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Model test of boson mappings

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Methods of boson mapping are tested in calculations for a simple model system of four protons and four neutrons in single-j distinguishable orbits. Two-body terms in the boson images of the fermion operators are considered. Effects of the seniority $v=4$ states are thus included. The treatment of unphysical states and the influence of boson space truncation are particularly studied. Both the Dyson boson mapping and the seniority boson mapping as dictated by the similarity transformed Dyson mapping do not seem to be simply amenable to truncation. This situation improves when the one-body form of the seniority image of the quadrupole operator is employed. Truncation of the boson space is addressed by using the effective operator theory with a notable improvement of results.

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I. INTRODUCTION

Studies of boson mappings are motivated by the occurrence of bosonlike features in many-fermion systems. Phenomenologically described by various models, such features are believed to be microscopically understandable by transforming the fermion problem into the boson one. An extensive review on boson mappings with a particular emphasis on the nuclear shell model problem has appeared recently [1].

The fermion system can be mapped onto the boson one in various ways. As far as the boson images reproduce the commutation relations of the bifermion operators and an appropriate relation between the fermion and boson vacua is defined, the mapping is exact and the fermion results are correctly obtained in the physical boson sector. The behavior of unphysical (spurious) states can differ in different mappings. It is clearly desirable to find mappings where spurious and physical states are well separated, with the lowest-lying states physical, rather than to have to deal with mappings where these states are mixed, although in the latter case one could still derive procedures to identify spurious states.

Another aspect which can demonstrate differences between boson mappings is linked to truncation of the boson space. Since the dimension of the ideal boson space is usually larger than that of the corresponding original fermion space, one usually has to resort to truncation of the space in practice. Some mappings then do not work well in such a restricted space and therefore their suitability for understanding fermion systems is limited.

One should judge the applicability of a boson mapping by comparing its results with a particular physical situation. In the nuclear structure case, such a comparison is, however, complicated by rather insufficient knowledge of the effective nuclear Hamiltonian. Therefore, exactly soluble simple models which could serve as tools for studying, testing, and comparing boson mappings [2,3] play an important role.

In a previous paper [3], we have investigated boson mappings for a simple fermion system of four protons and four neutrons in single-j distinguishable orbits. A monopole pairing interaction between like nucleons and a quadrupole-quadrupole interaction between unlike nucleons has been considered. This system is rather simply amenable to numerical treatment in both the original fermion form and the mapped boson version. However, the dimensions of matrices in the boson description can be relatively large.

It was shown in our study [3] that in a direct application of the generalized Dyson boson mapping (DBM), numerous spurious states appear and are spread among physical states. Furthermore, physical and unphysical states are mixed when the boson space is truncated. Direct application of the DBM thus seems to be of little value for practical purposes in the nuclear structure calculations. On the other hand, if the seniority boson mapping (SBM) in its lowest order is applied, calculations in the truncated *sd* space reproduce exact fermion results quite well and no problems with spurious states appear.

Nevertheless, some questions and extensions which have not been discussed in Ref. [3] remain. An extension of the SBM to treat matrix elements for seniority $v=4$ states is still required. An application of the SBM in a more complete boson space than the *sd* space of Ref. [3] should also be investigated. Furthermore a generalization of the model to include more complicated interaction between like nucleons could reveal some aspects of boson mappings more distinctly. Due to its quasispin SU(2) algebraic structure, the monopole pairing interaction used in Ref. [3] obscures or simplifies some features of the mapping when the boson space is truncated. Finally, a new approach to the truncation of the boson space seems desirable. Here an approach based on effective operator techniques is introduced.

The paper is organized as follows. In Sec. II, the exactly solvable model system to be studied is described. In Sec. III, the Dyson boson mapping is reviewed. Section IV is devoted to the seniority boson mapping and in Sec.

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V effective operator theory is applied to the seniority boson mapping. Conclusions are summarized in Sec. VI.

II. MODEL

The model system consists of four protons and four neutrons, each kind of nucleon occupying a single-j shell. The proton and neutron orbits are assumed to be distinguishable and the isospin degree of freedom is not taken into account. In some cases, we also present results for a system of four identical nucleons.

As an interaction between like nucleons, we consider the monopole pairing force (MPI) ($\rho=\pi$ for protons and $\rho=\nu$ for neutrons)

$$V_\rho^{\mathrm{mono}} = -G_\rho S_\rho^\dagger S_\rho \qquad (1)$$

or the surface delta interaction (SDI)

$$V_\rho^{\mathrm{SDI}} = -\tfrac{1}{2}4\pi G_\rho^{\mathrm{SDI}} \sum (1/\hat{\lambda})\langle j_\rho \| Y_\lambda \| j_\rho\rangle \langle j_\rho \| Y_\lambda \| j_\rho \rangle \times (A_{\rho,\lambda}^\dagger \tilde{A}_{\rho,\lambda})_0^{(0)} \; . \qquad (2)$$

The interaction between protons and neutrons is taken to be the quadrupole-quadrupole interaction

$$V_{\pi\nu} = -F_{\pi\nu} Q_\pi \cdot Q_\nu \; , \qquad (3)$$

with

$$F_{\pi\nu} = (4\pi/5) k_{\pi\nu} / \langle r_\pi^2 \rangle \langle r_\nu^2 \rangle \; . \qquad (4)$$

For definitions of the bifermion pair operators $A_{\rho,\lambda}$ and S_ρ and of the quadrupole operator Q_ρ, see Ref. [3].

III. DYSON BOSON MAPPING

The Dyson boson images of operators introduced in the preceding section are easily deduced by applying the relations given in Ref. [3].

First, we consider four nucleons interacting through the MPI in orbit j. Fermion states can be classified according to the seniority quantum number v. The lowest state is the $v=0, J=0$ state. Then, degenerate states with $v=2, J=0,2,\ldots,2j-1$ follow. The $v=4$ states lie at zero binding energy. In the Dyson boson mapping, physical boson states reproduce, of course, the fermion spectrum. Unphysical boson states have zero energy and are degenerate with the physical $v=4$ states. Truncation of the boson space is possible and the same spectrum is still obtained in the truncated space. Only degeneracies of levels decrease appropriately. The truncation works, however, due to the SU(2) symmetry of the monopole pairing interaction and the consequent validity of the skeletonization procedure discussed by Kim and Vincent [4].

