Ward-Takahashi identities for the radiative axial-vector vertex and low-energy theorem for the pion electroproduction

Koichi Ohta
Institute of Physics, University of Tokyo, Komaba, Tokyo 153, Japan
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Using the assumption of a partially conserved axial-vector current and the conservation law for the electromagnetic current, we rederive a low-energy theorem for the pion electroproduction. In contrast to the traditional approach, we include all effects of off-shell nucleons and pions. We parametrize the axial-vector current in its most general form and apply the minimal substitution prescription to obtain the radiative axial-vector vertex that is required for gauge invariance. We split the full radiative axial-vector vertex into the isolated-pole contribution, the minimal-coupling interaction, and the remainder. The Ward-Takahashi identities are translated into the constraints on the pion electroproduction amplitude. The rigorous low-energy theorem is obtained for the zero four-momentum pion electroproduction. It is found that the off-shell matrix elements of the axial-vector current affect the isolated-pole term and the gauge term so that the low-energy theorem is at variance with the standard one given in the literature.

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I. INTRODUCTION

The assumption of a partially conserved axial-vector current (PCAC) [1] was intensively investigated in the 1960s, and low-energy theorems were derived from PCAC supplemented by current commutation relations [2–6] (for a recent reference, see Ref. [7]). The PCAC hypothesis relates any hadronic process in which a zero four-momentum pion is emitted to the process in the absence of the pion. Among others, the low-energy theorem for pion photoproduction and electroproduction was derived following the first attempt of Nambu and Schrauner [8]. In these studies, effects of off-shell nucleons and pions are not considered. However, even if we are interested in processes of physical particles, it is unavoidable to consider off-shell processes. This can be illustrated by the simplest example, the Born diagram: The internal particles inevitably go off their mass shells, while all external particles are on shell. In investigating the interactions of off-shell particles, the Ward-Takahashi (WT) equation [9] plays an essential role. The importance of the WT identity for the electromagnetic current was recognized by Berends and West [10] concerning the Born approximation in pion electroproduction. The additional extra term that was introduced by Fubini, Nambu, and Wataghin [11] to restore gauge invariance was explained by the nonvanishing contribution from the off-shell current matrix elements. Naus, Koch, and Friar [12] studied the effect of electromagnetic and strong form factors including the PCAC constraint. However, the effect of the off-shell matrix elements of the axial-vector current has never been investigated.

Since the nuclear constituents are composite particles which possess finite space-time extension, we have to treat particles which have form factors at vertices where interactions take place and particles go off shell. The WT equations are valid not only for elementary particles, but also for composite particles. The presence of hadronic and electromagnetic form factors of interacting hadrons should not violate the WT equation, which is a direct manifestation of gauge invariance. The electromagnetic current operator associated with the hadronic form factors should arise in such a way that the WT equation is satisfied. This holds true for nonconserved currents. The axial-vector current form factors should not violate the WT equation, which results from the PCAC relation. It is customary to utilize the axial-vector current with two form factors, the axial-vector form factor and the induced pseudoscalar form factor, for processes involving off-shell nucleons. But this immediately violates the WT identity for the axial-vector current. Since low-energy theorems given in the literature are based on this assumption, it is necessary to reexamine its validity. The primary purpose of this paper is to rederive a low-energy theorem for zero four-momentum pion electroproduction in the most rigorous way. The present work is motivated by recent experiments of π^0 photoproduction close to the threshold [13], but derivation of the low-energy theorem at the threshold will be published in a separate paper.

In Sec. II we derive WT equations for the radiative axial-vector vertex using the PCAC assumption for the axial-vector current and the gauge invariance for the electromagnetic current. Since two currents are acting, two WT equations are obtained. Consistency of the two equations is examined. It is shown that for on-shell nucleons our results coincide with the ones given in the literature. The generalized Born term (the isolated-pole term) is calculated for the radiative axial-vector vertex and subtracted from the WT equations. In Sec. III we write down the most general form of the axial-vector current and extract the electromagnetic current operator on the basis of the minimal-substitution prescription. The isolated-pole term and minimal-coupling interaction are subtracted from the total radiative axial-vector vertex to obtain the constraints on the pion electroproduction amplitude. In Sec. IV we take the limit of a zero four-momentum pion and obtain a rigorous low-energy theorem for the pion electroproduction amplitude. Comparison with the conventional result is made in detail. The relationship to the Kroll-Ruderman theorem [14] is discussed. Finally, in Sec. V, we give a brief summary.

II. WARK-TAKAHASHI IDENTITIES

A. Basic equations

We consider the four-point Green's function

$$\langle T[\psi(x')\bar{\psi}(x)j_\alpha^{Ai}(y)j_\mu(z)]\rangle=\int d^4\xi'\,d^4\xi\,S_F'(x'-\xi')M_{\alpha\mu}^i(\xi'\xi:yz)S_F'(\xi-x)\ , \tag{2.1}$$

where ψ and $\bar{\psi}$ are the nucleon field operators, j_α^{Ai} is the isovector axial-vector current operator with i being the isospin index, j_μ is the electromagnetic current operator, and

$$S_F'(x'-x)=\langle T[\psi(x')\bar{\psi}(x)]\rangle \tag{2.2}$$

is the nucleon propagator. The four-point vertex operator $M_{\alpha\mu}^i$ describes radiative weak interactions (e.g., radiative μ capture of the nucleon). Since two currents are involved, we can derive two WT identities. One of them follows from taking a derivative with respect to the position of the axial-vector current operator,

$$\begin{aligned}(\partial/\partial y_\alpha)&\langle T[\psi(x')\bar{\psi}(x)j_\alpha^{Ai}(y)j_\mu(z)]\rangle\\ &=-\delta(x_0'-y_0)\langle T[[\psi(x'),j_0^{Ai}(y)]\bar{\psi}(x)j_\mu(x)]\rangle-\delta(x_0-y_0)\langle T[\psi(x')[\bar{\psi}(x),j_0^{Ai}(y)]j_\mu(x)]\rangle\\ &\quad-\delta(z_0-y_0)\langle T[\psi(x')\bar{\psi}(x)[j_\mu(z),j_0^{Ai}(y)]]\rangle+\langle T[\psi(x')\bar{\psi}(x)\partial_\alpha j_\alpha^{Ai}(y)j_\mu(z)]\rangle\ .\end{aligned} \tag{2.3}$$

The four-divergence of the axial-vector current is proportional to the pion field ϕ_π^i, as given by the PCAC hypothesis,

$$\partial_\alpha j_\alpha^{Ai}(y)=f_\pi m_\pi^2\phi_\pi^i(y)\ , \tag{2.4}$$

where f_π is the pion decay constant and m_π is the pion mass. Inserting into (2.3) the PCAC relation (2.4), the equal-time commutation relations with respect to the axial-charge density j_0^{Ai},

$$[\psi(x'),j_0^{Ai}(y)]\delta(x_0'-y_0)=\tau^i\gamma_5\psi(x')\delta(x'-y)\ , \tag{2.5}$$

$$[\bar{\psi}(x),j_0^{Ai}(y)]\delta(x_0-y_0)=\bar{\psi}(x')\gamma_5\tau^i\delta(x-y)\ , \tag{2.6}$$

and that of the chiral SU(2)×SU(2) algebra,

$$[j_\mu(z),j_0^{Ai}(y)]\delta(y_0-z_0)=ie\epsilon^{3ij}j_\mu^{Aj}(y)\delta(y-z)\ , \tag{2.7}$$

we find the WT equation in the form

$$\begin{aligned}(\partial/\partial y_\alpha)\langle T[\psi(x')\bar{\psi}(x)j_\alpha^{Ai}(y)j_\mu(z)]\rangle=&-\langle T[\tau^i\gamma_5\psi(x')\bar{\psi}(x)j_\mu(z)]\rangle\delta(x'-y)-\langle T[\psi(x')\bar{\psi}(x)\gamma_5\tau^i j_\mu(z)]\rangle\delta(x-y)\\ &-ie\epsilon^{3ij}\langle T[\psi(x')\bar{\psi}(x)j_\mu^{Aj}(y)]\rangle\delta(z-y)+f_\pi m_\pi^2\langle T[\psi(x')\bar{\psi}(x)\phi_\pi^i(y)j_\mu(z)]\rangle\ .\end{aligned} \tag{2.8}$$

The first two terms in (2.8) can be expressed in terms of the three-point vertex function for the electromagnetic interaction of the nucleon,

$$\begin{aligned}&\langle T[\psi(x')\bar{\psi}(x)j_\mu(z)]\rangle\\ &\quad=i\int d^4\xi'\,d^4\xi\,S_F'(x'-\xi')j_\mu(\xi'\xi:z)S_F'(x-\xi)\ .\end{aligned} \tag{2.9}$$

We use the same notation j_μ for the current and vertex operators for simplicity. The electromagnetic vertex operator satisfies the celebrated WT identity, which can be written in the form

$$(p'-p)_\mu j_\mu(p',p)=e_N[S_F'^{-1}(p')-S_F'^{-1}(p)]\ , \tag{2.10}$$

where we have made the Fourier decomposition

$$\begin{aligned}&j_\mu(x'x:z)\\ &\quad=-\frac{i}{(2\pi)^8}\int d^4p'\,d^4p\,e^{ip'\cdot(x'-z)+ip\cdot(z-x)}j_\mu(p',p)\end{aligned} \tag{2.11}$$

and

$$S_F'(x'-x)=-\frac{i}{(2\pi)^4}\int d^4p\,e^{ip\cdot(x'-x)}S_F'(p)\ . \tag{2.12}$$

In (2.10), e_N is the nucleon charge operator

$$e_N=\tfrac{1}{2}e(1+\tau_3)\ . \tag{2.13}$$

The third term in (2.3) can be expressed in terms of the vertex operator for the axial-vector current,

$$\begin{aligned}&\langle T[\psi(x')\bar{\psi}(x)j_\alpha^{Ai}(y)]\rangle\\ &\quad=i\int d^4\xi'\,d^4\xi\,S_F'(x'-\xi)j_\alpha^{Ai}(\xi'\xi:y)S_F'(x-\xi)\ .\end{aligned} \tag{2.14}$$

The PCAC relation (2.4) and equal-time commutation relations (2.5) and (2.6) lead to the WT identity for the axial-vector current,

$$(\partial/\partial y_\alpha)\langle T[\psi(x')\bar\psi(x)j_\alpha^{Ai}(y)]\rangle = -\langle T[\tau^i\gamma_5\psi(x')\bar\psi(x)]\rangle\delta(x'-y) - \langle T[\psi(x')\bar\psi(x)\gamma_5\tau^i]\rangle\delta(x-y) + f_\pi m_\pi^2\langle T[\psi(x')\bar\psi(x)\phi_\pi^i(y)]\rangle \; . \tag{2.15}$$

The three-point Green's function in this equation defines the vertex function for the interaction between nucleon and pion,

$$\langle T[\psi(x')\bar\psi(x)\phi_\pi^i(y)]\rangle = -\int d^4\xi'\, d^4\xi\, d^4\eta\, S_F'(x'-\xi')\Gamma^i(\xi'\xi:\eta)S_F'(\xi-x)\Delta_F'(y-\eta) \; , \tag{2.16}$$

where the pion propagator is defined by

$$\delta^{ij}\Delta_F'(y'-y) = \langle T[\phi_\pi^i(y')\phi_\pi^j(y)]\rangle \; . \tag{2.17}$$

In momentum representation,

$$j_\alpha^{Ai}(x'x:y) = -\frac{i}{(2\pi)^8}\int d^4p'\, d^4p\, e^{ip'\cdot(x'-y)+ip\cdot(y-x)} j_\alpha^{Ai}(p',p) \; , \tag{2.18}$$

$$\Gamma^i(x'x:y) = -\frac{i}{(2\pi)^8}\int d^4p'\, d^4p\, e^{ip\cdot(x'-y)+ip\cdot(y-x)}\Gamma^i(p',p) \; , \tag{2.19}$$

$$\Delta_F'(y'-y) = -\frac{i}{(2\pi)^4}\int d^4q\, e^{iq\cdot(y'-y)}\Delta_F'(q) \; , \tag{2.20}$$

one obtains

$$(p-p')_\alpha j_\alpha^{Ai}(p',p) = -\tau^i[S_F'^{-1}(p')\gamma_5 + \gamma_5 S_F'^{-1}(p)] - if_\pi m_\pi^2\Delta_F'(p-p')\Gamma^i(p',p) \; . \tag{2.21}$$

