Multichannel scattering with nonlocal and confining potentials. I. General theory

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A general discussion of nonrelativistic multichannel scattering for nonlocal potentials is presented. The approach is based on the extensive use of the Fredholm determinants that are associated with the integral equations occurring at various points of the theory. Special attention is paid to situations where confining potentials are present, as in nonrelativistic quark models of hadron-hadron interactions. Some standard results of multichannel scattering theory for local potentials and one-channel theory for nonlocal ones are generalized in this context, including Levinson's theorem.

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I. INTRODUCTION

The purpose of this paper is to present a general discussion of multichannel scattering by nonlocal potentials in situations where certain channels are permanently confined. Our interest in this problem arose as a consequence of the recent development of nonrelativistic quark-cluster models for the baryon-baryon interaction [1]. When formulated within the framework of the resonating group method, these models provide instructive examples of such a situation. The nonlocality of the potential, then, appears as a consequence of the Pauli principle, and the confining potentials are explicitly introduced in the model so as to prevent the existence of scattering states in which colored fragments would fly apart. From the point of view of scattering theory, many interesting questions are raised by these models. What is the role of the states that are forbidden by the Pauli principle? Do these models exhibit continuum bound states? What becomes of Levinson's theorem?

While investigating these matters, we constantly needed general results from the theory of multichannel scattering by nonlocal potentials. There is a vast literature on single-channel scattering by such potentials, but, to the best of our knowledge, the many-channel situation has been studied for local potentials only [2–4]. A systematic discussion of these problems thus seemed to be in order, and in view of its general nature we thought it should be published as a separate paper. The analysis of quark-cluster models for the baryon-baryon interaction, which provides a concrete illustration of results derived here, is to be found in a second paper [5], the scope of which is much narrower, since it deals to a large extent with problems that are very specific to the resonating group method.

In Sec. II we establish a certain number of general results for the situation where there are no permanently confined channels. As a first step, we derive a relation between the Jost matrices and the Fredholm determinants for physical and regular solutions. Although this result is a generalization of the one that holds for many-channel problems with local interactions, it is shown to imply identical relations between the open-channel part of the S matrix and the Fredholm determinants for physical solutions. After establishing the analyticity properties of the latter for a certain class of nonlocal potentials, we study bound states embedded in the continuum, show that, contrary to what happens with local potentials, their occurrence is not limited to energies lower than the one at which all channels become open, and, finally, derive the corresponding generalized Levinson theorem. Amongst earlier works on nonlocal potentials, those that are most closely related with our own deal with this generalized theorem [6,7] and with the connection between Jost functions and Fredholm determinants [8–10].

Section III deals with systems having permanently confined channels. This problem has already been studied for the case of local interactions by Dashen, Healy, and Muzinich [11,12], who considered a two-channel model where one of the channels is confined. Surprisingly enough, their work does not seem to have attracted the attention it deserved from the various groups that have developed nonrelativistic quarks models for the nucleon-nucleon interaction. This may be due to the excessive simplicity of the model they have considered. We generalize their analysis to situations involving an arbitrary number of channels and nonlocal potentials. In order to use results derived in Sec. II, where the approach is based on the Jost matrices, we do not tackle the problem in the same fashion as they did: We define an effective potential acting in the scattering channels only, by projecting out the confined channels. Although our treatment is not mathematically as rigorous as theirs, it presents at least the advantage of using a procedure that is familiar in the contexts of the optical model [13] and the many-body problem [14].

II. MULTICHANNEL SCATTERING WITH NONLOCAL POTENTIALS IN THE ABSENCE OF CONFINED CHANNELS

A. Definitions

Let us consider a nonrelativistic multichannel scattering problem where the channel components $\phi_\alpha(E,r)$, $\alpha=1,\ldots,N$, are determined by a set of coupled Schrödinger equations,

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$$\frac{\hbar^2}{2\mu_\alpha}\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}\right]\phi_\alpha(E,r)+\sum_{\beta=1}^{N}\int_0^\infty V_{\alpha\beta}(r,r')\phi_\beta(E,r')dr'=(E-E_\alpha^0)\phi_\alpha(E,r) ,$$

with nonlocal kernels $V_{\alpha\beta}(r,r')$. In these equations E is the total energy of the system, while μ_α and E_α^0 are the reduced mass and the internal energy, respectively, of the fragments in channel α. This system of equations may be rewritten more compactly in a matrix notation,

$$\frac{\hbar^2}{2}M^{-1}\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}\right]\phi(E,r)+\int_0^\infty V(r,r')\phi(E,r')dr'=(E\mathbf{1}-E^0)\phi(E,r) , \tag{1}$$

$\mathbf{1}$ being the $N\times N$ unit matrix. The elements of the column vector $\phi(E,r)$ are the N amplitudes $\phi_\alpha(E,r)$, and M and E^0 are diagonal matrices with elements μ_α and E_α^0, respectively. We have assumed, for the sake of simplicity, that all interactions are central, so that the orbital quantum number l is the same for all components. When this condition is not fulfilled, it is a simple matter to introduce the diagonal matrix L of channel angular momenta. Defining the quantities

$$K^2\equiv\frac{2}{\hbar^2}M(E\mathbf{1}-E^0) , \tag{2}$$

$$\Psi(K,r)\equiv M^{-1/2}\phi(E,r) ,$$

and

$$U(r,r')\equiv\frac{2}{\hbar^2}M^{1/2}V(r,r')M^{1/2} ,$$

one can write Eq. (1) in the form

$$\mathcal{L}_0(K,r)\psi(K,r)+\int_0^\infty U(r,r')\psi(K,r')dr'=0 , \tag{3}$$

where

$$\mathcal{L}_0(K,r)=\left[-\frac{d^2}{dr^2}+\frac{l(l+1)}{r^2}\right]\mathbf{1}-K^2 .$$

In the following, the potential $U(r,r')$ will be assumed to be *real* and *symmetric*, that is,

$$U_{\alpha\beta}(r,r')=U_{\beta\alpha}(r',r) . \tag{4}$$

When the problem is formulated in terms of integral equations, it proves useful to define a matrix Ψ of solutions, which has in column β the vector solution $\psi^{(\beta)}$ corresponding to the entrance channel β. The *physical solution* $\Psi^+(K,r)$ and its conjugate $\Psi^-(K,r)$ obey the integral equation

$$\Psi^\pm(K,r)=u(Kr)+\int_0^\infty\int_0^\infty G_0^\pm(K;r,r')U(r',r'')\times\Psi^\pm(K,r'')dr'\,dr'' , \tag{5}$$

where the free Green's kernel

$$G_0^\pm(K;r,r')=-K^{-1}u(Kr_<)w^\pm(Kr_>) , \tag{6}$$

is a diagonal matrix, since the matrices $u(Kr)$ and $w^\pm(Kr)$ are themselves diagonal: The elements of $u(Kr)$ are Riccati-Bessel functions $u(K_\alpha r)$ involving the elements K_α of the matrix K, while those of $w^\pm(K_\alpha r)$ are Riccati-Hankel functions $w^\pm(K_\alpha r)=v(K_\alpha r)\pm iu(K_\alpha r)$, $v(K_\alpha r)$ being a Riccati-Neumann function. As everywhere else, the angular momentum label l has been systematically dropped.

