ has reached an $R\gg r_0$ asymptotic form. On the other side, R_0 is chosen such that $R_0\sqrt{|E|}\ll 1$ and $R_0\alpha\ll 1$. The leading asymptotic behavior of χ_0 in this region of R is [13]

$$\chi_0 \propto \sin[|s_0|\ln(R/R_0)+\Delta_{\rm in}] \tag{12}$$

(we omit the dependence on the other variables), where $|s_0|$ is a certain known number, and the phase $\Delta_{\rm in}$ is determined by the interaction of three particles at small distances $r_{ik}\sim r_0$. Since we only deal with the configuration space $R>R_0$, the phase $\Delta_{\rm in}$ is a parameter of our theory. The boundary condition referred to sets a certain fixed value for the phase $\Delta_{\rm in}$ or, equivalently, for the logarithmic derivative of χ_0 with respect to R at $R=R_0$.

As long as we restrict our consideration to the linear correction, the energy dependence of the phase $\Delta_{\rm in}$ can be neglected. Indeed, let us expand $\Delta_{\rm in}$ in powers of $r_0\sqrt{|E|}$. The leading term is of order of unity. The estimate for a first correction is r_0^2E because the Schrödinger equation at $R<R_0$ contains only an integer (first) power of E. The correction is seen to be quadratic in r_0 and can therefore be omitted. Thus, the phase $\Delta_{\rm in}$ is an independent three-body parameter which, like the two-body parameters α and r_0, determines the solution of our three-body problem.

III. LINEAR CORRECTIONS TO THE BINDING ENERGY AND SCATTERING LENGTH

The small perturbation on the right-hand side of boundary condition (10) gives rise to corrections to various three-body quantities. In this section we derive linear corrections to the three-particle binding energy and particle-pair scattering length.

We start with the binding energy. We write the energy and the bound-state wave function in the form $E=E_0+\delta E$, $\chi_0=\chi_0^{(0)}+\chi_0^{(1)}$, where δE and $\chi_0^{(1)}$ are corrections proportional to r_0. For $\chi_0^{(0)}$ and $\chi_0^{(1)}$ we have from (9) and (10)

$$(\Delta+E_0)\chi_0^{(0)}=0 \ , \tag{13}$$

$$\left[\frac{\partial}{\partial r}+\alpha\right]_{r=0}\chi_0^{(0)}+\frac{8}{\sqrt{3}\rho}\chi_0^{(0)}\left[\frac{\sqrt{3}\rho}{2},\frac{\rho}{2}\right]=0 \ , \tag{14}$$

$$(\Delta+E_0)\chi_0^{(1)}=-\delta E\chi_0^{(0)} \ , \tag{15}$$

$$\left[\frac{\partial}{\partial r}+\alpha\right]_{r=0}\chi_0^{(1)}+\frac{8}{\sqrt{3}\rho}\chi_0^{(1)}\left[\frac{\sqrt{3}\rho}{2},\frac{\rho}{2}\right] =\frac{r_0}{2}\left[-\frac{\partial^2}{\partial r^2}+\alpha^2\right]_{r=0}\chi_0^{(0)} \ . \tag{16}$$

According to the standard methods of solving the boundary-condition problems, Eq. (15) should be multiplied by $\chi_0^{(0)}$ on the left and integrated over r and ρ at $R>R_0$. The integral should then be transformed so as to make the Laplacian operate on $\chi_0^{(0)}$ instead of $\chi_0^{(1)}$. Equation (13) is used to calculate the result of the operation. Combinations of the boundary values of the functions $\chi_0^{(0)}$, $\chi_0^{(1)}$, and of their derivatives, which arise in the course of the transformation of the integral, should then be expressed in terms of the perturbation by using boundary conditions (14) and (16). As a result, one can obtain an explicit expression for δE.

In our problem this approach does not work directly. The reason is that the normalization condition for χ_0 has a nonstandard form. To see this, consider the normalization of Ψ. According to (5),

$$\langle\Psi|\Psi\rangle=3\langle\Psi|\chi\rangle \ . \tag{17}$$

Since at $R>R_0$ the function χ depends only on the length of the vector $\mathbf{r}$, the contribution from the distances $R>R_0$ to (17) contains only the projection of Ψ onto the s state of a pair. Introducing the projection operator P_0 onto the s state and defining $\Psi_0=r\rho P_0\Psi$, we obtain that the normalization at $R>R_0$ is determined by the integral

$$\int_{R>R_0}dr\,d\rho\,\Psi_0(r,\rho)\chi_0(r,\rho) \tag{18}$$

instead of the standard expression $\int dr\,d\rho\,\chi_0^2$.

