Relation between $E2$ and orbital $M1$ transition strengths using a $Q \cdot Q$ interaction: Further developments

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An expression, previously derived, which relates the energy-weighted sum rule for orbital magnetic dipole ($M1$) excitations to summed electric quadrupole ($E2$) strength, is further developed. It is shown that with a quadrupole-quadrupole interaction, the energy-weighted $M1$ strength is proportional to the difference in summed isoscalar and isovector $E2$ strengths.

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In the same issue of Physical Review C (December 1991) there are two articles that relate the energy-weighted orbital $M1$ strength to the summed $E2$ strength [1, 2]. The current authors used a simple quadrupole-quadrupole ($Q \cdot Q$) interaction between nucleons; the results of Heyde and De Coster were obtained in the framework of the interacting boson model (IBM-2). The correlation between orbital $M1$ and $E2$ strengths in deformed nuclei was first pointed out by Ziegler *et al.* [3] and Rangacharyulu *et al.* [4]. Previous work on $M1$ sum rules was reported by Halemane *et al.* [5] and by Zamick *et al.* [6].

We now find that the main results of our previous work [Eqs. (3.9) and (3.11)] can be further developed to give sharper expressions for the relation between summed orbital $M1$ and summed $E2$ strengths. Before proceeding to this, let us first list some errata in the previous paper [1]:

(i) In Eq. (2.4), the minus sign should be removed. Thus χ=0.1324 following Eq. (2.5). (ii) In the right-hand side of Eq. (3.7), the coefficient $(6+2m^2)$ of the $Y_{2,m}(i)Y_{2,-m}(j)$ term should be $(12+2m^2)$. (iii) In the right-hand side of Eq. (3.8), the factor $18\sqrt{5}$ should be 24. (iv) In the right-hand side of Eq. (3.9), the factor 135 should be replaced by $36\sqrt{5}$. (v) In Eq. (3.10), a minus sign should be added to the right-hand side.

All these changes do not affect any of the numerical results presented in that paper [1].

We are now ready for the further developments. As in the previous paper [1], we use a pure $Q \cdot Q$ interaction:

$$V_q = -\sqrt{5}\chi \sum_{i>j}^{N_V} r_i^2 r_j^2 [Y_i^2 \times Y_j^2]^{L=0} \equiv -\chi \sum_{i>j}^{N_V} v_q(i,j), \tag{1}$$

where N_V represents the number of active nucleons. In shell model calculations, N_V refers to the number of nucleons in the model space used. The most realistic value for N_V is A, the mass number of the nucleus under consideration. We choose χ to be $240/A^{5/3}b^4$ with b determined by $\hbar\omega = \hbar^2/mb^2 = 45A^{-1/3} - 25A^{-2/3}$. In the above equation, we have defined $v_q(i,j)$ as

$$v_q(i,j) = \sqrt{5} r_i^2 r_j^2 [Y_i^2 \times Y_j^2]^{L=0}. \tag{2}$$

The method of double commutators for evaluating the energy-weighted sum rule for $M1$ excitations is amply described in Ref. [1]. The value of the energy-weighted sum rule for orbital $M1$ transitions is

$$\begin{aligned} S_{M1}^{ew}(\text{orbital}) &= \frac{9\sqrt{5}\chi}{8\pi} \langle 0_g^+ | \sum_{i,j=1} (t_{z,i} - t_{z,j})^2 r_i^2 r_j^2 [Y_i^2 \times Y_j^2]^{L=0} | 0_g^+ \rangle \\ &= \frac{9\chi}{16\pi} \langle 0_g^+ | \sum_{i,j=1} (1 - 4 t_{z,i} t_{z,j}) v_q(i,j) | 0_g^+ \rangle , \end{aligned} \tag{3}$$

where $t_{z,i} = \pm\frac{1}{2}$.

The $E2$ strength for the transition from a ground state $|0_g^+\rangle$ to an excited state $|2_f^+\rangle$ is given by

$$B(E2; e_p, e_n)_{g\to f} = |\langle 2_f^+ \| \sum_{i=1}^{N_V} e_i r_i^2 Y_i^2 \| 0_g^+ \rangle|^2, \tag{4}$$

where the double bars signify a reduced matrix element *à la* Edmonds.