The four-point Green's function, the last term in (2.3), defines the vertex operator for the pion electroproduction,

$$\langle T[\psi(x')\bar\psi(x)\phi_\pi^i(y)j_\mu(x)]\rangle = i\int d^4\xi'\, d^4\xi\, d^4\eta\, S_F'(x'-\xi')M_\mu^i(\xi'\xi:\eta z)S_F'(\xi-x)\Delta_F'(y-\eta) \; . \tag{2.22}$$

Using the momentum representation

$$M_{\alpha\mu}^i(x'x:yz) = -\frac{i}{(2\pi)^{12}}\int d^4p'\, d^4p\, d^4k\, e^{ip'\cdot(x'-y)+ip\cdot(y-x)+ik\cdot(y-z)}M_{\alpha\mu}^i \; , \tag{2.23}$$

$$M_\mu^i(x'x:yz) = -\frac{i}{(2\pi)^{12}}\int d^4p'\, d^4p\, d^4k\, e^{ip'\cdot(x'-y)+ip\cdot(y-x)+ik\cdot(y-z)}M_\mu^i \; , \tag{2.24}$$

we get

$$q_\alpha M_{\alpha\mu}^i = -S_F'^{-1}(p')\gamma_5 S_F'(p+k)\tau^i j_\mu(p+k,p) - j_\mu(p',p'-k)\tau^i S_F'(p'-k)\gamma_5 S_F'^{-1}(p) - ie\epsilon^{3ij}j_\mu^{Aj}(p',p) - if_\pi m_\pi^2\Delta_F'(q)M_\mu^i \; , \tag{2.25}$$

where p and p' are the nucleon momenta before and after the interaction, respectively, k is the photon momentum, and $q=p+k-p'$ is the momentum carried by the axial-vector current. In $M_{\alpha\mu}^i$ and M_μ^i, we suppress momenta p', p, and k.

Next, consider the derivative of (2.1) with respect to the photon position z,

$$(\partial/\partial z_\mu)\langle T[\psi(x')\bar\psi(x)j_\alpha^{Ai}(y)j_\mu(z)]\rangle$$
$$= -\delta(x_0'-z_0)\langle T[[\psi(x'),j_0(z)]\bar\psi(x)j_\alpha^{Ai}(y)]\rangle - \delta(x_0-z_0)\langle T[\psi(x')[\bar\psi(x),j_0(z)]j_\alpha^{Ai}(y)]\rangle$$
$$-\delta(y_0-z_0)\langle T[\psi(x')\bar\psi(x)[j_\alpha^{Ai}(y),j_0(z)]]\rangle + \langle T[\psi(x')\bar\psi(x)j_\alpha^{Ai}(y)\partial_\mu j_\mu(z)]\rangle \; , \tag{2.26}$$

where j_0 is the charge-density operator. Using the current conservation $\partial_\mu j_\mu(z)=0$, the equal-time commutation relations

$$[\psi(x'),j_0(z)]\delta(x_0'-z_0) = e_N\psi(x')\delta(x'-z) \; , \tag{2.27}$$

$$[\bar\psi(x),j_0(z)]\delta(x_0-z_0) = -\bar\psi(x')e_N\delta(x-z) \; , \tag{2.28}$$

and the chiral algebra

$$[j_\alpha^{Ai}(y),j_0(z)]\delta(y_0-z_0) = -ie\epsilon^{3ij}j_\alpha^{Aj}(y)\delta(y-z) \; , \tag{2.29}$$

we obtain another WT equation

$$(\partial/\partial z_\mu)\langle T[\psi(x')\bar{\psi}(x)j_\alpha^{Ai}(y)j_\mu(z)]\rangle = -e_N\langle T[\psi(x')\bar{\psi}(x)j_\alpha^{Ai}(y)]\rangle\delta(x'-z)+\langle T[\psi(x')\bar{\psi}(x)j_\alpha^{Ai}(y)]\rangle e_N\delta(x-z)$$
$$+ie\epsilon^{3ij}\langle T[\psi(x')\bar{\psi}(x)j_\alpha^{Aj}(y)]\rangle\delta(y-z) , \qquad (2.30)$$

which becomes, in momentum space,

$$k_\mu M^i_{\alpha\mu} = e_N S_F'^{-1}(p')S_F'(p'-k)j_\alpha^{Ai}(p'-k,p)-j_\alpha^{Ai}(p',p+k)S_F'(p+k)S_F'^{-1}(p)e_N - ie\epsilon^{3ij}j_\alpha^{Aj}(p',p) . \qquad (2.31)$$

B. Consistency of the two WT identities

We are now provided with two divergences $k_\mu M_{\alpha\mu}$ and $q_\alpha M_{\alpha\mu}$. Therefore we can calculate $q_\alpha k_\mu M_{\alpha\mu}$ in two distinct ways: From $k_\mu M_{\alpha\mu}$, we get

$$q_\alpha k_\mu M^i_{\alpha\mu} = e_N S_F'^{-1}(p')S_F'(p'-k)q_\alpha j_\alpha^{Ai}(p'-k,p)-q_\alpha j_\alpha^{Ai}(p',p+k)S_F'(p+k)S_F'^{-1}(p)e_N - ie\epsilon^{3ij}q_\alpha j_\alpha^{Aj}(p',p) \qquad (2.32)$$

and, from $q_\alpha M_{\alpha\mu}$,

$$q_\alpha k_\mu M^i_{\alpha\mu} = -S_F'^{-1}(p')\gamma_5 S_F'(p+k)\tau^i k_\mu j_\mu(p+k,p)-k_\mu j_\mu(p',p'-k)\tau^i S_F'(p'-k)\gamma_5 S_F'^{-1}(p)$$
$$-ie\epsilon^{3ij}k_\mu j_\mu^{Aj}(p',p)-if_\pi m_\pi^2\Delta_F'(q)k_\mu M^i_\mu . \qquad (2.33)$$

Equations (2.32) and (2.33) must coincide. We can check this in the following way. We make use of the WT equations

$$q_\alpha j_\alpha^{Ai}(p'-k,p) = -\tau^i[S_F'^{-1}(p'-k)\gamma_5+\gamma_5 S_F'^{-1}(p)]-if_\pi m_\pi^2\Delta_F'(q)\Gamma^i(p'-k,p) , \qquad (2.34)$$

$$q_\alpha j_\alpha^{Ai}(p',p+k) = -\tau^i[S_F'^{-1}(p')\gamma_5+\gamma_5 S_F'^{-1}(p+k)-if_\pi m_\pi^2\Delta_F'(q)\Gamma^i(p',p+k) , \qquad (2.35)$$

in Eq. (2.32), and

$$k_\mu j_\mu(p+k,p) = e_N[S_F'^{-1}(p+k)-S_F'^{-1}(p)] , \qquad (2.36)$$

$$k_\mu j_\mu(p',p'-k) = e_N[S_F'^{-1}(p')-S_F'^{-1}(p'-k)] , \qquad (2.37)$$

in Eq. (2.33). Equating (2.32) and (2.33) and using

$$(q-k)_\alpha j_\alpha^{Ai}(p',p) = -\tau^i[S_F'^{-1}(p')\gamma_5+\gamma_5 S_F'^{-1}(p)]-if_\pi m_\pi^2\Delta_F'(q-k)\Gamma^i(p',p) , \qquad (2.38)$$

we solve for $k_\mu M^i_\mu$ to obtain

$$k_\mu M^i_\mu = e_N S_F'^{-1}(p')S_F'(p'-k)\Gamma^i(p'-k,p)-\Gamma^i(p',p+k)S_F'(p+k)S_F'^{-1}(p)e_N - ie\epsilon^{3ij}\Delta_F'^{-1}(q)\Delta_F'(q-k)\Gamma^i(p',p) . \qquad (2.39)$$

This is nothing but the WT equation for the pion electroproduction amplitude [16,17]. This can be derived directly from

$$(\partial/\partial z_\mu)\langle T[\psi(x')\bar{\psi}(x)\phi_\pi^i(y)j_\mu(z)]\rangle = -e_N\langle T[\psi(x')\bar{\psi}(x)\phi_\pi^i(y)]\rangle\delta(x'-z)+\langle T[\psi(x')\bar{\psi}(x)\phi_\pi^i(y)]\rangle e_N\delta(x-z)$$
$$+ie\epsilon^{3ij}\langle T[\psi(x')\bar{\psi}(x)\phi_\pi^i(y)]\rangle\delta(y-z) , \qquad (2.40)$$

where the current conservation is applied and (2.27), (2.28), and

$$[\phi_\pi^i(y),j_0(z)]\delta(y_0-z_0) = -ie\epsilon^{3ij}\phi_\pi^j(y)\delta(y-z) \qquad (2.41)$$

are inserted. We can conclude that once $M^i_{\alpha\mu}$ satisfies the requirement of PCAC and gauge invariance, M^i_μ satisfies the gauge invariance automatically.

C. On-shell nucleons

The WT equations (2.25) and (2.31) are generalizations of the ones given by Adler and Dothan [4] for on-shell nucleons. Indeed, Eqs. (2.25) and (2.31) are reduced to

$$q_\alpha M^i_{\alpha\mu} = -ie\epsilon^{3ij}j_\mu^{Aj}(p',p)-if_\pi m_\pi^2\Delta_F'(q)M^i_\mu , \qquad (2.42)$$

$$k_\mu M^i_{\alpha\mu} = -ie\epsilon^{3ij}j_\alpha^{Aj}(p',p), \qquad (2.43)$$

for nucleons on the mass shell since $S_F'^{-1}(p)u(p)=0$ and $\bar{u}(p')S_F'^{-1}(p')=0$ can be used when (2.25) and (2.31) are sandwiched between Dirac spinors $\bar{u}(p')$ and $u(p)$.

Equation (2.42) is usually derived from the PCAC relation in the presence of the electromagnetic field A_α,

$$\partial_\alpha j_\alpha^{Ai}(y) = e\epsilon^{3ij}A_\alpha(y)j_\alpha^{Aj}(y)+f_\pi m_\pi^2\phi_\pi^i(y) . \qquad (2.44)$$

Taking the matrix element between the one-nucleon state $|p'\rangle$ and one-nucleon–one-photon state $|pk\rangle$ and using the pion source function defined by

$$(-\partial_\alpha^2+m_\pi^2)\phi_\pi^i(y) = J_\pi^i(y) , \qquad (2.45)$$

we find

$$\langle p'|\partial_\alpha j_\alpha^{Ai}|pk\rangle = e\epsilon^{3ij}\langle p'|j_\alpha^{Ai}|p\rangle \varepsilon_\alpha(k) + f_\pi m_\pi^2 \frac{1}{q^2+m_\pi^2}\langle p'|J_\pi^i|pk\rangle \ , \quad (2.46)$$

where $\varepsilon_\alpha(k)$ is the photon polarization vector. Since the current matrix elements are related to $M^i_{\alpha\mu}$ and M^i_μ, apart from trivial normalization factors, as

$$\bar{u}(p')M^i_{\alpha\mu}u(p)\varepsilon_\mu(k) = \langle p'|j_\alpha^{Ai}|pk\rangle \ , \quad (2.47)$$

$$\bar{u}(p')M^i_\mu u(p)\varepsilon_\mu(k) = \langle p'|J_\pi^i|pk\rangle \ , \quad (2.48)$$

we get the effective operator valid for on-shell nucleons,

$$q_\alpha M^i_{\alpha\mu} = ie\epsilon^{3ij} j_\mu^{Aj}(p',p) - if_\pi m_\pi^2 \frac{1}{q^2+m_\pi^2} M^i_\pi \ , \quad (2.49)$$

which differs from the rigorous one (2.42) in that, instead of the full pion propagator $\Delta'_F(q)$, the free one $(q^2+m_\pi^2)^{-1}$ enters. They must coincide in the limit of on-shell pions, $q^2 \to -m_\pi^2$.