The normalization (18) will be used in our derivation of δE. We multiply Eq. (15) by $\Psi_0^{(0)}$ on the left and then follow the standard approach. This modification proves to be enough to derive δE. After integration over r and ρ at $R>R_0$ and transformation of the integral we have

$$\int_{R>R_0}dr\,d\rho\,\chi_0^{(1)}(\Delta+E_0)\Psi_0^{(0)}+\int_{R_0}^{\infty}dR\left[\frac{\partial\Psi_0^{(0)}}{\partial r}\chi_0^{(1)}-\Psi_0^{(0)}\frac{\partial\chi_0^{(1)}}{\partial r}\right]_{r=0} +R_0\int_0^{\pi/2}d\beta\left[\frac{\partial\Psi_0^{(0)}}{\partial R}\chi_0^{(1)}-\Psi_0^{(0)}\frac{\partial\chi_0^{(1)}}{\partial R}\right]_{R=R_0}=-\delta E\int_{R>R_0}dr\,d\rho\,\Psi_0^{(0)}\chi_0^{(0)} \ , \tag{19}$$

where a polar angle $\beta=\arctan(r/\rho)$ is introduced [14]. The integral over two boundaries appear as a result of transformation of the initial integral. Boundary condition (11) was also used. The result of applying the operator $\Delta+E_0$ in (19) to the function $\Psi_0^{(0)}$ can be found from Eq. (13) and relation (5). The calculation is done in Appendix B, with the result

$$(\Delta+E_0)\Psi_0^{(0)}=-\frac{8}{\sqrt{3}R^2}(\Psi_0^{(0)})_{\beta=0}\delta\left[\beta-\frac{\pi}{3}\right] \ . \tag{20}$$

Using also boundary conditions (7) and (16), as well as the fact that $(\Psi_0^{(0)})_{r=0}=(\chi_0^{(0)})_{r=0}$, we find that the sum of the first two terms on the left-hand side of (19) is equal to

$$-\frac{r_0}{2}\int_{R_0}^{\infty}dR\left[\chi_0^{(0)}\left[-\frac{\partial^2}{\partial r^2}+\alpha^2\right]\chi_0^{(0)}\right]_{r=0} \ . \tag{21}$$

It is directly expressible in terms of the boundary-condition perturbation. For the shift δE we have

$$\delta E=\frac{(r_0/2)\int_{R_0}^{\infty}dR\left[\chi_0^{(0)}\left[-\frac{\partial^2}{\partial r^2}+\alpha^2\right]\chi_0^{(0)}\right]_{r=0}+R_0\int_0^{\pi/2}d\beta\left[\Psi_0^{(0)}\frac{\partial\chi_0^{(1)}}{\partial R}-\frac{\partial\Psi_0^{(0)}}{\partial R}\chi_0^{(1)}\right]_{R=R_0}}{\int_{R>R_0}dr\,d\rho\,\Psi_0^{(0)}\chi_0^{(0)}} \ . \tag{22}$$

The meaning of the second term in the numerator becomes clear if one compares Eq. (22) with the standard quantum-mechanical expression for the energy shift. Suppose one solves a two-dimensional problem with a potential $V_0(R)$ perturbed by another potential $V_1(R)$, such that both potentials operate at $R>R_0$, and with an energy-independent boundary condition imposed on the logarithmic derivative of the wave function at $R=R_0$. The energy shift for this problem is

$$\delta E=\frac{\int_{R_0}^{\infty}dR\,RV_1(R)(F^{(0)})^2+R_0\left[F^{(0)}\frac{dF^{(1)}}{dR}-\frac{dF^{(0)}}{dR}F^{(1)}\right]_{R=R_0}}{\int_{R_0}^{\infty}dR\,R\,(F^{(0)})^2} \ , \tag{23}$$

where $F^{(0)}$ is the wave function for the potential V_0, and $F^{(1)}$ is the wave-function correction proportional to the perturbation V_1. The presence of the second term in the numerator is due to the fact that the logarithmic derivative of the wave function at $R=R_0$ may be given an arbitrary small variation. If one chooses not to change it, the second term disappears, and the standard expression for the shift δE comes out.

The origin of the second term in the numerator of Eq. (22) appears to be the same, although its form is more complicated due to the dependence of the wave functions on the polar angle β. If we are not interested in the energy shift due to a variation of the boundary condition at $R=R_0$, this term can be omitted. As a result, we have for the shift

$$\delta E=\frac{(r_0/2)\int_{R_0}^{\infty}dR\left[\chi_0^{(0)}\left[-\frac{\partial^2}{\partial r^2}+\alpha^2\right]\chi_0^{(0)}\right]_{r=0}}{\int_{R>R_0}dr\,d\rho\,\Psi_0^{(0)}\chi_0^{(0)}} . \tag{24}$$

The normalization integral in (24) can also be expressed in terms of $\chi_0^{(0)}$ (Appendix B). The question, what form of the perturbation $V_1(R)$ this shift corresponds to, will be answered later.

Now we proceed to the second part of this section, the derivation of the shift of the particle-pair scattering length. For this case $E=-\alpha^2$. The equations and boundary conditions for $\chi_0^{(0)}$ and $\chi_0^{(1)}$ have the form

$$(\Delta-\alpha^2)\chi_0^{(0)}=0 , \tag{25}$$

$$\left[\frac{\partial}{\partial r}+\alpha\right]_{r=0}\chi_0^{(0)}+\frac{8}{\sqrt{3}\rho}\chi_0^{(0)}\left[\frac{\sqrt{3}\rho}{2},\frac{\rho}{2}\right]=0 , \tag{26}$$

$$(\Delta-\alpha^2)\chi_0^{(1)}=0 , \tag{27}$$

$$\left[\frac{\partial}{\partial r}+\alpha\right]_{r=0}\chi_0^{(1)}+\frac{8}{\sqrt{3}\rho}\chi_0^{(1)}\left[\frac{\sqrt{3}\rho}{2},\frac{\rho}{2}\right]=\frac{r_0}{2}\left[-\frac{\partial^2}{\partial r^2}+\alpha^2\right]_{r=0}\chi_0^{(0)} . \tag{28}$$