The S matrix is defined by the asymptotic behavior of the physical solution,

$$\Psi^+(K,r)\underset{r\to\infty}{\longrightarrow}\frac{i}{2}[w^-(Kr)-w^+(Kr)K^{-1/2}S(K)K^{1/2}] . \tag{7}$$

The *regular solution* $\Phi(K,r)$ obeys the integral equation

$$\Phi(K,r)=K^{-1}u(Kr)+\int_0^\infty\int_0^\infty G_0(K;r,r')U(r',r'')\times\Phi(K,r'')dr'\,dr'' , \tag{8}$$

with the free Green's kernel

$$G_0(K;r,r')=-\frac{1}{2i}K^{-1}[w^-(Kr)w^+(Kr')-w^+(Kr)w^-(Kr')]\theta(r-r') . \tag{9}$$

It is defined by the boundary condition

$$\Phi(K,r)\underset{r\to 0}{\longrightarrow}K^{-1}u(Kr) .$$

The *Jost matrices* $\mathcal{F}_\pm(K)$ are defined by the asymptotic behavior of the regular solution

$$\Phi(K,r)\underset{r\to\infty}{\longrightarrow}-\frac{1}{2i}\left[w^-(Kr)K^{-1}\mathcal{J}_+(K)-w^+(Kr)K^{-1}\mathcal{J}_-(K)\right] , \tag{10}$$

namely,

$$\mathcal{J}_\pm(K)=\mathbf{1}+\int_0^\infty\int_0^\infty w^\pm(Kr')U(r',r'')\times\Phi(K,r'')dr'\,dr'' . \tag{11}$$

It is easy to show that Eqs. (7), (10), and (11) imply the following identities:

$$\mathcal{F}_-(K)=\mathcal{F}_+(-K) , \tag{12}$$

$$\Phi(K,r)=\Psi^+(K,r)K^{-1}\mathcal{F}_+(K) ,$$

$$S(K)=K^{-1/2}\mathcal{F}_-(K)[\mathcal{F}_+(K)]^{-1}K^{1/2} . \tag{13}$$

Let us rewrite Eq. (5) in the form

$$\int_0^\infty [\delta(r-r')1-\mathcal{R}^\pm(K;r,r')]\Psi^\pm(K,r')dr'=u(Kr)\ , \tag{14}$$

with

$$\mathcal{R}^\pm(K;r,r')=\int_0^\infty G_0^\pm(K;r,r'')U(r'',r')dr''\ , \tag{15}$$

and let us give the position variables the discrete values $r_1,\dots,r_T$ in such a way as to make T and r_T arbitrarily large while keeping $\Delta r_i \equiv r_{i+1}-r_i$ arbitrarily small. Thus, Eq. (14) becomes

$$\sum_{\gamma=1}^{N}\sum_{j=1}^{T}[\delta_{\alpha\gamma}\delta_{ij}-\mathcal{R}^\pm_{\alpha\gamma}(K;r_i,r_j)\Delta r_j]\Psi^\pm_{\gamma\beta}(K;r_j)=u_{\alpha\beta}(Kr_i)\ .$$

The solution of this linear system of equations involves the determinant of the matrix $\mathbf{1}-\boldsymbol{R}^\pm(K)$,

$$D^\pm(K)=\det[\mathbf{1}-\boldsymbol{R}^\pm(K)]\ , \tag{16}$$

the elements of which are labeled by the double index $b\equiv(\alpha,i)$ which specifies a channel and a position,

$$[\mathbf{1}-\boldsymbol{R}^\pm(K)]_{bb'}\equiv\delta_{\alpha\alpha'}\delta_{ii'}-\mathcal{R}^\pm_{\alpha\alpha'}(K;r_i,r_{i'})\Delta r_{i'}\ . \tag{17}$$

One has, in a standard fashion, the expansion

$$D^\pm(K)=1+\sum_{n=1}^{TN}\frac{(-1)^n}{n!}D^\pm_{(n)}(K)\ , \tag{18}$$

where

$$D^\pm_{(n)}=\sum_{b_1,\dots,b_n}\begin{vmatrix}R^\pm_{b_1b_1} & \cdots & R^\pm_{b_1b_n}\\ \cdots & \cdots & \cdots\\ R^\pm_{b_nb_1} & \cdots & R^\pm_{b_nb_n}\end{vmatrix}=\sum_{\substack{\alpha_1,\dots,\alpha_n\\ i_1,\dots,i_n}}\begin{vmatrix}\mathcal{R}^\pm_{\alpha_1\alpha_1}(r_{i_1},r_{i_1}) & \cdots & \mathcal{R}^\pm_{\alpha_1\alpha_n}(r_{i_1},r_{i_n})\\ \cdots & \cdots & \cdots\\ \mathcal{R}^\pm_{\alpha_n\alpha_1}(r_{i_n},r_{i_1}) & \cdots & \mathcal{R}^\pm_{\alpha_n\alpha_n}(r_{i_n},r_{i_n})\end{vmatrix}\Delta r_{i_1}\cdots\Delta r_{i_n}\ . \tag{19}$$

The Fredholm determinants $D^\pm$ corresponding to the solutions $\Psi^\pm$ are obtained by taking the limit $\Delta r_i\to 0$, namely, by performing the substitution $\sum_i \Delta r_i \to \int dr$ in (19).

Similarly, the Fredholm determinant corresponding to the regular solution $\Phi(K,r)$ is

$$D(K)=\det[\mathbf{1}-\boldsymbol{R}(K)]\ , \tag{20}$$

the relation between $\boldsymbol{R}(K)$ and

$$\mathcal{R}(K;r,r')=\int_0^\infty G_0(K;r,r'')U(r'',r')dr''\ ,$$

being analogous to the one between $\boldsymbol{R}^\pm$ and $\mathcal{R}^\pm$.

B. S matrix and Fredholm determinants

Lemma I. One always has

$$\det[\mathcal{F}_\pm(K)]=D^\pm(K)/D(K)\ . \tag{21}$$

Proof. The matrix identity

$$G_0^\pm(K;r,r')=G_0(K;r,r')-K^{-1}u(Kr)w^\pm(Kr')\ ,$$

jointly with Eq. (8), entails

$$\delta(r-r')1-\mathcal{R}^\pm(K;r,r')=\int_0^\infty[\delta(r-r'')1-\mathcal{R}(K;r,r'')][\delta(r''-r')1+\mathcal{B}^\pm(K;r'',r')]dr''\ , \tag{22}$$

where

$$\mathcal{B}^\pm(K;r,r')=\Phi(K,r)\int_0^\infty w^\pm(Kr'')U(r'',r')dr''\ .$$

Discretizing the position variables as earlier, one sees that Eq. (22) implies

$$D^\pm(K)=D(K)\det[\mathbf{1}+\boldsymbol{B}^\pm(K)]\ , \tag{23}$$

where $\boldsymbol{B}^\pm$ stands in the same relation to $\mathcal{B}^\pm$ as $\boldsymbol{R}^\pm$ to $\mathcal{R}^\pm$ in Eq. (17).

Now, use will be made of the following identity, which can easily be proved from well-known properties of determinants: If X_i and Y_i ($i=1,\dots,q$) are $p\times p$ matrices and Z is defined as

$$Z_{ab}=\sum_{\gamma=1}^{p}(X_i)_{\alpha\gamma}(Y_j)_{\gamma\beta}\ ,$$

$a\equiv(\alpha,i)$ and $b\equiv(\beta,j)$ being double indices, then one has

$$\det(\mathbf{1}_{pq\times pq}+Z)=\det\left[\mathbf{1}_{p\times p}+\sum_{i=1}^{q}Y_iX_i\right]\ .$$

Identifying $\boldsymbol{B}^\pm(K)$ with Z, one gets immediately

$$\det[\mathbf{1}+\boldsymbol{B}^\pm(K)]=\det[\mathcal{J}_\pm(K)]\ ,$$

which, inserted in Eq. (23), yields the relation (21).

Lemma I constitutes a generalization of a result obtained by Warke and Bhaduri [8] for symmetric nonlocal potentials in the one-channel case. Nowhere, however, did we have to assume that $U(r,r')$ is symmetric in the process of derivation.

If the potential is nonsymmetric, according to the result of Ref. [10], one should have, in the denominator of Eq. (21) the Fredholm determinant Δ associated with the Jost solutions, which becomes identical with D for symmetric potentials only. The disagreement, still, is only apparent and follows from the fact that the Jost functions can be defined either from the asymptotic behavior of the regular solutions, as was done here, or from the Jost solutions at $r=0$, as was done in Ref. [10]. The two definitions are identical for symmetric potentials only.