For the SDI, the situation is different. Here, the states can still be classified according to the seniority quantum number. The states of the same seniority are, however, no longer degenerate. The exact fermion spectrum is shown in the first column of Fig. 1 for $j=\frac{9}{2}$. In the full DBM treatment, the physical spectrum is reproduced and the spurious states again lie at zero energy. In the truncated space, the DBM picture deteriorates as shown in the second column of Fig. 1. The physical and un-

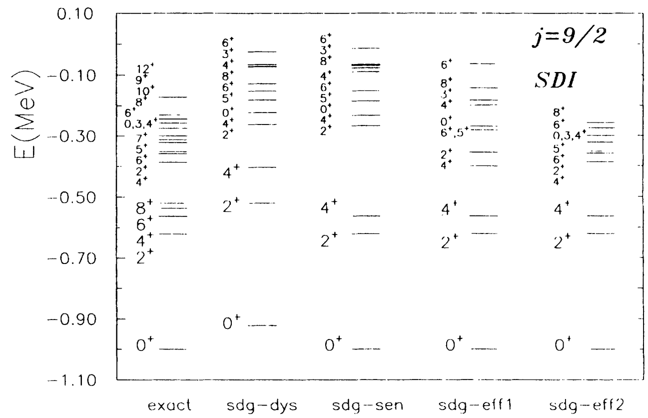

FIG. 1. Energy spectrum of the boson image of the SDI Hamiltonian for the case of four fermions occupying the $\frac{9}{2}$ level. The strength of the interaction is $G^{\mathrm{SDI}}=0.1$ MeV. In the first column, the exact results are shown. In columns 2–5, the results of calculation in the *sdg* space using, respectively, the DBM, the SBM, $H_{\mathrm{eff1}}^{\mathrm{SDI}}$, and $H_{\mathrm{eff2}}^{\mathrm{SDI}}$ are displayed. The spurious states lie at the energy 0 MeV and are not shown.

physical states are mixed by the truncation procedure. The energies of states originally corresponding to the physical states are shifted up. The states corresponding to the original spurious states remain at the zero energy.

The effect of truncation can also be seen clearly if the Majoranalike operator [5,6] (called the Park operator in the following) is taken into account. The Park operator is an image of the fermion operator

$$\hat{S}_F = n^2 - n - 2\sum \hat{\lambda} (A_\lambda^\dagger \tilde{A}_\lambda)_0^{(0)} \; , \qquad (5)$$

which is identically zero. Here n is the fermion number operator. The Dyson image $\hat{S}_D$ of $\hat{S}_F$ is easily deduced by using the formulas of Ref. [3]. When the operator $\Lambda\hat{S}_D$ is added to the Dyson Hamiltonian, the energies of unphysical states are shifted by 12Λ in our case of four nucleons.

The use of the Park operator becomes worthless when the boson space is truncated [7]. In the truncated space, even the lowest-lying states are mixtures of the unphysical and physical components. The Park operator shifts the whole spectrum so that the relation to the original fermion problem is completely destroyed. This is true in the case of both the MPI and the SDI.

Even more complications occur in the proton-neutron system. When the neutron-proton interaction is switched off, the spectrum is simply given by superimposing the proton and neutron spectra. The seniority classification is still relevant and the unphysical boson states have $E_\pi=0$ and/or $E_\nu=0$ and lie in the upper part of spectrum. For nonzero quadrupole-quadrupole strength $k_{\pi\nu}$, the seniority classification is no longer valid. It appears that with increasing strength $k_{\pi\nu}$, the energy of unphysical states decreases more steeply than the energy of the physical states. At some stage, an unphysical state becomes the lowest-lying one in the boson spectrum (see Figs. 1 and 2 of Ref. [3]). Moreover, truncation of the boson space disturbs the task considerably and the Park operator is again of no use. Both the presence of unphysical states in the lower part of spectrum and the impossibility to truncate make the DBM rather impractical for the treatment of problems with the multipole-multipole interactions between unlike nucleons.

IV. SENIORITY BOSON MAPPING

Reasons which complicate application of the Dyson mapping are made transparent by considering the DBM image of the pair creation operator $S^\dagger$ (see Ref. [3] again for details in notation), namely,

$$(S^\dagger)_D = s^\dagger[1-(2/\Omega)N+(1/\Omega)n_s] - (1/\Omega)\sum_{\lambda\neq 0}\hat{\lambda}(B^\dagger_\lambda B^\dagger_\lambda)^{(0)}s + \sqrt{2/\Omega}\sum_{\lambda_i\neq 0}\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3 \begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3 \\ j & j & j\end{Bmatrix} \times[(B^\dagger_{\lambda_1}B^\dagger_{\lambda_2})^{(\lambda_3)}\tilde{B}_{\lambda_3}]^{(0)} . \quad (6)$$

The fermion state with $v=0$ is obtained by an application of the operator $(S^\dagger)^N$ onto the fermion vacuum, N being half the number of nucleons. As follows from Eq. (6), the corresponding Dyson mapped state for $N>1$ contains a complicated mixture of components of bosons with all possible angular momentum quantum numbers. A similar situation occurs for $v=2$ states. On the other hand, it would seem natural to map the $v=0$ state onto a boson state composed of N bosons with $\lambda=0$ only and the $v=2,L$ states onto boson states containing $N-1$ bosons with $\lambda=0$ and one boson with angular momentum $\lambda=L$. Such a natural connection holds only for the bra vectors in the DBM. The bra $v=0$ state can be constructed by the DBM image of the pair annihilation operator S

$$(S)_D=\sqrt{\Omega}s . \quad (7)$$

One, therefore, attempts to construct a mapping in which both bra and ket images of the $v=0$ and 2 states would be simple. Such a mapping is called a seniority boson mapping (SBM). Unitary realizations of the seniority mapping have extensively been studied by Klein and collaborators [8]. In the lowest order, that approach gives results identical to the well-known Otsuka-Arima-Iachello (OAI) [9] method of bosonization.

Starting from the nonunitary Dyson picture, the seniority image of the pair annihilation operator S would be given as in Eq. (7). An image of the pair creation operators would be

$$(S^\dagger)_{\text{sen}}=\sqrt{\Omega}s^\dagger[1+(1/\Omega)s^\dagger s-(2/\Omega)N] . \quad (8)$$

Of course, images of the bifermion operators in the SBM should be in accordance with the respective bifermion commutation relations. Equation (8) is constructed in a proper way.

Generally, the above task of finding a (nonunitary) seniority mapping is not defined uniquely. Leading order terms in the seniority images of bifermion operators have been given in Ref. [3]. These follow also from an exact and full realization of the seniority mapping given by Geyer [10]. This realization is discussed in some detail in the following.

In Geyer's approach, referred to also as the seniority mapping from the similarity transformed Dyson mapping, the SBM operators Θ_{sen} and the DBM operators Θ_D are related by a similarity transformation

$$\Theta_{\text{sen}}=Z\Theta_D Z^{-1} . \quad (9)$$

For the transformation matrix, a series expansion has been given [10]

$$Z^{-1}=\sum_{k=0}^{\infty}\left[\frac{1}{\hat{H}_0-H_0}W\right]^k\wedge , \quad (10)$$

in which H_0 and W are diagonal and nondiagonal parts, respectively, of the Dyson image of the MPI, and the symbol $\wedge$ denotes the positional operator [11] determining at which position the operator $\hat{H}_0$ is evaluated. To the extent that we work with finite systems, it is sufficient to consider only a finite number of terms in expansion (10). For our model case of four identical nucleons, the $k=0$, 1, and 2 terms should be taken into account in the expansion of Z^{-1}. The operator Z is found from the relation $ZZ^{-1}=1$.