Employing the method described, we multiply Eq. (27) by $\Psi_0^{(0)}$ on the left and integrate over r,ρ from R_0 to $\bar{R}$, where $\bar{R}$ is a certain large radius much greater than the extent of the bound pair α^{-1}. We will tend $\bar{R}$ to infinity in the end of the derivation. After transformation of the integral and use of the boundary conditions, we obtain

$$-\frac{r_0}{2}\int_{R_0}^{\bar{R}}dR\left[\chi_0^{(0)}\left[-\frac{\partial^2}{\partial r^2}+\alpha^2\right]\chi_0^{(0)}\right]_{r=0}+R_0\int_0^{\pi/2}d\beta\left[\frac{\partial\Psi_0^{(0)}}{\partial R}\chi_0^{(1)}-\Psi_0^{(0)}\frac{\partial\chi_0^{(1)}}{\partial R}\right]_{R=R_0}=\bar{R}\int_0^{\pi/2}d\beta\left[\frac{\partial\Psi_0^{(0)}}{\partial R}\chi_0^{(1)}-\Psi_0^{(0)}\frac{\partial\chi_0^{(1)}}{\partial R}\right]_{R=\bar{R}} . \tag{29}$$

We show now that the integral on the right-hand side can be expressed in terms of the shift of the scattering length. Indeed, at $R=\bar{R}$ we can use large-distance expressions for the wave functions in the integrand. The form of χ_0 at large distances between the particle and the pair is determined by the particle-pair scattering length,

$$\chi_0\approx\left[\frac{\sqrt{3}\rho}{2}-A\right]\exp(-\alpha r) . \tag{30}$$

Here $\exp(-\alpha r)$ is proportional to the wave function of the pair, A is the scattering length, and $\sqrt{3}\rho/2$ is the relative distance between the particle and the pair, according to our definition of ρ. The particular choice of overall normalization in (30) is not important and was made so as to simplify the calculations. Writing A in the form $A=A_0+\delta A$, where δA is the scattering-length shift proportional to r_0, we have from (30) for the asymptotic forms of $\chi_0^{(0)}$ and $\chi_0^{(1)}$:

$$\chi_0^{(0)}\approx\left[\frac{\sqrt{3}\rho}{2}-A_0\right]\exp(-\alpha r) , \tag{31}$$

$$\chi_0^{(1)}\approx-\delta A\exp(-\alpha r) . \tag{32}$$

Due to the fact that these functions rapidly decrease with r, the integration over β on the right-hand side of Eq. (29) involves only a narrow strip $\beta\lesssim(\bar{R}\alpha)^{-1}$. In this region the second and third terms of decomposition (5) are exponentially small. For $\Psi_0^{(0)}\approx\chi_0^{(0)}$ the integral becomes

$$\bar{R}\int_0^{\pi/2}d\beta\left[\frac{\partial\Psi_0^{(0)}}{\partial R}\chi_0^{(1)}-\Psi_0^{(0)}\frac{\partial\chi_0^{(1)}}{\partial R}\right]_{R=\bar{R}}\approx\int_0^{\infty}dr\left[\frac{\partial\chi_0^{(0)}}{\partial\rho}\chi_0^{(1)}-\chi_0^{(0)}\frac{\partial\chi_0^{(1)}}{\partial\rho}\right]_{\rho=\bar{R}} . \tag{33}$$

Substituting expressions (31) and (32), we obtain $-(\sqrt{3}/4)(\delta A/\alpha)$, in accordance with our claim.

Thus, the shift of the scattering length is

$$\delta A=\frac{2}{\sqrt{3}}r_0\alpha\int_{R_0}^{\bar{R}}dR\left[\chi_0^{(0)}\left[-\frac{\partial^2}{\partial r^2}+\alpha^2\right]\chi_0^{(0)}\right]_{r=0}+\frac{4}{\sqrt{3}}R_0\alpha\int_0^{\pi/2}d\beta\left[\Psi_0^{(0)}\frac{\partial\chi_0^{(1)}}{\partial R}-\frac{\partial\Psi_0^{(0)}}{\partial R}\chi_0^{(1)}\right]_{R=R_0} . \tag{34}$$

The integral over β has the same origin and structure as that in Eq. (22). In fact, these two integrals are directly related to one other. Since the energy E in the Schrödinger equation can be neglected at $R \sim R_0$, the functions $\Psi_0^{(0)}$ and $\chi_0^{(1)}$ in both integrals are the same (to a numerical factor). Hence the integrals themselves differ only by a factor. If the integral in (22) is omitted, the integral in (34) must be omitted also. Tending $\bar{R}$ to infinity, we finally have for the shift of the scattering length

$$\delta A = \frac{2}{\sqrt{3}} r_0 \alpha \int_{R_0}^{\infty} dR \left[\chi_0^{(0)} \left[-\frac{\partial^2}{\partial r^2} + \alpha^2 \right] \chi_0^{(0)} \right]_{r=0} . \tag{35}$$

If one chooses another normalization of asymptotic form (30), expression (35) needs to be divided by the square of the normalization coefficient.

It is worth noting that expression (24) and (35) for δE and δA are consistent with one another. The shift δE is determined by an average of the perturbation operator over the bound-state wave function; accordingly, the shift δA involves the average of the same operator over the scattering wave function.

