Multichannel scattering with nonlocal and confining potentials. II. Application to a nonrelativistic quark model of the *NN* interaction

F. Vidal* and J. LeTourneux
Laboratoire de Physique Nucléaire, Université de Montréal, Case Postale 6128, Succ. A, Montréal, Québec, Canada H3C 3J7
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The generalized Levinson theorem for systems involving nonlocal and confining potentials is illustrated by a nonrelativistic quark model for the *NN* interaction, formulated within the framework of the resonating group (RG) method. In order to apply the results of the preceding paper to this problem, we first show how the system of integro-differential equations of the RG method can be transformed into a system of coupled Schrödinger equations with nonlocal potentials. The forbidden states arising in the many-channel wave function from the Pauli principle are then shown to play the same role as normalizable bound states, as far as the generalized Levinson theorem is concerned. Numerical results are presented for the $l=0$ partial wave in the $(ST)=(01)$ channel.

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I. INTRODUCTION

Levinson's theorem [1] has enjoyed a somewhat special status in the long history of the quantum theory of scattering. As more and more complex situations were considered, it underwent a sequence of generalizations that enabled it to express in a very condensed fashion deep relations between the scattering matrix and the spectral properties of the corresponding Hamiltonian [2]. In its simplest form, it relates the phase shift $\delta_l(E)$ for potential scattering in partial wave l at $E=0$ and ∞ with the numbers $n_l^{\rm b}$ and $n_l^{\rm hb}$ of bound and half-bound states in that partial wave,

$$\delta_l(0)-\delta_l(\infty)=\pi(n_l^{\rm b}+n_l^{\rm hb}/2)\ . \tag{1}$$

In one of its most extended versions, derived some years ago by Dashen, Healy, and Muzinich [3], it deals with two-channel scattering by local potentials in the presence of a permanently confined channel. In the preceding paper [4] (henceforth referred to as I), this analysis was generalized by following a somewhat different approach to situations involving nonlocal potentials and an arbitrary number of channels. For such multichannel problems, the quantity that is involved in Levinson's theorem is no longer the phase shift $\delta_l(E)$, but the real function $\Delta_l(E)$, which is defined as follows:

$$\det[S_l^{\rm op}(E)]=\exp[2i\,\Delta_l(E)]\ , \tag{2}$$

$S_l^{\rm op}(E)$ being the open-channel partial-wave S matrix, namely, the submatrix corresponding to all the channels that are open when the relative energy in the entrance channel is E. If the function $\Delta_l(E)$ is defined to be continuous, so that it has no discontinuities of $n\pi$ at isolated points, the theorem reads

$$\Delta_l(0)-\Delta_l(E)=\pi(n_l^{\rm b}+n_l^{\rm hb}/2-n_l^i)+\mathcal{O}(E^{-\alpha}),\quad \alpha>0\ , \tag{3}$$

for $E\to\infty$ and $E_l^i<E<E_l^{i+1}$. In this expression, $n_l^{\rm b}$ is the total number of negative and positive energy bound states for the complete set of coupled Schrödinger equations, whereas n_l^i is the total number of confined states up to energy E when confined and scattering channels are decoupled, the energies of these decoupled confined states being denoted by E_l^n $(n=1,\ldots,i,\ldots)$. The inequality accompanying Eq. (3) should be understood as follows: the equation holds only for energies E, which are not in a vicinity of order $(E_l^i)^{-\alpha}$ of any of the confined system energies E_l^i, a condition that can be fulfilled only when the spacing $E_l^{i+1}-E_l^i$ decreases much more slowly than $(E_l^i)^{-1/2}$.

The purpose of this paper is to provide a concrete illustration of this generalized Levinson theorem. We will consider for that purpose a nonrelativistic quark model for the nucleon-nucleon (*NN*) interaction formulated within the framework of the resonating group (RG) method, a subject on which much work has been done lately [5]. This model features one of the ingredients necessary to provide such an illustration: a phenomenological confining interaction between colored clusters.