When the potential is local, Eq. (8), which defines $\Phi(K,r)$, reduces to an integral equation of the Volterra type. Consequently, $D(K)=1$ and one recovers the well-known result [4],

$$\det[\mathcal{F}_{\pm}(K)]=D^{\pm}(K) \quad \text{(local potentials)} . \tag{24}$$

Lemma II. The S matrix is symmetric,

$$S_{\alpha\beta}(K)=S_{\beta\alpha}(K) .$$

Proof. Using Eqs. (5) and (7) we get

$$S(K)=1-2iK^{-1/2}\left[\int_0^\infty\int_0^\infty u(Kr)U(r,r')\Psi^+(K,r')dr\,dr'\right]K^{-1/2} .$$

Writing the solution of Eq. (5) as a Neumann series,

$$\Psi^+(K,r)=u(Kr)+\int_0^\infty\int_0^\infty G_0^+(K;r,r')U(r',r'')u(Kr'')dr'dr''+\cdots ,$$

and using the fact that besides being diagonal, the Green's kernel has the property

$$G_0^+(K;r,r')=G_0^+(K;r',r) , \tag{25}$$

we see immediately that condition (4) entails the symmetry of the S matrix.

Theorem I. The S matrix is completely determined by $D^+(K)$,

$$S_{\alpha\alpha}(K)=\frac{D^+(\ldots,-K_\alpha,\ldots)}{D^+(K)} , \tag{26}$$

$$S_{\alpha\beta}(K)=S_{\alpha\alpha}(K)S_{\beta\beta}(K) - \frac{D^+(\ldots,-K_\alpha,\ldots,-K_\beta,\ldots)}{D^+(K)} \quad (\alpha\neq\beta) . \tag{27}$$

Proof. These relations have the same form as in the case of local potentials [4] (they are often referred to as the LeCouteur-Newton relations) and since their proof proceeds along much the same lines, we will just indicate the differences between the two situations. One of them lies in the denominator $D(K)$ which appears in Eq. (21). Since $G_0(K;r,r')$, Eq. (9), is an even function of each one of the K_α's, $D(K)$ by its very definition, Eq. (20), shares the same property,

$$D(\ldots,-K_\alpha,\ldots)=D(\ldots K_\alpha,\ldots) . \tag{28}$$

Following Newton's demonstration [4], we rewrite

$$S(K)=K^{-1/2-l}\mathcal{F}'_-(K)\mathcal{F}'_+(K)^{-1}K^{1/2+l} ,$$

where l denotes the partial wave, and

$$\mathcal{F}'_\pm(K)=K^l\mathcal{F}_\pm(K)K^{-l} .$$

Since $\Phi(K,r)K^{-l}$ is an even function of all the K_α's, it follows from Eq. (11) that $[\mathcal{F}'_\pm(K)]_{\alpha\beta}$ is an even function of all the elements of K except K_α. This property, along with (12) and (21), suffices to demonstrate the identity (26) as in Ref. [4], since, as a consequence of (28), the determinants occurring in the numerator and the denominator of (26) cancel each other.

The proof of (27) requires moreover that the S matrix be symmetric. This is ensured by Lemma II.

The usefulness of identities (26) and (27) stems from the fact that while the determinant $D^+(K)$ is well defined at energies where some of the channels are closed, the matrix $\mathcal{F}_+(K)$ need not be so. It is then more useful to use them than (13).

So far, we have treated the K_α's as independent variables. It follows from Eq. (2), however, that they are not so. It will prove useful in the following to consider the matrix K as a function of the single variable

$$k\equiv K_1 , \tag{29}$$

that is, the momentum in channel 1 which has the lowest internal energy.

Theorem I implies two important corollaries which have already been established in the context of local potentials [2–4] and which therefore we shall simply recall.

Let $S^{\mathrm{op}}(k)$ be the submatrix of S which refers to open channels only.

Corollary 1. The submatrix $S^{\mathrm{op}}(k)$ is unitary,

$$S^{\mathrm{op}}(k)[S^{\mathrm{op}}(k)]^\dagger=1 \quad (k \text{ real}) . \tag{30}$$

Therefore,

$$\det[S^{\mathrm{op}}(k)]=\exp[2i\Delta^+(k)] , \tag{31}$$

where $\Delta^+(k)$ is a real function.

Corollary 2.

$$\Delta^+(k)=-\arg D^+(k) \quad (k \text{ real}) . \tag{32}$$

C. Analyticity properties of D^+

The analyticity properties of the Fredholm determinant D^+ here can be demonstrated for "regular" potentials. It has been shown [3] for local potentials that if the condition

$$\int_0^\infty r|U_{\alpha\beta}(r)|e^{2br}dr < \infty \tag{33}$$

is fulfilled for all values of α and β for some $b > 0$, then $D^+(K)$ is analytical in all the half planes $\text{Im}K_\alpha > -b$ (the K_α's being provisionally considered as independent variables). The analogous result for nonlocal potentials is, however, more difficult to state. We will only show that for a certain class of nonlocal potentials condition (33) can be replaced by

$$\int_0^\infty \int_0^\infty r^{1-|s|}e^{br}|U_{\alpha\beta}(r,r')|r'^{|s|}e^{br}dr\,dr' < \infty \ , \tag{34}$$

for some $b > 0$ and for at least one value of s such that $|s| \le 1$. In the following, potentials obeying condition (34) for arbitrary $b > 0$ will be referred to as "regular."

Lemma III. For the class of nonlocal potentials that can be approximated arbitrarily closely by separable potentials of rank P,

$$U(r,r') \approx \sum_{j=1}^{P} W_j(r) \otimes \overline{Y}_j(r') \ ,$$

where $W_j(r)$ and $Y_j(r')$ are N-component vectors, and the overbar denotes a transposition, the Fredholm determinant $D^+(K)$ is analytic in the half planes $\text{Im}K_\alpha > -b$ when condition (34) is fulfilled.

Proof. For a separable potential [15] of rank P, the terms $D^+_{(n)}(K)$ in the expansion (18) vanish identically for $n > P$. One finds therefore

$$D^+(K) = \det[M(K)] \ ,$$

where $M(K)$ is a $P \times P$ matrix, the elements of which are given by

$$M_{ij}(K) = \delta_{ij} - \int_0^\infty \int_0^\infty \overline{Y}_i(r) G_0^+(K;r,r') W_j(r')dr\,dr' \ .$$

Since $D^+(K)$ can be expressed as the determinant of a finite matrix, its analyticity properties are determined by those of the matrix elements $M_{ij}(K)$.

Using the inequality

$$|[G_0^+(K_\alpha;r,r')]_{\alpha\alpha}| \le \frac{r_<}{1+|K_\alpha r_<|}\exp(|\text{Im}K_\alpha r_<| - \text{Im}K_\alpha r_>) \ , \tag{35}$$

which holds for all partial waves, and $r_< \le r^{|s|}r'^{1-|s|}$, where $|s| \le 1$, one shows easily that all the $M_{ij}(K)$'s are finite in the half planes $\text{Im}K_\alpha > -b$, as long as the elements of W_j and Y_j satisfy

$$\int_0^\infty r^{|s|}|Y_{j,\alpha}(r)|e^{br}dr < \infty$$

and

$$\int_0^\infty r^{1-|s|}|W_{j,\alpha}(r)|e^{br}dr < \infty \ ,$$

for all values of α and j and at least one value of $|s| \le 1$. These inequalities will surely be satisfied if all elements of U fulfill condition (34). The same condition (with $b > 0$) also ensures the existence of the derivatives of $D^+(K)$ with respect to the K_α's in the half planes $\text{Im}K_\alpha > -b$.

Lemma IV. If $D^+(K)$ is an analytic function of the K_α's for $\text{Im}K_\alpha \ge 0$ ($\alpha = 1, \ldots, N$), then $D^+(k)$, in the complex k plane, is an analytic function of k on the first sheet for $\text{Im}k \ge 0$.

Proof. In the complex k plane, $D^+(k)$ has branching points on the real axis at points k_α such that $K_\alpha(k) = 0$ ($\alpha \ne 1$). Setting up the cuts from $-\infty$ to $-k_\alpha$ and from $+k_\alpha$ to $+\infty$ ($\alpha \ne 1$) just below the real axis, one sees easily that the first sheet of the k plane is mapped on the half planes $\text{Im}K_\alpha \ge 0$ ($\alpha \ne 1$). Lemma IV thus follows.