The seniority image of the MPI is, of course, the diagonal matrix H_0 (this is the starting point in Geyer's method). For the SDI Hamiltonian, the seniority image is obtained from the DBM image H_D^{SDI} by

$$H^{\text{SDI}}_{\text{sen}}=H^{\text{SDI}}_D+H^{\text{SDI}}_D\frac{1}{\hat{H}_0-H_0}W\wedge-\frac{1}{\hat{H}_0-H_0}W\wedge H^{\text{SDI}}_D-\frac{1}{\hat{H}_0-H_0}W\wedge H^{\text{SDI}}_D\frac{1}{\hat{H}_0-H_0}W\wedge , \quad (11)$$

which gives the final expression

$$H_{\rm sen}^{\rm SDI}=H_0-2\pi G^{\rm SDI}\sum_{\lambda\neq 0}(1/\hat{\lambda}^2)\langle j\|Y_\lambda\|j\rangle\langle j\|Y_\lambda\|j\rangle$$
$$\times\Bigg[B_\lambda^\dagger\cdot\tilde{B}_\lambda-\frac{2}{\Omega}n_sB_\lambda^\dagger\cdot\tilde{B}_\lambda-\frac{4}{\Omega-2}\hat{\lambda}^2\sum_{\lambda_1\neq 0}\left[\frac{1}{\Omega}(\delta_{\lambda\lambda_1}/\hat{\lambda}_1^2-1)+\begin{Bmatrix} j & j & \lambda_1\\ j & j & \lambda\end{Bmatrix}\right]n_sB_{\lambda_1}^\dagger\cdot\tilde{B}_{\lambda_1}$$
$$+\hat{\lambda}^2\frac{1}{\Omega(\Omega-1)}n_s(n_s-1)-(1/\Omega)s^\dagger s^\dagger\tilde{B}_\lambda\cdot\tilde{B}_\lambda$$
$$+4\left[\frac{1}{2\Omega}\right]^{1/2}\sum_{\lambda_i\neq 0}\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda\\ j & j & j\end{Bmatrix}s^\dagger[B_{\lambda_1}^\dagger(\tilde{B}_{\lambda_2}\tilde{B}_\lambda)^{(\lambda_1)}]_0^{(0)}-\frac{1}{\Omega(\Omega-1)}\sum_{\lambda_1\neq 0}B_{\lambda_1}^\dagger\cdot B_{\lambda_1}^\dagger\tilde{B}_\lambda\cdot\tilde{B}_\lambda$$
$$-\frac{4}{\Omega-2}\sum_{K,\lambda_i\neq 0}\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3\hat{\lambda}\hat{K}\begin{Bmatrix}\lambda_1 & \lambda_2 & K\\ j & j & j\end{Bmatrix}\begin{Bmatrix}\lambda_3 & \lambda & K\\ j & j & j\end{Bmatrix}[(B_{\lambda_1}^\dagger B_{\lambda_2}^\dagger)^{(K)}(\tilde{B}_{\lambda_3}\tilde{B}_\lambda)^{(K)}]_0^{(0)}$$
$$-2\sum_{\lambda_i\neq 0}\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3\hat{\lambda}\hat{K}\begin{Bmatrix} j & \lambda_3 & j\\ \lambda_1 & K & \lambda_2\\ j & \lambda & j\end{Bmatrix}[(B_{\lambda_1}^\dagger B_{\lambda_2}^\dagger)^{(K)}(\tilde{B}_{\lambda_3}\tilde{B}_\lambda)^{(K)}]_0^{(0)}\Bigg]\ . \tag{12}$$

For the quadrupole operator, using an expression analogous to Eq. (11) we get the SBM image as

$$Q_{\rm sen}=\sqrt{1/5}\langle j\|r^2Y_2\|j\rangle\Bigg[\sqrt{2/\Omega}\left[s^\dagger\tilde{d}+\left[1-\frac{n_s}{\Omega-1}\right]d^\dagger s\right]-2\left[1-\frac{2n_s}{\Omega-2}\right]\sum_{\lambda_1\lambda_2\neq 0}\hat{\lambda}_1\hat{\lambda}_2\begin{Bmatrix}\lambda_1 & \lambda_2 & 2\\ j & j & j\end{Bmatrix}(B_{\lambda_1}^\dagger\tilde{B}_\lambda)_2^{(2)}$$
$$-\frac{1}{\Omega-2}\sum_{\lambda_i\neq 0}\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ j & j & j\end{Bmatrix}[(B_{\lambda_1}^\dagger B_{\lambda_2}^\dagger)^{(\lambda_3)}\tilde{B}_{\lambda_3}]_0^{(0)}\tilde{d}$$
$$+\sqrt{2/\Omega}\sum_{\lambda\neq 0}\hat{\lambda}\left[\frac{1}{\Omega-1}-\frac{1}{\Omega-2}\left[1-2\delta_{\lambda 2}/5-2\Omega\begin{Bmatrix} j & j & 2\\ j & j & \lambda\end{Bmatrix}\right]\right](B_\lambda^\dagger B_\lambda^\dagger)_0^{(0)}\tilde{d}s$$
$$-\frac{2\sqrt{2\Omega}}{\Omega-2}\sum_{\lambda_i\lambda\neq 0}\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3\hat{\lambda}\left[\begin{Bmatrix} j & \lambda_2 & j\\ 2 & \lambda & \lambda_3\\ j & \lambda_1 & j\end{Bmatrix}-\delta_{\lambda_1 2}\delta_{\lambda_2\lambda_3}\frac{(-1)^\lambda}{5\Omega\hat{\lambda}_2}+\frac{1+(-1)^\lambda}{\Omega-2}\begin{Bmatrix}\lambda_1 & \lambda_2 & \lambda\\ j & j & j\end{Bmatrix}\begin{Bmatrix}\lambda_3 & \lambda & 2\\ j & j & j\end{Bmatrix}\right]$$
$$\times[(B_{\lambda_1}^\dagger B_{\lambda_2}^\dagger)^{(\lambda)}\tilde{B}_{\lambda_3}]^{(2)}s\Bigg]\ . \tag{13}$$

One immediately sees that, unlike the DBM in which states are complicated and operators simple, we have in the SBM simple images of states but complicated images of operators.

Now, the SBM calculations are performed. When the SBM Hamiltonian is diagonalized in the full boson space, the results obtained are identical to those of the DBM presented in Ref. [3]. Of course, this is to be expected as the Dyson Hamiltonian and the seniority Hamiltonian are connected by a similarity transformation and therefore have identical spectra. The SBM thus does not remove problems connected with the presence of the unphysical states in the low-lying part of spectrum when the quadrupole-quadrupole interaction is switched on.

Differences with results obtained by the DBM and the SBM appear when the boson space is truncated. In the third column of Fig. 1, the SBM results in the truncated *sdg* space are shown. The 0_1, 2_1, and 4_1 states are given at exact positions in the truncated SBM calculations since the SBM has been constructed in such a way that the boson images of these states are not affected by the truncation of the boson space. The above result was thus to be expected. Of course, the 6_1 and 8_1 states are outside the *sdg*-boson space and are not present in the boson spectrum. An agreement for the group of the $v=4$ states is of the same quality as that in the DBM.