The quantities e_p and e_n are the probe-dependent charges. For electroexcitation we have e_p=1 and e_n=0. The value of the summed $E2$ strength can be obtained from Eq. (4) by using closure:

$$S_{E2}(e_p,e_n) \equiv \sum_f B(E2;e_p,e_n)_{g\to f}$$
$$= \sqrt{5}\langle 0_g^+|\sum_{i,j=1}^{N_V} e_ie_jr_i^2r_j^2[Y_i^2\times Y_j^2]^{L=0}|0_g^+\rangle$$
$$= \langle 0_g^+|\sum_{i,j=1}^{N_V} e_ie_jv_q(i,j)|0_g^+\rangle. \qquad (5)$$

By comparing Eqs. (3) and (5) we get the main result:

$$S_{M1}^{ew}(\text{orbital}) = \frac{9\chi}{16\pi}[\,S_{E2}(e_p{=}1,e_n{=}1) - S_{E2}(e_p{=}1,e_n{=}-1)], \qquad (6)$$

where $M1$ strength is in units of μ_N^2 and $E2$ strength in units of $e^2\,\text{fm}^4$. We thus see that we have related the energy-weighted sum rule for orbital $M1$ strength to the *difference* of the *summed* isoscalar ($e_p=1, e_n=1$) and *summed* isovector ($e_p{=}1, e_n{=}-1$) $E2$ strengths.

We can further relate the energy-weighted sum rule for the orbital $M1$ strength to the ground state energy. To this end, let us divide the summed $E2$ strength into two terms ($i\neq j$ and $i=j$) as

$$S_{E2}(e_p,e_n) = \langle 0_g^+|\sum_{i\neq j}^{N_V} e_ie_jv_q(i,j) + \sum_{i=1}^{N_V} e_ie_iv_q(i,i)|0_g^+\rangle$$
$$= 2\langle 0_g^+|\sum_{i>j}^{N_V} e_ie_jv_q(i,j)|0_g^+\rangle + \frac{5}{4\pi}\langle 0_g^+|\sum_{i=1}^{N_V} e_i^2r_i^4|0_g^+\rangle$$
$$= S'_{E2}(e_p,e_n) + S''_{E2}. \qquad (7)$$

Clearly the second term S''_{E2} (with $i{=}j$) is common to the isoscalar and the isovector $E2$ strengths and does not contribute to the right-hand side of Eq. (6). We can thus rewrite Eq. (6) as follows:

$$S_{M1}^{ew}(\text{orbital}) = \frac{9\chi}{16\pi}\,[S'_{E2}(1,1) - S'_{E2}(1,-1)]\,. \qquad (8)$$

It is easy to see that $S'_{E2}(1,1)$ is proportional to the ground state potential energy for the N_V *valence* nucleons:

$$S'_{E2}(1,1) = -\frac{2}{\chi}E_g \qquad (9)$$

with

$$E_g = \langle 0_g^+|V_q|0_g^+\rangle = \chi\langle 0_g^+|\sum_{i>j} v_q(i,j)|0_g^+\rangle. \qquad (10)$$

To evaluate $S'_{E2}(1,-1)$, we introduce the following operator in isospin space:

$$O_\mu^\lambda(1,2) = \sum_{m_1,m_2}\langle 1,m_1,1,m_2|\lambda,\mu\rangle t_{m_1}(1)t_{m_2}(2), \qquad (11)$$

where the quantity with "⟨ ⟩" is a Clebsch-Gordan coefficient, $t_{\pm1}{=}\mp\frac{1}{\sqrt{2}}t_\pm$ and $t_0{=}t_z$. For $\lambda{=}2$, $\mu{=}0$ and $\lambda{=}0$, $\mu{=}0$, we have

$$O_0^2(1,2) = \frac{1}{\sqrt{6}}[\,t_{+1}(1)t_{-1}(2) + 2t_z(1)t_z(2) + t_{-1}(1)t_{+1}(2)], \qquad (12)$$

$$O_0^0(1,2) = \frac{1}{\sqrt{3}}[\,t_{+1}(1)t_{-1}(2) - t_z(1)t_z(2) + t_{-1}(1)t_{+1}(2)] \equiv -\frac{1}{\sqrt{3}}\boldsymbol{t}_1\cdot\boldsymbol{t}_2. \qquad (13)$$

From the above two equations we have

$$t_z(1)t_z(2) = \frac{1}{3}\left[\sqrt{6}O_0^2(1,2) - \sqrt{3}O_0^0(1,2)\right] = \frac{\sqrt{6}}{3}O_0^2 + \frac{1}{3}\boldsymbol{t}_1\cdot\boldsymbol{t}_2. \qquad (14)$$

Therefore we obtain

$$S'_{E2}(1,-1) = \langle 0_g^+|\sum_{i\neq j}^{N_V} 4t_z(i)t_z(j)v_q(i,j)|0_g^+\rangle$$
$$= \frac{2}{3}\langle 0_g^+|\sum_{i<j}\left[4\sqrt{6}O_0^2(i,j) + 4\boldsymbol{t}_i\cdot\boldsymbol{t}_j\right]v_q(i,j)|0_g^+\rangle. \qquad (15)$$

The expectation of the O_0^2 term in the above equation vanishes for a ground state with isospin zero (e.g., ^{20}Ne). However it has a nonzero value for a ground state with a nonzero isospin (e.g., ^{22}Ne).