The rest of this section concerns with the form of the perturbation $V_1(R)$. The form of the interaction $V_0(R)$ is well known [6,7]: at the distances $r_0 \ll R \ll \alpha^{-1}$ it is an attraction of $1/R^2$ type with a known strength $|s_0|^2$. The wave function $\chi_0^{(0)}$ factorizes in this region, $\chi_0^{(0)} = F^{(0)}(R)\phi(\beta)$, with the radial wave function $F^{(0)}$ satisfying the equation of motion with the potential $V_0(R)$:

$$\left[-\frac{d^2}{dR^2} - \frac{d}{RdR} - \frac{|s_0|^2}{R^2} \right] F^{(0)}(R) = E_0 F^{(0)}(R) . \tag{36}$$

To find the form of $V_1(R)$, we rewrite the matrix elements in (24) and (35) using Eqs. (13) and (25) for $\chi_0^{(0)}$. At the distances of interest, $r_0 \ll R \ll \alpha^{-1}$, we employ the above factorization of the wave function and Eq. (36). We obtain

$$\begin{aligned} \int dR \left[\chi_0^{(0)} \left[-\frac{\partial^2}{\partial r^2} + \alpha^2 \right] \chi_0^{(0)} \right]_{r=0} &\approx \int dR \left[\chi_0^{(0)} \left[\frac{\partial^2}{\partial \rho^2} + E_0 \right] \chi_0^{(0)} \right]_{r=0} \\ &= -\phi^2(0) \int dR\, F^{(0)} \left[\frac{|s_0|^2}{R^2} + \frac{d}{R\, dR} \right] F^{(0)} \\ &= -\phi^2(0) \left[\left. \frac{(F^{(0)})^2}{2R} \right|_{R_0}^{\sim \alpha^{-1}} + \int dR \frac{|s_0|^2 + \frac{1}{2}}{R^2} (F^{(0)})^2 \right] . \end{aligned} \tag{37}$$

The term that involves the boundary values comes from the distances $R \sim r_0$ and $R \sim \alpha^{-1}$ and can be omitted. Therefore, the contribution of the distances $r_0 \ll R \ll \alpha^{-1}$ to the matrix element of the perturbation is

$$-\frac{1}{2}\phi^2(0) \left[|s_0|^2 + \frac{1}{2} \right] \int dR\, R \left[\frac{r_0}{R^3} \right] (F^{(0)})^2 , \tag{38}$$

where the phase space is written in the form $dR\, R$ appropriate for our two-dimensional formulation of the problem. Expression (38) shows that $V_1(R)$ has the form r_0/R^3 and is negative. This forms looks natural on the dimensional grounds [9] because the linear perturbative correction to the interaction $1/R^2$ must have the form r_0/R^3 in the region where no dimensional parameters are available.

At the distances $R \sim r_0$ the correction r_0/R^3 becomes comparable with the leading term $1/R^2$. If the distances are greater than the extent of the pair, $R \gg \alpha^{-1}$, the perturbation $V_1(R)$, like $V_0(R)$, dies out. Indeed, at these distances the function $\chi_0^{(0)}$ in matrix elements (24) and (35) describes an independent motion of the particle and the pair. Since the wave function of the pair is proportional to $\exp(-\alpha r)$ and because the application of the perturbation operator $-(\partial^2/\partial r^2) + \alpha^2$ to this function gives zero, the distances $R \gg \alpha^{-1}$ do not contribute to the matrix elements of perturbation.

To summarize, we have come up to the following qualitative picture of the effective interaction of three resonantly interacting particles. There is the potential $1/R^2$ plus a correction r_0/R^3, both acting at the distances $r_0 \lesssim R_0 \lesssim \alpha^{-1}$. The potential $1/R^2$ and the correction r_0/R^3 are attractive for symmetric 0^+ three-particle states.

IV. CORRELATION BETWEEN THE BINDING ENERGY AND SCATTERING LENGTH

Expressions (24) and (35) solve the problem of finding the linear shifts. Yet the solution is still incomplete. By this we mean that these expressions determine the shifts δE and δA as functions of the phase $\Delta_{\text{in}}^{(0)}$ of asymptotic expression (12) for $\chi_0^{(0)}$ [15]. This phase is not an observable. As was stated in Sec. II, the phase Δ_{in} characterizes the interaction of three particles at small distances $R \sim r_0$.

It is more useful to have our solution cast in a form determined by a certain observable instead of the phase $\Delta_{\text{in}}^{(0)}$. We can choose, for example, the particle-pair scattering length A. With this choice, in the leading approximation in expansion parameters (1), instead of two functions $E_0(\Delta_{\text{in}}^{(0)})$ and $A_0(\Delta_{\text{in}}^{(0)})$ we will have a correlation $E_0(A_0)$ between the binding energy and scattering length. The account of the corrections $\delta E(\Delta_{\text{in}}^{(0)})$ and

$\delta A(\Delta_{in}^{(0)})$ found in Sec. III will modify this correlation. It becomes $E(A)$, where $E=E_0+\delta E$, $A=A_0+\delta A$. In this section we calculate the difference between $E_0(A)$ and $E(A)$, a linear shift in the correlation.