The RG method provides a very clean microscopic description of scattering and reactions between composite systems [6,7]. It involves no spurious degrees of freedom and fully implements the Pauli principle. Indeed, many of the distinctive features of the method, such as the nonlocality of its kernels, are direct consequences of that principle. Because of their nonlocal overlap kernels, the RG equations cannot be considered Schrödinger equations any more than their solutions, the RG amplitudes can be considered wave functions. The results derived in I cannot therefore be applied directly to the RG method: one must first show that the RG equations can be reduced to Schrödinger equations with nonlocal potentials. In showing this, one must deal properly with another

*Present address: INRS-Energie, C.P. 1020, Varennes, Québec, Canada J3X 1S2.

consequence of the Pauli principle, namely, the existence of forbidden states for systems formed of two clusters described by harmonic oscillators with equal length parameters. These states are normalizable relative motion amplitudes for which the total wave function vanishes identically. Although they obviously cannot be realized physically, they will be seen to play the same part as any normalizable bound state as far as the generalized Levinson theorem is concerned, a result that is fully consistent with the observations made by Swan of several special cases [8]. The problem will be further complicated by the fact that these forbidden states occur in a many-channel wave function, a situation that does not appear to have been investigated in detail so far.

Of course, one point should be emphasized: the number of forbidden states, just like the number of confined channels, depends on the choice of configurations that are retained in a given RG calculation. These numbers characterize a given level of description beyond which they lose, therefore, all physical meaning. Forbidden states disappear, for instance, when the clusters have different harmonic oscillator parameters or when their internal structure is described by a wave function that goes beyond the simplest shell-model configuration with harmonic-oscillator wave functions. Similarly, as emphasized by Oka and Yazaki in their review article [5], confined hidden-colored channels are not necessary to describe globally colorless multiquark states, as long as one takes into account a sufficiently large number of unconfined channels involving only color-singlet clusters.

The plan of the paper is the following. The RG method is briefly reviewed in Sec. II, mainly for the sake of establishing the notation. We show in the following section how the RG equations can be transformed into Schrödinger equations with nonlocal potentials. The forbidden states associated with the six-quark configurations that are retained in our model are constructed explicitly in Sec. IV. Finally, in the last section we illustrate by means of numerical calculations the generalized Levinson theorem for systems involving forbidden states.

II. THE RG METHOD

A. The RG equation and its kernels

The starting point of the RG method is the following ansatz for the wave function describing a two-cluster N-channel scattering process:

$$\Psi(\xi_a,\xi_b,\mathbf{r})=\sum_{\gamma=1}^{N}\mathcal{A}\{[\phi_a(\xi_a)\phi_b(\xi_b)]_\gamma\chi_\gamma(\mathbf{r})\}\ , \tag{4}$$

where $[\phi_a(\xi_a)\phi_b(\xi_b)]_\gamma$ is the product of the internal wave functions for the two clusters in channel γ, while $\chi_\gamma(\mathbf{r})$ is the amplitude for their relative motion. The operator $\mathcal{A}$ ensures complete antisymmetrization of Ψ. In order to avoid constantly writing channel indices, it will prove useful in the following to consider quantities like χ or $[\phi_a\phi_b]$ as N-component column vectors, the corresponding row vectors being denoted by $\bar{\chi}$ or $\overline{[\phi_a\phi_b]}$, respectively. With this notation, Eq. (4) becomes

$$\Psi=\mathcal{A}\{\overline{[\phi_a\phi_b]}\chi\}\ .$$

The relative motion amplitude $\chi(r)$ is determined by the RG equation,

$$\int[H(\mathbf{r},\mathbf{r}')-\mathcal{E}N(\mathbf{r},\mathbf{r}')]\chi(\mathbf{r}')d\mathbf{r}'=0\ , \tag{5}$$

which is obtained by projecting the Schrödinger equation

$$(\mathcal{H}-\mathcal{E})\Psi=0 \tag{6}$$

on the subspace of states where the internal wave function $[\phi_a\phi_b]_\gamma$ in each channel is frozen. In Eq. (6), $\mathcal{H}$ and $\mathcal{E}$ are, respectively, the total Hamiltonian and the total energy in the c.m. frame of reference. In the RG equation, the energy and overlap kernels, $H(\mathbf{r},\mathbf{r}')$ and $N(\mathbf{r},\mathbf{r}')$, are $N\times N$ matrices defined as follows:

$$\begin{Bmatrix}H(\mathbf{r},\mathbf{r}')\\N(\mathbf{r},\mathbf{r}')\end{Bmatrix}=\int[\phi_a(\xi_a)\phi_b(\xi_b)]\delta(\mathbf{r}-\mathbf{r}'')\begin{Bmatrix}\mathcal{H}\\1\end{Bmatrix}\times\mathcal{A}\{\overline{[\phi_a(\xi_a)\phi_b(\xi_b)]}\delta(\mathbf{r}''-\mathbf{r}')\}\times d\xi_a d\xi_b d\mathbf{r}''\ . \tag{7}$$