In a system with two valence protons and two valence neutrons (e.g., ^{8}Be, ^{20}Ne), we have

$$\langle 0_g^+|\sum_{i<j} 4\boldsymbol{t}_i\cdot\boldsymbol{t}_jv_q(i,j)|0_g^+\rangle = -\langle 0_g^+|\sum_{i<j} v_q(i,j)|0_g^+\rangle \qquad (\text{for } N_V=4,\ T_g=0). \quad (16)$$

So $S'_{E2}(1,-1)$ is also proportional to E_g:

$$S'_{E2}(1,-1) = \frac{2}{3\chi}E_g \qquad (17)$$

and Eq. (8) becomes

$$S_{M1}^{ew}(\text{orbital}) = -\frac{9}{16\pi}\left[2E_g + \frac{2}{3}E_g\right] = -\frac{3}{2\pi}E_g \qquad (\text{for } N_V=4,\ T_g=0). \quad (18)$$

As an application of the relation (18), we perform a shell model calculation for the ground state in ^{20}Ne using the $Q \cdot Q$ interaction with χ=0.1646 MeV/fm^4. We treat this nucleus as an inert ^{16}O core plus four active $1s$-$0d$ nucleons (N_V=4). The calculated ground state energy is −19.0615 MeV. According to Eq. (18), the energy-weighted sum rule for the orbital $M1$ strength (in this case the spin $M1$ strength vanishes) is 9.101 MeVμ_N^2. This is the same as the value given by the shell model calculation in which we calculate the energy-weighted $M1$ sum rule by explicitly summing up the contributions $(E_f - E_g)B(M1)_{0_g^+ \to 1_f^+}$ from all the final 1^+ states in ^{20}Ne.

According to Eqs. (9) and (17) we have

$$S'_{E2}(1,1) = 231.59\ e^2\text{fm}^4, \quad S'_{E2}(1,-1) = -77.20\ e^2\text{fm}^4.$$

The summed isoscalar and isovector $E2$ strengths $S_{E2}(1,1)$ and $S_{E2}(1,-1)$ can be obtained by adding to $S'_{E2}(1,1)$ and $S'_{E2}(1,-1)$ the common term S''_{E2} which appeared in Eq. (7). We have $\langle \sum_{i=1}^{N_V} r_i^4 \rangle = 667.13\,\text{fm}^4$. Therefore

$$\begin{aligned} S_{E2}(1,1) &= S'_{E2}(1,1) + \frac{5}{4\pi} \times 667.13 \\ &= 497.03\ e^2\text{fm}^4, \end{aligned} \tag{19}$$

$$\begin{aligned} S_{E2}(1,-1) &= S'_{E2}(1,-1) + \frac{5}{4\pi} \times 667.13 \\ &= 188.24\ e^2\text{fm}^4. \end{aligned} \tag{20}$$

For a ground state with a nonzero isospin, the right-hand side of Eq. (15) can be expressed as a product of the two-body matrix elements (TBME) of the transition operator and the two-body transition density (TBTD) from the ground state to the ground state of the nucleus considered. The TBTD can be obtained from the shell model program.

A little thought on the matter shows that it is theoretically more satisfying to relate the energy-weighted orbital $M1$ strength to a difference in summed $E2$ strengths (isovector-isoscalar) rather than to the summed $E2$ strength itself. For example, either in the limit of no $Q \cdot Q$ interaction at all or in the case of the closed shell nucleus, one can still have $E2$ transitions but not $M1$ transitions with $\Delta E = 2\hbar\omega$. Of course in the above limits, the isoscalar and isovector $E2$ strengths are identical so the difference is zero. Even in the $0\hbar\omega$ space, when one approaches the vibrational limit, although $B(M1)$ vanishes, $B(E2)$ does not—the dominantly isoscalar transition to the one-phonon 2^+ state is rather large. One needs some other term to make the vanishing of $M1$ strength consistent with the nonvanishing of $E2$ strength. The isovector strength plays this role.

It should be emphasized that the relation (6) is valid as long as the left-hand side and the right-hand side are treated on the same footing. For example, one can treat ^{20}Ne as a system of 20 nucleons instead of only four valence nucleons as we did in the shell model calculation. One would then introduce the multiple-particle, multiple-hole configuration mixing to the ground state and both the left-hand side and the right-hand side of Eq. (6) would have new values.