D. Bound states

According to the theory of integral equations, the homogeneous equation associated with Eq. (5),

$$\chi^+(k,r) = \int_0^\infty \int_0^\infty G_0^+(k;r,r')U(r',r'')\chi^+(k,r'')dr'dr'' \ , \tag{36}$$

has a nontrivial solution only when k takes a value k_0 such that $D^+(k_0) = 0$. Inversely, a zero of $D^+(k)$ implies a solution of Eq. (36). The solution $\chi^+(k,r)$ has the form of a column vector, since it involves no entrance channel.

As a consequence of the analyticity properties of $D^+(k)$ described in the previous section, the zeros of $D^+(k)$ are of integer order except perhaps at branching points $k = \pm k_\alpha$ ($\alpha = 2, \ldots, N$), where $D^+(k)$ is not analytic. At such points, for instance at $k = k_\gamma$ ($\gamma \ne 1$), it is easy to show that $D^+(k)$ has a zero of order $m/2$ if D^+, expressed as a function of K_γ only, has a zero of order m at $K_\gamma = 0$.

Lemma V. The set of solutions of Eq. (36) such that $\chi^+(k,0) = \chi^+(k,\infty) = 0$ coincides with the set of bound states of the system.

Proof. Applying $G_0^\pm(k;r'',r')$ to the left of the Schrödinger equation, (3), and integrating twice by parts over r from 0 to ∞, one sees using

$$\mathcal{L}_0(K,r)G_0^\pm(K;r,r'') = -1\delta(r-r'') \tag{37}$$

and Eq. (25) that solutions of Eq. (3) such that

$$\psi(k,0) = \psi(k,\infty) = (d/dr)\psi(k,r)|_{r=\infty} = 0 \ ,$$

namely, bound states, obey the homogeneous integral equation (36). Conversely, applying the operator $\mathcal{L}_0(K,r)$ to both members of Eq. (36) and using Eq. (37), one finds that solutions of (36) are equally solutions of (3).

Lemma VI. Bound states occur only for real values of k^2 and any two of them with $k = k_a$ and k_b ($k_a \ne k_b$) satisfy the "orthogonality" relation

$$(\psi(k_b)|\psi(k_a)) \equiv \int_0^\infty [\overline{\psi}(k_b,r)]^* M \psi(k_a,r)dr = 0 \ . \tag{38}$$

Proof. As long as the potential is real and symmetric, and

$$\left[\overline{\psi}^*(k_b,r)\frac{d}{dr}\psi(k_a,r) - \overline{\psi}(k_a,r)\frac{d}{dr}\psi^*(k_b,r)\right]\Bigg|_{r=0}^{r=\infty} = 0 \ , \tag{39}$$

a condition which is satisfied by bound states, two solutions of Eq. (3) with $k = k_a$ and k_b can easily be shown to obey the relation

$$[(k_b^2)^* - k_a^2](\psi(k_b)|\psi(k_a)) = 0 \ .$$

Lemma VI directly follows from this relation.

Of course, for degenerate levels, solutions can always be defined so as to be orthogonal in the sense of Eq. (38).

Lemma VII. Solutions χ^+ of Eq. (36) which occur in the half plane $\mathrm{Im}k \geq 0$ vanish at both $r=0$ and $r=\infty$, except perhaps when they are located at points $k=k_\alpha$ ($\alpha=1,\ldots,N$).

Proof. The very definition of $G_0^+(k;r,r')$ determines the behavior of χ^+ at the origin, $\chi^+(k,0)=0$. Inspection of Eq. (36) shows that $\chi_1^+(k,r)$, the component of $\chi^+(k,r)$ in channel 1, cannot vanish for $r\to\infty$ if $\mathrm{Im}k<0$, which suffices to prove that no bound state can lie in this half plane. It can also easily be shown that, if the potential is regular, $\chi_\alpha^+(k,r)$ vanishes at infinity if $\mathrm{Im}k>0$ or if k is real and under the αth threshold. Let us see what happens to $\chi_\beta^+(k,r)$ when k is above the βth threshold. One can easily show, using Eqs. (5), (36), (25), and (4), that [16]

$$\int_0^\infty \int_0^\infty \bar{u}^{(\beta)}(Kr)U(r,r')\chi^+(k,r')dr\,dr' = 0 \ , \qquad (40)$$

where $u^{(\beta)}(Kr)$ is the βth column of the matrix $u(Kr)$. Now, (36) yields

$$\chi_\beta^+(k,r) \underset{r\to\infty}{\to} -K_\beta^{-1}w^+(K_\beta r)\int_0^\infty \int_0^\infty \bar{u}^{(\beta)}(Kr')U(r',r'') \times\chi^+(k,r'')dr'dr'' \ . \qquad (41)$$

Insertion of (40) in (41) shows immediately that both χ_β^+ and $d\chi_\beta^+/dr$ vanish at infinity for symmetric potentials, except perhaps when $k=k_\beta$ because of the factor K_β^{-1} in (41).

It has been proved for local potentials that bound states on the real k axis occur only for $k<k_N$, k_N being the value of k beyond which the N channels are open (the so-called bound states embedded in the continuum). We will now see that this restriction disappears for nonlocal potentials. Then, one still has the relation [4]

$$\det[\mathcal{F}_+(k)]\neq 0, \quad \text{for } k\geq k_N \ , \qquad (42)$$

but while Eq. (24) implied $D^+(k)\neq 0$ and no bound states for $k\geq k_N$, Eq. (21) holds for nonlocal potentials and (42) may still be true even if $D^+(k)=0$, as long as $D(k)$ has a zero of the same order for the same value of k. For the $N=1$ case, one recovers the well-known result according to which only nonlocal potentials can have bound states for real values of k: these are the continuum bound states (CBS's).

Starting from the definition of $G_0^\pm(K;r,r')$, one can show that $D^+(K)=[D^+(-K^*)]^*$ as long as the potential is real. Since $K_\alpha(-k^*)=-[K_\alpha(k)]^*$, one can write

$$D^+(k)=[D^+(-k^*)]^* \ . \qquad (43)$$

This implies that if $D^+(k_0)=0$ for a real k_0, then $D^+(-k_0)=0$. These two symmetrical zeros on the real axis correspond to the same bound state.

Lemma VIII. (a) To a zero of order m of $D^+(k)$ in the half plane $\mathrm{Im}k\geq 0\neq\pm k_\alpha$ ($\alpha=1,\ldots,N$), correspond m linearly independent bound states. (b) The converse of (a) is also true.

Proof. Lemma VI and the proof of Lemma VII show that the zeros of $D^+(k)$ which are in $\mathrm{Im}k\geq 0$ occur for real values of k^2 and correspond to bound states, except perhaps at $k=\pm k_\alpha$.

Assume that $D^+(k)$ has a zero of order m at $k=k_0\neq\pm k_\alpha$, where k_0^2 is real and $\mathrm{Im}k_0\geq 0$. At least one bound solution of Schrödinger's equation (3) is associated with this zero. We shall call it $\psi_1(r)$ and assume [17] that $(\psi_1|\psi_1)=1$. Let $\psi(k,r)$ be a solution of the same equation for $k\neq k_0$, which does not necessarily vanish as $r\to\infty$. It may thus be a solution in the continuum. Even then condition (39) is fulfilled by ψ_1 and ψ, since both ψ_1 and its first derivative vanish at infinity, as we have seen. One may thus get from Eq. (38),

$$(\psi_1(k_0)|\psi(k))=0 \quad (k\neq k_0) \ . \qquad (44)$$

Consequently, $\psi(k,r)$ is equally a solution of the modified Schrödinger equation,

$$\mathcal{L}_0(k,r)\psi(k,r)+\int_0^\infty U_1(r,r')\psi(k,r')dr'=0 \ , \qquad (45)$$

where

$$U_1(r,r')=U(r,r')+\alpha_1 M\psi_1(r)\otimes\bar{\psi}_1(r')M \ , \qquad (46)$$