D. Constraints on non-Born terms

The radiative weak vertex operator $M^i_{\alpha\beta}$ can be decomposed into the generalized Born term (the isolated-pole term) and the remainder,

$$M^i_{\alpha\mu} = B^i_{\alpha\mu} + \Delta M^i_{\alpha\mu} \ , \quad (2.50)$$

where the Born term $B^i_{\alpha\mu}$ consists of three diagrams, as indicated in Fig. 1; the electromagnetic current interacts with the nucleon before and after the axial-vector current and with the virtual pion emitted by the nucleon. In momentum space $B^i_{\alpha\beta}$ is explicitly given by

$$\begin{aligned} B^i_{\alpha\mu} &= j_\alpha^{Ai}(p',p+k)S'_F(p+k)j_\mu(p+k,p) \\ &\quad + j_\mu(p',p'-k)S'_F(p'-k)j_\alpha^{Ai}(p'-k,p) \\ &\quad + if_\pi q_\alpha \Delta'_F(q) j_\mu^{\pi ij}(q,q-k)\Delta'_F(q-k)\Gamma^j(p',p) \ , \end{aligned} \quad (2.51)$$

where the electromagnetic vertex of the pion is defined by

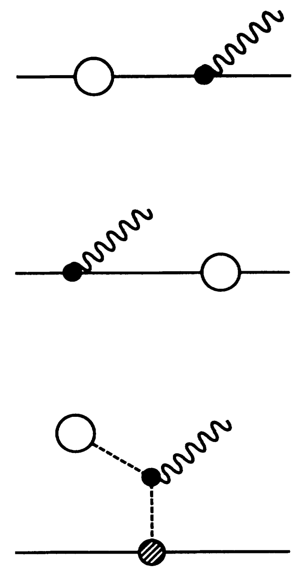

FIG. 1. Born approximation diagrams for the radiative axial-vector vertex. The axial-vector coupling is denoted by the open circle, and the electromagnetic interaction is denoted by the solid circle. The hatched circle is for the pion nucleon vertex.

$$\langle T[\phi_\pi^i(y')\phi_\pi^i(y)j_\mu(z)]\rangle = i\int d^4\eta'\, d^4\eta\, \Delta'_F(y'-\eta') j_\mu^{\pi ij}(\eta'\eta : z)\Delta'_F(y-\eta) \ , \quad (2.52)$$

with the Fourier decomposition

$$j_\mu^{\pi ij}(y'y:z) = -\frac{i}{(2\pi)^8}\int d^4q'\, d^4q\, e^{iq'\cdot(y'-z)+iq\cdot(z-y)} j_\mu^{\pi ij}(q',q) \ . \quad (2.53)$$

First, we evaluate $q_\alpha B^i_{\alpha\mu}$. Using the WT equations (2.34) and (2.35) for the axial-vector vertex of the nucleon and inserting the result into the WT equation (2.25) for $M^i_{\alpha\mu}$, one finds a constraint on $\Delta M^i_{\alpha\mu}$,

$$\begin{aligned} q_\alpha \Delta M^i_{\alpha\mu} &= -ie\epsilon^{3ij} j_\mu^{Aj}(p',p) - if_\pi m_\pi^2 \Delta'_F(q)\Delta M^i_\mu + \tau^i\gamma_5 j_\mu(p+k,p) + j_\mu(p',p'-k)\tau^i\gamma_5 \\ &\quad - if_\pi(q^2+m_\pi^2)\Delta'_F(q) j_\mu^{\pi ij}(q,q-k)\Delta'_F(q-k)\Gamma^j(p',p) \ . \end{aligned} \quad (2.54)$$

We have split the virtual pion electroproduction operator M^i_μ into the Born term and the rest,

$$M^i_\mu = B^i_\mu + \Delta M^i_\mu \ . \quad (2.55)$$

The Born term again consists of two nucleon-pole diagrams and a pionic pole diagram. In momentum space B^i_μ is given by

$$B^i_\mu = \Gamma^i(p',p+k)S'_F(p+k)j_\mu(p+k,p) + j_\mu(p',p'-k)S'_F(p'-k)\Gamma^i(p'-k,p) + j_\mu^{\pi ij}(q,q-k)\Delta'_F(q-k)\Gamma^j(p',p) \ . \quad (2.56)$$

The divergence of B^i_μ is calculated using the WT identities for the electromagnetic vertex of the nucleon [Eqs. (2.36) and (2.37)] and that of the pion,

$$(q'-q)_\mu j_\mu^{\pi ij}(q',q) = -ie\epsilon^{3ij}[\Delta_F'^{-1}(q') - \Delta_F'^{-1}(q)] \ . \quad (2.57)$$

The WT identity for the pion electroproduction (2.39) leads to [17]

$$k_\mu \Delta M^i_\mu = -ie\epsilon^{3ij}\Gamma^j(p',p) + e_N \Gamma^i(p'-k,p) - \Gamma^i(p',p+k)e_N \ . \tag{2.58}$$

In a similar manner, we calculate the four-dimensional divergence of $B^i_{\alpha\mu}$,

$$k_\mu B^i_{\alpha\mu} = j^{Ai}_\alpha(p',p+k)[S'_F(p) - S'_F(p+k)]S'^{-1}_F(p)e_N + e_N S'^{-1}_F(p')[S'_F(p'-k) - S'_F(p')]j^{Ai}_\alpha(p'-k,p) - ie\epsilon^{3ij}if_\pi q_\alpha[\Delta'_F(q-k) - \Delta'_F(q)]\Gamma^j(p',p) \ . \tag{2.59}$$

From (2.23), one of the constraints on $\Delta M^i_{\alpha\mu}$ follows:

$$k_\mu \Delta M^i_{\alpha\mu} = -ie\epsilon^{3ij} j^{Aj}_\alpha(p',p) + e_N j^{Ai}_\alpha(p'-k,p) - j^{Ai}_\alpha(p',p+k)e_N - ie\epsilon^{3ij}if_\pi q_\alpha[\Delta'_F(q) - \Delta'_F(q-k)]\Gamma^j(p',p) \ . \tag{2.60}$$

Equations (2.54) and (2.60) are our basic equations.

III. GAUGE INVARIANCE

A. Pion pole and nonpole contributions

Electrodynamics is made invariant by introducing the gauge field such that a gradient of the nucleon field is allowed to appear only in the form of the covariant derivative

$$\partial_\mu \psi(x) \to [\partial_\mu - ie_N A_\mu(x)]\psi(x) \ . \tag{3.1}$$

Once a nonradiative interaction is given as a function of p and p' in momentum space, we can transform them into differential operators acting on $\psi(x)$ and $\bar\psi(x)$, respectively, in position space. Then it is straightforward to derive the corresponding current operator by demanding gauge invariance [17].

The gauge invariance requirement does not determine the whole radiative weak vertex operator $M^i_{\alpha\mu}$ but some part of it. To see this we split j^{Ai}_α into pion nonpole and pole terms:

$$j^{Ai}_\alpha(p',p) = \bar j^{Ai}_\alpha(p',p) + if_\pi(p-p')_\alpha \Delta'_F(p-p')\Gamma^i(p',p) \ . \tag{3.2}$$

From the momentum dependence of each term, we get the contribution to $M^i_{\alpha\mu}$,

$$G^i_{\alpha\mu} = \bar G^i_{\alpha\mu} + ie\epsilon^{3ij}\delta_{\alpha\mu} if_\pi \Delta'_F(q-k)\Gamma^j(p',p) + if_\pi q_\alpha \Delta'_F(q) G^i_\mu \ . \tag{3.3}$$

The first term $\bar G^i_{\alpha\mu}$ is obtained from $\bar j^{Ai}_\alpha(p',p)$, as will be shown in Sec. III C. The second term comes from the minimal substitution of $(p-p')_\alpha$ in (3.2) and describes the radiative pion decay process. The last term is from the pion-nucleon vertex $\Gamma^i(p',p)$ and describes virtual pion electroproduction. G^i_μ is also given in Sec. III C. The three terms are graphically indicated in Fig. 2. There exists the contribution coming from the minimal substitution of $\Delta'_F(p-p')$ in (3.2),

$$G^{\pi i}_{\alpha\mu} = ie\epsilon^{3ij} if_\pi q_\alpha \frac{(2q-k)_\mu}{q^2-(q-k)^2} \times [\Delta'_F(q) - \Delta'_F(q-k)]\Gamma^j(p',p) \ , \tag{3.4}$$

which also describes the radiative pion decay process.

We treat $G^{\pi i}_{\alpha\mu}$ separately because the pion-pole term already appeared in the Born term $B^i_{\alpha\mu}$ [Eq. (2.51)]. They differ from each other by the term which is gauge invariant by itself. This can easily be seen since both have the same divergence owing to the WT identity (2.57). To show this more explicitly, we note that the electromagnetic vertex of pion has the structure

$$j^{\pi ij}_\mu(q',q) = -ie\epsilon^{3ij} j^\pi_\mu(q',q) \ . \tag{3.5}$$

The most general form of the electromagnetic vertex of the pion is

$$j^\pi_\mu(q',q) = \left[(q'+q)_\mu - \frac{q'^2-q^2}{(q'-q)^2}(q'-q)_\mu\right]A + (q'+q)_\mu B \ , \tag{3.6}$$

where A and B are functions of q^2, q'^2, and $(q'-q)^2$. Evaluating its divergence, we find

$$(q'-q)_\mu j^\pi_\mu(q',q) = (q'^2-q^2)B \ . \tag{3.7}$$

The WT identity (2.57) fixes B,

$$B = \frac{1}{q'^2-q^2}[\Delta'^{-1}_F(q') - \Delta'^{-1}_F(q)] \ , \tag{3.8}$$

so that

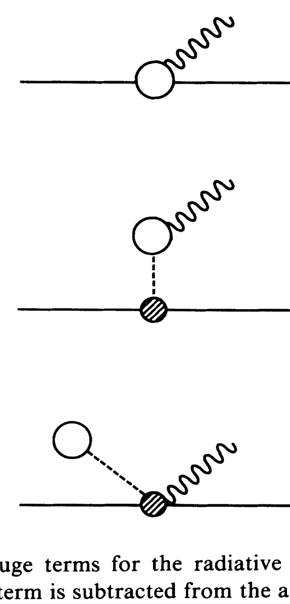

FIG. 2. Gauge terms for the radiative axial-vector vertex. The pion pole term is subtracted from the axial-vector coupling denoted by the open circle.

$$j_\mu^\pi(q',q)=(F_\pi-1)(q'+q)_\beta\left[\delta_{\beta\mu}-\frac{(q'-q)_\beta(q'-q)_\mu}{(q'-q)^2}\right]+\frac{(q'+q)_\mu}{q'^2-q^2}[\Delta_F'^{-1}(q')-\Delta_F'^{-1}(q)]\ , \tag{3.9}$$

where the pion electromagnetic form factor $F_\pi=A+1$ is introduced. Thus one sees that the pionic interaction term in $B^i_{\alpha\mu}$ is the sum of $G^{\pi i}_{\alpha\mu}$ and the remainder $\Delta G^{\pi i}_{\alpha\mu}$,

$$if_\pi q_\alpha\Delta_F'(q)j_\mu^{\pi ij}(q,q-k)\Delta_F'(q-k)\Gamma^j(p',p)=G^{\pi i}_{\alpha\mu}+\Delta G^{\pi i}_{\alpha\mu}\ , \tag{3.10}$$

where $\Delta G^{\pi i}_{\alpha\mu}$ is of the form

$$\Delta G^{\pi i}_{\alpha\mu}=-2ie\epsilon^{3ij}if_\pi q_\alpha q_\beta(F_\pi-1)\left[\delta_{\beta\mu}-\frac{k_\beta k_\mu}{k^2}\right]\times\Delta_F'(q)\Delta_F'(q-k)\Gamma^j(p',p)\ . \tag{3.11}$$

Clearly, $k_\mu\Delta G^{\pi i}_{\alpha\mu}=0$. For pointlike pions, $F_\pi=1$, and hence $\Delta G^{\pi i}_{\alpha\mu}=0$; namely, the minimal substitution reproduces the pionic Born term. It is a matter of course that since the total gauge invariance cannot determine form factors, we got the pionic Born term with $F_\pi=1$. It is now clear that $G^{\pi i}_{\alpha\mu}$ is already contained in the Born term and that it should not be a new contribution to $\Delta M^i_{\alpha\mu}$. The non-Born contribution to $M^i_{\alpha\mu}$ thus takes the form

$$\Delta M^i_{\alpha\mu}=G^i_{\alpha\mu}+R^i_{\alpha\mu}, \tag{3.12}$$

where $R^i_{\alpha\mu}$ collects all contributions other than the Born and gauge terms.