We note that the appearance of a correlation and its subsequent modification also occur in the two-body problem under conditions (1). In the zero-range approximation the correlation between the two-particle binding energy and the scattering length a,

$$\epsilon_0(a)=a^{-2} , \tag{39}$$

is well known. This correlation is modified in the next, linear in r_0, approximation to become

$$\epsilon(a)=\epsilon_0(a)+\Delta\epsilon(a)=a^{-2}+r_0 a^{-3} . \tag{40}$$

Our correlation $E_0(A)$ is a direct analog of correlation (39). To find a correction $\Delta E(A)$ analogous to the correction $\Delta\epsilon(a)$, we write $E(A)$ in a form similar to (40):

$$E(A)=E_0(A)+\Delta E(A) . \tag{41}$$

From this definition we can obtain the following for $\Delta E(A)$:

$$\Delta E(A)=\delta E-\left[\frac{dE_0}{dA}\right]\delta A . \tag{42}$$

There is an interesting physics explaining the necessity of the subtraction term in expression (42). Since the perturbation r_0/R^3 is singular at small distances, the main contribution to δE and δA comes from distances $R\sim R_0$ [16]. As was pointed in Sec. II, the effect of any interaction at the small distances $R \ll (|E|^{-1/2},\alpha^{-1})$ can be absorbed into the phase Δ_{in}. Therefore, the effect of the singular part of the shifts is equivalent to resetting the value of $\Delta_{in}^{(0)}$. The variation of $\Delta_{in}^{(0)}$ just moves a point on the (E,A) plane along the correlation curve $E_0(A)$ of the leading approximation. On the other side, the shift $\Delta E(A)$ is a deviation from this curve. Therefore, the small-distance contributions to this shift must cancel, which is accomplished by the subtraction term in (42).

Since the expressions for δE and δA have been derived in Sec. III, the only unknown quantity in (42) is the derivative dE_0/dA. Its calculation can also be done using the method of Sec. III. Setting $r_0=0$ there, let us give a small variation to the phase $\Delta_{in}^{(0)}$. The quantities E_0 and A_0 will change by some δE_0 and δA_0, respectively. The ratio $\delta E_0/\delta A_0$ is the derivative dE_0/dA that we are looking for.

Let us find δE_0 and δA_0. From expression (22) we have

$$\delta E_0=\frac{R_0\int_0^{\pi/2}d\beta\left[\Psi_0^{(0)}\frac{\partial\chi_0^{(1)}}{\partial R}-\frac{\partial\Psi_0^{(0)}}{\partial R}\chi_0^{(1)}\right]_{R=R_0}}{\int_{R>R_0}dr\,d\rho\,\Psi_0^{(0)}\chi_0^{(0)}} , \tag{43}$$

where, for this case, $\chi_0^{(1)}$ is a correction to $\chi_0^{(0)}$ produced by the variation of the phase $\Delta_{in}^{(0)}$. From (34)

$$\delta A_0=\frac{4}{\sqrt{3}}R_0\alpha\int_0^{\pi/2}d\beta\left[\Psi_0^{(0)}\frac{\partial\chi_0^{(1)}}{\partial R}-\frac{\partial\Psi_0^{(0)}}{\partial R}\chi_0^{(1)}\right]_{R=R_0} . \tag{44}$$

As was pointed out in Sec. III, at $R\sim R_0$ the bound-state and scattering wave functions differ only by a factor. Therefore, the angular integrals in (43) and (44) are related to one another. If we denote the square of the factor by $Z+\delta Z$, where δZ is proportional to the variation of the phase $\Delta_{in}^{(0)}$, we have for the integrals in (43) and (44)

$$\int_0^{\pi/2}d\beta\left[\Psi_0^{(0)}\frac{\partial\chi_0^{(1)}}{\partial R}-\frac{\partial\Psi_0^{(0)}}{\partial R}\chi_0^{(1)}\right]_b=Z\int_0^{\pi/2}d\beta\left[\Psi_0^{(0)}\frac{\partial\chi_0^{(1)}}{\partial R}-\frac{\partial\Psi_0^{(0)}}{\partial R}\chi_0^{(1)}\right]_s+\frac{\delta Z}{2}\int_0^{\pi/2}d\beta\left[\Psi_0^{(0)}\frac{\partial\chi_0^{(0)}}{\partial R}-\frac{\partial\Psi_0^{(0)}}{\partial R}\chi_0^{(0)}\right]_s , \tag{45}$$

where b and s stand for the bound-state and scattering wave functions, respectively. Due to the boundary condition at $R=R_0$, the second term on the right-hand side disappears. Therefore, the integrals for the bound and scattering state differ only by the factor Z.

Taking this into account, we find the following for the derivative dE_0/dA:

$$\frac{dE_0}{dA}=\frac{\sqrt{3}Z/4\alpha}{\int_{R>R_0}dr\,d\rho\,\Psi_0^{(0)}\chi_0^{(0)}} , \tag{46}$$

with $Z=(\chi_{0b}^{(0)}/\chi_{0s}^{(0)})^2_{R=R_0}$.

We now use this expression to finish our calculation of the shift $\Delta E(A)$. Substituting the expressions for δE, δA, and dE_0/dA into (42), we obtain

$$\Delta E(A)=\frac{(r_0/2)\int_0^{\infty}dR\left[\chi_0^T\left[-\frac{\partial^2}{\partial r^2}+\alpha^2\right]\chi_0^T-Z\chi_0^0\left[-\frac{\partial^2}{\partial r^2}+\alpha^2\right]\chi_0^0\right]_{r=0}}{\int_0^{\infty}dr\,d\rho\,\Psi_0^T\chi_0^T} . \tag{47}$$

To avoid confusion with notations, we switched from $\chi_0^{(0)}$ to χ_0^T for the bound-state wave function and to χ_0^0 for the scattering wave function, respectively. Since the main contributions to the integrals come from the distances $R \sim (|E|^{-1/2}, \alpha^{-1})$, we extended the integrations to $R=0$. For Z we can use the asymptotic ratio $(\chi_0^T/\chi_0^0)^2_{R\to 0}$.