We now give a crude estimate of the effects of ΔN=2 mixing on the summed $B(E2)$ strength. We first write down the energy-weighted sum rule for the $E2$ strength. With the simple interaction V_q which contains no isospin dependence, we have equal isoscalar and isovector sum rules (λ=2):

$$\begin{aligned} &\sum_f (E_f - E_g) B(E2;1,1) \\ &\quad = \sum_f (E_f - E_g) B(E2;1,-1) \\ &\quad = \frac{\hbar^2}{2m} \frac{\lambda(2\lambda+1)^2}{4\pi} \langle Ar^{2\lambda-2} \rangle = \frac{\hbar^2}{2m} \frac{50}{4\pi} \langle Ar^2 \rangle, \end{aligned} \tag{21}$$

where $\langle Ar^2 \rangle$ is the mean square matter radius.

In the absence of correlations, the isoscalar and isovector particle-hole $J^\pi = 2^+$ states are at excitation energies of $2\hbar\omega$. However an isospin-independent $Q \cdot Q$ interaction will bring the isoscalar quadrupole state down to $\sqrt{2}\hbar\omega$ but leave the isovector state at $2\hbar\omega$. (With more realistic interactions, as noted by Bohr and Mottelson [7], the isovector quadrupole state goes up in energy to about $3.1\hbar\omega$. However in this work, we will carry through with the consequences of our simpler model.) We get the isoscalar and isovector strengths by dividing the energy-weighted strength by $\sqrt{2}\hbar\omega$ and $2\hbar\omega$, respectively. For $\langle Ar^2 \rangle$ we take the oscillator result: $\langle Ar^2 \rangle = \sum b^2$, where b is the oscillator length, $\sum$ is the sum of $(N + \frac{3}{2})$ over occupied orbits with N the number of quanta. We obtain

$$\text{isoscalar strength} = \sqrt{2}\frac{25}{8\pi} \sum b^4,$$

$$\text{isovector strength} = \frac{25}{8\pi} \sum b^4.$$

For ^{16}O, $\sum$ is equal to 36 and b is taken to be 1.726 fm. We find the summed $2\hbar\omega$ strength is 449.45 e^2 fm^4 (this is comparable to the $0\hbar\omega$ strength in ^{20}Ne) and the corresponding isovector strength is 317.81 e^2 fm^4. The change in the right-hand side of Eq. (6) due to the $\Delta E = 2\hbar\omega$ $E2$ strengths is $2.91\mu_N^2$.

The ΔN=2 contribution to the orbital $M1$ strength has been considered in Refs. [8–11]. We wish to pursue this point in the near future. We also intend to readdress the sum rule technique with more realistic interactions. Even for a $Q \cdot Q$ interaction, one can make things more realistic by introducing a $\boldsymbol{t}_1 \cdot \boldsymbol{t}_2$ term.

Note added in proof. In the SU(3) limit, the quantity S_{E2}(1,1) in Eq. (6) consists of a single 2^+ state. This is the $J = 2^+$ member of the ground state rotational band. Likewise, the sum $S_{E2}(1,-1)$ involves only one term which can be regarded as the $J = 2^+$ member of the $K = 1$ "scissors mode" rotational band.

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[1] L. Zamick and D.C. Zheng, Phys. Rev. C **44**, 2522 (1991).
[2] K. Heyde and C. De Coster, Phys. Rev. C **44**, R2262 (1991).
[3] W. Ziegler, C. Rangacharyulu, A. Richter, and C. Spieler, Phys. Rev. Lett. **65**, 2515 (1990).
[4] C. Rangacharyulu, A. Richter, H.J. Wörtche, W. Ziegler, and R.F. Casten, Phys. Rev. C **43**, R949 (1991).
[5] T.R. Halemane, A. Abbas, and L. Zamick, J. Phys. G **7**, 1639 (1981).
[6] L. Zamick, A. Abbas, and T.R. Halemane, Phys. Lett. **103B**, 87 (1981).
[7] A. Bohr and B.R. Mottelson, *Nuclear Structure* (Benjamin, New York, 1975), Vol. 2, pp. 508–513.
[8] E. Garrido, E. Moya de Guerra, P. Sarriguren, and J.M. Udias, Phys. Rev. C **44**, 1250 (1991).
[9] I. Hamamoto and C. Magnusson, Phys. Lett. B **260**, 6 (1991).
[10] D. Zawischa, M. Macfarlane, and J. Speth, Phys. Rev. C **42**, 1461 (1990).
[11] A. Faessler, R. Najarov, and F.G. Scholtz, Nucl. Phys. **A515**, 237 (1990).