α_1 being an arbitrary real number. Notice that $U_1(r,r')$ is symmetric if $U(r,r')$ is. The bound state $\psi_1(r)$ is equally a solution of the modified Schrödinger equation (45), with $k^2=k_0^2+\alpha_1$. Since the second term on the right-hand side of (46) is a separable potential of rank 1, one can show that the Fredholm determinant D_1^+ associated with U_1 is

$$D_1^+(k)=D^+(k)\left[1-\alpha_1\int_0^\infty\int_0^\infty\bar{\psi}_1(r)MG^+(k;r,r')M\psi_1(r')dr\,dr'\right] \ , \qquad (47)$$

where $G^+(k;r,r')$ obeys the equation

$$\mathcal{L}_0(K,r)G^+(k;r,r')+\int_0^\infty U(r,r'')G^+(k;r'',r')dr'' = -1\delta(r-r') \ .$$

The double integral in (47) is simply $(k^2-k_0^2)^{-1}$ and, consequently,

$$D_1^+(k)=D^+(k)\frac{k^2-(k_0^2+\alpha_1)}{k^2-k_0^2} \ . \qquad (48)$$

Thus $D_1^+(k)$ has a zero of order 1 at $k=\sqrt{k_0^2+\alpha_1}$ and another of order $m-1$ at $k=k_0$. The bound state, let us say $\psi_2(r)$, with normalization $(\psi_2|\psi_2)=1$, which is a solution of Eq. (45) and corresponds to the zero of $D_1^+(k)$ at $k=k_0$, cannot be identical with $\psi_1(r)$ since, according to

Lemma VI, they are orthogonal. Furthermore, this orthogonality implies that $\psi_2(r)$ is also a solution of the original Schrödinger equation (3) for $k=k_0$. If one repeats the same argument for the zero of order $m-1$ as for the zero of order m, starting from Eq. (45), and so on, one identifies m bound states $\psi_i(r)$ $(i=1,\ldots,m)$ which are mutually orthogonal and obey Eq. (3) for $k=k_0$. Statement (a) of Lemma VIII is thus proved.

Statement (b) is easily proved by reversing the previous argument: Given m degenerate bound states at $k=k_0$, it follows that $D^+(k)$ must have a zero of order $m'\geq m$ at $k=k_0$. Using statement (a) we conclude that $m'=m$.

If there is a bound state ψ_1 at a branching point $k=k_\gamma$ or at $k=0$, the components of which satisfy

$$\int_0^\infty r^{1/2}|\psi_{1,\alpha}(r)|dr<\infty \;, \tag{49}$$

then $U_1(r,r')$, Eq. (46), satisfies condition (34) with $b=0$ and $s=\frac{1}{2}$ and, according to the proof of Lemma III, $D_1^+(k)$, Eq. (48), is not singular at $k=k_\gamma$. Consequently, the zero of $D^+(k)$ at $k=k_\gamma$ is at least simple when $\gamma\neq 1$ and double when $\gamma=1$. We cannot conclude that the converse statement is also true however.

Condition (49) is certainly too severe in general since for a local potential in the one-channel case it is well known that a double zero of $D^+(k)$ at $k=0$ occurs provided the bound state is normalizable [4]. If it is not normalizable, one has a "half-bound state" and the zero of D^+ at $k=0$ is simple.

In a similar way, we shall define a half-bound state as a solution associated to a simple zero of $D^+(k)$ at $k=0$ or a zero of order $\frac{1}{2}$ at $k=k_\gamma$, $\gamma\neq 1$. However, we have not been able to prove that the condition of normalizability is sufficient to distinguish bound states from half bound states in the present situation.

E. The generalized Levinson theorem

Let us introduce a function $\Delta(k)$ which differs from the phase $\Delta^+(k)$, Eq. (32), only by the fact that it is defined to be continuous at points where $\Delta^+(k)$ undergoes a discontinuity of π; this occurs for real values of k at which $D^+(k)$ has a zero of odd multiplicity.

Theorem II. Let n^b and n^{hb} denote the total number of bound and half-bound states. Then,

$$\Delta(0)-\Delta(\infty)=\pi(n^b+n^{hb}/2) \;. \tag{50}$$

Proof. The proof of this result can be carried out in a standard fashion, and for this reason it will be outlined only. Since $D^+(k)$ is analytic in the upper half of the k plane, one has

$$\oint d\ln D^+(k)=2\pi i\sum_{i=1}^{n}\mu_i \;, \tag{51}$$

for the contour shown in Fig. 1; n is the number of zeros of $D^+(k)$ in the half plane $\mathrm{Im}k>0$, the μ_i's being their respective multiplicities. The contour goes from $-R$ to $+R$ on the real axis, avoiding all branching points and possible zeros on this axis, and it is closed by a semicircle of radius R. In the limit $R\to\infty$, the contour encloses all

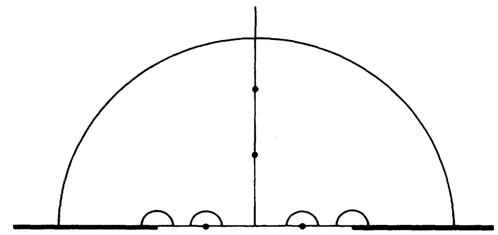

FIG. 1. Schematic representation of the integration contour C in the complex k plane. On the real axis, C avoids branching points k_α (inner ends of bold lines) and zeros of $D^+(k)$ (bold points).

zeros of $D^+(k)$ in $\mathrm{Im}k>0$ and no contribution comes from the integral over the half circle, since, for $\mathrm{Im}k>0$,

$$|D^+(k)-1|=\mathcal{O}(|k|^{-1}) \text{ as } |k|\to\infty \;, \tag{52}$$

as can easily be shown, using the inequality

$$|[G_0^+(K_\alpha;r,r')]_{\alpha\alpha}|<C|k|^{-1} \text{ as } |k|\to\infty \;, \tag{53}$$

which follows from (35). The integrand is perfectly well defined on the rest of the contour, if one takes the precaution of placing the cuts just beneath the real axis. Taking Eq. (43) into account, one gets, for the contour integral,

$$2i\Delta(0)-2i\Delta(\infty)-i\pi\sum_{i=1}^{n'}\mu_i' \;, \tag{54}$$

where n' and μ_i' are the quantities analogous to n and μ_i for the zeros of $D^+(k)$ on the real axis. Equations (51) and (54) yield

$$\Delta(0)-\Delta(\infty)=\pi(s+s_0/2) \;, \tag{55}$$

where s is the number of zeros of $D^+(k)$ on the semiaxes $\mathrm{Im}k>0$ and $\mathrm{Re}k\geq 0$ $(k\neq k_\alpha,\ \alpha=1,\ldots,N)$, and s_0 is the number of zeros of $D^+(k)$ at points $k=k_\alpha$ $(\alpha=1,\ldots,N)$, the zeros being counted according to their multiplicity.

Theorem II follows from (55), Lemma VIII and the definition we gave of half-bound states at the end of Sec. II D.

III. MODIFICATIONS DUE TO CONFINED CHANNELS

A. Definitions

We shall now consider the situation where scattering can take place in N_S of the N channels, while the remaining N_C channels are permanently confined. Labeling them from 1 to N_S and from N_S+1 to N, respectively, we can express the vector ψ in terms of two vectors ψ_S and ψ_C,

$$\psi=\begin{bmatrix}\psi_s\\ \psi_c\end{bmatrix} \;.$$

Similarly,

$$U=\begin{bmatrix} U_{SS} & U_{SC} \\ U_{CS} & U_{CC} \end{bmatrix},\quad K^2=\begin{bmatrix} K_s^2 & 0 \\ 0 & K_c^2 \end{bmatrix},$$

where U_{AB} ($A,B=S$ or C) denotes an $N_A \times N_B$ submatrix, while K_S^2 and K_C^2 are square diagonal matrices of dimensions N_S and N_C, respectively.