Expression (47) is our final answer for the linear shift of the correlation between the binding energy and scattering length. Let us now show the correspondence of (47) with an expression for this shift obtained in Ref. [11] in a rather different way.

We rewrite Eq. (47) taking into account Eqs. (13) and (25) for χ_0^T and χ_0^0:

$$\Delta E(A) = \frac{(r_0/2)\int_0^\infty d\rho \left[\chi_0^T\left[\frac{\partial^2}{\partial\rho^2}+E_0+\alpha^2\right]\chi_0^T - Z\chi_0^0\frac{\partial^2}{\partial\rho^2}\chi_0^0\right]_{r=0}}{\int_0^\infty dr\, d\rho\, \Psi_0^T\chi_0^T} . \tag{48}$$

The approach of Ref. [11] is formulated in terms of momentum components $\chi_T(p)$ and $\chi_0(p)$ related to χ_0^T and χ_0^0 in the following way [6]:

$$\chi_0^T(r,\rho) = \frac{2}{\pi}\int_0^\infty dp\, \chi_T(p)\sin\left[\frac{\sqrt{3}}{2}p\rho\right]\exp\left[-r\left[\frac{3p^2}{4}-E_0\right]^{1/2}\right] , \tag{49}$$

$$\chi_0^0(r,\rho) = \frac{2}{\pi}\int_0^\infty dp\left[\frac{\pi}{2p}\delta(p)+\chi_0(p)\right]\sin\left[\frac{\sqrt{3}}{2}p\rho\right]\exp\left[-r\left[\frac{3p^2}{4}+\alpha^2\right]^{1/2}\right] . \tag{50}$$

The term $(\pi/2p)\delta(p)$ is the free wave of the scattering wave function, and the coefficient in front of integral (50) is chosen such that the free wave coincides with that determined by Eq. (31). The factors $\sqrt{3}/2$ in (49) and (50) reflect the fact that the momentum conjugated to the coordinate ρ is $\sqrt{3}p/2$.

The substitution of these expressions into Eq. (48) gives the shift $\Delta E(A)$ obtained in Ref. [11] (see remark [17]).

V. CONCLUSION

The results of this paper can be summarized as follows. It has been shown by explicit calculations in coordinate space that the effect of force range can be described by a correction r_0/R^3 to the well-known effective interaction $1/R^2$ existing in systems of three resonantly interacting particles. Like the interaction $1/R^2$, the correction is universal. The correction does not destroy correlations between three-particle observables caused by the interaction $1/R^2$ yet modifies them. The modification of the well-known correlation between the binding energy of three particles and the particle-pair scattering length is explicitly calculated, and the result is given by expression (47).

ACKNOWLEDGMENTS

I thank E. G. Adelberger, W. C. Haxton, E. M. Henley, and M. N. McDermott for their hospitality which made possible the completion of this work. The support of the Nuclear Theory Group, Nuclear Physics Laboratory, and Institute for Nuclear Theory, University of Washington, is gratefully acknowledged.

APPENDIX A: BOUNDARY CONDITION FOR THE THREE-BODY WAVE FUNCTION

In this appendix we derive boundary condition (8) for the three-body wave function. The idea is first demonstrated for a simpler case of two particles.

We start with some definitions. Let ϕ_0 be the solution of the two-body Schrödinger equation for a state with the energy $-\alpha^2$,

$$(T+V)\phi_0 = -\alpha^2\phi_0 . \tag{A1}$$

The solution at a close energy E and in the region $r\sim r_0$ can be presented in the form

$$\phi = \phi_0 + (E+\alpha^2)\phi_1 , \tag{A2}$$

where the second term is a small correction. The equation for ϕ_1 follows from the Schrödinger equation:

$$(T+V)\phi_1 = \phi_0 . \tag{A3}$$

The estimate of ϕ_1 is $\phi_1 \sim r_0^2\phi_0$.

To derive boundary condition (3) using these definitions, we first calculate the logarithmic derivative of the wave function $r\phi$ at the boundary of the interaction region. Denoting the boundary by r_0', we have

$$\left[\frac{(r\phi)'}{r\phi}\right]_{r=r_0'} = \left[\frac{(r\phi_0)'+(E+\alpha^2)(r\phi_1)'}{r\phi_0+(E+\alpha^2)r\phi_1}\right]_{r=r_0'} .$$

The second term in the denominator is a quadratic correction in expansion parameters (1) and may be omitted. Yet we keep the second term in the numerator because the condition $\alpha r_0 \ll 1$ implies that the derivative $(r\phi_0)'_{r=r_0'}$ is abnormally small. Thus,

$$\left[\frac{(r\phi)'}{r\phi}\right]_{r=r_0'} = \left[\frac{(r\phi_0)'}{r\phi_0}\right]_{r=r_0'} + (E+\alpha^2)\left[\frac{(r\phi_1)'}{r\phi_0}\right]_{r=r_0'} . \tag{A4}$$

