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Approach to the relativistic extended Thomas-Fermi expansion for Green's functions, phase-space densities, and densities

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We present an extension of the semiclassical Thomas-Fermi model to relativistic systems. These are obtained by application of the gradient expansion scheme on the Wigner transformed Dyson equation. Explicitly we give the expansion of the Green's functions, phase-space densities, and densities for a system of nucleons in a vector and scalar potential to second order.

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I. INTRODUCTION

In recent years the interest in the investigation of nuclear systems has shifted strongly towards a relativistic approach (see, for instance, Refs. [1–3]). However, only little research utilizing a semiclassical expansion of the relativistic theory, which was sometimes successful in the simplification of nonrelativistic calculations in atomic and nuclear physics (see, for instance, Refs. [4–9]), has been published (see Refs. [1,10–13]). Generally the advantage of the semiclassical approach is the avoidance of wave-function calculations by utilizing densities, which in many cases makes the calculations easier. In particular, sometimes one is only interested in the total energy of the system or the density distribution. Therefore it seems natural to search for methods that bypass the wave-function scheme. An additional argument for the investigation of the semiclassical expansion is the fact that all one-body quantities can be expressed in terms of the one-body density matrix. If one is capable of obtaining a semiclassical expansion in powers of $\hbar$ of the Wigner transformed density matrix, i.e., the phase-space density, one gets a systematic expansion, which in lowest-order corresponds to the Thomas-Fermi treatment, which we will deduce later for illustration. The arguments and motivations are rather similar to the nonrelativistic case, and we will not repeat them in detail (see Ref. [6]). Furthermore, the gradient expansion is of interest in relativistic transport theories, where Wigner transformed Green's functions also occur.

In the relativistic case one expects several complications, which make the formalism more difficult [1,11–13]. First one faces relativistic corrections due to the Dirac structure of the approach. Furthermore, one has to deal with position-dependent Dirac masses, which are absent in the standard nonrelativistic theory. Examples are given in Sec. III and Appendix A, where we encounter, for instance, at least three kinds of phase-space densities, etc. For these reasons one restricts oneself to the relativistic Thomas-Fermi method in numerical calculations [1]. In the atomic case the semiclassical relativistic extended Thomas-Fermi (RETF) expansion for particles in an external potential was obtained by either a tricky ansatz for the higher-order Green's functions [14,15] or "window" procedures [15] (elimination of the Dirac sea, but tedious resummation procedures enter for the phase-space density). An alternative procedure was proposed in Ref. [11], where one expands the Bloch equation for the propagator $\exp(-\beta H)$ and treats the coupled differential equations in a recursive scheme. However, the solution already becomes quite involved in the nonrelativistic case [6]. In this contribution we want to present a pure algebraic method, which more closely resembles the standard nonrelativistic scheme as described, for instance, by Grammaticos and Voros [6]. It involves only straightforward but tedious algebraic methods, and the residuum calcules. Furthermore, it has the advantage that it can be generalized to interacting particles. The paper is organized as follows. In the next section we describe the general G-function expansion by utilizing the Wigner transformed Dyson equation. From the $\hbar$ expansion of the G function one can obtain by complex integration the phase-space density. Integration over the momentum then gives the density. The energy density emerges in a similar manner. The explicit expansion for the G function, which is the key expression for the wanted densities, is calculated up to second order in the third section. The resulting densities are given, in detail, in Appendix A. In order to illustrate the procedure within a Dirac description we rederive the TF approximation in Appendix B as the first-order approximation of the Hartree-Fock (HF) theory or by minimizing the energy density with respect to the densities, respectively.

II. GENERAL THEORY

The basic quantity for the semiclassical expansion is the Wigner transform of the one-particle propagator, defined as (we use the conventions of Bjorken and Drell [16]; $|N\rangle$ denotes the ground state)

$$G(R,p)=\int d^4r\,\exp(ipr/\hbar)G(R+r/2,R-r/2)$$
$$=\int d^4r\{\exp(ipr/\hbar)(-i)\langle N|T[\Psi(R+r/2)\bar{\Psi}(R-r/2)]|N\rangle\}\ . \tag{2.1}$$

The Green's function obeys the Dyson equation, which has in the standard space-time representation the form

$$\{\hbar c(i\gamma^\mu\partial_\mu=m_N c/\hbar)_{12}-\Sigma(1,2)\}G(2,1')=\hbar c\delta(1,1')\ , \tag{2.2}$$

where we sum or integrate over all doubly occurring variables ($1{:}=x_1^\mu$, spinor index, etc.). After performing the WT one obtains the Dyson equation in the mixed position-momentum representation the following form ($H\to H-\mu N$):

$$\left[c\gamma^\mu\left[p_\mu+\frac{i\hbar}{2}\frac{\partial}{\partial R'^\mu}\right]+\mu\gamma^0-m_N c^2-\Sigma\left[R,p+\frac{i\hbar}{1}\frac{\partial}{\partial R'}\right]\right]_{12}G_{21'}\left[R',p'-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial R}\right]\Bigg|_{R=R'}^{p=p'}$$
$$=(G^{\mathrm{TF}})_{12}^{-1}(R,p)\exp\left[i\frac{\hbar}{2}\left[\frac{\vec{\partial}}{\partial R}\cdot\frac{\overleftarrow{\partial}}{\partial p}-\frac{\overleftarrow{\partial}}{\partial R}\cdot\frac{\vec{\partial}}{\partial p}\right]\right]G_{21'}(R,p)=\hbar c\delta_{11'}\ . \tag{2.3}$$

Arrows pointing to the right imply that the differential operator acts on the quantities to the right, etc.