Unlike the DBM case, the seniority form of the Park operator seems to be applicable to the lowest seniority states in the truncated space as motivated below. Using a procedure analogous to Eq. (11), we obtain the SBM image of the operator (5) as

$$\hat{S}_{\rm sen}=4N(N-1)-4n_s(n_s-1)-8n_s(N-n_s)+2/\Omega\sum_{\lambda\neq0}s^\dagger s^\dagger\tilde{B}_\lambda\cdot\tilde{B}_\lambda$$
$$-8\left[\frac{1}{2\Omega}\right]^{1/2}\sum_{K,\lambda_i\neq0}\hat{\lambda}_1\hat{\lambda}_2\hat{K}\begin{Bmatrix}\lambda_1&\lambda_2&K\\ j&j&j\end{Bmatrix}s^\dagger[B_K^\dagger(\tilde{B}_{\lambda_1}\tilde{B}_{\lambda_2})^{(K)}]_0^{(0)}+\frac{2}{\Omega(\Omega-1)}\sum_{\lambda_i\neq0}B^\dagger_{\lambda_1}\cdot B^\dagger_{\lambda_1}\tilde{B}_{\lambda_2}\cdot\tilde{B}_{\lambda_2}$$
$$+\frac{8}{\Omega-2}\sum_{K,\lambda_i\neq0}\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3\hat{\lambda}_4\hat{K}\begin{Bmatrix}\lambda_1&\lambda_2&K\\ j&j&j\end{Bmatrix}\begin{Bmatrix}\lambda_3&\lambda_4&K\\ j&j&j\end{Bmatrix}[(B^\dagger_{\lambda_1}B^\dagger_{\lambda_2})^{(K)}(\tilde{B}_{\lambda_3}\tilde{B}_{\lambda_4})^{(K)}]_0^{(0)}$$
$$+4\sum_{\lambda_i\neq0}\hat{\lambda}_1\hat{\lambda}_2\hat{\lambda}_3\hat{\lambda}_4\hat{K}\begin{Bmatrix}j&\lambda_3&j\\ \lambda_1&K&\lambda_2\\ j&\lambda_4&j\end{Bmatrix}[(B^\dagger_{\lambda_1}B^\dagger_{\lambda_2})^{(K)}(\tilde{B}_{\lambda_3}\tilde{B}_{\lambda_4})^{(K)}]_0^{(0)}\ . \qquad (14)$$

Even in the truncated space, the SBM images of the $v=0,2$ states are not affected by the action of the Park operator $\hat{S}_{\rm sen}$. This is again in agreement with our construction of the SBM images of these states. On the other hand, for the states with two bosons with nonzero angular momenta λ_1 and λ_2, we find

$$(\lambda_1\lambda_2|\hat{S}_{\rm sen}|\lambda_1\lambda_2)\neq0\ . \qquad (15)$$

The physical SBM image $|v=4)$ of the fermion $v=4$ state is given as a combination of components containing two-boson states $|\lambda_1\lambda_2)$ in such a way that

$$(v=4|\hat{S}_{\rm sen}|v=4)=0\ . \qquad (16)$$

Here, the SBM differs from the OAI method [9]. In the OAI, the image of the $v=4$ state is defined as a state with two $\lambda=2$ bosons. The OAI operators are then constructed to be consistent with this definition. In the SBM, the operators are first obtained and afterwards the physical boson states in a straightforward way in principle, at least. In this respect, the seniority mapping seems to be more natural than the OAI method which may be difficult to extend when an enlarged boson space with $\lambda=4$ bosons is considered.

In Fig. 2, the SBM calculations for the proton-neutron system in the truncated *sdg* space are shown. The dependence of the lowest-energy eigenstates on the quadrupole-quadrupole strength $k_{\pi\nu}$ is displayed. Remember that in this case the seniority is not conserved. With increasing strength $k_{\pi\nu}$, the boson states of the prevalently unphysical character become the lowest-lying states in spectra for $k_{\pi\nu}>0.8$ MeV. The mixture of the $|\lambda_1\neq0,\lambda_2\neq0)$ states in the low-lying physical states increases and consequently the SBM description deteriorates in the truncated space. Nevertheless, the states which correspond to the exact physical states can still be traced among the boson eigenstates. The SBM Park operator is also of some use in the truncated space. It shifts up the $|\lambda_1\neq0,\lambda_2\neq0)$ components. Then the ground state eigenvalue, for example, agrees quite well with the exact one due to the small admixtures of those components. On the whole, however, difficulties discussed in the DBM case are not cured satisfactorily in the SBM.

Note that in our previous paper [3], calculations had also been performed with the Hermitian lowest-order form of the seniority quadrupole operator. This form, denoted $Q_{\rm ob}$, is obtained from Eq. (13) by inserting for n_s the value $N-1$, discarding two-body terms, and taking

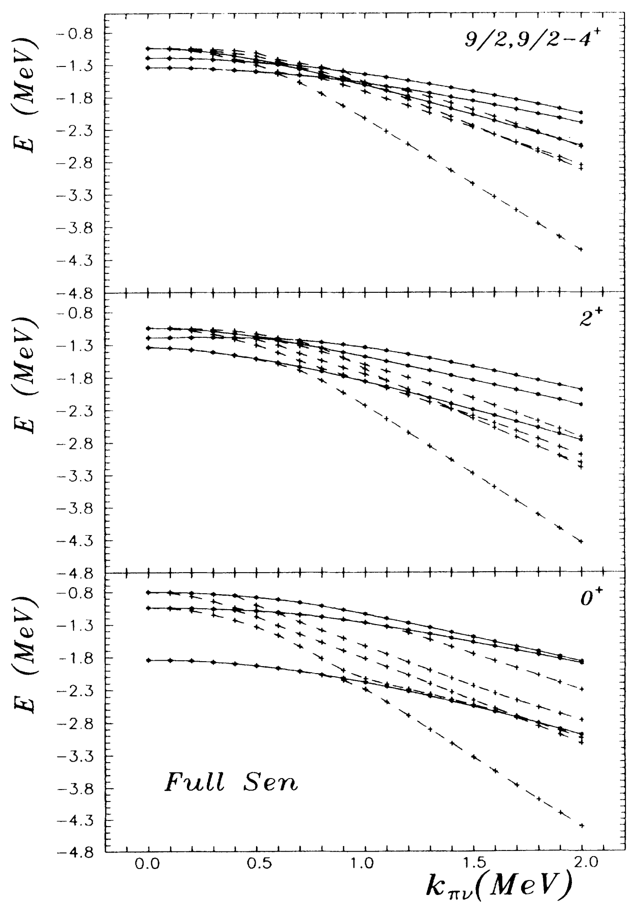

FIG. 2. The dependence of the lowest states 0^+, 2^+, and 4^+ on the strength of the proton-neutron quadrupole-quadrupole interaction for the system of four protons and four neutrons on the levels $j_\pi=\frac{9}{2}^+$, $j_\nu=\frac{9}{2}^-$. The MPI with the strengths $G_\pi=0.13$ MeV and $G_\nu=0.10$ MeV is considered among identical nucleons. The full lines connect the exact fermion results. The dashed lines are obtained by using the full SBM Hamiltonian in the *sdg* space.