The main feature of the confining potential U_{CC} lies in the fact that the Schrödinger equation, when restricted to the confined channels,

$$\mathcal{L}_0(K_C(\kappa_j),r)\varphi_{js}(r)+\int_0^\infty U_{CC}(r,r')\varphi_{js}(r')dr'=0\ , \qquad (56)$$

has no scattering states (the index s, $1\le s\le \sigma_j$, labels possibly linearly independent degenerate solutions). Its spectrum is assumed to be discrete, lower semibounded, and to contain an infinite set of positive eigenvalues occurring at points $k=\kappa_i$ such that κ_i^2 is real. This occurs, for instance, when the diagonal elements of the nonlocal potential $U_{CC}(r,r')$ turn into local harmonic-oscillator potentials for r and r' going to infinity, while the nondiagonal elements remain finite for all values of r and r'.

It proves useful to define the $\kappa_i^{(W)}$'s which are related to the potential $U_{CC}+W$ in the same way as the κ_i's are related to U_{CC} in Eq. (56), W being an arbitrary nonconfining potential acting within the subspace of confined channels only. Obviously, $\kappa_i=\kappa_i^{(W)}|_{W=0}$. The integral equation of the N-channel problem can then be written, for $k\neq\kappa_i^{(W)}$, in the following manner:

$$\Psi^+(k,r)=\Psi_0^+(k,r)+\int_0^\infty\int_0^\infty \mathcal{G}_W^+(k;r,r')U_W(r',r'')\times\Psi^+(k,r'')dr'dr''\ , \qquad (57)$$

where Ψ_0^+ is a diagonal matrix containing the $N_S\times N_S$ diagonal matrix $u(Kr)$ in its upper part and zeros elsewhere. Moreover,

$$U_W=\begin{bmatrix} U_{SS} & U_{SC} \\ U_{CS} & -W \end{bmatrix},\quad \mathcal{G}_W^+=\begin{bmatrix} G_0^+ & 0 \\ 0 & G_C^{(W)} \end{bmatrix},$$

and $G_C^{(W)}(k;r,r')$ is the Green's kernel obeying

$$\mathcal{L}_0(K_C,r)G_C^{(W)}(K_c;r,r')+\int_0^\infty [U_{CC}(r,r'')+W(r,r'')]G_C^{(W)}(K_C;r'',r')dr''=-1\delta(r-r')\ . \qquad (58)$$

The integral equation (57) has been written in such a way that U_W does not contain the confining potential U_{CC}, so that the Freholm determinant D_W^+ of the N-channel problem should exist. It is still expressed by Eqs. (16) and (17) with

$$\mathcal{R}^+(k;r,r')\equiv\int_0^\infty \mathcal{G}_W^+(k;r,r'')U_W(r'',r')dr''\ .$$

It is worth pointing out that although the various D_W^+'s depend on the choice of W, they constitute equivalent definitions of the Fredholm determinant associated with the physical solutions. For instance, when the full system has bound solutions, the corresponding zeros are located at the same point in all of them, since the energies of the bound states are independent of W. They have the following relation with $D^+\equiv D_W^+|_{W=0}$,

$$D^+(k)=d_W(k)D_W^+(k)\ , \qquad (59)$$

where $d_W(k)$ is the Fredholm determinant defined by Eqs. (16) and (17) with

$$\mathcal{R}^+(K;r,r')\equiv\int_0^\infty G_C(K;r,r'')W(r'',r')dr''\ ,$$

G_C being $G_C^{(W)}|_{W=0}$. Since $d_W(k)$ is a real function, the continuous phase $\Delta(k)$ of D_W^+ is independent of W.

It is not difficult to show that D^+ can also be expressed by means of Eqs. (16) and (17) with

$$\mathcal{R}^+(K;r,r')\equiv\int_0^\infty G_0^+(K;r,r'')\mathcal{U}(r'',r')dr''\ ,$$

where the matrix $\mathcal{R}^+$ is now defined in the subspace of scattering channels only and in the space of positions, while

$$\mathcal{U}(k;r,r')=U_{SS}(r,r')+\int_0^\infty\int_0^\infty U_{SC}(r,r'')G_C(K_C;r'',r''')\times U_{CS}(r''',r')dr''dr'''\ . \qquad (60)$$

is interpreted as an effective potential in the subspace of scattering channels.

This interpretation of $\mathcal{U}$ becomes obvious when one projects the complete system of equations on the subspace of scattering channels only, as is frequently done when studying the many-body problem [14] or compound nuclear reactions [13]. The equation for the confined channels can be written, when $k\neq\kappa_i$,

$$\psi_C(k,r)=\int_0^\infty\int_0^\infty G_C(k;r,r')U_{CS}(r',r'')\psi_S(k,r'')dr'dr''\ . \qquad (61)$$

Inserting (61) in the equation for the scattering channels, one gets the reduced set of equations

$$\mathcal{L}_0(K_S,r)\psi_s(k,r)+\int_0^\infty \mathcal{U}(k;r,r')\psi_s(k,r')dr'=0\ . \qquad (62)$$

Since $U_{AB}(r,r')$ obeys the symmetry relation (4), one can show readily from Eqs. (58) and (60) that $G_C(k;r,r')$ and $\mathcal{U}(k;r,r')$ have the same property. There is only one difference between the system of equations (62) and the one considered in Sec. II: Now $\mathcal{U}(k;r,r')$ is a k-dependent potential.

B. Miscellaneous results

Lemma IX. At a point $k=\pm\kappa_j$, $D^+(k)$ generally has a pole, the order of which is $\le\sigma_j$.

Proof. One can easily show that the Green's kernel has the following behavior in the vicinity of κ_j:

$$G_C(k;r,r')=\tilde{G}_C(k;r,r')+\sum_{s=1}^{\sigma_j}\frac{\varphi_{js}(r)\otimes\bar{\varphi}_{js}(r')}{k^2-\kappa_j^2},$$

where $\tilde{G}_C$ is a real regular function at $k=\pm\kappa_j$. The φ_{js}'s are normalized according to $(\varphi_{js}|\varphi_{js})=1$, where the special product (|) involves only the submatrix M_C of M, which is associated with the confined channels. Inserting this in Eq. (60), one gets

$$\mathcal{U}(k;r,r')=\tilde{\mathcal{U}}(k;r,r')+\sum_{s=1}^{\sigma_j}\frac{A_{js}(r)\otimes\bar{A}_{js}(r')}{k^2-\kappa_j^2}, \quad (63)$$

where $\tilde{\mathcal{U}}$ is the nonsingular part of the potential in the same vicinity, and

$$A_{js}(r)=\int_0^\infty U_{SC}(r,r')\varphi_{js}(r')dr'. \quad (64)$$

Let us denote by $\sigma_j'\le\sigma_j$ the number of nonvanishing functions $A_{js}(r)$ for a given j, and let us assume, moreover, that the degenerate functions φ_{js} have been chosen in such a way that σ_j' be minimal, namely that the A_{js}'s be linearly independent. Since the second term on the right-hand side of (63) is a separable potential of rank σ_j', one has, in the vicinity of $k=\pm\kappa_j$,

$$D^+(k)=\tilde{D}^+(k)\det[T_j^+(k)], \quad (65)$$

where $\tilde{D}^+(k)$ is the Fredholm determinant associated with the regular potential $\tilde{\mathcal{U}}$ only and T_j^+ is a $\sigma_j'\times\sigma_j'$ matrix. Its elements are

$$[T_j^+]_{uu'}=\delta_{uu'}-\frac{1}{k^2-k_j^2}\int_0^\infty\int_0^\infty\bar{A}_{ju}(r)\tilde{G}^+(k;r,r')A_{ju'}(r')dr\,dr',$$

where the vectors $A_{ju}(r)$ are nonzero and $\tilde{G}^+$ is the Green's kernel that obeys the Lippmann-Schwinger equation

$$\tilde{G}^+(k;r,r')=G_0^+(k;r,r')+\int_0^\infty\int_0^\infty G_0^+(k;r,r'')\tilde{\mathcal{U}}(k;r'',r''')\tilde{G}^+(k;r''',r')dr'''dr''.$$

Since $\tilde{D}^+(k)$ is regular at $k=\pm\kappa_j$, a singularity of $D^+(k)$ at these points can only come from $\det(T_j^+)$, with an order $\sigma_j'\le\sigma_j$. No singularities may come from $\tilde{G}^+$, since they would be exactly canceled by zeros of $\tilde{D}^+(k)$.