Knowledge of the zeroth-order (Thomas-Fermi) G function G^{TF} and of the functional dependence of Σ with respect to G is now, in principle, sufficient for the determination of the semiclassical expansion. The relevant expansion of the G functions, phase-space densities, and densities is given by the following scheme [$G(R,p)=G(\mathbf{R},p)$]:

$$G(\mathbf{R},p)=\hbar^4\sum_{j=0}^{\infty}\hbar^j G^{(j)}(\mathbf{R},p)\ , \tag{2.4a}$$

$$n(\mathbf{R},\mathbf{p})=\hbar^3\sum_{j=0}^{\infty}\hbar^j n^{(j)}(\mathbf{R},\mathbf{p})$$
$$=\hbar^3\sum_{j=0}^{\infty}\hbar^j\int\frac{dp_0}{2\pi i}e^{ip_0\eta}G^{(j)}(\mathbf{R},p)\ , \tag{2.4b}$$

$$n(\mathbf{R})=\sum_{j=0}^{\infty}\hbar^j n^{(j)}(\mathbf{R})=\sum_{j=0}^{\infty}\hbar^j\int\frac{d^3p}{(2\pi)^3}n^{(j)}(\mathbf{R},\mathbf{p})\ . \tag{2.4c}$$

The so-called extended Thomas-Fermi (ETF) approximation [4,6] is restricted to local external potentials, i.e.,

$$\Sigma(x,x')=\Sigma\left[\frac{\mathbf{x}+\mathbf{x}'}{2}\right]\delta^4(x-x')\ . \tag{2.5}$$

The same structure is encountered in the Hartree approximation [1,12]. For simplicity we will only treat the case $N=Z$ with a Hamiltonian of the standard structure [1,2] ($\Sigma_\mu=\delta_{\mu 0}V(\mathbf{R})$; $c^2M{:}=c^2m_N+\Sigma_s(\mathbf{R})$):

$$H=c\boldsymbol{\alpha}\cdot\mathbf{p}+\beta c^2M(\mathbf{R})+V(\mathbf{R})\ . \tag{2.6}$$

($N\neq Z$ would lead to $\mu\to\mu\pm\nu/2$; $\Sigma_0\to\Sigma_0\mp\Sigma_{\tau 0}$.) The zeroth-order (TF) solution of Eq. (2.3) corresponds to the local-density approximation (nuclear matter) solution [1,2,12].

$$G^{\mathrm{TF}}(\mathbf{R},p)\equiv\hbar^4G^{(0)}(\mathbf{R},p)$$
$$=\hbar c\frac{c\gamma^\mu k_\mu+Mc^2}{c^2k^2-M^2c^4}=:\hbar c\frac{c\gamma^\mu k_\mu+Mc^2}{N} \tag{2.7}$$

with ($\mathbf{k}\equiv\mathbf{p}$)

$$ck_0{:}=cp_0+\mu-V(\mathbf{R})\ . \tag{2.8}$$

The detailed pole structure is given by [$\epsilon{:}=(c^2\mathbf{p}^2+M^2c^4)^{1/2}$]

$$\frac{1}{N}=\left\{\frac{1}{2\epsilon}\frac{1}{ck_0-\epsilon+i\eta\,\mathrm{sgn}(\epsilon-\epsilon_F)}-\frac{1}{2\epsilon}\frac{1}{ck_0+\epsilon-i\eta}\right\}\ . \tag{2.9}$$

The poles correspond to the single-particle energies:

$$\hbar\omega(\mathbf{R},\mathbf{p})=V(\mathbf{R})\pm\epsilon(\mathbf{R},\mathbf{p})-\mu\ . \tag{2.10}$$

The negative energies describe the energy momentum relation for the antiparticles.

For the energy one obtains

$$E=-i\,\mathrm{Tr}\left\{\int d^3R\int\frac{d^4p}{(2\pi\hbar)^4}e^{ip_0\eta}\gamma^0H\left[\mathbf{R},\mathbf{p}+\frac{\hbar}{2i}\frac{\partial}{\partial\mathbf{R}'}\right]G\left[\mathbf{R}',p'-\frac{\hbar}{2i}\frac{\overleftarrow{\partial}}{\partial\mathbf{R}}\right]\Bigg|_{\mathbf{R}=\mathbf{R}'}^{p=p'}\right\}$$
$$=-i\,\mathrm{Tr}\left\{\int d^3R\int\frac{d^4p}{(2\pi\hbar)^4}\gamma^0(cp_0+\mu)G(\mathbf{R},p)\right\}\ . \tag{2.11}$$

The last expression was obtained by use of the Dyson equation (2.3). The kinetic energy is given by

$$T=-i\,\mathrm{Tr}\int d^3R\int\frac{d^4p}{(2\pi\hbar)^4}e^{ip_0\eta}\left\{\left[c\gamma\left[\mathbf{p}+\frac{\hbar}{2i}\frac{\partial}{\partial\mathbf{R}}\right]+m_Nc^2-\gamma^0m_Nc^2\right]G(\mathbf{R},p)\right\}\ . \tag{2.12}$$

The interpretation of the integrand as a local kinetic-energy density may be misleading, since one actually already calculates in the TF approximation the expectation value of the free Hamiltonian (minus rest mass) in a "free" relativistic Fermi gas with a position and density-dependent Dirac mass. For that reason, for instance, the Dirac energy is no monotonic function of the density [17]. A final point is the calculation of the energy density for a mass operator emerging from a self-consistent many-body procedure. In such a case one has to subtract from H the term $\frac{1}{2}[\Sigma_s(\mathbf{R})+\gamma^0 V(\mathbf{R})]$, i.e., the meson contributions.