the Hermitian conjugate according to the well-known prescription [12]. For the *sd* space, the resulting image is identical with the OAI expression [9]. In the full boson space, the above procedure leads to

$$Q_{\rm ob}=\sqrt{1/5}\langle j\|r^2Y_2\|j\rangle\left[\sqrt{2/\Omega}\left[1-\frac{N-1}{\Omega-1}\right]^{1/2}(s^\dagger\tilde{d}+d^\dagger s)-2\left[1-\frac{2N-2}{\Omega-2}\right]\sum_{\lambda_1\lambda_2\neq 0}\hat{\lambda}_1\hat{\lambda}_2\begin{Bmatrix}\lambda_1 & \lambda_2 & 2\\ j & j & j\end{Bmatrix}(B^\dagger_{\lambda_1}\tilde{B}_{\lambda_2})^{(2)}\right]. \tag{17}$$

Expression (17) represents the simplest form of the seniority image of the quadrupole operator which exactly reproduces the fermion matrix elements for $v=0$ and 2 states.

It has been found (see Fig. 3 of Ref. [3]) that using $Q_{\rm ob}$ in the seniority image of the Hamiltonian, one gets results which reasonably reproduce the exact fermion results even in the truncated *sd* space. Moreover, the spurious states do not appear in the lower part of the spectrum. In the present study the same conclusions are also reached in the full boson space and in the truncated *sdg* space. Results in the *sdg* space are shown in Fig. 3 for the MPI. One immediately sees a principle difference with the calculations of Fig. 2. Unphysical boson states among the lowest states do not move with the increasing strength $k_{\pi\nu}$. In the restricted boson space and also in the full boson space, states have a mixed structure of the physical and unphysical components (the mapping with $Q_{\rm ob}$ is not exact). Nevertheless, those states which approximate the fermion states lie below states with a dominant unphysical component.

The lowest-lying 0_1 and 2_1 states are well approximated in boson calculations in the whole range of displayed values of $k_{\pi\nu}$. For $k_{\pi\nu}<0.9$, the state 4_1 is also described reasonably (here the dashed line cannot be distinguished from the full one in the figure scale). On the other hand, for $k_{\pi\nu}>0.9$, the lowest 4_1 state is that with a dominant $v=4$ component. It is not described well in the boson calculations. Also a description of excited states of a given spin deteriorates for higher values of $k_{\pi\nu}$. Nevertheless, an identification of the exact and boson eigenstates can be made essentially by following the order of levels. The correctness of the above identification can be confirmed from calculated matrix elements of the quadrupole operator (see our present Fig. 4 and also Table II of Ref. [3]). Finally, we note that the quality of results for

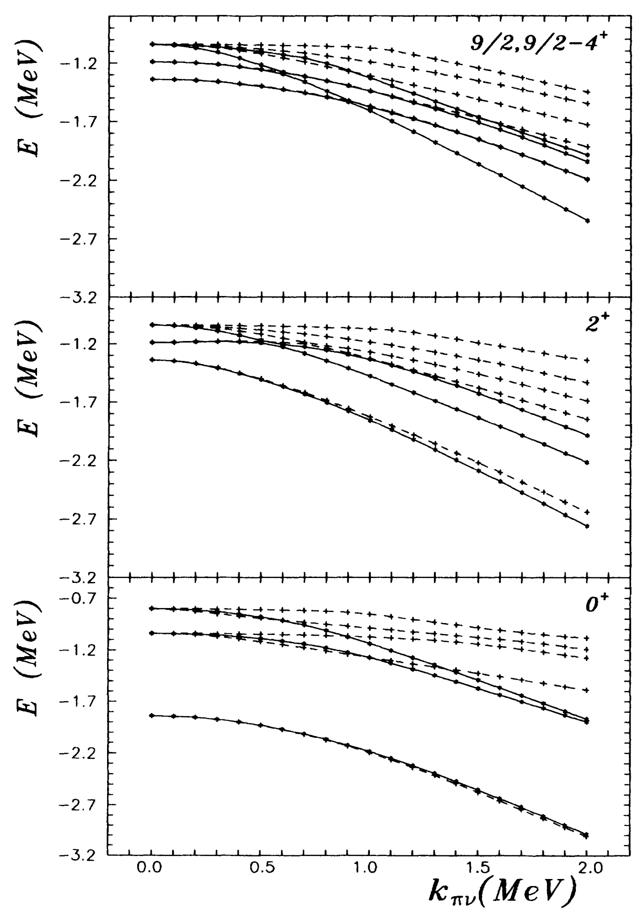

FIG. 3. The same as Fig. 2, but the dashed lines are the boson calculation obtained by using the SBM Hamiltonian with the one-body form of the seniority quadrupole operator (17) in the *sdg* space.

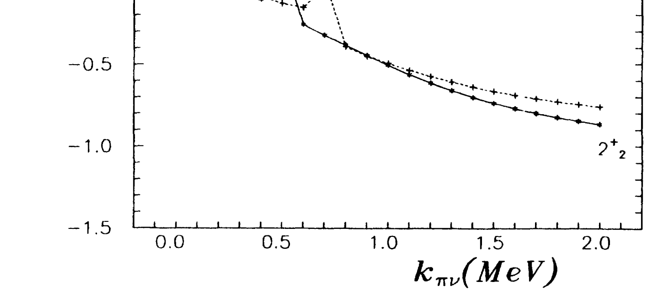

FIG. 4. The dependence of the diagonal matrix elements of the proton quadrupole operator for the states 2_i^+ $(i=1,2,3)$ on the strength of the quadrupole-quadrupole interaction for calculation of Fig. 3. The full lines connect the exact fermion results. The dashed lines are obtained with the one-body form of the seniority quadrupole operator (17). The factor $\sqrt{5/4\pi}\langle r^2\rangle$ is not included in the displayed dimensionless quantity $\langle Q\rangle$.

the 0_1 and 2_1 states is very weakly sensitive to the truncation of the boson space to either the *sdg* space or the *sd* space. Of course, for the description of the 4_1 state, the *g* boson is of vital importance.

In Fig. 5, results analogous to Fig. 3 are shown for the SDI. Qualitative conclusions are the same as those discussed above. Perhaps, differences between the exact and boson results cases are more pronounced in the SDI than in the MPI calculations. This can be understood rather simply because even for $k_{\pi\nu}=0$, the energies of the $v=4$ states are not exactly reproduced in the truncated space for the SDI case.