The potential $\mathcal{U}$ will be said to be "regular" if it satisfies condition (34) for some arbitrary $b>0$ and for a value of $s\le1$, except at points $k=\pm\kappa_i$. We shall moreover require that $\kappa_{i+1}-\kappa_i>\kappa_i^{-2+\varepsilon}$ where $\varepsilon>0$, as $\kappa_i\to\infty$. It is easy to show that such potentials exist. Take, for instance, the situation where there is only one confined channel with the local confining potential $U_{CC}(r)=0$ for $r<R$, and ∞ for $r\ge R$. One then has, in the $l=0$ wave,

$$G_C(K_C;r,r')=-\frac{1}{K_C\sin(K_CR)}\sin(K_Cr_<)\times\sin[K_C(R-r_>)],$$

where $r_<\ge0$ and $r_>\le R$, and $\mathcal{U}$ is easily seen to be perfectly well defined in the whole complex k plane except at points k for which $K_C(k)R=n\pi$ ($n=0,\dots,\infty$) [18]. In particular, when k is very large without being in the immediate vicinity of any of the values $k=\pm\kappa_i$, one then has

$$\mathcal{U}(k;r,r')=U_{SS}(r,r')+\mathcal{O}(k^{-\alpha}), \quad (66)$$

where $\alpha\ge0$.

Most of the discussion of Sec. II can be applied to Eq. (62), as long as one takes properly into account the fact that the various quantities defined there do usually not exist at points $k=\pm\kappa_i$.

It should be noticed that Lemma VI and, therefore, Lemma VIII are not directly relevant to the reduced problem (62), because the potential $\mathcal{U}$ is k dependent, which prevents the normalizable solutions of Eq. (62) to be mutually orthogonal. Since, however, a solution of the full N-channel problem corresponds to each solution of Eq. (62), these lemmas and their generalization can be proved quite directly by considering the full N-channel problem instead of the reduced one. Thus, solutions of Eq. (62) can be associated with solutions of the full problem for which Lemma VI holds, and Lemma VIII can be restated in the following way:

Lemma VIII′. (a) To a zero of order m of $D^+(k)$ in the half plane $\mathrm{Im}k\ge0$, $k\ne\kappa_i$ and $k\ne k_\alpha$ ($\alpha=1,\dots,N$) correspond m linearly independent bound states of the full N-channel problem. (b) The converse of (a) is also true.

Proof. At points $k\ne\kappa_i$, $D^+(k)$ is well defined and Lemma VIII can be demonstrated here almost in the same way as in Sec. II. One difference in the present situation is that the Fredholm determinant D_1^+ corresponding to the potential U_1, Eq. (46), now becomes

$$D_1^+(k)=D^+(k)\frac{k^2-(k_0^2+\alpha_1)}{k^2-k_0^2}\left[1-\alpha_1\int_0^\infty\int_0^\infty\bar{\psi}_{1_C}(r)M_CG_C(k;r,r')M_C\psi_{1_C}(r')dr\,dr'\right]^{-1},$$

ψ_{1_C} being the C part of the bound state ψ_1. The confining part of U_1 is then defined as $U_{1_{CC}}=U_{CC}+\alpha_1|\psi_{1_C}\rangle\langle\psi_{1_C}|$ and the ensuing shift of the poles of D^+ can be shown to be carried out by the real factor between brackets.

Statement (b) of Lemma VIII′ can be proved in the same way as in Lemma VIII.

The following Lemma extends Lemma VIII′ to points $k=\pm\kappa_i$.

Lemma X. (a) If $D^+(k)$ contains a factor $(k^2-\kappa_j^2)^{-p_j}$, $\kappa_j \neq k_\alpha$ $(\alpha=1,\ldots,N)$, then the full N-channel problem has $m_j=\sigma_j-p_j$ bound states at $k=\kappa_j$. (b) The converse of (a) is also true.

Proof. In order to deal with the points $k=\pm\kappa_j$ we shall make use of D_W^+ for $W\neq 0$. When W is properly chosen, D_W^+ is well defined at $k=\pm\kappa_j$, contrarily to D^+, since its poles occur at points $k=\pm\kappa_i^{(W)}$. It is particularly illuminating to use

$$W(r,r')\equiv \sum_{s=1}^{\sigma_j} \beta_s M_C \varphi_{js}(r)\otimes\overline{\varphi}_{js}(r')M_C ,$$

where the coefficients β_s are nonzero. While shifting the eigenvalues of Eq. (56) from κ_j^2 to $\kappa_j^2+\beta_s$, this potential leaves those of the full problem unaffected. One then obtains from (59)

$$D^+(k)=\frac{\prod_{s=1}^{\sigma_j}[k^2-(\kappa_j^2+\beta_s)]}{(k^2-\kappa_j^2)^{\sigma_j}} D_w^+(k) . \quad (67)$$

Now, since D^+ contains a factor $(k^2-\kappa_j^2)^{-p_j}$, D_w^+ must contain a factor $(k^2-\kappa_j^2)^{\sigma_j-p_j}$. But from Lemma IX we know that $p_j\leq\sigma_j$. Therefore, D_W^+ must have a zero of order $m_j=\sigma_j-p_j$ at $k=\pm\kappa_j$. Lemma VIII′ can be used here for D_W^+ since it is well defined at $k=\pm\kappa_j$. While shifting the eigenvalues of the Schrödinger equation restricted to confined channels from κ_i^2 to $\kappa_i^2+\beta_s$, the potential W obviously leaves the eigenvalues of the full coupled problem unaffected. We conclude that the full N-channel problem has m_j bound states at $k=\pm\kappa_j$.

The converse of statement (a) is easily proved by reversing the argument.

One can as well formulate Lemmas VIII′ and X in terms of the solutions of Eq. (62) alone. The formulation of Lemma X is, however, more intricate.

In order to do this, let us first put all sets of degenerate bound states of the full N-channel problem in a form such that the states are mutually orthogonal and linearly independent, while all the nonzero S parts (ψ_S) are linearly independent. This can be achieved through the following prescription. First, find arbitrary linear combinations of the states such that all the nonzero S parts are linearly independent. Second, in order to get a set of mutually orthogonal states, use the Schmidt orthogonalization method, beginning with the states for which the S part is zero.

Whenever a state has a vanishing S part, its C part (ψ_C) is a solution of Eq. (56) at $k=\kappa_j$ for some j. Moreover, the term coupling the S and C channels must then vanish,

$$\int_0^\infty U_{SC}(r,r')\psi_C(r')dr'=0 . \quad (68)$$

Such a confined state ψ_C is thus completely decoupled from the scattering channels. Conversely, if a solution φ_{js} of (56) at $k=\kappa_j$ is such that the corresponding $A_{js}(r)$, Eq. (64), vanishes, the state $\psi=\binom{0}{\varphi_{js}}$ is a solution of the full N-channel problem at the same value of k. It may be worth stressing that this possibility is a consequence of our considering nonlocal potentials: If U_{SC} were local, condition (68) would be satisfied only in the trivial situation $U_{SC}(r)=0$. Let us note finally that bound states occurring at $k\neq\pm\kappa_i$ all have linearly independent nonvanishing S parts.

We thus conclude that as long as $k\neq\pm\kappa_i$, there is a one-to-one correspondence between solutions of Eq. (62) and those of the full N-channel problem. Therefore, Lemma VIII′ also holds for the solutions of Eq. (62) since it has been established for the full problem.

Let us denote by m_j' the number of solutions of Eq. (62) at $k=\kappa_j$ (with, obviously, $\psi_S\neq 0$) and by g_j the number of solutions of the full N-channel problem with a zero S part. One then has $m_j'=m_j-g_j$ and $\sigma_j'=\sigma_j-g_j$. It is thus possible to express p_j in terms of m' in Lemma X, namely, $p_j=\sigma_j'-m_j'$.