The $\hbar$ expansion follows from the energy expressions by inserting of the G-function expansion (2.4a). In the following section we will explicitly give the G-function expansion up to second order, which is the relevant ingredient for the calculation of the different densities, which are obtainable in the next steps by p integration.

III. EXPANSION UP TO SECOND ORDER IN $\hbar$

The RETF expansion of the G function can be obtained from a straightforward but tedious evolution of the Dyson equation (2.3). With the Wigner operator

$$\overleftrightarrow{\Lambda}:=\overleftarrow{\nabla}_R\cdot\overrightarrow{\nabla}_P-\overrightarrow{\nabla}_R\cdot\overleftarrow{\nabla}_P \tag{3.1}$$

one gets ($\hbar=c=1$)

$$G^{(1)}(\mathbf{R},p)=\frac{1}{2i}G^{(0)}(\mathbf{R},p)\{G^{\mathrm{TF}}(\mathbf{R},p)^{-1}\overleftrightarrow{\Lambda}G^{(0)}(\mathbf{R},p)\}=\frac{1}{N^2}\left[\sigma^{\mu i}k_\mu\frac{\partial}{\partial R^i}M-\gamma^0\sigma^{ji}k_j\frac{\partial}{\partial R^i}V+\sigma^{0i}M\frac{\partial}{\partial R^i}V\right], \tag{3.2}$$

$$\begin{aligned}G^{(2)}(\mathbf{R},p)&=G^{(0)}(\mathbf{R},p)\left[\frac{1}{8}[G^{\mathrm{TF}}(\mathbf{R},p)^{-1}\overleftrightarrow{\Lambda}^2G^{(0)}(\mathbf{R},p)]+\frac{1}{2i}[G^{\mathrm{TF}}(\mathbf{R},p)^{-1}\overleftrightarrow{\Lambda}G^{(1)}(\mathbf{R},p)]\right]\\&=G_s^{(2)}(\mathbf{R},p)+\gamma^0G_0^{(2)}(\mathbf{R},p)-\boldsymbol{\gamma}\mathbf{G}^{(2)}(\mathbf{R},p)+\gamma^0\boldsymbol{\gamma}\mathbf{G}_0(\mathbf{R},p)\end{aligned} \tag{3.3}$$

with

$$\begin{aligned}G_s^{(2)}(\mathbf{R},p)=&-\frac{3}{4N^2}\Delta M-\frac{1}{N^3}[M^2\Delta M+Mk_0\Delta V+\mathbf{k}\cdot\nabla(\mathbf{k}\cdot\nabla M)+2M(\nabla V)^2+2M(\nabla M)^2+2k_0(\nabla M\cdot\nabla V)]\\&-\frac{1}{N^4}[2Mk_0\mathbf{k}\cdot\nabla(\mathbf{k}\cdot\nabla V)+2M^2\mathbf{k}\cdot\nabla(\mathbf{k}\cdot\nabla M)-2M(\mathbf{k}\cdot\nabla V)^2+2\epsilon^2M(\nabla V)^2+2M(\mathbf{k}\cdot\nabla M)^2\\&+2M^3(\nabla M)^2+4M^2k_0(\nabla M\cdot\nabla V)]\ ,\end{aligned} \tag{3.4a}$$

$$\begin{aligned}G_0^{(2)}(\mathbf{R},p)=&-\frac{1}{4N^2}\Delta V-\frac{1}{N^3}[k_0M\Delta M+\epsilon^2\Delta V+\mathbf{k}\cdot\nabla(\mathbf{k}\cdot\nabla V)+2M(\nabla M\cdot\nabla V)]\\&-\frac{2}{N^4}[\epsilon^2\mathbf{k}\cdot\nabla(\mathbf{k}\cdot\nabla V)+Mk_0\mathbf{k}\cdot\nabla(\mathbf{k}\cdot\nabla M)+k_0\epsilon^2(\nabla V)^2-k_0(\mathbf{k}\cdot\nabla V)^2+k_0M^2(\nabla M)^2\\&+k_0(\mathbf{k}\cdot\nabla M)^2+2M\epsilon^2(\nabla M\cdot\nabla V)]\ ,\end{aligned} \tag{3.4b}$$

$$\begin{aligned}\mathbf{G}^{(2)}(\mathbf{R},p)=&-\frac{1}{N^3}[M\mathbf{k}\Delta M+k_0\mathbf{k}\Delta V+2\mathbf{k}(\nabla V)^2-2\nabla V(\mathbf{k}\cdot\nabla V)+2\nabla M(\mathbf{k}\cdot\nabla M)]\\&-\frac{2}{N^4}[k_0\mathbf{k}\mathbf{k}\cdot\nabla(\mathbf{k}\cdot\nabla V)+M\mathbf{k}\mathbf{k}\cdot\nabla(\mathbf{k}\cdot\nabla M)+\epsilon^2\mathbf{k}(\nabla V)^2+\mathbf{k}(\mathbf{k}\cdot\nabla V)^2-2\mathbf{k}^2\nabla V(\mathbf{k}\cdot\nabla V)+2\mathbf{k}^2\nabla M(\mathbf{k}\cdot\nabla M)\\&+M^2\mathbf{k}(\nabla M)^2-\mathbf{k}(\mathbf{k}\cdot\nabla M)^2+2Mk_0\mathbf{k}(\nabla M\cdot\nabla V)]\ ,\end{aligned} \tag{3.4c}$$

$$\mathbf{G}_0^{(2)}(\mathbf{R},p)=-\frac{2}{N^3}[\nabla M(\mathbf{k}\cdot\nabla V)-\nabla V(\mathbf{k}\cdot\nabla M)]\ . \tag{3.4d}$$