This expression contains two parameters describing the interaction of two particles in the region $r\sim r_0$. Since $r\phi_0$

has the form of $\exp(-\alpha r)$ outside the interaction region, the parameter $[(r\phi_0)'/r\phi_0]_{r=r_0'}$ is equal to $-\alpha$. We denote the second parameter $[(r\phi_1)'/r\phi_0]_{r=r_0'}$, which has the dimension of length, by l; its estimate is $l \sim r_0$. Introducing these notations in (A4), we have

$$\left[\frac{(r\phi)'}{r\phi}\right]_{r=r_0'} = -\alpha + l(E+\alpha^2) \ . \tag{A5}$$

Now, following the standard procedure, we match (A5) with the logarithmic derivative of a solution outside the interaction region. Denoting this solution by ψ and expanding in $r_0'\sqrt{|E|}$, we obtain

$$\left[\frac{(r\psi)'}{r\psi}\right]_{r=0} + r_0'\left\{\frac{(r\psi)''}{r\psi} - \left[\frac{(r\psi)'}{r\psi}\right]^2\right\}_{r=0} = -\alpha + l(E+\alpha^2) \ . \tag{A6}$$

To linear terms in r_0', the second term in the curly braces is equal to $-\alpha^2$, as follows from expression (A6) itself. In addition, the Schrödinger equation for ψ enables us to rewrite the energy E as $-[(r\psi)''/r\psi]_{r=0}$. Therefore, the factors multiplying r_0' and l differ only in their sign. Combining the terms with r_0' and l into one term and denoting the sum $r_0'+l$ by $r_0/2$, we finally obtain

$$\left[\frac{(r\psi)'}{r\psi}\right]_{r=0} = -\alpha + \frac{r_0}{2}\left\{-\left[\frac{(r\psi)''}{r\psi}\right]_{r=0} + \alpha^2\right\} .$$

This is boundary condition (3).

Consider now the three-body problem. First, we construct an analog of expansion (A2). We write the solution in the region $r \sim r_0$, $\rho \gg r_0$ in the form

$$\Phi = \Phi_0 + \Phi_1 \ ,$$

where the function Φ_0 is the solution in the leading approximation in expansion parameters (1). As is known [6], it factorizes, $\Phi_0 = \phi_0(r)A_0(\rho)$. For the correction Φ_1 we have from the Schrödinger equation:

$$(T_r + V)\Phi_1 = (E + \alpha^2 - T_\rho)\phi_0 A_0 \ ,$$

where T_r and T_ρ are the kinetic-energy operators for motion along the variables $\mathbf{r}$ and $\boldsymbol{\rho}$, respectively. Using Eq. (A3) we find that this equation is satisfied by the function $\Phi_1 = \phi_1(r)A_1(\rho)$, with $A_1 = (E+\alpha^2 - T_\rho)A_0$. Therefore, at $r \sim r_0$ [cf. Eq. (A2)]

$$\Phi = \phi_0 A_0 + \phi_1(E + \alpha^2 - T_\rho)A_0 \ . \tag{A7}$$

Following the two-body case, we now calculate the logarithmic derivative of Φ at r_0' and match it with the logarithmic derivative of an external solution. Denoting the latter by Ψ, we have

$$\left[\frac{(r\Psi)'}{r\Psi}\right]_{r=0} + r_0'\left\{\frac{(r\Psi)''}{r\Psi} - \left[\frac{(r\Psi)'}{r\Psi}\right]^2\right\}_{r=0} = -\alpha + l(E + \alpha^2 - A_0^{-1}T_\rho A_0) \ . \tag{A8}$$

The second term in the curly braces is again equal to $-\alpha^2$. As compared with (A6), the right-hand side of (A8) contains a new term $A_0^{-1}T_\rho A_0$. To calculate it, we apply the operator $T_r + V$ to both parts of (A7). Using Eqs. (A1) and (A3) and neglecting high-order terms, we find

$$A_0^{-1}(E - T_\rho)A_0 = (\phi_0 A_0)^{-1}(T_r + V)\Phi \approx \Phi^{-1}(T_r + V)\Phi \ .$$

Let us calculate this expression at the boundary $r = r_0'$. Here V can be omitted and Φ replaced by Ψ. Introducing $r\Psi$ and expanding in $r_0'\sqrt{|E|}$, we obtain

$$A_0^{-1}(E - T_\rho)A_0 = (\Psi^{-1}T_r\Psi)_{r=r_0'} \approx -\left[\frac{(r\Psi)''}{r\Psi}\right]_{r=0} .$$

After substitution of this expression into (A8), we again arrive at the result that the factors multiplying r_0' and l differ only in their sign. Combining the terms with r_0' and l and introducing r_0, we finally obtain

$$\left[\frac{(r\Psi)'}{r\Psi}\right]_{r=0} = -\alpha + \frac{r_0}{2}\left\{-\left[\frac{(r\Psi)''}{r\Psi}\right]_{r=0} + \alpha^2\right\} .$$

This is boundary condition (8).

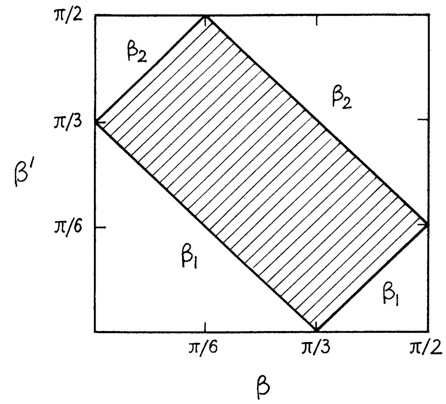

FIG. 1. Limits of integration.