C. The generalized Levinson theorem

Lemma XI. When k is large and passes through a point κ_j, for which $\sigma_j=1$, the phase $\Delta(k)$ increases continuously by π within an interval of order κ_j^{-2}. Outside this interval, $\Delta(k)=\mathcal{O}(k^{-\alpha})$ (mod $n\pi$) where $\alpha>0$.

Proof. Even though $D^+(k)$ is singular at points $k=\kappa_i$, the continuous phase $\Delta(k)$ and its first derivative remain perfectly well defined at those points since the leading singular factor $(k^2-\kappa_j^2)^{-1}$ appears in front of both the real and the imaginary parts of $D^+(k)$.

Let us consider the Fredholm determinant $D^+(k)$ in the vicinity of κ_j for $\kappa_j\to\infty$. Assuming that (66) holds, one can easily show that if U_{SS} is regular, $\widetilde{G}^+(k;r,r')\approx G_0^+(k;r,r')$ and $|\widetilde{D}^+(k)-1|=\mathcal{O}(k^{-\alpha})$, $\alpha>0$, as $k\to\infty$. Thus, for $\sigma_j=1$, (65) becomes, near κ_j,

$$D^+(k)\approx\frac{\widetilde{D}^+(k)}{k^2-\kappa_j^2}\left[k^2-\kappa_j^2-\int_0^\infty\int_0^\infty \overline{A}_j(r)G_0^+(k;r,r')A_j(r')dr\,dr'\right] . \quad (69)$$

As pointed out in Ref. [11], the behavior of the phase $\Delta(k)$ near κ_j is determined by the complex zero of $D^+(k)$ near the real point $k=\kappa_j$, say, at $k=\overline{k}_j$, as for the usual resonances. It follows from the inequality (53) that the modulus of the double integral in (69) decreases at least as fast as k^{-1}. Therefore, $\mathrm{Re}\overline{k}_j\leq\kappa_j+|\mathcal{O}(\kappa_j^{-2})|$ and $|\mathrm{Im}\overline{k}_j|\leq|\mathcal{O}(\kappa_j^{-2})|$, which indicates that, when k passes through κ_j, the continuous phase $\Delta(k)$ varies by π over an interval of order κ_j^{-2}. Moreover, this variation is positive since, by rewriting the double integral in (69), for $k>0$, as

$$\mathrm{P}\int_0^\infty \frac{|\widetilde{A}_j(p)|^2}{k^2-p^2}dp - i\frac{\pi}{2}k^{-1}|\widetilde{A}_j(k)|^2 ,$$

where $P\int$ stands for the Cauchy principal value integral, and

$$\tilde{A}_j(k)=\left[\frac{2}{\pi}\right]^{1/2}\int_0^\infty A_j(r)u(kr)dr ,$$

we see that $\mathrm{Im}\bar{k}_j<0$. Equation (64) also entails that, outside an interval of order κ_j^{-2} about κ_j, $\Delta(k)=\mathcal{O}(k^{-\alpha})$ (mod $n\pi$).

The $\sigma_j>1$ case is more difficult to deal with, since $D^+(k)$ involves the determinant of the $\sigma_j\times\sigma_j$ matrix T_j^+ (assuming $g_j=0$). We expect, as in the $\sigma_j=1$ case, $\Delta(k)$ to be determined by the σ_j zeros of $\det(T_j^+)$ near κ_j. We know moreover from Sec. II D that no zero of $D^+(k)$ at a point $k=\bar{k}_j$ can lie in $\mathrm{Im}k>0$, when $\mathrm{Re}\bar{k}_j\neq 0$. If t_j amongst the σ_j zeros are on the real axis, $\Delta(k)$ will increase by $\pi(\sigma_j-t_j)$ as k passes through κ_j within an interval of order κ_j^{-2} since a zero on the real axis only causes a discontinuity of π, which, from the definition of $\Delta(k)$, is not taken into account.

Theorem III. For large values of k such that

$$\kappa_j+|\mathcal{O}(\kappa_j^{-2})|<k<\kappa_{j+1}-|\mathcal{O}(\kappa_{j+1}^{-2})| ,$$

one has the relation

$$\Delta(0)-\Delta(k)=\pi(n^b+n^{\mathrm{hb}}/2-n^j)+\mathcal{O}(k^{-\alpha}) , \quad (70)$$

where n^b is the number of bound states of the full N-channel problem, n^{hb} the number of half-bound states, and n^j the total number of solutions of Eq. (56) occurring at points $k^2=\kappa_i^2$ such that $\kappa_i^2\leq\kappa_j^2$.

Proof. The regularity of $\mathcal{U}$ ensures that $D^+(k)$ is an analytic function in the half plane $\mathrm{Im}k\geq 0$ (see Sec. II C), except at $k=\pm\kappa_i$ and at branching points.

Let us come back to the integral on the left-hand side of (51). Besides avoiding zeros and branching points of $D^+(k)$ on the real axis, the contour C now must also go around the poles $k=\pm\kappa_i$; the radius R of the semicircle is located between κ_j and κ_{j+1} and is larger than any of the zeros of $D^+(k)$ on the real axis. A given pole of $D^+(k)$ is easily shown to give a contribution of opposite sign to that of a zero of $D^+(k)$ of the same order.

As long as $\mathcal{U}$ is regular and $|k|$, while being large, is not close to any of the κ_i's, condition (52) holds and the semicircle of radius $k=R$ gives a vanishing contribution to the integral. As indicated by Lemma XI this contribution is significant only in an interval of order κ_i^{-2} near the κ_i's, where it can be shown to be discontinuous.

One thus has

$$\Delta(0)-\Delta(k)=\pi(s+s_0/2-q_j)+\mathcal{O}(k^{-\alpha}) , \quad (71)$$

where s and q_j are, respectively, the number of zeros and poles of $D^+(k)$ on the semiaxis $\mathrm{Im}k>0$ and the segment $0<\mathrm{Re}k\leq\kappa_j$ ($k\neq k_\alpha$, $\alpha=1,\ldots,N$), and s_0 is the number of zeros of $D^+(k)$ at points $k=k_\alpha$ ($\alpha=1,\ldots,N$), the zeros and poles being counted according to their multiplicity.

Using Lemmas VIII′ and X, one can see that

$$s+s_0/2=n_b'+\sum_{\substack{i=1\\(m_i>\sigma_i)}}^{j}(m_i-\sigma_i)+n^{\mathrm{hb}}/2 ,$$

where n_b' is the number of bound states at points other than the κ_i's, and

$$q_j=\sum_{\substack{i=1\\(\sigma_i\geq m_i)}}^{j}(\sigma_i-m_i)$$

Now, since

$$n^j=\sum_{i=1}^{j}\sigma_i$$

$$n_b=n_b'+\sum_{i=1}^{j}m_i ,$$

the right-hand sides of (70) and (71) are equal.

The discussion of this section would have been almost identical, had we considered a local potential U instead of a nonlocal one. In particular, Theorem III coincides with the corresponding result of Ref. [11] when $N_S=1$. The main reason for this is that even when the complete potential U is local, the effective potential $\mathcal{U}$ is nonlocal: Nonlocality is thus an unavoidable feature of scattering problems with confined channels. In particular, in the light of the present discussion, the CBS's observed in Ref. [11] appear to be a natural consequence of the nonlocality of $\mathcal{U}$.

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[15] The proof that $\mathcal{L}^2$ kernels can be approximated arbitrarily closely by finite-rank kernels can be found, for instance, in F. Smithies, *Integral Equations* (Cambridge University Press, Cambridge, England, 1970).
[16] Equation (40) is only true when k is above the βth threshold, since otherwise the elements $S_{\gamma\beta}$ ($\gamma=1,\ldots,N$) of the S matrix generally have a pole at the position of the bound state and, consequently, the corresponding elements of Ψ^+, used in obtaining Eq. (40), do not exist [see, for instance, J. R. Taylor, *Scattering Theory,* (Wiley, New York, 1972), Sec. 20-d].
[17] It may be worth pointing out that we have not proved that this is always possible: Nothing excludes the pathological possibility that ψ might vanish at $r=\infty$ and still not be square integrable.
[18] In Refs. [11,12], a similar conclusion has been shown to hold for harmonic-oscillator potentials.