The phase-space density is now obtainable from Eqs. (2.4b), (2.7), (2.9), (3.2), and (3.3) by complex integration. Crucial in this context is the pole structure of $G^{(0)}$ and N, respectively, which is given in Eq. (2.9). It permits the decomposition of N^{-n} in powers of $(k_0\mp\epsilon)$. The contributions corresponding to the poles at $k_0=-\epsilon$ are neglected in the further procedure. The general space density is obtainable by integration over the momentum. Here, one is mainly interested in the scalar and baryonic density, which can be extracted from the general density by performing the corresponding traces. For instance, for the baryon density one obtains ($G^{(1)}$ is traceless)

$$n_B(\mathbf{R})=\frac{-i}{(2\pi)^4}\mathrm{Tr}\left[\gamma^0\int d^4p\,e^{ip_0\eta}[G^{(0)}(\mathbf{R},p)+G^{(1)}(\mathbf{R},p)+G^{(2)}(\mathbf{R},p)]\right], \tag{3.5}$$

which yields the following expression for each kind of nucleons ($x_F:=\epsilon_F/p_F$):

$$n_B(\mathbf{R})=\frac{p_F^3}{3\pi^2}+\frac{1}{24\pi^2}\left[\left[-2x-2\ln\frac{x_F+1}{x_F-1}\right]\Delta V-2(x_F^2-1)p_F\frac{\Delta M}{M}\right.$$
$$\left.+(2-x_F^2)\frac{(\nabla M)^2}{p_F}-(x_F^2-3)\frac{(\nabla V)^2}{p_F}-2x(x_F^2-3)\frac{(\nabla V\cdot\nabla M)}{M}\right], \tag{3.6}$$

which agrees with Ref. [15] for $M^*=m_N$. The other densities and the energy density can be calculated by similar procedures and are given explicitly in Appendix A.

As a final remark we would like to mention that the RETF formalism developed above on the basis of a relativistic field theoretical scheme is the equivalent of the ETF method in nonrelativistic physics (see, for instance, Ref. [6]). However, due to the more complicated relativistic ingredients caused by the Dirac structure, it involves substantially more effort as in the nonrelativistic case in the numerical many-body treatment, which is presently under investigation. This feature is not unexpected, since also the wave-function scheme is much more complicated in the relativistic treatment (see, for instance, Refs. [2,18]).

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APPENDIX A: PHASE-SPACE DENSITIES, DENSITIES, ENERGY DENSITY, KINETIC ENERGY DENSITY

1. Phase-space densities

The Wigner function follows according to Eq. (2.4b) as

$$n^{(m)}(\mathbf{R},\mathbf{p})=\frac{1}{(2\pi i)}\int dp^0\exp(ip_0\eta)G^{(m)}(\mathbf{R},p)\ . \tag{A1}$$

The different kinds of phase-space densities for each kind of nucleon (no isospin trace) are defined by ($n_0\equiv n_B$)

$$n_0(\mathbf{R},\mathbf{p})=\mathrm{Tr}[\gamma^0 n(\mathbf{R},\mathbf{p})]\ , \tag{A2a}$$

$$n_s(\mathbf{R},\mathbf{p})-\mathrm{Tr}[n(\mathbf{R},\mathbf{p})]\ , \tag{A2b}$$

$$n_v(\mathbf{R},\mathbf{p})=\mathrm{Tr}[\boldsymbol{\gamma}\cdot\mathbf{p}n(\mathbf{R},\mathbf{p})]\ . \tag{A2c}$$

Evaluation of (A1) and (A2) leads to [$\Theta(x)$ denotes the step function, $|\mathbf{p}|\equiv q$ angle averaging over $\mathbf{R}$ is implied]

$$n_0^{(0)}(\mathbf{R},\mathbf{p})=2\Theta(-\omega(\mathbf{R},\mathbf{p}))\ , \tag{A3a}$$

$$n_0^{(2)}(\mathbf{R},\mathbf{p})=4\Delta V\left[\frac{q^2}{36}\frac{1}{(2\epsilon)^2}\frac{\partial^2\delta(\omega)}{\partial\omega^2}+\left[\frac{1}{8}\frac{1}{2\epsilon}-\frac{1}{6}\frac{q^2}{(2\epsilon)^3}\right]\frac{\partial\delta(\omega)}{\partial\omega}\right.$$
$$\left.+\left[\frac{2}{3}\frac{q^2}{(2\epsilon)^4}-\frac{1}{2}\frac{1}{(2\epsilon)^2}\right]\delta(\omega)+\left[\frac{4}{3}\frac{q^2}{(2\epsilon)^5}-\frac{1}{(2\epsilon)^3}\right]\Theta(-\omega)\right]$$
$$+4M\Delta M\left[\frac{1}{18}\frac{q^2}{(2\epsilon)^3}\frac{\partial^2\delta(\omega)}{\partial\omega^2}+\left[\frac{1}{4}\frac{1}{(2\epsilon)^2}-\frac{1}{3}\frac{q^2}{(2\epsilon)^4}\right]\frac{\partial\delta(\omega)}{\partial\omega}+\left[\frac{2}{3}\frac{q^2}{(2\epsilon)^5}-\frac{1}{2}\frac{1}{(2\epsilon)^3}\right]\delta(\omega)\right]$$
$$+4(\nabla V)^2\left[\left[\frac{1}{24}\frac{1}{(2\epsilon)}-\frac{q^2}{18}\frac{1}{(2\epsilon)^3}\right]\frac{\partial^2\delta(\omega)}{\partial\omega^2}+\left[\frac{1}{3}\frac{q^2}{(2\epsilon)^4}-\frac{1}{4}\frac{1}{(2\epsilon)^2}\right]\frac{\partial\delta(\omega)}{\partial\omega}+\left[\frac{1}{2}\frac{1}{(2\epsilon)^3}-\frac{2}{3}\frac{q^2}{(2\epsilon)^5}\right]\delta(\omega)\right]$$
$$+4(\nabla M)^2\left[\frac{1}{6}\frac{1}{(2\epsilon)^3}\frac{\partial^2\delta(\omega)}{\partial\omega^2}-\frac{1}{(2\epsilon)^4}\frac{\partial\delta(\omega)}{\partial\omega}+\frac{2}{(2\epsilon)^5}\delta(\omega)\right](M^2+\tfrac{1}{3}q^2)$$
$$+4M(\nabla V)\cdot(\nabla M)\left[\frac{1}{6}\frac{1}{(2\epsilon)^2}\frac{\partial^2\delta(\omega)}{\partial\omega^2}-\frac{1}{(2\epsilon)^3}\frac{\partial\delta(\omega)}{\partial\omega}+\frac{4}{(2\epsilon)^4}\delta(\omega)+\frac{8}{(2\epsilon)^5}\Theta(-\omega)\right], \tag{A3b}$$