They are Hermitian,

$$N_{\alpha\beta}(\mathbf{r},\mathbf{r}')=N_{\beta\alpha}(\mathbf{r}',\mathbf{r})\ , \tag{8a}$$

$$H_{\alpha\beta}(\mathbf{r},\mathbf{r}')=H_{\beta\alpha}(\mathbf{r}',\mathbf{r})\ , \tag{8b}$$

and they split into direct (d) and exchange (e) parts,

$$N(\mathbf{r},\mathbf{r}')=1\delta(\mathbf{r}-\mathbf{r}')+N^{(e)}(\mathbf{r},\mathbf{r}')\ , \tag{9a}$$

$$H(\mathbf{r},\mathbf{r}')=H^{(d)}(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')+H^{(e)}(\mathbf{r},\mathbf{r}')\ , \tag{9b}$$

where

$$H^{(d)}(\mathbf{r})=-\frac{\hbar^2}{2}M^{-1}\Delta+E^0+H^{(d)}_{\rm int}(\mathbf{r})\ . \tag{10}$$

In this expression, M is a diagonal matrix, the elements of which are the reduced masses

$$\mu_\gamma=\frac{N_aN_b}{N_a+N_b}m\ ,$$

N_a and N_b being the number of particles in clusters a and b of channel γ, while m is the mass of each one of these particles. Similarly, E^0 is the diagonal matrix formed with the internal energies E_γ of the fragments in the various channels. Finally, $H^{(d)}_{\rm int}(\mathbf{r})$ is the local part of the interaction between clusters.

B. Forbidden states

Let us introduce the eigenvectors $\zeta_\alpha(\mathbf{r})$ of the overlap kernel,

$$\int N(\mathbf{r},\mathbf{r}')\zeta_\alpha(\mathbf{r}')d\mathbf{r}'=(1+\lambda_\alpha)\zeta_\alpha(\mathbf{r})\ , \tag{11}$$

the eigenvalues of which are semipositive definite [7]. Thus, the eigenvalues of $N^{(E)}$ obey the relation

$$\lambda_\alpha\geq-1\ , \tag{12}$$

whereas its eigenvectors form an orthogonal set

$$\int \bar{\xi}_\alpha(\mathbf{r})\xi_{\alpha'}(\mathbf{r})d\mathbf{r}=\delta_{\alpha\alpha'} . \tag{13}$$

When many eigenvectors correspond to a given λ_α, they can always be defined in such a way as to be orthogonal to each other. Even though they do not necessarily form a complete set, the following expansion can be written:

$$N(\mathbf{r},\mathbf{r}')=1\delta(\mathbf{r}-\mathbf{r}')+\sum_\alpha \lambda_\alpha\xi_\alpha(\mathbf{r})\otimes\bar{\xi}_\alpha(\mathbf{r}') . \tag{14}$$

When the internal wave functions of both clusters are described by harmonic-oscillator wave functions with equal length parameters, λ_α can take the value -1. The overlap kernel then has a vanishing eigenvalue

$$\int N(\mathbf{r},\mathbf{r}')\xi_i(\mathbf{r}')d\mathbf{r}'=0 . \tag{15}$$

In the following, we shall label such eigenvalues with latin indices, as well as the corresponding eigenvectors, which are called forbidden states, since they yield total wave functions that vanish identically as a consequence of the Pauli principle,

$$\mathcal{A}\{[\phi_a(\xi_a)\phi_b(\xi_b)]\xi_i(\mathbf{r})\}=0 . \tag{16}$$

This entails

$$\int H(\mathbf{r},\mathbf{r}')\xi_i(\mathbf{r}')d\mathbf{r}'=0 . \tag{17}$$

As a consequence, the forbidden states ξ_i are trivial solutions of Eq. (5) for all values of $\mathcal{E}$.

III. TRANSFORMATION OF THE RG EQUATION INTO A SCHRÖDINGER EQUATION

The total wave function Ψ and the relative motion amplitude χ do not have the same norm, since

$$\langle\Psi|\Psi\rangle=\langle\chi|N|\chi\rangle .$$

For this reason, one cannot consider χ as a wave function or the RG equation it obeys as a Schrödinger equation. It is thus necessary to show that Eq. (5) can be transformed into a Schrödinger equation before beginning to extract illustrations of the generalized Levinson theorem from RG calculations. This can be done in the following manner.