We now want to separate the radiative weak vertex into pion nonpole and pole parts,

$$M^i_{\alpha\mu}=\overline{M}^i_{\alpha\mu}+if_\pi q_\alpha\Delta_F'(q)M^i_\mu\ . \tag{3.13}$$

The full amplitude for the virtual pion electroproduction is decomposed as

$$M^i_\mu=B^i_\mu+\Delta M^i_\mu=B^i_\mu+G^i_\mu+R^i_\mu\ , \tag{3.14}$$

where G^i_μ, which entered (3.3), is obtained from Γ^i using the minimal-substitution method and R^i_μ is all the rest. Since the Born term for the radiative weak vertex (2.51) is rewritten using the nonpole axial-vector current $\overline{j}^{Ai}_\alpha$ as

$$B^i_{\alpha\mu}=\overline{B}^i_{\alpha\mu}+if_\pi q_\alpha\Delta_F'(q)B^i_\mu\ , \tag{3.15}$$

with Eq. (2.56) for B^i_μ and

$$\overline{B}^i_{\alpha\mu}=\overline{j}^{Ai}_\alpha(p',p+k)S_F'(p+k)j_\mu(p+k,p)+j_\mu(p',p'-k)S_F'(p'-k)\overline{j}^{Ai}_\alpha(p'-k,p)\ , \tag{3.16}$$

$\overline{M}^i_{\alpha\mu}$ turns out to be

$$\overline{M}^i_{\alpha\mu}=\overline{B}^i_{\alpha\mu}+ie\epsilon^{3ij}\delta_{\alpha\mu}if_\pi\Delta_F'(q-k)\Gamma^j(p',p)+\overline{G}^i_{\alpha\mu}+\overline{R}^i_{\alpha\mu}\ . \tag{3.17}$$

It is found that $R^i_{\alpha\mu}$ is of the form

$$R^i_{\alpha\mu}=\overline{R}^i_{\alpha\mu}+if_\pi q_\alpha\Delta_F'(q)R^i_\mu\ . \tag{3.18}$$

Everything except $\overline{R}^i_{\alpha\mu}$ and R^i_μ is known explicitly.

B. Axial-vector vertex

The axial-vector vertex function of the nucleon,

$$j^{Ai}_\alpha(p',p)=\tau^i j^A_\alpha(p',p)\ , \tag{3.19}$$

is parametrized in the most general form

$$\begin{aligned}j^A_\alpha(p',p)={}&iF^A_1\gamma_\alpha\gamma_5-F^A_2\gamma_5(p-p')_\alpha+iF^A_3\sigma_{\alpha\beta}(p-p')_\beta\gamma_5\\&+(i\gamma\cdot p'+m)[iF^A_4\gamma_\alpha\gamma_5-F^A_5\gamma_5(p-p')_\alpha+iF^A_6\sigma_{\alpha\beta}(p-p')_\beta\gamma_5]\\&+[iF^A_7\gamma_\alpha\gamma_5-F^A_8\gamma_5(p-p')_\alpha+iF^A_9\sigma_{\alpha\beta}(p-p')_\beta\gamma_5](i\gamma\cdot p+m)\\&+(i\gamma\cdot p'+m)[iF^A_{10}\gamma_\alpha\gamma_5-F^A_{11}\gamma_5(p-p')_\alpha+iF^A_{12}\sigma_{\alpha\beta}(p-p')_\beta\gamma_5](i\gamma\cdot p+m)\ ,\end{aligned} \tag{3.20}$$

with m being the nucleon mass. The 12 form factors are functions of p'^2, p^2, and $(p-p')^2$. The pion pole part of the axial-vector current contributes to F^A_2, F^A_5, F^A_8, and F^A_{11}, as is seen from (3.2). Since the fully off-shell pion-nucleon vertex function

$$\Gamma^i(p',p)=\tau^i\Gamma(p',p) \tag{3.21}$$

is put in the form

$$\Gamma(p',p)=i\gamma_5 g_1+i\gamma_5 g_2(i\gamma\cdot p+m)+(i\gamma\cdot p'+m)i\gamma_5 g_3+(i\gamma\cdot p'+m)i\gamma_5 g_4(i\gamma\cdot p+m)\ , \tag{3.22}$$

with g_1, g_2, g_3, and g_4 being form factors, F^A_2, F^A_5, F^A_8, and F^A_{11} can be separated into nonpole and pole parts:

$$F^A_2=\overline{F}^A_2+f_\pi\Delta_F'(p-p')g_1\ , \tag{3.23}$$

$$F^A_5=\overline{F}^A_5+f_\pi\Delta_F'(p-p')g_3\ , \tag{3.24}$$

$$F^A_8=\overline{F}^A_8+f_\pi\Delta_F'(p-p')g_2\ , \tag{3.25}$$

$$F^A_{11}=\overline{F}^A_{11}+f_\pi\Delta_F'(p-p')g_4\ . \tag{3.26}$$

From (2.21) one sees that the pion nonpole part $\overline{j}^{Ai}_\alpha(p',p)=\tau^i\overline{j}^A_\alpha(p',p)$ satisfies

$$(p-p')_\alpha \bar{j}^A_\alpha(p',p) = -S_F'^{-1}(p')\gamma_5 - \gamma_5 S_F'^{-1}(p) + if_\pi[(p-p')^2+m_\pi^2]\Delta_F'(p-p')\Gamma(p',p) \ . \tag{3.27}$$

Using the fully dressed nucleon propagator parametrized in the form

$$S_F'^{-1}(p) = i\gamma\cdot pG(p^2) + mF(p^2), \tag{3.28}$$

the WT identity (3.27) imposes the four constraints

$$F_1^A + (p-p')^2\bar{F}_5^A - (p^2+m^2)F_{10}^A = G(p'^2) - f_\pi[(p-p')^2+m_\pi^2]\Delta_F'(p-p')g_3 \ , \tag{3.29}$$

$$F_1^A + (p-p')^2\bar{F}_8^A - (p'^2+m^2)F_{10}^A = G(p^2) - f_\pi[(p-p')^2+m_\pi^2]\Delta_F'(p-p')g_2 \ , \tag{3.30}$$

$$F_4^A + F_7^A + 2mF_{10}^A + (p-p')^2\bar{F}_{11}^A = -f_\pi[(p'-p)^2+m_\pi^2]\Delta_F'(p-p')g_4 \ , \tag{3.31}$$

$$\begin{aligned} 2mF_1^A - (p-p')^2\bar{F}_2^A + (p'^2+m^2)F_4^A + (p^2+m^2)F_4^A \\ = m[G(p'^2) - G(p^2) - F(p'^2) + F(p^2)] + f_\pi[(p-p')^2+m_\pi^2]\Delta_F'(p-p')g_1 \ . \end{aligned} \tag{3.32}$$

One can eliminate $\bar{F}_2^A$, $\bar{F}_5^A$, $\bar{F}_8^A$, and $\bar{F}_{11}^A$ to express $\bar{j}_\alpha^A$ in the form

$$\begin{aligned} \bar{j}_\alpha^A(p',p) = & -\frac{(p-p')_\alpha}{(p-p')^2}[S_F'^{-1}(p')\gamma_5 + \gamma_5 S_F'^{-1}(p)] - if_\pi\frac{(p-p')_\alpha}{(p-p')^2}[(p-p')^2+m_\pi^2]\Delta_F'(p-p')\Gamma(p',p) \\ & + iF_1^A\left[\gamma_\alpha - \frac{(p-p')_\alpha}{(p-p')^2}\gamma\cdot(p-p')\right]\gamma_5 + iF_3^A\sigma_{\alpha\beta}(p-p')_\beta\gamma_5 \\ & + (i\gamma\cdot p'+m)\left\{iF_4^A\left[\gamma_\alpha - \frac{(p-p')_\alpha}{(p-p')^2}\gamma\cdot(p-p')\right]\gamma_5 + iF_6^A\sigma_{\alpha\beta}(p-p')_\beta\gamma_5\right\} \\ & + \left\{iF_7^A\left[\gamma_\alpha - \frac{(p-p')_\mu}{(p-p')^2}\gamma\cdot(p-p')\right]\gamma_5 + iF_9^A\sigma_{\alpha\beta}(p-p')_\beta\right\}(i\gamma\cdot p+m) \\ & + (i\gamma\cdot p'+m)\left\{iF_{10}^A\left[\gamma_\alpha - \frac{(p-p')_\alpha}{(p-p')^2}\gamma\cdot(p-p')\right]\gamma_5 + iF_{12}^A\sigma_{\alpha\beta}(p-p')_\beta\gamma_5\right\}(i\gamma\cdot p+m). \end{aligned} \tag{3.33}$$

The pole at $(p-p')^2=0$ arises from the WT constraint on the axial-vector form factors.

We shall retain the most general forms for the axial-vector current and the strong vertex in the development that follows in the subsequent subsections. To conclude the present subsection, we want to compare the rigorous expressions (3.27) and (3.33) with the standard ones. To this end, we replace the nucleon and pion propagators with the free ones. For on-shell nucleons (3.32) is reduced to

$$2mF_1^A - (p-p')^2\bar{F}_2^A = f_\pi g_1 \ , \tag{3.34}$$

where all form factors become functions of $(p-p')^2$. In the limit $(p-p')^2=0$, we get the Goldberger-Treiman relation [15]

$$2mF_1^A(0) = f_\pi g_1(0). \tag{3.35}$$

It should be noted that (3.35) is valid independently of parametrization of the axial-vector current and pion-nucleon vertex. If we choose the pseudoscalar coupling for the pion-nucleon vertex, $g_2=g_3=g_4=0$. No information is available concerning the axial-vector current form factors except for F_1^A and F_2^A. However, as a result of the WT constraints, we cannot put all other form factors equal to zero. To be consistent with (3.31), we can choose $F_4^A=F_7^A=F_{10}^A=\bar{F}_{11}^A=0$. On the other hand, (3.29) and (3.30) give

$$\bar{F}_5^A = \bar{F}_8^A = \frac{1-F_1^A}{(p-p')^2} \ , \tag{3.36}$$

which never vanish (to avoid this pole, we must abandon the assumption of $g_2=g_3=0$). Therefore the axial-vector current becomes

$$\begin{aligned} \bar{j}_\alpha^A(p-p') = & iF_1^A\left[\gamma_\alpha - \frac{(p-p')_\alpha}{(p-p')^2}\gamma\cdot(p-p')\right]\gamma_5 \\ & + \frac{(p-p')_\alpha}{(p-p')^2}i\gamma\cdot(p-p')\gamma_5 \\ & - \left[\bar{F}_2^A - 2m\frac{F_1^A-1}{(p-p')^2}\right](p-p')_\alpha\gamma_5 \ . \end{aligned} \tag{3.37}$$

When we sandwich (3.37) between $\bar{u}(p')$ and $u(p)$, we get the standard result

$$\begin{aligned} & \bar{u}(p')\bar{j}_\alpha^A(p-p')u(p) \\ & = \bar{u}(p')[iF_1^A\gamma_\alpha\gamma_5 - \bar{F}_2^A(p-p')_\alpha\gamma_5]u(p) \ . \end{aligned} \tag{3.38}$$

If the axial-vector current appears in a diagram with one of the nucleon legs off the mass shell, the term that disappeared in (3.38) makes difference. This point is essentially important for our discussion since in Born diagrams the intermediate nucleon goes inevitably off the mass shell so that the term usually ignored affects $R^{i}_{\alpha\mu}$ even for on-shell nucleons. To see this, consider the divergence of (3.37),

$$(p-p')_\alpha \bar{j}^{A}_{\alpha}(p-p') = i\gamma\cdot(p-p')\gamma_5 - 2m\gamma_5 + f_\pi g_1 \gamma_5 , \tag{3.39}$$

which follows directly from (3.27) with the approximations of the free nucleon and pion propagators and the pseudoscalar coupling. This is to be contrasted with the standard assumption

$$(p-p')_\alpha[iF^{A}_{1}\gamma_\alpha\gamma_5 - \bar{F}^{A}_{2}(p-p')_\alpha\gamma_5] = F^{A}_{1}[i\gamma\cdot(p-p')\gamma_5 - 2m\gamma_5] + f_\pi g_1\gamma_5 . \tag{3.40}$$

The WT equation tells us that the divergence of the axial-vector current does not depend on the axial-vector current form factors. Of course, the term proportional to F^{A}_{1} in (3.40) vanishes when sandwiched between the Dirac wave functions, but it gives a contribution different from (3.39) for off-shell nucleons.