$$n_s^{(0)}(\mathbf{R},\mathbf{p})=2\frac{M}{\epsilon}\Theta(-\omega(\mathbf{R},\mathbf{p}))\ , \tag{A4a}$$

$$\begin{aligned}
n_s^{(2)}(\mathbf{R},\mathbf{p})=4M\Delta V&\left[\frac{q^2}{18}\frac{1}{(2\epsilon)^3}\frac{\partial^2\delta(\omega)}{\partial\omega^2}+\left[\frac{1}{4}\frac{1}{(2\epsilon)^2}-\frac{1}{3}\frac{q^2}{(2\epsilon)^4}\right]\frac{\partial\delta(\omega)}{\partial\omega}+\left[\frac{2}{3}\frac{q^2}{(2\epsilon)^5}-\frac{1}{2}\frac{1}{(2\epsilon)^3}\right]\delta(\omega)\right]\\
&+4\Delta M\left[\frac{1}{9}\frac{q^2M^2}{(2\epsilon)^4}\frac{\partial^2\delta(\omega)}{\partial\omega^2}+\left[\frac{1}{6}\frac{q^2}{(2\epsilon)^3}+\frac{M^2}{2}\frac{1}{(2\epsilon)^3}-\frac{4}{3}\frac{q^2M^2}{(2\epsilon)^5}\right]\frac{\partial\delta(\omega)}{\partial\omega}\right.\\
&\qquad+\left[\frac{3}{4}\frac{1}{(2\epsilon)^2}-\frac{q^2}{(2\epsilon)^4}-\frac{3M^2}{(2\epsilon)^4}+\frac{20}{3}\frac{q^2M^2}{(2\epsilon)^6}\right]\delta(\omega)\\
&\qquad\left.+\left[\frac{3}{2}\frac{1}{(2\epsilon)^3}-\frac{2q^2}{(2\epsilon)^5}-\frac{6M^2}{(2\epsilon)^5}+\frac{40}{3}\frac{q^2M^2}{(2\epsilon)^7}\right]\Theta(-\omega)\right]\\
&+4(\nabla M)\cdot(\nabla V)\left[\frac{M^2}{3}\frac{1}{(2\epsilon)^3}\frac{\partial^2\delta(\omega)}{\partial\omega^2}+\left[\frac{1}{2}\frac{1}{(2\epsilon)^2}-\frac{2M^2}{(2\epsilon)^4}\right]\frac{\partial\delta(\omega)}{\partial\omega}+\left[\frac{4M^2}{(2\epsilon)^5}-\frac{1}{(2\epsilon)^3}\right]\delta(\omega)\right]\\
&+4M(\nabla V)^2\left[\left[\frac{2}{9}\frac{q^2}{(2\epsilon)^4}+\frac{1}{3}\frac{M^2}{(2\epsilon)^4}\right]\frac{\partial^2\delta(\omega)}{\partial\omega^2}+\frac{4}{3}\frac{q^2}{(2\epsilon)^5}\frac{\partial\delta(\omega)}{\partial\omega}+\left[-\frac{20}{3}\frac{q^2}{(2\epsilon)^6}-\frac{1}{(2\epsilon)^4}\right]\delta(\omega)\right.\\
&\qquad\left.+\left[-\frac{40}{3}\frac{q^2}{(2\epsilon)^7}-\frac{2}{(2\epsilon)^5}\right]\Theta(-\omega)\right]\\
&+4M(\nabla M)^2\left[\left[\frac{1}{9}\frac{q^2}{(2\epsilon)^4}+\frac{1}{3}\frac{M^2}{(2\epsilon)^4}\right]\frac{\partial^2\delta(\omega)}{\partial\omega^2}+\left[\frac{1}{(2\epsilon)^3}-\frac{4}{3}\frac{q^2}{(2\epsilon)^5}-\frac{4M^2}{(2\epsilon)^5}\right]\frac{\partial\delta(\omega)}{\partial\omega}\right.\\
&\qquad\left.+\left[\frac{20}{3}\frac{q^2}{(2\epsilon)^6}+\frac{20M^2}{(2\epsilon)^6}-\frac{6}{(2\epsilon)^4}\right]\delta(\omega)+\left[\frac{40}{3}\frac{q^2}{(2\epsilon)^7}+\frac{40M^2}{(2\epsilon)^7}-\frac{12}{(2\epsilon)^5}\right]\Theta(-\omega)\right]\ ,
\end{aligned} \tag{A4b}$$