Let us first define the operator

$$\Lambda(\mathbf{r},\mathbf{r}')=1\delta(\mathbf{r}-\mathbf{r}')-\sum_{i=1}^{n_f}\xi_i(\mathbf{r})\otimes\bar{\xi}_i(\mathbf{r}') , \tag{18}$$

which projects outside the subspace of the n_f forbidden states ξ_i, and the modified overlap kernel

$$N_0(\mathbf{r},\mathbf{r}')=1\delta(\mathbf{r}-\mathbf{r}')+\sum_{\lambda_\alpha\neq-1}\lambda_\alpha\xi_\alpha(\mathbf{r})\otimes\bar{\xi}_\alpha(\mathbf{r}') , \tag{19}$$

which differs from $N(\mathbf{r},\mathbf{r}')$ by the fact that forbidden states are excluded from the summation on the right-hand side. While N cannot be inverted in the presence of forbidden states, because of its vanishing eigenvalues, N_0 is not plagued with this problem, and its inverse square root is readily expressed as

$$N_0^{-1/2}(\mathbf{r},\mathbf{r}')=1\delta(\mathbf{r}-\mathbf{r}')+\sum_{\lambda_\alpha\neq-1}[s_\alpha(1+\lambda_\alpha)^{-1/2}-1]\times\xi_\alpha(\mathbf{r})\otimes\bar{\xi}_\alpha(\mathbf{r}') , \tag{20}$$

up to the sign indeterminancy, which is characteristic of the extraction of square roots, and which manifests itself here in the fact that s_α can take the values ±1.

Quite obviously, one has

$$\int N_0(\mathbf{r},\mathbf{r}'')\Lambda(\mathbf{r}'',\mathbf{r}')d\mathbf{r}''=N(\mathbf{r},\mathbf{r}') \tag{21}$$

and

$$\int H(\mathbf{r},\mathbf{r}'')\Lambda(\mathbf{r}'',\mathbf{r}')d\mathbf{r}''=H(\mathbf{r},\mathbf{r}') \tag{22}$$

as a consequence of Eq. (17). It is therefore possible to rewrite Eq. (5) as follows:

$$\int[H(\mathbf{r},\mathbf{r}')-\mathcal{E}N_0(\mathbf{r},\mathbf{r}')]\Lambda(\mathbf{r}',\mathbf{r}'')\chi(\mathbf{r}'')d\mathbf{r}'d\mathbf{r}''=0 , \tag{23}$$

or defining

$$\tilde{H}(\mathbf{r},\mathbf{r}')=\int N_0^{-1/2}(\mathbf{r},\mathbf{r}'')H(\mathbf{r}'',\mathbf{r}''')\times N_0^{-1/2}(\mathbf{r}''',\mathbf{r}')d\mathbf{r}''d\mathbf{r}''' \tag{24}$$

and

$$\tilde{\chi}(\mathbf{r})=\int N^{1/2}(\mathbf{r},\mathbf{r}')\chi(\mathbf{r}')d\mathbf{r}' \tag{25}$$

in the form

$$\int\tilde{H}(\mathbf{r},\mathbf{r}')\tilde{\chi}(\mathbf{r}')d\mathbf{r}'=\mathcal{E}\tilde{\chi}(\mathbf{r}) , \tag{26}$$

where use has been made of the identity

$$\int N_0^{1/2}(\mathbf{r},\mathbf{r}'')\Lambda(\mathbf{r}'',\mathbf{r}')=\int\Lambda(\mathbf{r},\mathbf{r}'')N_0^{1/2}(\mathbf{r}'',\mathbf{r}')=N^{1/2}(\mathbf{r},\mathbf{r}') . \tag{27}$$

The renormalized relative motion amplitude $\tilde{\chi}(\mathbf{r})$ now has the same norm as the total wave function Ψ, obeys a Schrödinger-like equation [9], and can consequently be regarded as a wave function. It differs from $\chi(\mathbf{r})$, at short distance only, for values of r of the same order as the range of N, namely, as the diameter of the clusters. Since the asymptotic behavior of both functions is the same, they define the same S matrix and all the information relevant to the scattering processes can be extracted directly from the RG amplitude $\chi(\mathbf{r})$.