C. Minimal replacement

The result of the minimal substitution applied to the pion pole term is given in Sec. III A. In that expression [Eq. (3.4)], one notes that the current contains $\Delta'_F(q)$ as well as $\Delta'_F(q-k)=\Delta'_F(p-p')$, while the nonradiative vertex contains only $\Delta'_F(p-p')$. This holds true for any gauge terms. $\bar{G}^{i}_{\alpha\mu}$ is written in terms of $\bar{j}^{A}_{\alpha}(p',p)$, $\bar{j}^{A}_{\alpha}(p'-k,p)$, and $\bar{j}^{A}_{\alpha}(p',p+k)$. In $\bar{j}^{A}_{\alpha}(p',p)$ the form factors depend on p'^2, p^2, and $(p-p')^2=(q-k)^2$, while in $\bar{j}^{A}_{\alpha}(p'-k,p)$ they depend on $(p'-k)^2$, p^2, and $(p-p'+k)^2=q^2$, and in $\bar{j}^{A}_{\alpha}(p',p+k)$ they depend on p'^2, $(p+k)^2$, and q^2. In addition to these current operators, there appears $\bar{j}^{A}_{\alpha}(p',p)$, in which the form factors are evaluated at p'^2, p^2, and q^2. We distinguish this current by writing it as $\bar{j}^{A0}_{\alpha}(p',p)$.

The minimal substitution (3.1) produces currents given by differences of nonradiative vertex functions evaluated at momenta q and $q-k$, p' and $p'-k$, and $p+k$ and p. The axial-vector vertex has momentum dependence through the form factors $i\gamma\cdot p$, $i\gamma\cdot p'$, $(p-p')_\alpha$, and $\sigma_{\alpha\beta}(p-p')_\beta$. Since we are interested in the lowest-order electromagnetic current in e, we can carry out the minimal substitution of each factor independently. The result of the minimal replacement for $\bar{j}^{Ai}_{\alpha}$ is

$$\bar{G}^{i}_{\alpha\mu} = ie\epsilon^{3ij}\tau^j \frac{(2q-k)_\mu}{q^2-(q-k)^2}\delta_q \bar{j}^{A}_{\alpha}(p',p) - e_N\tau^i \frac{(2p'-k)_\mu}{p'^2-(p'-k)^2}\delta_{p'}\bar{j}^{A}_{\alpha}(p',p) - \tau^i e_N \frac{(2p+k)_\mu}{(p+k)^2-p^2}\delta_p \bar{j}^{A}_{\alpha}(p',p)$$
$$+ ie\epsilon^{3ij}\tau^j\mathcal{H}_{\alpha\mu}(p',p) - ie_N\tau^i\gamma_\mu\mathcal{H}_{1\alpha}(p'-k,p) - i\tau^i e_N\mathcal{H}_{2\alpha}(p',p+k)\gamma_\mu . \tag{3.41}$$

The first three terms are brought about by the momentum dependence of the form factors and the last three terms by the additional momentum dependence in $\bar{j}^{A}_{\alpha}$. In (3.41), $\delta_q\bar{j}^{A}_{\alpha}(p',p)$ means the difference of the axial-vector currents in which all the form factors are evaluated at q^2 and $(q-k)^2$ while p'^2 and p^2 are fixed. Namely, it is of the same form as (3.20), in which F^{A}_{1} is replaced by $F^{A}_{1}(q^2,p'^2,p^2)-F^{A}_{1}((q-k)^2,p'^2,p^2)$, etc. Similarly, $\delta_{p'}\bar{j}^{A}_{\alpha}(p',p)$ stands for the difference of the axial-vector currents with the form factors at p'^2 and $(p'-k)^2$, and $\delta_p\bar{j}^{A}_{\alpha}(p',p)$ for that between p'^2 and $(p'-k)^2$. We can express these differences in terms of $\bar{j}^{A}_{\alpha}$ in the forms

$$\delta_q\bar{j}^{A}_{\alpha}(p',p) = \bar{j}^{A0}_{\alpha}(p',p) - \bar{j}^{A}_{\alpha}(p',p) - \mathcal{H}_{\alpha\beta}(p',p)k_\beta , \tag{3.42}$$

$$\delta_{p'}\bar{j}^{A}_{\alpha}(p',p) = \bar{j}^{A0}_{\alpha}(p',p) - \bar{j}^{A}_{\alpha}(p'-k,p) - i\gamma\cdot k\mathcal{H}_{1\alpha}(p'-k,p) , \tag{3.43}$$

$$\delta_p\bar{j}^{A}_{\alpha}(p',p) = \bar{j}^{A}_{\alpha}(p',p+k) - \bar{j}^{A0}_{\alpha}(p',p) - \mathcal{H}_{2\alpha}(p',p+k)i\gamma\cdot k , \tag{3.44}$$

where

$$\mathcal{H}_{1\alpha}(p',p) = \frac{\delta\bar{j}^{A}_{\alpha}(p',p)}{\delta(i\gamma\cdot p'+m)}$$
$$= iF^{A}_{4}\gamma_\alpha\gamma_5 - \bar{F}^{A}_{5}\gamma_5(p-p')_\alpha + iF^{A}_{6}\sigma_{\alpha\beta}(p-p')_\beta\gamma_5$$
$$+ [iF^{A}_{10}\gamma_\alpha\gamma_5 - \bar{F}^{A}_{11}\gamma_5(p-p')_\alpha + iF^{A}_{12}\sigma_{\alpha\beta}(p-p')_\beta\gamma_5](i\gamma\cdot p+m) , \tag{3.45}$$

$$\mathcal{H}_{2\alpha}(p',p) = \frac{\delta\bar{j}^{A}_{\alpha}(p',p)}{\delta(i\gamma\cdot p+m)}$$
$$= iF^{A}_{7}\gamma_\alpha\gamma_5 - \bar{F}^{A}_{8}\gamma_5(p-p')_\alpha + iF^{A}_{9}\sigma_{\alpha\beta}(p-p')_\beta\gamma_5$$
$$+ (i\gamma\cdot p'+m)[iF^{A}_{10}\gamma_\alpha\gamma_5 - \bar{F}^{A}_{11}\gamma_5(p-p')_\alpha + iF^{A}_{12}\sigma_{\alpha\beta}(p-p')_\beta\gamma_5] , \tag{3.46}$$

$$\mathcal{H}_{\alpha\beta}(p',p)=\frac{\delta\bar{j}^{\,A}_{\alpha}(p',p)}{\delta(p-p')_{\beta}}$$
$$=-\bar{F}^{\,A}_{2}\delta_{\alpha\beta}\gamma_5+iF^{A}_{3}\sigma_{\alpha\beta}\gamma_5+(i\gamma\cdot p'+m)(-\bar{F}^{\,A}_{5}\delta_{\alpha\beta}\gamma_5+iF^{A}_{6}\sigma_{\alpha\beta}\gamma_5)$$
$$+(-\bar{F}^{\,A}_{8}\delta_{\alpha\beta}\gamma_5+iF^{A}_{9}\sigma_{\alpha\beta}\gamma_5)(i\gamma\cdot p+m)+(i\gamma\cdot p'+m)(-\bar{F}^{\,A}_{11}\delta_{\alpha\beta}\gamma_5+iF^{A}_{12}\sigma_{\alpha\beta}\gamma_5)(i\gamma\cdot p+m)\ . \tag{3.47}$$

The derivatives are taken with respect to $i\gamma\cdot p'+m$, $i\gamma\cdot p+m$, or $(p-p')_{\beta}$, while all the form factors are kept fixed. By the use of Eqs. (3.42)–(3.44), it is easy to see that

$$k_{\mu}\bar{G}^{\,i}_{\alpha\mu}=-ie\epsilon^{3ij}\bar{j}^{\,Aj}_{\alpha}(p',p)+e_N\bar{j}^{\,Ai}_{\alpha}(p'-k,p)-\bar{j}^{\,Ai}_{\alpha}(p',p+k)e_N\ . \tag{3.48}$$

$\bar{j}^{\,A0}_{\alpha}(p',p)$ does not contribute to the divergence $k_{\mu}\bar{G}^{\,i}_{\alpha\mu}$. Since j^{A}_{α} and $\bar{j}^{\,A}_{\alpha}$ have identical structure, the minimal current obtained from j^{A}_{α} has the same form as (3.41) with $\bar{F}^{\,A}_{2}$, $\bar{F}^{\,A}_{5}$, $\bar{F}^{\,A}_{8}$, and $\bar{F}^{\,A}_{11}$ replaced by F^{A}_{2}, F^{A}_{5}, F^{A}_{8}, and F^{A}_{11}, respectively. It is satisfying to see that the difference between the two currents coincides with the sum of the last two terms in (3.3) and $G^{\pi i}_{\alpha\mu}$ given in the preceding subsection.

The pion electroproduction amplitude G^{i}_{μ} obtained from Γ^{i} using the minimal replacement is given in a previous paper [17]. For later use we quote the result

$$G^{i}_{\mu}=ie\epsilon^{3ij}\tau^{j}\frac{(2q-k)_{\mu}}{q^2-(q-k)^2}\delta_q\Gamma(p',p)-e_N\tau^{i}\frac{(2p'-k)_{\mu}}{p'^2-(p'-k)^2}\delta_{p'}\Gamma(p',p)$$
$$-\tau^{i}e_N\frac{(2p+k)_{\mu}}{(p+k)^2-p^2}\delta_p\Gamma(p',p)+ie_N\tau^{i}\gamma_{\mu}\mathcal{G}_1(p'-k,p)-i\tau^{i}e_N\mathcal{G}_2(p',p+k)\gamma_{\mu}\ , \tag{3.49}$$

where

$$\delta_q\Gamma(p',p)=\Gamma^{0}(p',p)-\Gamma(p',p)\ , \tag{3.50}$$

$$\delta_{p'}\Gamma(p',p)=\Gamma^{0}(p',p)-\Gamma(p'-k,p)-i\gamma\cdot k\,\mathcal{G}_1(p'-k,p)\ , \tag{3.51}$$

$$\delta_{p}\Gamma(p',p)=\Gamma(p',p+k)-\Gamma^{0}(p',p)-\mathcal{G}_2(p',p+k)i\gamma\cdot k \tag{3.52}$$

are defined in the same way as above, and

$$\mathcal{G}_1(p',p)=\frac{\delta\Gamma(p',p)}{\delta(i\gamma\cdot p'+m)}=i\gamma_5g_3+i\gamma_5g_4(i\gamma\cdot p+m)\ , \tag{3.53}$$

$$\mathcal{G}_2(p',p)=\frac{\delta\Gamma(p',p)}{\delta(i\gamma\cdot p+m)}=i\gamma_5g_2+(i\gamma\cdot p'+m)i\gamma_5g_4\ . \tag{3.54}$$

In Eqs. (3.50)–(3.52), $\Gamma^{0}(p',p)$ denotes the vertex function in which the form factors are evaluated at p'^2, p^2, and q^2 instead of $(q-k)^2$. The four-divergence of G^{i}_{μ} is easily calculated as

$$k_{\mu}G^{i}_{\mu}=-ie\epsilon^{3ij}\Gamma^{j}(p',p)+e_N\Gamma^{i}(p'-k,p)$$
$$-\Gamma^{i}(p',p+k)e_N\ . \tag{3.55}$$

From Eq. (2.60) it follows that

$$k_{\mu}R^{\,i}_{\mu}=0\ . \tag{3.56}$$

This states the requirement of gauge invariance: The pion electroproduction amplitude consists of the Born term B^{i}_{μ}, the gauge term G^{i}_{μ}, and the remainder $R^{\,i}_{\mu}$, which must be separately gauge invariant.