$$n_v^{(0)}(\mathbf{R},\mathbf{p})=2\frac{q^2}{\epsilon}\Theta(-\omega(\mathbf{R},\mathbf{p}))\ , \tag{A5a}$$

$$\begin{aligned}
n_v^{(2)}(\mathbf{R},\mathbf{p})=4\Delta V&\left[\frac{q^4}{18}\frac{1}{(2\epsilon)^3}\frac{\partial^2\delta(\omega)}{\partial\omega^2}+\left[\frac{1}{4}\frac{q^2}{(2\epsilon)^2}-\frac{q^4}{3}\frac{1}{(2\epsilon)^4}\right]\frac{\partial\delta(\omega)}{\partial\omega}\left[\frac{2}{3}\frac{q^4}{(2\epsilon)^5}-\frac{1}{2}\frac{q^2}{(2\epsilon)^3}\right]\delta(\omega)\right]\\
&+4M\Delta M\left[\frac{1}{9}\frac{q^4}{(2\epsilon)^4}\frac{\partial^2\delta(\omega)}{\partial\omega^2}+\left[\frac{1}{2}\frac{q^2}{(2\epsilon)^3}-\frac{4}{3}\frac{q^4}{(2\epsilon)^5}\right]\frac{\partial\delta(\omega)}{\partial\omega}+\left[\frac{20}{3}\frac{q^4}{(2\epsilon)^6}-\frac{3q^2}{(2\epsilon)^4}\right]\delta(\omega)\right.\\
&\qquad\left.+\left[\frac{40}{3}\frac{q^4}{(2\epsilon)^7}-\frac{6q^2}{(2\epsilon)^5}\right]\Theta(-\omega)\right]\\
&+4M(\nabla M\cdot\nabla V)\left[\frac{q^2}{3}\frac{1}{(2\epsilon)^3}\frac{\partial^2\delta(\omega)}{\partial\omega^2}-2q^2\frac{1}{(2\epsilon)^4}\frac{\partial\delta(\omega)}{\partial\omega}+\frac{4q^2}{(2\epsilon)^5}\delta(\omega)\right]\\
&+4(\nabla V)^2\left[\left[-\frac{1}{9}\frac{q^4}{(2\epsilon)^4}+\frac{1}{12}\frac{q^2}{(2\epsilon)^2}\right]\frac{\partial^2\delta(\omega)}{\partial\omega^2}+\left[-\frac{1}{3}\frac{q^2}{(2\epsilon)^3}+\frac{4}{3}\frac{q^4}{(2\epsilon)^5}\right]\frac{\partial\delta(\omega)}{\partial\omega}\right.\\
&\qquad\left.+\left[-\frac{20}{3}\frac{q^4}{(2\epsilon)^6}+\frac{q^2}{(2\epsilon)^4}\right]\delta(\omega)+\left[-\frac{40}{3}\frac{q^4}{(2\epsilon)^7}+\frac{2q^2}{(2\epsilon)^5}\right]\Theta(-\omega)\right]\\
&+4(\nabla M)^2\left[\left[\frac{q^4}{9}\frac{1}{(2\epsilon)^4}+\frac{q^2M^2}{3}\frac{1}{(2\epsilon)^4}\right]\frac{\partial^2\delta(\omega)}{\partial\omega^2}+\left[\frac{q^2}{3}\frac{1}{(2\epsilon)^3}-\frac{4}{3}\frac{q^4}{(2\epsilon)^5}-4q^2M^2\frac{1}{(2\epsilon)^5}\right]\frac{\partial\delta(\omega)}{\partial\omega}\right.\\
&\qquad\left.+\left[\frac{20}{3}\frac{q^4}{(2\epsilon)^6}+20q^2M^2\frac{1}{(2\epsilon)^6}-\frac{2q^2}{(2\epsilon)^4}\right]\delta(\omega)+\left[\frac{40}{3}\frac{q^4}{(2\epsilon)^7}+40q^2M^2\frac{1}{(2\epsilon)^7}-\frac{4q^2}{(2\epsilon)^5}\right]\Theta(-\omega)\right]\ .
\end{aligned} \tag{A5b}$$

2. Densities

The densities emerge from the phase-space densities by momentum integration [see Eq. (2.4c)]. It follows that [the baryon density n_0 is given in expression (3.6)]

$$n_s^{(0)}(\mathbf{R})=\frac{1}{2\pi^2}\left[Mp_F\epsilon_F-\frac{M^3}{2}\ln\left[\frac{x_F+1}{x_F-1}\right]\right], \tag{A6a}$$

$$\begin{aligned}n_s^{(2)}(\mathbf{R})=\frac{1}{24\pi^2}\Bigg\{&-\Delta V\frac{2M}{p_F}+\Delta M\left[3\ln\left[\frac{x_F+1}{x_F-1}\right]-2x_F\right]-\frac{(\nabla M)^2}{M}x_F^2(x_F^2+2)\\&-(\nabla M)\cdot(\nabla V)\frac{2}{p_F}(2+x_F^2)-\frac{(\nabla V)^2}{M}-x_F(x_F^2+1)\Bigg\}.\end{aligned} \tag{A6b}$$

$$n_v^{(0)}(\mathbf{R})=\frac{1}{8\pi^2}\left[5p_F^3\epsilon_F-3p_F\epsilon_F^3+\frac{3}{2}M^4\ln\left[\frac{x_F+1}{x_F-1}\right]\right], \tag{A7a}$$

$$\begin{aligned}n_v^{(2)}(\mathbf{R})=\frac{1}{24\pi^2}\Bigg\{&-2M\Delta M\ln\left[\frac{x_F+1}{x_F-1}\right]-4p_F\Delta V+\left[3x-\ln\left[\frac{x_F+1}{x_F-1}\right]\right](\nabla V)^2\\&+\left[3x_F-\frac{1}{2}\ln\left[\frac{x_F+1}{x_F-1}\right]\right](\nabla M)^2+6\frac{M}{p_F}(\nabla M)\cdot(\nabla V)\Bigg\}.\end{aligned} \tag{A7b}$$