One often finds in the literature an equation analogous to Eq. (26), with $\tilde{H}$ replaced by

$$\tilde{\tilde{H}}(\mathbf{r},\mathbf{r}')=\int N^{-1/2}(\mathbf{r},\mathbf{r}'')H(\mathbf{r}'',\mathbf{r}''')\times N^{-1/2}(\mathbf{r}''',\mathbf{r}')d\mathbf{r}''d\mathbf{r}''' . \tag{28}$$

It should be pointed out that even when $N^{-1/2}$ does not exist, the quantity $\tilde{\tilde{H}}$ can still be defined [10] if one first computes the right-hand side of (27) for clusters having unequal parameters and then carefully takes the limit where both parameters become equal. When this is actually done, the resulting equations give a more complete description of multichannel scattering than Eq. (26), since

the limiting process introduces a coupling to some new configurations describing excited internal states that are not included in the initial wave function (4).

The amplitudes $\chi(\mathbf{r})$ obtained by solving the RG equation (5) contain the same physical information as the wave functions $\tilde{\chi}(\mathbf{r})$, which are solutions of the Schrödinger equation (26). They differ in the following respect, however. It was mentioned previously that the forbidden states are trivial solutions of the RG equation at all energies. Moreover, they can be admixed to an arbitrary extent into the amplitudes $\chi(\mathbf{r})$, this extent being determined by the details of the numerical procedure used to solve the RG equation (5) and having absolutely no physical consequence whatsoever, since it leaves the total wave function (4) unmodified. A different situation holds for solutions of the Schrödinger equation (26). It follows immediately from Eqs. (17), (20), and (24) that the forbidden states $\zeta_i(\mathbf{r})$ are solutions at $\mathcal{E}=0$ only,

$$\int \tilde{H}(\mathbf{r},\mathbf{r}')\zeta_i(\mathbf{r}')d\mathbf{r}'=0 , \tag{29}$$

and from Eqs. (24), (25), and (27) that they are orthogonal to the physical wave function at an arbitrary energy,

$$\int \bar{\zeta}_i(\mathbf{r})\tilde{\chi}(\mathbf{r})d\mathbf{r}=0 . \tag{30}$$

It may be worth pointing out that although the energy $\mathcal{E}=0$ at which the forbidden states are solutions of Eq. (26) is perfectly well defined within the generalized formulation of the RG method leading to Eq. (26), this energy nevertheless is completely devoid of any physical meaning: one could shift it arbitrarily to $\mathcal{E}=\alpha_i$ by making the transformation

$$\tilde{H}(\mathbf{r},\mathbf{r}')\rightarrow\tilde{H}(\mathbf{r},\mathbf{r}')-\sum_{i=1}^{n_f}\alpha_i\zeta_i(\mathbf{r})\otimes\bar{\zeta}_i(\mathbf{r}') ,$$

which would affect neither the physical spectrum of $\tilde{H}$ nor its physical wave functions. Let us point out finally that (30) can easily be shown to express the orthogonality of solutions of the Schrödinger equation belonging to different energies.

Performing a partial-wave expansion for the wave function $\tilde{\chi}(\mathbf{r})$, or the amplitude $\chi(\mathbf{r})$, one can solve Eq. (26) or (5) separately for each value of the orbital quantum number l when all interactions are central. The matrix equation (25) then reduces a set of coupled nonlocal Schrödinger equations for various partial waves,

$$\int_0^\infty \tilde{H}_l(r,r')\tilde{\chi}_l(r')dr'=\mathcal{E}(E)\tilde{\chi}_l(r) .$$

They have precisely the same form as those that were used at the starting point of I. As in the Introduction, E is the relative energy in the entrance channel. All the results derived in I can therefore be applied to calculations performed within the framework of the RG method. As far as the generalized Levinson theorem is concerned, the forbidden states $\zeta_i(\mathbf{r})$ play the same part as any other bound state as a consequence of the orthogonality relations (30), in spite of the fact that no physical states of the system correspond to them. Eq. (3) then reads

$$\Delta_l(0)-\Delta_l(E)=\pi(n_l^f+n_l^{\mathrm{b}}+n_l^{\mathrm{hb}}/2-n_l^i) +\mathcal{O}(E^{-\alpha}),\quad \alpha>0 , \tag{31}$$

for $E\rightarrow\infty$ and $E_l^i<E<E_l^{i+1}$. The E_l^i's form the spectrum of the equation

$$\int_0^\infty \tilde{H}_{\mathrm{cc},l}(r,r')\varphi_{l,i}(r')dr'=\mathcal{E}(E_l^i)\varphi_{l,i}(r) , \tag{32}$$

where $\tilde{H}_{\mathrm{cc}}$ is the submatrix of $\tilde{H}$ corresponding to the confined channels only.