D. WT identities

We now derive the WT constraints on $R^{\,i}_{\alpha\mu}$. First, consider $k_{\mu}R^{\,i}_{\alpha\mu}$. To this end, we calculate the divergence of (3.3). Using (3.48) and (3.55), we find

$$k_{\mu}G^{i}_{\alpha\mu}=-ie\epsilon^{3ij}j^{Aj}_{\alpha}(p',p)+e_Nj^{Ai}_{\alpha}(p'-k,p)-j^{Ai}_{\alpha}(p',p+k)e_N-ie\epsilon^{3ij}if_{\pi}q_{\alpha}[\Delta'_F(q)-\Delta'_F(q-k)]\Gamma^{j}(p',p)\ , \tag{3.57}$$

where $\bar{j}^{\,Ai}_{\alpha}$ is eliminated in favor of j^{Ai}_{α} [Eq. (3.2)]. Therefore the constraint (2.60) leads to

$$k_{\mu}R^{\,i}_{\alpha\mu}=0 \tag{3.58}$$

or, as a result of (3.18) and (3.56),

$$k_{\mu}\bar{R}^{\,i}_{\alpha\mu}=0\ . \tag{3.59}$$

The residual amplitude for the radiative axial-vector vertex must be gauge invariant by itself, as was expected.

Next, we turn to the calculation of $q_{\alpha}R^{\,i}_{\alpha\mu}$. For that purpose it is important to note that $\bar{j}^{\,A0}_{\alpha}(p',p)$ does not satisfy the WT identity (3.27) because it is evaluated at $q\neq p-p'$. We calculate $q_{\alpha}\bar{j}^{\,A0}_{\alpha}(p',p)$ explicitly using (3.29)–(3.32),

$$q_\alpha \bar{j}\,^{\alpha}_{A0}(p',p) = -S_F'^{-1}(p')\gamma_5 - \gamma_5 S_F'^{-1}(p) - if_\pi(q^2+m_\pi^2)\Delta_F'(q)\Gamma^0(p',p) + \mathcal{D}^0 \ . \tag{3.60}$$

The extra term $\mathcal{D}^0$ is of the form

$$\mathcal{D}^0 = iF_1^A\gamma\cdot k\gamma_5 + (i\gamma\cdot p'+m)iF_4^A\gamma\cdot k\gamma_5 + iF_7^A\gamma\cdot k\gamma_5(i\gamma\cdot p+m) + (i\gamma\cdot p'+m)iF_{10}^A\gamma\cdot k\gamma_5(i\gamma\cdot p+m) \ . \tag{3.61}$$

We are using the convention that if the momentum dependence is not explicit for a quantity in momentum space, it depends on three momenta p', p, and k. In $\mathcal{D}^0$ the form factors are functions of p'^2, p^2, and $q^2=(p+k-p')^2$. Note that $\mathcal{D}^0=0$ at $k=0$. For $\bar{j}\,^A_\alpha(p',p)$, (2.38) holds. Thus

$$q_\alpha \bar{j}\,^{A0}_\alpha(p',p) - q_\alpha \bar{j}\,^A_\alpha(p',p) = -if_\pi(q^2+m_\pi^2)\Delta_F'(q)\Gamma^0(p',p) + if_\pi[(q-k)^2+m_\pi^2]\Delta_F'(q-k)\Gamma(p',p) + \mathcal{D}^0 - k_\alpha \bar{j}\,^A_\alpha(p',p) \ . \tag{3.62}$$

Because of the relations (3.42) and (3.50), we find

$$\begin{aligned} q_\alpha\delta_q \bar{j}\,^A_\alpha(p',p) = &-if_\pi(q^2+m_\pi^2)\Delta_F'(q)\delta_q\Gamma(p',p) + if_\pi(q^2+m_\pi^2)[\Delta_F'(q-k)-\Delta_F'(q)]\Gamma(p',p) \\ &- if_\pi[q^2-(q-k)^2]\Delta_F'(q-k)\Gamma(p',p) - \mathcal{D}^0 - k_\alpha\bar{j}\,^A_\alpha(p',p) - q_\alpha k_\beta \mathcal{H}_{\alpha\beta}(p',p) \ . \end{aligned} \tag{3.63}$$

In exactly the same way, we calculate the four-divergences:

$$q_\alpha\delta_{p'}\bar{j}\,^A_\alpha(p',p) = -[S_F'^{-1}(p') - S_F'^{-1}(p'-k)]\gamma_5 - if_\pi(q^2+m_\pi^2)\Delta_F'(q)\delta_{p'}\Gamma(p',p) + \mathcal{D}^0 - i\gamma\cdot k\mathcal{F}_1(p'-k,p) \ , \tag{3.64}$$

$$q_\alpha\delta_p\bar{j}\,^A_\alpha(p',p) = -\gamma_5[S_F'^{-1}(p+k) - S_F'^{-1}(p)] - if_\pi(q^2+m_\pi^2)\Delta_F'(q)\delta_p\Gamma(p',p) - \mathcal{D}^0 - \mathcal{F}_2(p+k,p)i\gamma\cdot k \ , \tag{3.65}$$

where

$$\mathcal{F}_1(p'-k,p) = q_\alpha\mathcal{H}_{1\alpha}(p'-k,p) + if_\pi(q^2+m_\pi^2)\Delta_F'(q)\mathcal{G}_1(p'-k,p) \ , \tag{3.66}$$

$$\mathcal{F}_2(p',p+k) = q_\alpha\mathcal{H}_{2\alpha}(p',p+k) + if_\pi(q^2+m_\pi^2)\Delta_F'(q)\mathcal{G}_2(p',p+k) \ . \tag{3.67}$$

The terms containing the nucleon propagators in Eqs. (3.64) and (3.65) can be written in terms of the four-divergence of the electromagnetic current j_μ using the WT identities (2.36) and (2.37). The form factors $\bar{F}\,^A_5$, $\bar{F}\,^A_8$, and $\bar{F}\,^A_{11}$ can be eliminated from $\mathcal{F}_1$ and $\mathcal{F}_2$ using the WT constraints (3.29)–(3.31),

$$\mathcal{F}_1(p'-k,p) = f_1(p'-k,p) + q_\alpha h_{1\alpha}(p'-k,p) \ , \tag{3.68}$$

$$\mathcal{F}_1(p',p+k) = f_2(p',p+k) + q_\alpha h_{2\alpha}(p',p+k) \ , \tag{3.69}$$

with

$$f_1(p'-k,p) = [F_1^A - (p^2+m^2)F_{10}^A - G((p'-k)^2)]\gamma_5 + (F_4^A+F_7^A+2mF_{10}^A)\gamma_5(i\gamma\cdot p+m) \ , \tag{3.70}$$

$$f_2(p',p+k) = [F_1^A - (p'^2+m^2)F_{10}^A - G((p+k)^2)]\gamma_5 + (i\gamma\cdot p'+m)(F_4^A+F_7^A+2mF_{10}^A)\gamma_5 \ , \tag{3.71}$$

$$h_{1\alpha}(p'-k,p) = iF_4^A\gamma_\alpha\gamma_5 + iF_{10}^A\gamma_\alpha\gamma_5(i\gamma\cdot p+m) \ , \tag{3.72}$$

$$h_{2\alpha}(p',p+k) = iF_7^A\gamma_\alpha\gamma_5 + (i\gamma\cdot p'+m)iF_{10}^A\gamma_\alpha\gamma_5 \ . \tag{3.73}$$

It should be noted that the strong form factors are canceled out. This step of eliminating $\bar{F}\,^A_5$, $\bar{F}\,^A_8$, and $\bar{F}\,^A_{11}$ is of essential importance because $\mathcal{F}_1$ and $\mathcal{F}_2$ are written in terms of the form factor F_1^A, which does not vanish for on-shell nucleons.

The calculation of $q_\alpha\bar{G}\,^i_{\alpha\mu}$ is tedious but straightforward. After some manipulation we are led to

$$\begin{aligned} q_\alpha\bar{G}\,^i_{\alpha\mu} = &-if_\pi(q^2+m_\pi^2)\Delta_F'(q)G^i_\mu - ie\epsilon^{3ij}\bar{j}\,^{Aj}_\mu(p',p) - ie\epsilon^{3ij}if_\pi(2q-k)_\mu\Delta_F'(q-k)\Gamma^j(p',p) \\ &+ ie\epsilon^{3ij}if_\pi(q^2+m_\pi^2)\frac{(2q-k)_\mu}{q^2-(q-k)^2}[\Delta_F'(q-k)-\Delta_F'(q)]\Gamma^j(p',p) \\ &+ ie\epsilon^{3ij}\tau^j\left[J_\mu - \frac{(2q-k)_\mu}{q^2-(q-k)^2}k\cdot J\right] + \frac{(2p'-k)_\mu}{p'^2-(p'-k)^2}k\cdot j(p',p'-k)\tau^i\gamma_5 \\ &- ie_N\tau^i\left[\gamma_\mu - \frac{(2p'-k)_\mu}{p'^2-(p'-k)^2}\gamma\cdot k\right]\mathcal{F}_1(p'-k,p) + \tau^i\gamma_5\frac{(2p+k)_\mu}{(p+k)^2-p^2}k\cdot j(p+k,p) \\ &- i\tau^i e_N\mathcal{F}_2(p',p+k)\left[\gamma_\mu - \frac{(2p+k)_\mu}{(p+k)^2-p^2}\gamma\cdot k\right] \\ &+ \left[ie\epsilon^{3ij}\tau^j\frac{(2q-k)_\mu}{q^2-(q-k)^2} - e_N\tau^i\frac{(2p'-k)_\mu}{p'^2-(p'-k)^2} + \tau^i e_N\frac{(2p+k)_\mu}{(p+k)^2-p^2}\right]\mathcal{D}^0 \ , \end{aligned} \tag{3.74}$$

where we introduced the notation

$$J_\mu = iF_1^A\gamma_\mu\gamma_5 - iF_3^A\sigma_{\mu\nu}k_\nu\gamma_5 + (i\gamma\cdot p' + m)(iF_4^A\gamma_\mu\gamma_5 - iF_6^A\sigma_{\mu\nu}k_\nu\gamma_5)$$
$$+ (iF_7^A\gamma_\mu\gamma_5 - iF_9^A\sigma_{\mu\nu}k_\nu\gamma_5)(i\gamma\cdot p + m) + (i\gamma\cdot p' + m)(iF_{10}^A\gamma_\mu\gamma_5 - iF_{12}^A\sigma_{\mu\nu}k_\nu\gamma_5)(i\gamma\cdot p + m) . \qquad (3.75)$$

The form factors are functions of p'^2, p^2, and $(q-k)^2$. Since the form factors in J_μ and $\mathcal{D}^0$ are evaluated at different momenta, $k\cdot J \neq \mathcal{D}^0$, unless $k=0$. The induced pseudoscalar terms cancel out and do not appear in J_μ.