Another possible form of the boson quadrupole operator is obtained by considering the Hermitian one-body form of Eq. (13) with the n_s dependence retained. This prescription agrees in the *sd* space with the Otsuka-Arima-Iachello-Talmi (OAIT) expression [13]. The results of calculations with this form are close to those with the full SBM image.

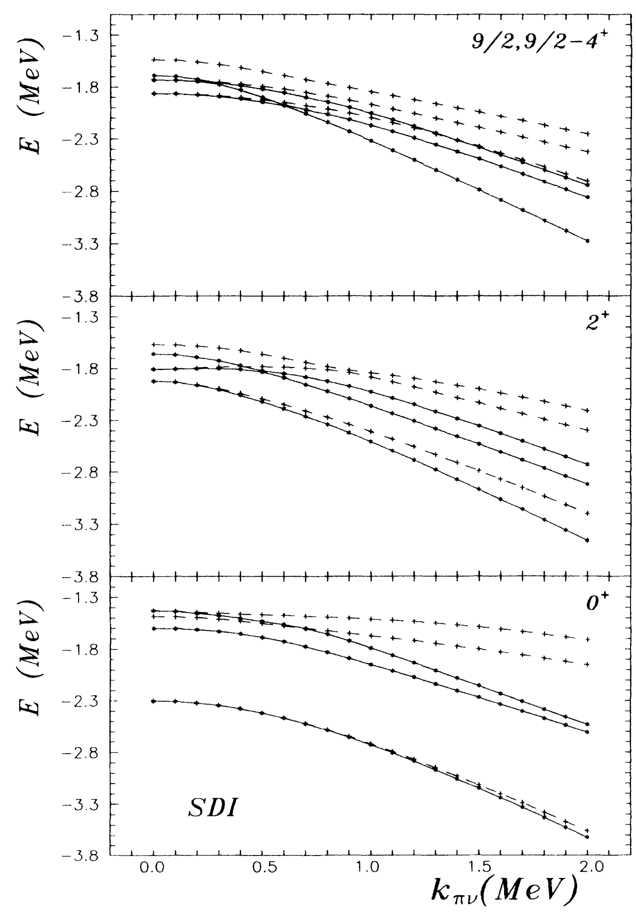

FIG. 5. The dependence of the lowest states 0^+, 2^+, and 4^+ on the strength of the proton-neutron quadrupole-quadrupole interaction for the system of four protons and four neutrons on the levels $j_\pi=\frac{9}{2}^+$, $j_\nu=\frac{9}{2}^-$. The SDI with the strengths $G_\pi^{\rm SDI}=0.13$ MeV and $G_\nu^{\rm SDI}=0.10$ MeV is considered among identical nucleons. The full lines connect the exact fermion results. The dashed lines are the boson calculation obtained by using the SBM Hamiltonian with the one-body form of the seniority quadrupole operator (17) in the *sdg* space.

We thus see that when the lowest-order expression $Q_{\rm ob}$ (OAI prescription) is employed in the mapped Hamiltonian, the results are in better agreement with the exact calculations than those obtained with the full SBM image of Q or with the more general OAIT formula.

Let us discuss this point in more detail. Obviously, the strong decrease of the energy of unphysical states with the increasing $k_{\pi\nu}$ strength is connected with the large Q matrix elements between unphysical states. A boson mapping is exact as far as it reproduces fermion matrix elements for physical states. The matrix elements between unphysical states can, however, differ for different but still exact boson mappings. The one-body form $Q_{\rm ob}$ of the quadrupole operator discussed above exactly reproduces the $v=0$ and 2 matrix elements. The required reproduction of the $v=4$ elements is not exactly satisfied in this approach, but, as previous results suggest (see Fig. 3), the matrix elements for the physical $v=4$ states are given reasonably well with the approximate $Q_{\rm ob}$ form. On the other hand, the matrix elements of $Q_{\rm ob}$ between unphysical states are suppressed in comparison with the full SBM and OAIT quadrupole operator (this can be seen from the n_s dependence in the OAIT expression). As a result, the unphysical states do not appear in the lower part of spectrum when $k_{\pi\nu}$ is increased.

Remember, however, that the present calculational procedure with $Q_{\rm ob}$ goes somewhat beyond the OAI approach. Namely, the two-body terms in the image of $H^{\rm SDI}$, which take into account the presence of the $v=4$ states, are not obtained straightforwardly within the OAI.

Another comment should be made in connection with the use of the SBM image of $H^{\rm SDI}$ (12). Terms of the type $s^\dagger B^\dagger BB$ and $s^\dagger s^\dagger BB$ which have no Hermitian counterparts appear in Eq. (12) and also in the image $\hat{S}_{\rm sen}$ [Eq. (14)] of the Park operator. As long as we require the $v=0$ state to be built by s bosons only and $v=2$ states to be built by s bosons and one $\lambda\neq0$ boson then these terms connect physical $v=0,2$ bra states with some unphysical ket states. This follows from the fact that $H^{\rm SDI}$ as well as the Park operator is diagonal in seniority in the fermion space. These terms therefore connect unphysical ket states with the physical bra states and have no effect on the spectrum in the full space calculations. Also in the truncated space, their effect is very small when the QQ interaction is switched on and null otherwise. Discarding these terms, we slightly change the mapping from Geyer's similarity transformed form, but the mapping is still exact. In fact, we do not consider these terms in obtaining the present results.

V. EFFECTIVE OPERATOR APPROACH

In the preceding section we have found that the SBM gives a correct picture of the $v=0$ and 2 states in the truncated space. On the other hand, the $v=4$ states are influenced by the truncation procedure and are not satisfactorily described in the *sdg*-boson space. To remedy this deficiency and to take into account the effect of states outside the truncated model space in general, one usually resorts to effective operator theory [14]. In the nuclear physics field, effective operator theory is frequently used to find the effective shell model interaction [15]. Its formalism is, however, quite general and can be directly applied to the present case. Actually, effective operator techniques have been used in connection with the microscopic derivation of the interacting boson model [16].

The full space is divided into the model truncated space and its complement in the full space. Introducing the projection operators $\mathcal{P}$ and $\mathcal{Q}$ onto the model space and onto the complement space, respectively, one obtains an equation for the projected part of the wave function

$$\mathcal{P}(H_0+V_{\text{eff}})\mathcal{P}|\psi\rangle=E\mathcal{P}|\psi\rangle \ , \qquad (18)$$

where

$$V_{\text{eff}}=V+V\mathcal{Q}\frac{1}{E-H_0-\mathcal{Q}V\mathcal{Q}}\mathcal{Q}V \qquad (19)$$

is the energy-dependent effective interaction acting in the model space. In the above, H_0 denotes the part of the Hamiltonian H diagonal in the $\mathcal{P}+\mathcal{Q}$ decomposition and $V=H-H_0$.