IV. EXAMPLE: THE QUARK MODEL FOR THE *N-N* INTERACTION

A. Construction of the forbidden states

In nonrelativistic quark models of hadrons, the orbital and spin-isospin parts of the wave function of the nucleon N have symmetries [3] and {3}, respectively, while the color part is completely antisymmetric, [111]. When studying NN scattering, one may retain the set of six-quark configurations with orbital $[f]$ and spin-isospin $\{f'\}$ symmetries resulting from the outer product of the corresponding symmetries for the nucleon, namely,

$$[3]\times[3]=[6]+[42]+[33]+[51] ,$$

while the color part is in the singulet [222]. This yields the configurations listed in Table I for the even partial waves, having $(ST)=(01)$ or (10), and the odd ones, having $(ST)=(00)$ or (11). These states will be referred to as the symmetry basis. They are related by a unitary transformation to the physical basis, which displays at large cluster separations the configurations NN, $\Delta\Delta$, and CC ("hidden color" channel) for $(ST)=(01)$, (10), and (00), and NN, $N\tilde{\Delta}$ (antisymmetric), $\Delta\Delta$, CC_1, and CC_2 for $(ST)=(11)$. The states listed in Table I constitute the minimal basis necessary to construct asymptotically the NN configuration [11].

The RG overlap kernels $N(\mathbf{r},\mathbf{r}')$ for these configurations can be extracted from those computed by Harvey [11] in the generator coordinate formalism, by using one or another of various prescriptions [12]. They are particularly simple in the symmetry basis, where they form diagonal matrices as a consequence of the ortho-

TABLE I. Orbital $[f]$ and spin-isospin $\{f'\}$ symmetries in various (ST) channels.

(ST)	Configurations
(01),(10)	[6]{33}, [42]{33}, [42]{51}
(00)	[51]{42}, [33]{42}, [33]{6}
(11)	$[51]\{42\}_1$, $[51]\{42\}_2$, $[33]\{42\}_1$, $[33]\{42\}_2$, [33]{6}

gonality of states endowed with different symmetries. Labeling states of the symmetry basis with latin indices $a,b,\ldots$, one has

$$N_{ab}^{(\mathrm{sym})}(\mathbf{r},\mathbf{r}')=\delta_{ab}\frac{\alpha^3}{2}[\delta(\boldsymbol{\rho}-\boldsymbol{\rho}')-(-1)^{S+T}\delta(\boldsymbol{\rho}+\boldsymbol{\rho}') + C_{[f_a]}n(\boldsymbol{\rho},\boldsymbol{\rho}')] , \tag{33}$$

where

$$\alpha\equiv\left[\frac{3}{2b^2}\right]^{1/2} , \tag{34}$$

b being the length parameter of the harmonic-oscillator wave functions,

$$\boldsymbol{\rho}\equiv\alpha\mathbf{r} \tag{35}$$

and

$$n(\boldsymbol{\rho},\boldsymbol{\rho}')=\left[\frac{9}{8\pi}\right]^{3/2}\exp[-\tfrac{5}{8}(\rho^2+\rho'^2)] \times[\exp(\tfrac{3}{4}\boldsymbol{\rho}\cdot\boldsymbol{\rho}')-(-1)^{S+T}\exp(-\tfrac{3}{4}\boldsymbol{\rho}\cdot\boldsymbol{\rho}')] . \tag{36}$$

The numbers $C_{[f_a]}$ take values that depend only on the orbital symmetry $[f]$. They are equal to 9, -1, -3, and 3 for $[f]=[6]$, [42], [33], and [51], respectively. In the physical basis, the states of which will be labeled by indices $A,B,\ldots$, the overlap kernels are given by

$$N_{AB}^{(\mathrm{phys})}(\mathbf{r},\mathbf{r}')=\sum_{a,b}\mathcal{M}_{Aa}^{*}\mathcal{M}_{Bb}N_{ab}^{(\mathrm{sym})}(\mathbf{r},\mathbf{r}') . \tag{37}$$

The unitary matrices $\mathcal{M}$ relating the two bases,

$$\Phi_A^{(\mathrm{phys})}=\sum_a\mathcal{M}_{Aa}\Phi_a^{(\mathrm{sym})} , \tag{38}$$

are given in Harvey's paper [11].