The divergence of the last two terms in (3.3),

$$ie\epsilon^{3ij}if_\pi q_\mu \Delta_F'(q-k)\Gamma^j(p',p) + if_\pi q^2\Delta_F'(q)G_\mu^i \ , \qquad (3.76)$$

is added to (3.74) to obtain $q_\alpha G_{\alpha\mu}^i$. The WT constraint (2.54) is then translated into $q_\alpha R_{\alpha\mu}^i$. Finally, with the help of (3.18), one finds

$$q_\alpha \bar{R}_{\alpha\mu}^i = -if_\pi(q^2+m_\pi^2)\Delta_F'(q)R_\mu^i + 2ie\epsilon^{3ij}if_\pi(q^2+m_\pi^2)\Delta_F'(q)(F_\pi - 1)q_\alpha\left[\delta_{\alpha\mu} - \frac{k_\alpha k_\mu}{k^2}\right]\Delta_F'(q-k)\Gamma^j(p',p)$$
$$-ie\epsilon^{3ij}\tau^j\left[J_\mu - \frac{(2q-k)_\mu}{q^2-(q-k)^2}k\cdot J\right] + \left[J_{1\mu} - \frac{(2p'-k)_\mu}{p'^2-(p'-k)^2}k\cdot J_1\right]\tau^i + \tau^i\left[J_{2\mu} - \frac{(2p+k)_\mu}{(p+k)^2-p^2}k\cdot J_2\right]$$
$$-\left[ie\epsilon^{3ij}\tau^j\frac{(2q-k)_\mu}{q^2-(q-k)^2} - e_N\tau^i\frac{(2p'-k)_\mu}{p'^2-(p'-k)^2} + \tau^i e_N\frac{(2p+k)_\mu}{(p+k)^2-p^2}\right]\mathcal{D}^0 \ , \qquad (3.77)$$

where

$$J_{1\mu} = j_\mu(p',p'-k)\gamma_5 + ie_N\gamma_\mu\mathcal{F}_1(p'-k,p) \ , \qquad (3.78)$$

$$J_{2\mu} = \gamma_5 j_\mu(p+k,p) + ie_N\mathcal{F}_2(p',p+k)\gamma_\mu \ . \qquad (3.79)$$

The electromagnetic current can be parametrized in exactly the same way as the axial-vector current (3.20) (see Ref. [17]). The WT constraint (2.10) eliminates four form factors F_3, F_6, F_9, and F_{12} out of 12. The result is

$$j_\mu(p',p) = e_N\frac{(p'-p)_\mu}{(p'-p)^2}[S_F'^{-1}(p') - S_F'^{-1}(p)] + iF_1\left[\gamma_\mu - \frac{(p'-p)_\mu}{(p'-p)^2}\gamma\cdot(p'-p)\right] - iF_2\sigma_{\mu\nu}(p'-p)_\nu$$
$$+(i\gamma\cdot p'+m)\left\{iF_4\left[\gamma_\mu - \frac{(p'-p)_\mu}{(p'-p)^2}\gamma\cdot(p'-p)\right] - iF_5\sigma_{\mu\nu}(p'-p)_\nu\right\}$$
$$+\left\{iF_7\left[\gamma_\mu - \frac{(p'-p)_\mu}{(p'-p)^2}\gamma\cdot(p'-p)\right] - iF_8\sigma_{\mu\nu}(p'-p)_\nu\right\}(i\gamma\cdot p+m)$$
$$+(i\gamma\cdot p'+m)\left\{iF_{10}\left[\gamma_\mu - \frac{(p'-p)_\mu}{(p'-p)^2}\gamma\cdot(p'-p)\right] - iF_{11}\sigma_{\mu\nu}(p'-p)_\nu\right\}(i\gamma\cdot p+m) \ . \qquad (3.80)$$

The identities (3.59) and (3.77) are our main results. The residual quantities $\bar{R}_{\alpha\mu}^i$ and R_μ^i are unknown, but the gauge invariance tells us that they must be four-divergence free, and the assumption of PCAC relates the two quantities as given by (3.77).

IV. LOW-ENERGY THEOREM

A. Pion electroproduction in the soft-pion limit

Without other information we cannot go further to determine $\bar{R}_{\alpha\mu}^i$ and R_μ^i. In the limit of a zero four-momentum pion, however, we can solve the constraint. Since $\bar{R}_{\alpha\mu}^i$ and R_μ^i are free from singularity arising from the particle poles, they behave smoothly in the limit $9\to 0$. We can expand them in powers of q,

$$\bar{R}_{\alpha\mu}^i = \bar{R}_{\alpha\mu}^i(0) + O(q) \ , \qquad (4.1)$$

$$R_\mu^i = R_\mu^i(0) + q_\alpha\frac{\partial R_\mu^i}{\partial q_\alpha} + O(q^2). \qquad (4.2)$$

The WT identity (3.77) determines $R_\mu^i(0)$ uniquely,

$$R^i_\mu(0)=\frac{-i}{f_\pi}\left\{-ie\epsilon^{3ij}\tau^j\left[J_\mu-\frac{k_\mu}{k^2}k\cdot J\right]+\left[J_{1\mu}-\frac{(p'+p)_\mu}{p'^2-p^2}k\cdot J_1\right]\tau^i\right.$$
$$\left.+\tau^i\left[J_{2\mu}-\frac{(p'+p)_\mu}{p'^2-p^2}k\cdot J_2\right]+ie\epsilon^{3ij}\tau^i\left[\frac{k_\mu}{k^2}-\frac{(p'+p)_\mu}{p'^2-p^2}\right]\mathcal{D}^0\right\}. \quad (4.3)$$

In $J_{1\mu}$ and $J_{2\mu}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ become f_1 and f_2, respectively. Note that $k_\mu R^i_\mu(0)=0$ is satisfied.

To compare with the result given in the literature, we put the external nucleons on the mass shell. Since we have eliminated the nonpole form factors $\bar F^A_2$, $\bar F^A_5$, $\bar F^A_8$, and $\bar F^A_{11}$, Eq. (4.3) does not contain them explicitly. As is seen from (3.61), (3.70), (3.71), and (3.75), only F^A_1 and F^A_3 are left in (4.3). Owing to the symmetry under charge conjugation, $F^A_3=0$ on the mass shell. If we put $F=G=1$, we find

$$J_\mu=iF^A_1(k^2)\gamma_\mu\gamma_5\,, \quad (4.4)$$

$$J_{1\mu}=j_\mu(k)\gamma_5+ie_N[F^A_1(0)-1]\gamma_\mu\gamma_5\,, \quad (4.5)$$

$$J_{2\mu}=\gamma_5 j_\mu(k)+ie_N[F^A_1(0)-1]\gamma_5\gamma_\mu\,. \quad (4.6)$$

Furthermore,

$$\mathcal{D}^0=iF^A_1(0)\gamma\cdot k\gamma_5\,. \quad (4.7)$$

We assume that the electromagnetic current has two form factors F_1 and F_2. Each of them has isoscalar and isovector parts. Equation (3.80) becomes of the form [10,18]

$$j_\mu(k)=ie_N\frac{k_\mu}{k^2}\gamma\cdot k+ie(\tfrac{1}{2}F^S_1+\tfrac{1}{2}F^V_1\tau^3)\left[\gamma_\mu-\frac{k_\mu}{k^2}\gamma\cdot k\right]$$
$$-ie(\tfrac{1}{2}F^S_2+\tfrac{1}{2}F^V_2\tau^3)\sigma_{\mu\nu}k_\nu\,. \quad (4.8)$$

The isospin structure of the electroproduction matrix element is given by

$$R^i_\mu=\delta_{i3}R^{(+)}_\mu+\tfrac{1}{2}[\tau^i,\tau^3]R^{(-)}_\mu+\tau^iR^{(0)}_\mu\,. \quad (4.9)$$

The matrix element is further expanded in terms of six covariants O_j,

$$\varepsilon_\mu R^{(\pm,0)}_\mu=\sum_{j=1}^{6}R^{(\pm,0)}_jO_j\,. \quad (4.10)$$

In the limit $q=0$, only two terms survive,

$$O_1=-\varepsilon_\mu\sigma_{\mu\nu}k_\mu\gamma_5\,, \quad (4.11)$$

$$O_6=k^2\varepsilon_\mu\left[\gamma_\mu-\frac{k_\mu}{k^2}\gamma\cdot k\right]\gamma_5\,. \quad (4.12)$$

Our low-energy theorem is stated in the form

$$R^{(0)}_1=\frac{eg_1(0)}{2mF^A_1(0)}F^S_2(k^2)\,, \quad (4.13)$$

$$R^{(+)}_1=\frac{eg_1(0)}{2mF^A_1(0)}F^V_2(k^2)\,, \quad (4.14)$$

$$R^{(-)}_6=\frac{eg_1(0)}{2mF^A_1(0)}[F^A_1(k^2)-F^A_1(0)+1-F^V_1(k^2)]\frac{1}{k^2}\,, \quad (4.15)$$

where we have used the Goldberger-Treiman relation (3.35). The corrections arising from the virtual nucleon mass dependence of F^A_1, G, and F are given below by (4.44) and (4.45).

B. Comparison with the conventional result

Equations (4.13)–(4.15) differ from those given in the literature. To find out the origin of the discrepancy, let us follow the usual derivation of the low-energy theorem. It starts with the calculation of the Born term for the radiative axial-vector vertex,

$$B^i_{\alpha\mu}=j^{Ai}_\alpha(q)\frac{1}{i\gamma\cdot(p+k)+m}j_\mu(k)+j_\mu(k)\frac{1}{i\gamma\cdot(p'-k)+m}j^{Ai}_\alpha(q)$$
$$+if_\pi q_\alpha\frac{1}{q^2+m^2_\pi}j^{\pi ij}_\mu(q,q-k)\frac{1}{(q-k)^2+m^2_\pi}ig_1((q-k)^2)\tau^i\gamma_5\,. \quad (4.16)$$

This should be inserted between $\bar u(p')$ and $u(p)$. The three current matrix elements take the simplest forms:

$$j_\mu(k)=ie(\tfrac{1}{2}F^S_1+\tfrac{1}{2}F^V_1\tau^3)\gamma_\mu-ie(\tfrac{1}{2}F^S_2+\tfrac{1}{2}F^V_2\tau^3)\sigma_{\mu\nu}k_\nu\,, \quad (4.17)$$

$$j^{Ai}_\alpha(q)=\tau^i[iF^A_1(q^2)\gamma_\alpha\gamma_5-F^A_2(q^2)q_\alpha\gamma_5]\,, \quad (4.18)$$

$$j^{\pi ij}_\mu(q,q-k)=-ie\epsilon^{3ij}(2q-k)_\mu\,. \quad (4.19)$$

The divergence of (4.18) is directly calculated as

$$q_\alpha j_\alpha^{Ai}(q)=\tau^i F_1^A(q^2)(i\gamma\cdot q\gamma_5-2m\gamma_5)+f_\pi\frac{m_\pi^2}{q^2+m_\pi^2}g_1(q^2)\tau^i\gamma_5\ , \tag{4.20}$$

where the PCAC relation for the on-shell nucleons,

$$2mF_1^A(q^2)-q^2F_1^A(q^2)=f_\pi\frac{m_\pi^2}{q^2+m_\pi^2}g_1(q^2)\ , \tag{4.21}$$

was used to eliminate $F_2^A(q^2)$. The divergence of the Born term becomes

$$\begin{aligned} q_\alpha B_{\alpha\mu}^i = & -F_1^A(q^2)[\tau^i\gamma_5 j_\mu(k)+j_\mu(k)\tau^i\gamma_5] \\ & +f_\pi\frac{m_\pi^2}{q^2+m_\pi^2}\left[g_1(q^2)\tau^i\gamma_5\frac{1}{i\gamma\cdot(p+k)+m}j_\mu(k)+j_\mu(k)\frac{1}{i\gamma\cdot(p'-k)+m}g_1(q^2)\tau^i\gamma_5\right] \\ & +if_\pi\frac{q^2}{q^2+m_\pi^2}j_\mu^{\pi ij}(q,q-k)\frac{1}{(q-k)^2+m_\pi^2}ig_1((q-k)^2)\tau^j\gamma_5\ . \end{aligned} \tag{4.22}$$

Here we have dropped terms containing $i\gamma\cdot p'+m$ on the leftmost and $i\gamma\cdot p+m$ on the rightmost positions. The WT constraint (2.42) leads to

$$\begin{aligned} q_\alpha\Delta M_{\alpha\mu}^i = & -ie\epsilon^{3ij}j_\mu^{Ai}(q-k)-if_\pi\frac{m_\pi^2}{q^2+m_\pi^2}\Delta M_\mu^i+F_1^A(q^2)[\tau^i\gamma_5 j_\mu(k)+j_\mu(k)\tau^i\gamma_5] \\ & -if_\pi j_\mu^{\pi ij}(q,q-k)\frac{1}{(q-k)^2+m_\pi^2}ig_1((q-k)^2)\tau^j\gamma_5\ , \end{aligned} \tag{4.23}$$

where the Born approximation for the pion electroproduction amplitude,

$$B_\mu^i=ig_1(q^2)\tau^i\gamma_5\frac{1}{i\gamma\cdot(p+k)+m}j_\mu(k)+j_\mu(k)\frac{1}{i\gamma\cdot(p'-k)+m}ig_1(q^2)\tau_5+j_\mu^{\pi ij}(q,q-k)\frac{1}{(q-k)^2+m_\pi^2}ig_1((q-k)^2)\tau^i\gamma_5\ , \tag{4.24}$$

is subtracted from M_μ^i. Equation (4.23) is to be compared with the rigorous result (2.54). One immediately notices that the axial-vector form factor $F_1^A(q^2)$ appears in front of the third term. This is brought about by the use of the axial-vector current (4.18), which is valid only when it is sandwiched between the Dirac wave functions. The WT equation (2.21) states that the four-divergence of the axial-vector current should not depend on the axial-vector current form factors. For the moment, we leave Eq. (4.23) as it is and see its consequence.