We apply the theory described above to the SBM image of the SDI Hamiltonian (12). The model space is the space of states composed of s, d, and g bosons. States in the complement space contain at least one boson with $\lambda>4$. In the present model study, the whole procedure of the $\mathcal{P}+\mathcal{Q}$ decomposition and of finding effective operators is applied separately in the space of states with two bosons of nonzero λ, the physical section of which contains the $v=4$ states. This follows since the SDI Hamiltonian does not connect states of different seniority and the boson operator (12) does not change the number of s bosons (discarding the $s^\dagger B^\dagger BB$ and $s^\dagger s^\dagger BB$ terms as discussed in the preceding section). On the other hand, it makes no sense to improve the treatment of the $v=2$, $L>4$ states in the effective operator approach as these states lie outside the model space completely.

Two points need to be solved in the procedure of constructing the effective operator V_{eff}. First, the inverse of the operator $(E-H_0-\mathcal{Q}V\mathcal{Q})$ has to be found. One usually treats this problem perturbatively by expanding

$$\frac{1}{E-H_0-\mathcal{Q}V\mathcal{Q}}=\frac{1}{E-H_0}\sum_{n=0}^{\infty}\left[\mathcal{Q}V\mathcal{Q}\frac{1}{E-H_0}\right]^n \qquad (20)$$

and considering the first few terms in the above expansion. One frequently retains only the first $n=0$ term. We do account for the higher-order terms in (20) and truncate the expansion when convergence is reached ($n=9$ for the present SDI Hamiltonian is needed). In our model calculations, we can even afford the straightforward and actually simpler approach to invert the matrix $(E-H_0-\mathcal{Q}V\mathcal{Q})$ directly in the $\mathcal{Q}$ space.

The second problem is related to the presence of the energy E in the denominator of Eq. (19). Equation (18), is, in fact, a nonlinear eigenvalue problem. When E in the denominator is fixed at $E>E_{\text{exact}}$ of a particular state, the calculated energy from Eq. (18) is below the exact one, and vice versa. In principle, one can proceed by an iterative procedure to determine the eigenenergy for each state separately. We employ, however, the fact that the eigenenergies of the $v=4$ states are grouped close together in our case and take one fixed energy E in the denominator of Eq. (19) for all the $v=4$ states. The energy E is chosen between the energies of the 2_2^+ and 4_2^+ states. Then the 4_2^+ state lies slightly below the exact one and the other $v=4$ states lie above the exact ones after diagonalization. We denote the effective Hamiltonian obtained as outlined above with $H^{\text{SDI}}_{\text{effl}}$.

Results of the calculations for the four-nucleon system (denoted as *sdg*-effl) are shown in the fourth column of Fig. 1. A notable improvement is seen in comparison with the SBM results of the third column. It is also interesting that the states corresponding to the spurious ones in the exact calculation (not shown in the figure) remain at zero energy in calculations with the effective operator.

As presented above, it is possible to construct an effective Park operator $\hat{S}_{\text{eff}}$. Starting from $\hat{S}_{\text{sen}}$ (14), the $\hat{S}_{\text{eff}}$ operator is obtained as follows:

$$\hat{S}_{\text{eff}}=\hat{S}_{\text{sen}}+\hat{S}_1\mathcal{Q}\frac{1}{-\hat{S}_0-\mathcal{Q}\hat{S}_1\mathcal{Q}}\mathcal{Q}\hat{S}_1 \ , \qquad (21)$$

where $\hat{S}_0$ is the part of $\hat{S}_{\text{sen}}$ diagonal in the $\mathcal{P}+\mathcal{Q}$ decomposition and $\hat{S}_1=\hat{S}_{\text{sen}}-\hat{S}_0$. In the case of the operator (21), we have found a slow convergence of the perturbative expansion and a direct inversion of the operator in the denominator has to be used. Note, that the energy-like factor E in the denominator of Eq. (21) is set to zero and $\hat{S}_{\text{eff}}$ does not depend on it. The physical states are eigenstates of $\hat{S}_{\text{sen}}$ with the eigenvalue 0 and expression (21) is thus an exact form for them. In the case of unphysical states, the exact eigenvalues of $\hat{S}_{\text{sen}}$ are greater than zero. Correspondingly, the operator $\hat{S}_{\text{eff}}$ with the choice $E=0$ must have eigenvalues greater than zero when diagonalized in the $\mathcal{P}$ projected unphysical boson space. The operator $\hat{S}_{\text{eff}}$ thus identifies and separates the physical and unphysical states in the model truncated space. Adding $\hat{S}_{\text{eff}}$ to the $H^{\text{SDI}}_{\text{effl}}$, we observe that the unphysical states are shifted up and the states corresponding to the physical ones remain at the same position.

Of course, expression (21) is only meaningful when $1/\mathcal{Q}\hat{S}_{\text{sen}}\mathcal{Q}$ is a nonsingular operator. One can then proceed further. The nonsingularity of the above operator is equivalent to the property that there is no physical state which lies completely in the complement $\mathcal{Q}$ space. Consequently, the number of states in the model truncated $\mathcal{P}$ space must be greater than or equal to the number of physical states. One can then relate a state $\mathcal{P}|\psi_{\text{phys}}\rangle$ in the model space to each physical state $|\psi_{\text{phys}}\rangle$ such that two states are connected by

$$|\psi_{\text{phys}}\rangle=\left[1+\mathcal{Q}\frac{1}{-\hat{S}_0-\mathcal{Q}\hat{S}_1\mathcal{Q}}\mathcal{Q}\hat{S}_1\right]\mathcal{P}|\psi_{\text{phys}}\rangle \ . \qquad (22)$$

The states in the set $\mathcal{P}|\psi_{\text{phys}}\rangle$ are certainly linearly independent.

The SBM image Θ_{sen} of any fermion operator Θ_F does not scatter outside the physical space when it acts onto the physical ket state. One can then diagonalize Θ_{sen} separately in the physical space to obtain eigenvectors

$$\Theta_{\rm sen}|\psi_{{\rm phys},K}\rangle = K|\psi_{{\rm phys},K}\rangle \ . \tag{23}$$

Employing (23), it is easy to show that for the operator

$$\Theta_{\rm eff2} = \mathcal{P}\Theta_{\rm sen} + \mathcal{P}\Theta_{\rm sen}\mathcal{Q}\frac{1}{-\hat{S}_0 - \mathcal{Q}\hat{S}_1\mathcal{Q}}\mathcal{Q}\hat{S}_1 \tag{24}$$

implies that

$$\Theta_{\rm eff2}\mathcal{P}|\psi_{{\rm phys},K}\rangle = K\mathcal{P}|\psi_{{\rm phys},K}\rangle \ . \tag{25}$$

The operator $\Theta_{\rm eff2}$ has the same spectral properties in the space of states $\mathcal{P}|\psi_{\rm phys}\rangle$ as the original operator $\Theta_{\rm sen}$ in the whole physical space. One can use the operator $\Theta_{\rm eff2}$ as an effective operator in the truncated model space. Outside the physical space, the operator $\Theta_{\rm eff2}$ need not be a reasonable effective operator for $\Theta_{\rm sen}$. This is, however, irrelevant for our considerations. Note that the operator $\Theta_{\rm eff2}$ is energy independent.