APPENDIX B: EQUATION FOR THE WAVE FUNCTION $\Psi_0^{(0)}$

To derive the equation for $\Psi_0^{(0)}$, we use a known relation between Ψ_0 and χ_0 [18], which is obtained by averaging decomposition (5) over the angle between $\mathbf{r}_{12}$ and $\boldsymbol{\rho}_3$:

$$\Psi_0(\beta) = \chi_0(\beta) + \frac{4}{\sqrt{3}}\int_{\beta_1(\beta)}^{\beta_2(\beta)} d\beta' \, \chi_0(\beta') \ . \tag{B1}$$

Here all functions are taken at the same radius R, and the integration limits are shown in Fig. 1. Applying the operators $\Delta + E_0$ to both sides of (B1) and using explicit expressions for the functions $\beta_1(\beta)$ and $\beta_2(\beta)$, as well as Eqs. (11), (13), and the relation $\Psi_0^{(0)}(0) = \chi_0^{(0)}(0)$, we come to Eq. (20).

Using relation (B1) we can also obtain the expression for the normalization of $\Psi^{(0)}$ in terms of the wave function $\chi_0^{(0)}$,

$$\int_{R>R_0} dr\, d\rho\, \Psi_0^{(0)}\chi_0^{(0)} = \int_{R>R_0} dr\, d\rho (\chi_0^{(0)})^2 + \frac{4}{\sqrt{3}} \int_{R_0}^{\infty} dR\, R \int d\beta\, d\beta' \chi_0^{(0)}(R,\beta)\chi_0^{(0)}(R,\beta') \,,$$

where the angular integration is performed over the shaded region shown in Fig. 1.

---

[1] Y. E. Kim and A. Tubis, Annu. Rev. Nucl. Sci. **24**, 69 (1974); A. C. Phillips, Rep. Prog. Phys. **40**, 905 (1977).

[2] V. Efimov, Comments Nucl. Part. Phys. **19**, 271 (1990); S. K. Adhikari and K. L. Kowalski, *Dynamical Collision Theory and Its Applications* (Academic, Boston, 1991).

[3] V. Efimov, Nucl. Phys. **A362**, 45 (1981); **A378**, 581(E) (1982).

[4] G. V. Skornyakov and K. A. Ter-Martirosyan, Zh. Eksp. Teor. Fiz. **31**, 775 (1956) [Sov. Phys. JETP **4**, 648 (1957)].

[5] G. S. Danilov, Zh. Eksp. Teor. Fiz. **40**, 498 (1961) [Sov. Phys. JETP **13**, 349 (1961)].

[6] V. Efimov, Yad. Fiz. **12**, 1080 (1970) [Sov. J. Nucl. Phys. **12**, 589 (1971)].

[7] V. Efimov, Yad. Fiz. **29**, 1058 (1979) [Sov. J. Nucl. Phys. **29**, 546 (1979)].

[8] E. G. Tkachenko, Phys. Lett. **98B**, 328 (1981); L. Tomio, A. Delfino, and S. K. Adhikari, Phys. Rev. C **35**, 441 (1987); D. V. Shapoval and I. V. Simenog, Few-Body Syst. **8**, 145 (1990).

[9] V. Efimov and E. G. Tkachenko, Few-Body Syst. **4**, 71 (1988).

[10] G. S. Danilov, Zh. Eksp. Teor. Phys. **43**, 1424 (1962) [Sov. Phys. JETP **16**, 110 (1963)].

[11] V. Efimov, Phys. Rev. C **44**, 2303 (1991).

[12] L. D. Landau and E. M. Lifshits, *Quantum Mechanics: Non-Relativistic Theory,* 2nd ed. (Pergamon, Oxford, 1965).

[13] It is a general solution of the Schrödinger equation for χ_0 at the distances $r_0 \ll R \ll (|E|^{-1/2}, \alpha^{-1})$ [6,7].

[14] In Ref. [6], which relates to our discussion, this angle is denoted by α. We change the notation because α is engaged as the two-body parameter in boundary conditions (3) and (8).

[15] The phase correction $\Delta_{\rm in}^{(1)}$ was in effect chosen in Sec. III so as to put the integrals over β in Eqs. (22) and (34) to zero.

[16] Indeed, using asymptotic form (12) for the function $F^{(0)}$, one can see that the integral in matrix element (38) diverges at a low limit as $r_0 \int dR/R^2$.

[17] Although this calculation is straightforward, one particular point is worth mentioning here. The operator $\partial^2/\partial\rho^2$ in Eq. (48) acts on the wave functions on its right and cancels the free wave in χ_0^0. The free wave in the function χ_0^0 on the left gives rise to the integral $\int d\rho\, \delta(\rho)\rho\chi_0(\rho) = [\rho\chi_0(\rho)]_{\rho=0}$, which is equal to $-A$. This is how a nonintegral part of the shift $\Delta E(A)$ in [11] arises in the present formulation.

[18] V. Efimov, Yad. Fiz. **10**, 107 (1969) [Sov. J. Nucl. Phys. **10**, 62 (1970)]; H. P. Noyes, in *Three Body Problem in Nuclear and Particle Physics,* edited by J. S. C. McKee and P. M. Rolph (North-Holland, Amsterdam, 1970).