The task of finding the eigenvectors of $N^{(\mathrm{phys})}$ is greatly facilitated by the fact that $\mathcal{M}$ diagonalizes $N^{(\mathrm{phys})}$. Indeed, defining

$$v_A^{(a)}\equiv\mathcal{M}_{Aa} , \tag{39}$$

one has, in matrix notation,

$$N^{(\mathrm{phys})}(\mathbf{r},\mathbf{r}')v^{(a)}=N_{aa}^{(\mathrm{sym})}(\mathbf{r},\mathbf{r}')v^{(a)} , \tag{40}$$

with no summation over a on the right-hand side. The eigenvectors can therefore be written in the form

$$\xi_\alpha(\mathbf{r})=v^{(a)}F_\nu(\boldsymbol{\rho}) , \tag{41}$$

with $\alpha\equiv(a,\nu)$, $F_\nu(\boldsymbol{\rho})$ being such that

$$\int n(\boldsymbol{\rho},\boldsymbol{\rho}')F_\nu(\boldsymbol{\rho}')d\boldsymbol{\rho}'=2x_\nu F_\nu(\boldsymbol{\rho}) . \tag{42}$$

Then, Eq. (11) is verified, with

$$\lambda_\alpha=C_{[f_a]}x_\nu . \tag{43}$$

It is easy to show that

$$F_\nu(\boldsymbol{\rho})=c_{n_xn_yn_z}H_{n_x}(\rho_x) \times H_{n_y}(\rho_y)H_{n_z}(\rho_z)\exp(-\boldsymbol{\rho}^2/2) , \tag{44}$$

and that

$$x_\nu=(\tfrac{1}{3})^{n_x+n_y+n_z} . \tag{45}$$

The index ν stands for the three integers n_x, n_y, and n_z, the sum of which must have the same parity as $S+T+1$, and $c_{n_xn_yn_z}$ is a normalization constant. These results follow immediately from the relation

$$\left[\frac{9}{8\pi}\right]^{1/2}\exp\left[-\frac{5}{8}u^2\right]\int_{-\infty}^{\infty}\exp\left[-\frac{9}{8}u'^2\pm\frac{3}{4}uu'\right]H_m(u')du'=\left[\pm\frac{1}{3}\right]^m H_m(u)\exp(-u^2/2) ,$$

which can be established by recursion. The functions $F_\nu(\boldsymbol{\rho})$ are nothing but harmonic-oscillator wave functions. Working with spherical rather than Cartesian coordinates, one may write

$$F_\nu(\boldsymbol{\rho})=\frac{1}{r}R_{nl}(r)Y_l^m(\theta,\phi) , \tag{46}$$

$R_{nl}(r)$ being a radial harmonic-oscillator wave function,

$$R_{nl}(r)=C_{nl}\rho^{l+1}L_{n-1}^{l+1/2}(\rho^2)\exp(-\tfrac{1}{2}\rho^2) , \tag{47}$$

where C_{nl} is a normalization constant and the Laguerre polynomials are defined as in Ref. [13]. The indices n_x, n_y, and n_z have been replaced by n, l, and m, to which they are related as follows:

$$n_x+n_y+n_z=2n+l-2 . \tag{48}$$

When a partial wave expansion is performed for the overlap kernel

$$N(\mathbf{r},\mathbf{r}')=\sum_l\frac{2l+1}{4\pi}\frac{1}{rr'}N_l(r,r')P_l(\cos(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')) ,$$

Eq. (10) is replaced by the radial eigenvalue problem,

$$\int_0^\infty N_l(r,r')\xi_{\alpha,nl}(r')dr'=[1+C_{[f_a]}(\tfrac{1}{3})^{2n+l-2}]\xi_{\alpha,nl}(r) , \tag{49}$$

where

$$\xi_{\alpha,nl}(r)=v^{(a)}R_{nl}(r) . \tag{50}$$

We can now find the forbidden states $\xi_{\alpha,nl}(r)$ for which the eigenvalue vanishes. They can occur in two different situations only, namely, when

$$[f]=[42],\quad n=1,\quad l=0$$

or

$$[f]=[33],\quad n=1,\quad l=1 .$$

There are thus two of them for $(ST)=(01)$, (10), or (00), and three for $(ST)=(11)$. The forbidden states can readily be written down by using Eqs. (50), (39), and the unitary transformation matrices given in Ref. [11]. For instance, for the channels $(ST)=(01)$ or (10) they are

$$\xi_{1,1s}(r)=\frac{1}{3}\begin{bmatrix}-2\\ \sqrt{5}\\ 0\end{bmatrix}R_{1s}(r)\ ,$$
$$\xi_{2,1s}(r)=\frac{1}{\sqrt{45}}\begin{bmatrix}2\sqrt{5}\\ 4\\ -3\end{bmatrix}R_{1s}(r)\ , \tag{51}$$

the physical states in these column vectors being in the order NN, $\Delta\Delta$, and CC.