3. Energy density ($m\equiv m_N$)

One obtains from (2.11)

$$\begin{aligned}e^{(0,2)}(\mathbf{R})&=n_v^{(0,2)}(\mathbf{R})+\left[\left[1-\frac{\lambda}{2}\right]\Sigma_s(\mathbf{R})+m\right]n_s^{(0,2)}(\mathbf{R})+\left[1-\frac{\lambda}{2}\right]V(\mathbf{R})n_0^{(0,2)}(\mathbf{R})\\&=n_v^{(0,2)}(\mathbf{R})+\left[\left[1-\frac{\lambda}{2}\right]M+\frac{\lambda}{2}m\right]n_s^{(0,2)}(\mathbf{R})+\left[1-\frac{\lambda}{2}\right]Vn_0^{(0,2)}(\mathbf{R}).\end{aligned} \tag{A8}$$

$\lambda=0$ corresponds to a purely external potential; $\lambda=1$ describes the case, if meson contributions are included (see text).

4. Kinetic energy density

Evaluation of (2.12) gives

$$\tau^{(0)}(\mathbf{R})=\frac{1}{8\pi^2}\left[2p_F\epsilon_F^3-M^2\left[5-4\frac{m}{M}\right]p_F\epsilon_F+M^4\left[3-4\frac{m}{M}\right]\ln\left[\frac{p_F+\epsilon_F}{M}\right]\right]-mn_0^{(0)}(\mathbf{R}), \tag{A8a}$$

$$\begin{aligned}\tau^{(2)}(\mathbf{R})=\frac{1}{24\pi^2}\Bigg\{&\left[\left[3-(1+x_F^2)\frac{m}{M}\right]x_F-(3-x_F^2)\frac{m}{p_F}-\ln\left[\frac{x_F+1}{x_F-1}\right]\right](\nabla V)^2\\&+2m\left[x_F-\frac{p_F}{m}\left[2-(1-x_F^2)\frac{m}{M}\right]+\ln\left[\frac{x_F+1}{x_F-1}\right]\right]\Delta V\\&-\left[\frac{2m}{p_F}(2+x_F^2)+2x_F\frac{m}{M}(3-x_F^2)+6\frac{p_F}{M}(1-x_F^2)\right](\nabla M)\cdot(\nabla V)\\&-\left[\frac{m}{p_F}(2-x_F^2)-\left[3-2\frac{m}{M}\right]x_F+\frac{m}{M}x_F^3+\frac{1}{2}\ln\left[\frac{x_F+1}{x_F-1}\right]\right](\nabla M)^2\\&-2m\left[x_F+\frac{p_F}{M}(1-x_F^2)-\left[3-2\frac{M}{m}\right]\frac{1}{2}\ln\left[\frac{x_F+1}{x_F-1}\right]\right]\Delta M\Bigg\}.\end{aligned} \tag{A8b}$$

By use of the definition of the Fermi momentum $V(\mathbf{R})$ and its derivatives can be eliminated by means of $n_0(\mathbf{R})$ and M and their derivatives. For $M=m$ the relativistic atomic expansion is recovered [15], which reduces to the nonrelativistic limit for $p_F, V(\mathbf{R})\ll m$. In principle, one can also eliminate M (i.e., Σ_s) in favor of $n_s(\mathbf{R})$. However such a procedure is not applicable for a general relativistic mass operator, since its Dirac structure is more complicated as the general density structure [2,12].

APPENDIX B: RELATIVISTIC ETF APPROXIMATION

As a simple example, we give the derivation of the relativistic Thomas-Fermi theory defined as the first term of the $\hbar$ expansion of the relativistic Hartree-Fock approximation. For simplicity we restrict ourselves to the σ-ω model with $N=Z$. In the Hartree-Fock approximation one obtains for the self-energy in the Wigner representation the following expressions [12]:

$$\Sigma^{(\sigma)}(\mathbf{R},p)=-\frac{i}{(2\pi\hbar)^4}\int d^4R'\,d^4q\,g_\sigma^2\Delta^{(\sigma)}(R-R')\,\mathrm{Tr}G(\mathbf{R}',q)+\frac{i}{(2\pi\hbar)^4}\int d^4x\,d^4q\,\exp[ix(p-q)/\hbar]g_\sigma^2\Delta^{(\sigma)}(x)G(\mathbf{R},q) \quad \text{(B1a)}$$

$$=\int d^3R'\,v^{(\sigma)}(\mathbf{R}-\mathbf{R}')\,\mathrm{Tr}n(\mathbf{R}')+\frac{i}{(2\pi\hbar)^4}\int d^4q\,g_\sigma^2\frac{\hbar^2}{(p-q)^2-m_\sigma^2c_2+i\eta}G(\mathbf{R},q)\ , \quad \text{(B1b)}$$

$$\Sigma^{(\omega)}(\mathbf{R},p)=-\frac{i}{(2\pi\hbar)^4}\int d^4R'\,d^4q\,g_\omega^2\gamma^\mu\Delta_{\mu\nu}^{(\omega)}(R-R')\,\mathrm{Tr}[\gamma^\nu G(\mathbf{R}',q)]+\frac{i}{(2\pi\hbar)^4}\int d^4x\,d^4q\,\exp[ix(p-q)/\hbar]g_\omega^2\Delta_{\mu\nu}^{(\omega)}(x)\gamma^\mu G(\mathbf{R},q)\gamma^\nu \quad \text{(B2a)}$$

$$=\gamma^0\int d^3R'\,v^{(\omega)}(\mathbf{R}-\mathbf{R}')\,\mathrm{Tr}[\gamma^0 n(\mathbf{R}')]+\frac{i}{(2\pi\hbar)^4}\int d^4q(-g_\omega^2)\frac{\hbar^2}{(p-q)^2-m_\omega^2c^2+i\eta}\gamma^0G(\mathbf{R},q)\gamma^0\ . \quad \text{(B2b)}$$