Transforming in this way the SBM image of the SDI Hamiltonian, the operator $H^{\rm SDI}_{\rm eff2}$ is obtained with a spectrum shown in the last column of Fig. 1. All the $v=4$ eigenenergies are exactly reproduced in the truncated model *sdg* space for those spins for which the number of states in the model space is greater than or equal to the number of physical $v=4$ states. This is fulfilled for all spins in the present calculation except those of which no state can be constructed at all in the *sdg* space, e.g., 7^+, 9^+, 10^+, 12^+. The spurious states lie at the zero energy and can be shifted up by adding the effective Park operator (21).

Recall that the exact reproduction of the eigenenergies of $H^{\rm SDI}$ is connected to the fact that the SDI interaction conserves seniority and that the procedure of getting the effective Hamiltonian can be applied separately in the $v=4$ space. Using the procedure of Eq. (24) for the quadrupole operator $Q_{\rm sen}$ (13), we find the effective quadrupole operator $Q_{\rm eff2}$ which exactly reproduces the original fermion matrix elements in the truncated *sdg*-boson space for those states which are described by the $H^{\rm SDI}_{\rm eff2}$ Hamiltonian. Note, however, that there are matrix elements of Q in the fermion space that cannot be reproduced due to the absence of their counterparts in the truncated boson space—for example, the matrix elements where the states $|v=2,J\geq 6\rangle$ are involved. That means that the calculations in the truncated boson space using the image $H_{\rm eff2}$ of the full neutron-proton Hamiltonian no longer provide an exact description of the fermion results. Nevertheless, as seen from Fig. 6, the agreement is very good even for large values of the strength $k_{\pi\nu}$. The spurious states in the low-lying part of spectra are removed by adding the effective Park operator $\hat{S}_{\rm eff}$.

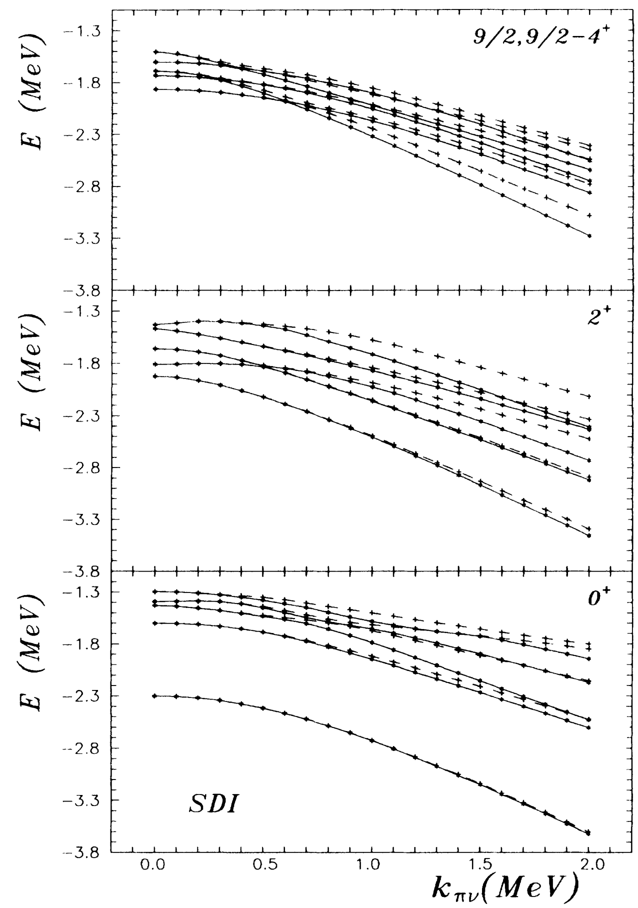

FIG. 6. The same as Fig. 5, but the dashed lines are the boson calculation obtained by using the effective Hamiltonian $H_{\rm eff2}$ in the *sdg* space. The unphysical states are removed by the $\hat{S}_{\rm eff}$ operators.

VI. CONCLUSION

To be of use in the derivation of models of collective nuclear structure, a boson mapping should (i) facilitate the separation and identification of the physical and unphysical states and (ii) provide a reasonable description of the problem in a truncated boson space. Both these features do not appear to be realized within the DBM. The SBM, in which the boson images of the $v=0,2$ fermion states are simple, aims at improving aspects of the boson space truncation. However, when the SBM as dictated by the similarity transformed DBM is used, the problem with the low-lying unphysical states recurs and cannot be treated by the SBM form of the Park operator in the truncated boson space.

Reasons for the behavior of spurious states in the DBM and in the similarity transformed DBM form are connected with the strong quadrupole matrix elements between unphysical states. When the quadrupole-quadrupole strength is increased, unphysical states become the lowest-lying ones. If one could change the mapping in the unphysical sector and weaken quadrupole matrix elements there, one should expect an improved description. Indeed, one approximation to such a modified mapping appears to result from considering only the lowest-order one-body terms in the seniority image of the quadrupole operator, which essentially is the OAI procedure.

A direct way to treat the truncation of the boson space is through effective operator theory. Using this formalism, we have found a notable improvement of results. The effective Park operator identifies unphysical states in the truncated space. Moreover, since eigenstates of the Park operator with a zero eigenvalue are physical states, it is possible to derive another form of the effective boson operators which is related to the original boson description in the physical subspace only and which reproduces fermion matrix elements in the truncated boson space.

Of course, the effective operator theory could be applied both within the DBM and the SBM. In the SBM, however, images of the $v=0$ and 2 states are simple and they are not affected by the truncation process. Consequently, it is sufficient to work out the effective operator theory only in the boson counterpart of the $v=4$ subspace.

As we consider a simple model system with the number of like nucleons being four, the mapping up to the $v=4$ states is all we need. For many-fermion systems, the $v \geq 4$ should be considered. The full SBM taking into account such states contains many-body terms and its derivation and application would not be easy. One may hope that those terms are of less importance and the present, then only approximate, formulas of the SBM with one- and two-body terms could then still be applicable. A similar comment applies to the use of effective operator theory. In the present model, we are able to employ it in an exact and complete form. In the case of many-fermion systems, one should resort to the lowest-order terms of the perturbative expansion.

Of course, an extension of the present methods to the case of several nondegenerate shells would be important. In a passage from the original fermion problem to the collective boson description, one could proceed in two ways: first truncate and then bosonize or vice versa. In both procedures, effective operator theory can be employed at the truncation step.

In the truncated fermion space, it is not, however, clear how to carry out the bosonization unless the truncated fermion operators close an algebra or the direct Marumori-like approach is applied. The former case could only occur under special conditions. In the latter method, one must know a solution of the fermion problem to some extent and the treatment of the unphysical states is not well under control. The approach studied in the present paper, in which the bosonization is first performed and only afterwards the important degrees of freedom are determined and the boson space truncated, seems to be more straightforward. In this way, the lowest-order one-body terms of the SBM operators have been obtained for the case of several nondegenerate shells in Ref. [17].

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