We now take the limit $q=0$ of (4.23). Substituting (4.21) into (4.18) and taking the limit $q=0$, one finds

$$j_\mu^{Aj}(-k)=-f_\pi m_\pi^2\frac{k_\mu}{k^2}\frac{1}{k^2+m_\pi^2}g_1(k^2)\tau^j\gamma_5+iF_1^A(k^2)\left[\gamma_\mu-\frac{k_\mu}{k^2}\gamma\cdot k\right]\tau^j\gamma_5\ , \tag{4.25}$$

where $i\gamma\cdot k\gamma_5=-2m\gamma_5$ is used because it is inserted between the Dirac spinors. If we follow the usual assumption that $\Delta M_{\alpha\mu}^i$ has no singularity at $q=0$, we may take $q_\alpha\Delta M_{\alpha\mu}^i=0$ in the limit $q=0$. Equation (4.23) then determines the pion electroproduction amplitude

$$\begin{aligned} \Delta M_\mu^i=\frac{1}{f_\pi}\Bigg\{ & -ie\epsilon^{3ij}F_1^A(k^2)\left[\gamma_\mu-\frac{k_\mu}{k^2}\gamma\cdot k\right]\tau^j\gamma_5-ie\epsilon^{3ij}f_\pi\frac{k_\mu}{k^2}ig_1(k^2)\tau^j\gamma_5 \\ & +F_1^A(0)[ie\epsilon^{3ij}\tau^jF_1^V(k^2)\gamma_\mu\gamma_5-F_2^S(k^2)\tau^i\sigma_{\mu\nu}k_\nu\gamma_5-F_2^V(k^2)\delta_{i3}\sigma_{\mu\nu}k_\nu\gamma_5]\Bigg\}\ . \end{aligned} \tag{4.26}$$

This does not satisfy the gauge-invariance requirement (2.58), however. Following Fubini, Nambu, and Wataghin [11], it is a common practice to add to the Born approximation the term

$$ie\epsilon^{3ij}\frac{k_\mu}{k^2}[F_1^V(k^2)-1]ig_1(0)\tau^j\gamma_5\ . \tag{4.27}$$

This term was originally introduced in an *ad hoc* way, but its origin is clarified by the additional off-shell term in (4.8),

$$ie_N\frac{k_\mu}{k^2}\gamma\cdot k-ie(\tfrac{1}{2}F_1^S+\tfrac{1}{2}F_1^V\tau^3)\frac{k_\mu}{k^2}\gamma\cdot k\ , \tag{4.28}$$

which is necessary for the electromagnetic current to satisfy the WT identity. The term (4.28), when inserted into the Born approximation, generates exactly (4.27), as was shown by Berends and West [10]. The addition of this term to the Born approximation entails a modification of ΔM_μ^i,

$$\Delta M_\mu^i=\frac{1}{f_\pi}\left\{-ie\epsilon^{3ij}[F_1^A(k^2)-F_1^A(0)F_1^V(k^2)]\left[\gamma_\mu-\frac{k_\mu}{k^2}\gamma\cdot k\right]\tau^j\gamma_5 -F_1^A(0)[F_2^S(k^2)\tau^i\sigma_{\mu\nu}k_\nu\gamma_5+F_2^V(k^2)\delta_{i3}\sigma_{\mu\nu}k_\nu\gamma_5]\right\}+ie\epsilon^{3ij}\frac{k_\mu}{k^2}[g_1(0)-g_1(k^2)]i\tau^j\gamma_5 . \tag{4.29}$$

Here $i\gamma\cdot k\gamma_5=-2m\gamma_5$ is again used. One can easily check that (4.29) satisfies the WT equation (2.58) or

$$k_\mu\Delta M_\mu^i=ie\epsilon^{3ij}[g_1(0)-g_1(k^2)]i\tau^j\gamma_5 . \tag{4.30}$$

The last term in (4.29) is not considered by Fubini, Nambu, and Wataghin [11] since the pion-nucleon vertex is assumed to be constant in their paper [19]. This term is necessary when the pion-nucleon vertex has the virtual pion mass dependence. The additional term is just the one arising from the minimal substitutions in the pion-nucleon vertex [17]

$$G_\mu^i=ie\epsilon^{3ij}\frac{k_\mu}{k^2}[g_1(0)-g_1(k^2)]i\tau^j\gamma_5 . \tag{4.31}$$

From $R_\mu^i=\Delta M_\mu^i-G_\mu^i$, we get the well-known result [8,2,3,5,6]

$$R_1^{(0)}=\frac{eg_1(0)}{2m}F_2^S(k^2) , \tag{4.32}$$

$$R_1^{(+)}=\frac{eg_1(0)}{2m}F_2^V(k^2) , \tag{4.33}$$

$$R_6^{(-)}=\frac{eg_1(0)}{2m}\left[\frac{F_1^A(k^2)}{F_1^A(0)}-F_1^V(k^2)\right]\frac{1}{k^2} . \tag{4.34}$$

It is now clear that (4.32)–(4.34) result from the use of the axial-vector current (4.18) for the Born term. In evaluating its four-divergence, we have to use, instead of (4.20), the WT equation (2.21) or

$$q_\alpha j_\alpha^{Ai}(q)=\tau^i(i\gamma\cdot q\gamma_g-2m\gamma_5)+f_\pi\frac{m_\pi^2}{q^2+m_\pi^2}g_1(q^2)\tau^i\gamma_5 , \tag{4.35}$$

which yields

$$R_\mu^i=\frac{1}{f_\pi}\left\{-ie\epsilon^{3ij}[F_1^A(k^2)-F_1^V(k^2)]\left[\gamma_\mu-\frac{k_\mu}{k^2}\gamma\cdot k\right]\tau^j\gamma_5 -F_2^S(k^2)\tau^i\sigma_{\mu\nu}k_\nu\gamma_5-F_2^V(k^2)\delta_{i3}\sigma_{\mu\nu}k_\nu\gamma_5\right\} . \tag{4.36}$$

This is not yet of the form of our low-energy limit. The discrepancy arises from the assumption $q_\alpha\Delta M_{\alpha\mu}^i=0$ in the limit $q=0$. The gauge term that has been obtained from the axial-vector current using the minimal-substitution prescription has the four-divergence, in the limit $q\to 0$,

$$q_\alpha\bar G_{\alpha\mu}^i=-ie\epsilon^{3ij}[F_1^A(0)-1]i\left[\gamma_\mu-\frac{k_\mu}{k^2}\gamma\cdot k\right]\tau^j\gamma_5 . \tag{4.37}$$

$\bar G_{\alpha\mu}^i$ does not tend to a constant, but has a singularity, so that $q_\alpha\bar G_{\alpha\mu}^i$ has a finite contribution at $q=0$. This singularity is due to the off-shell form factors $\bar F_5^A=\bar F_8^A=(1-F_1^A)/q^2$, which are necessary for the WT equation in the pseudoscalar-coupling model, although (4.37) itself is independent of models. As seen from (3.37) or

$$\bar j_\alpha^{Ai}(q)=\tau^i[iF_1^A(q^2)\gamma_\alpha\gamma_5-\bar F_2^A(q^2)q_\alpha\gamma_5] +\tau^i[1-F_1^A(q^2)]\frac{q_\alpha}{q^2}(i\gamma\cdot q\gamma_5-2m\gamma_5) , \tag{4.38}$$

the minimal substitution in the second term produces a radiative axial-vector vertex proportional to q_α/q^2, which has the nonvanishing divergence (4.37). As a result, we obtain an extra contribution to the pion electroproduction amplitude,

$$\frac{1}{f_\pi}ie\epsilon^{3ij}[F_1^A(0)-1]\left[\gamma_\mu-\frac{k_\mu}{k^2}\gamma\cdot k\right]\tau^j\gamma_5 , \tag{4.39}$$

which accounts for the second and third terms in Eq. (4.15).

To summarize this section, we found the two factors that were overlooked in the literature: (1) In calculating the Born term for the radiative weak interaction, we have to use the axial-vector current which contains off-shell form factors. (2) The additional off-shell term gives rise to a gauge term which has a nonvanishing four-divergence so that it affects the pion electroproduction amplitude in the limit $q=0$.

It should be stressed that the modification of the low-energy theorem does not violate the Kroll-Ruderman theorem [14]. The pseudoscalar Born approximation for the pion electroproduction gives, in the limit $q=0$, the dominant multipole amplitude

$$E_{0+}^{(-)}=\frac{eg_1(0)}{2m}F_1^V(k^2) . \tag{4.40}$$

The amplitude (4.15) predicted by our low-energy theorem is added to the Born approximation to give

$$E_{0+}^{(-)}=\frac{eg_1(0)}{2mF_1^A(0)}\{F_1^A(k^2)+[F_1^A(0)-1][F_1^V(k^2)-1]\} . \quad (4.41)$$

For the real photon with $k^2=0$, by virtue of $F_1^V(0)=1$, we recover the Kroll-Ruderman theorem

$$E_{0+}^{(-)}=\frac{eg_1(0)}{2m} . \quad (4.42)$$

It is noted that if it were not for the correction (4.39), the Kroll-Ruderman theorem would have been violated.

To conclude this section, we comment on the virtual nucleon mass dependence of the form factors. It is usual to assume that current form factors are functions of the momentum transfer squared, but, in principle, they depend also on p'^2 and p^2. This influences the low-energy theorem for on-shell nucleons. In the limit $p'^2=p^2$, we get the additional contribution

$$-i\frac{1}{f_\pi}\tau^i(p'+p)_\mu\frac{\partial}{\partial p^2}(iF_1^A\gamma\cdot k+mG-mF)\bigg|_{p^2=-m^2}\gamma_5 . \quad (4.43)$$

Namely, the extra term

$$\Delta R_1^{(0)}=\frac{eg_1(0)}{2F_1^A(0)}\frac{\partial}{\partial p^2}(2F_1^A-G+F)\bigg|_{p^2=-m^2} \quad (4.44)$$

arises from the virtual nucleon mass dependence of the axial-vector form factor and that of the nucleon propagator. The use of the full nucleon propagator also gives rise to a correction

$$\Delta R_6^{(-)}=\frac{eg_1(0)}{2mF_1^A(0)}[G(-m^2)-1]\frac{1}{k^2} , \quad (4.45)$$

which was neglected in deriving (4.15).

V. SUMMARY

We have reexamined the consequences of the PCAC hypothesis, gauge invariance, and current algebra on the radiative weak interaction and pion electroproduction. We did not use any approximation. All particles can be off their mass shells. The axial-vector and electromagnetic currents are parametrized in the most general forms with form factors which depend on the virtual nucleon masses and momentum transfer squared. The PCAC hypothesis and gauge invariance are expressed in the forms of the WT identities. We have split the radiative weak interaction and the pion electroproduction amplitudes into Bohn terms, gauge terms, and remainders. We wrote the Born terms in their most general form, allowing all participating particles to go off shell. We also wrote down explicitly the gauge terms using the minimal-substitution method. The low-energy theorem was obtained for the pion electroproduction amplitude. It is found that our low-energy theorem is at variance with the result given in the literature. The discrepancy arose from the off-shell matrix elements of the axial-vector current. They affect the radiative axial-vector vertex in two aspects: First, they enter the Born term in such a way that the axial-vector form factors do not appear in the divergence of the Born term. Second, they generate the gauge term which has a nonvanishing four-divergence in the low-energy limit. It is also shown that the Kroll-Ruderman theorem is exactly respected by our low-energy theorem. It is desired that our approach is applied to the derivation of the low-energy theorem for the pion photoproduction at the threshold. The result will be published in a separate paper.

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[19] Riazuddin and Lee [2] take into account the q^2 dependence of the pion-nucleon vertex and the k^2 dependence of the pion electromagnetic form factor. Their gauge term at $q=0$ is

$$ie\epsilon\frac{k_\mu}{k^2}[g_1(0)F_1^V(k^2)-g_1(k^2)F_\pi(k^2)]i\tau^j\gamma_5 ,$$

which is the sum of (4.27), (4.31), and the correction caused by the off-shell electromagnetic current interaction of the pion [10],

$$-ie\epsilon^{3ij}\frac{k_\mu}{k^2}[F_\pi(k^2)-1]ig_1(k^2)\tau^j\gamma_5 .$$