The second parts give the exchange contribution. $\Delta^{(\sigma)}(x)$ denotes the σ-meson propagator:

$$\Delta^{(\sigma)}(x)=\frac{1}{(2\pi)^4}\int d^4k\frac{e^{-ikx}}{k^2-\kappa_\sigma^2+i\eta}\ . \quad \text{(B3)}$$

The potential $v^\sigma(\mathbf{R})$ is given by

$$v^{(\sigma)}(\mathbf{R})=-\frac{g_\sigma^2}{4\pi}\frac{e^{-\kappa_\sigma|\mathbf{R}|}}{|\mathbf{R}|}\ . \quad \text{(B4)}$$

A similar expression holds for the ω propagator and $v^{(\omega)}(\mathbf{R})$. In static approximation $(p_0-q_0)^2 \ll m_M^2c^2$ one can perform in the second terms of (B1) and (B2) the q^0 integration which leads according to (2.4b) to the replacement $(i/2\pi\hbar)\int dq_0\,G(\mathbf{R},q)\rightarrow -n(\mathbf{R},\mathbf{q})$. Therefore one needs for the evaluation of the self-energies only the densities and phase-space densities, respectively. In the zeroth orer of the $\hbar$ expansion one can neglect the Fock terms, since they are, according to (B1) and (B2), at least of order $\hbar^2$.

The TF approximation is now defined by expression (2.7) for G^{TF}:

$$G^{\mathrm{TF}}(\mathbf{R},p)=\hbar c\frac{ck^\mu\gamma_\mu+Mc^2}{c^2k^2-M^2c^2} \quad \text{(B5)}$$

and

$$\Sigma^{(\sigma),\mathrm{TF}}(\mathbf{R})=\int d^3R'\,v^{(\sigma)}(\mathbf{R}-\mathbf{R}')n_s^{(0)}(\mathbf{R}')\ , \quad \text{(B6)}$$

$$\Sigma^{(\omega),\mathrm{TF}}(\mathbf{R})=\gamma^0V(\mathbf{R})=\gamma^0\int d^3R'\,v^{(\omega)}(\mathbf{R}-\mathbf{R}')n_B^{(0)}(\mathbf{R}')\ . \quad \text{(B7)}$$

The Fermi momentum $p_F(\mathbf{R})$ is defined via the single-particle relation (2.10)

$$V(\mathbf{R})+\epsilon(\mathbf{R},p_F(\mathbf{R}))-\mu=0\ . \quad \text{(B8)}$$

The set of equations (B6)–(B8) combined with the expressions for $n_B^{(0)}(\mathbf{R})$ $(=p_F^3/3\pi^2)$ (3.6) and $n^{(0)}(\mathbf{R})$ (A6a) define the first order in the $\hbar$ expansion, i.e., the Thomas-Fermi expression.

As expected, one can recover this result also in the traditional manner by minimizing the energy density, given by (A8) ($\lambda=1$ for the self-consistent case; see text) with the subsidiary condition of conserved baryon number, with respect to the "density." However, in the relativistic case one has to deal with three types of density variations, i.e., $\delta n_0^{(0)}$, $\delta n_s^{(0)}$, and $\delta n_v^{(0)}$.

Minimization of

$$e^{(0)}(\mathbf{R})-\mu n_0^{(0)}(\mathbf{R})=n_v^{(0)}(\mathbf{R})+[\tfrac{1}{2}\Sigma_s^{(0)}(\mathbf{R})+mc^2]n_s^{(0)}(\mathbf{R})+\tfrac{1}{2}V(\mathbf{R})n_0^{(0)}(\mathbf{R})-\mu n_0^{(0)}(\mathbf{R}) \quad \text{(B9)}$$

leads to the desired result:

$$\delta\{e^{(0)}(\mathbf{R})-\mu_0 n^{(0)}(\mathbf{R})\}=\delta n^{(0)}(\mathbf{R})\{\epsilon(\mathbf{p}_F(\mathbf{R}))+V(\mathbf{R})-\mu\}\ . \quad \text{(B10)}$$

This can be demonstrated by rewriting of (B9) utilizing

$$n^{(0)}(\mathbf{R})=\frac{1}{2}\int_0^{p_F}\frac{d^3p}{(2\pi\hbar)^3}\left[\gamma^0-\frac{c}{\epsilon}\gamma\cdot\mathbf{p}+\frac{Mc^2}{\epsilon}\right]\ , \quad \text{(B11)}$$

$$n_v^{(0)}(\mathbf{R})+Mc^2n_s^{(0)}(\mathbf{R})=\frac{1}{2(2\pi\hbar)^3}\,\mathrm{Tr}\int_0^{p_F}d^3p\,\epsilon(\mathbf{p},\Sigma_s^{(0)})\ , \quad \text{(B12)}$$

and neglecting the surface terms—which give no contribution to the total energy—in calculating $\delta\Sigma_s^{(0)}$ and δV via the differential form of (B6) and (B7). Higher-order corrections with respect to the density of the H approximation are partly incorporated in Ref. [11], where one starts directly from the nucleon-meson Lagrangian for the energy expression.

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