B. Numerical results

NN scattering was studied in the $(ST)=(01)$ and (10) channels by solving numerically the RG Eq. (5) for two clusters of three quarks coupled to these quantum numbers. Such states involve only the NN, $\Delta\Delta$, and CC configurations. The interaction between the quarks was described by potential Id of Ref. [14], which confines the C clusters at large distances through a local quadratic potential. The system of coupled equations was solved in the $l=0$ partial wave by the same method as in Ref. [15], namely, by a generalization of the Robertson-Friedrich [16] method to coupled channels, although solutions were obtained for a much larger range of energies. This yielded the open-channel S matrix, from which the function $\Delta_l(E)$ for $l=0$ was extracted through Eq. (2). This quantity is shown in Fig. 1, for the (01) channel in the energy range $0<E<3585$ MeV. The cusp just below 600 MeV occurs at the threshold energy for the $\Delta\Delta$ channel. Since the results for the (10) channel and their analysis are quite similar to those of the (01) channel, only the latter will be represented.

One sees on Fig. 1 that

$$\Delta(0)-\Delta(3585)\approx-\pi\ . \tag{52}$$

Consequently, according to Eq. (31), one has

$$n_0^i-(n_0^b+n_0^{hb}/2)=3 \tag{53}$$

for $E=3585$ MeV, since $n_f=2$ in the present case. The number n_0^i of confined bound states having an energy lower than this value of E can be determined by solving

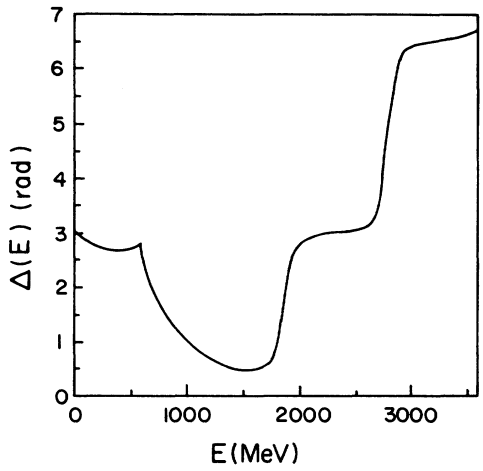

FIG. 1. Plot of the phase Δ as a function of the relative energy E for the (01) state.

Eq. (32) for the partial wave $l=0$. It turns out that $n_0^i=3$ because the energies E_0^j of the first four decoupled confined states are located at 618, 1830, 2815, and 3845 MeV, approximately, and these states are not degenerate.

Since $n_0^i=3$ for $E=3585$ MeV, one can conclude from Eq. (53) that $n_0^b=n_0^{hb}=0$: there is no bound or half-bound state. This result was to be expected, since the model considered here is known to yield essentially a repulsive short-range interaction.

As discussed in Ref. [3] and in I, the function $\Delta(E)$ exhibits a sharp increase of π, approximately, in the vicinity of the energies E_0^j at which confined bound states are located, as long as these energies are sufficiently large. Just as for ordinary resonances, this sharp increase is due to a pole of $S(E)$ at a complex energy $\bar{E}_j$ close to a point $E=E_0^j\to\infty$, with $\mathrm{Im}\bar{E}_j<0$ [16]. Whereas the quantity $D^+(E)$ is difficult to compute numerically, the function $S(E)$ can easily be obtained, and we studied it numerically in the complex plane. It turns out to have poles at $E=\bar{E}_2\approx1853-i51$ MeV and $\bar{E}_3\approx i31$ MeV.

The reason the phase $\Delta(E)$ does not undergo a rapid increase of approximately π in the vicinity of $E=E_1$ is that the S matrix has no pole close to E_1. This is not incompatible with the discussion of I, since the condition $E_1\to\infty$ is obviously not satisfied at $E_1=618$ MeV.

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