Born-approximation amplitudes for pion production in heavy-ion collisions

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Formal solutions of the second-order amplitude in the Born approximation to the quantal, microscopic many-body problem of pion production from the collision of equal-mass nuclei are presented where Δ-hole states are created in the intermediate nuclear states. The form factors containing nuclear structure information are solved in terms of particle-hole coefficients which describe any intermediate particle-hole state, regardless of whether it is coherent or incoherent. In this approach, the important tensor term has been included. A semiquantitative analysis is carried out which gives an indication of the approximate conditions under which pion production may be maximized under the excitation of Δ-hole states by the transition-spin, -isospin, one-pion-exchange interaction.

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I. INTRODUCTION

The subject of subthreshold pion production provokes a great deal of interest both theoretically and experimentally. On the theoretical side, for example, a number of authors [1–4] have proposed various independent-particle models which involve incoherent nucleon-nucleon collisions as a basis for interpretation. Other groups [5–8] have proposed various thermal models which assume a local "hot spot" or a cooperative multinucleon process. Furthermore, collective or coherent models have been proposed as playing a role in the interpretation, especially near absolute threshold [9–15]. A summary of typical experiments can be found in the excellent review article by Braun-Munzinger and Stachel [16], where it is pointed out that the observed cross sections cannot be fully understood in terms of any of the models yet proposed. Inclusive experiments [17–19] for the production of neutral pions suggest that either collective or coherent effects may be needed to get better agreement with data. However, a new generation of semiexclusive experiments [20] on charged pions in coincidence with either target and/or projectile fragments may shed greater light on the mechanisms that may be involved, but they are still in an early stage of development. A most exciting experiment has just been reported by Erazmus *et al.* [21] that strongly suggests the existence of a coherent subthreshold pion-production process.

This paper is the first in a series which presents the rather long and involved microscopic formalism for Δ production that for the first time under the present approach contains the tensor term of the one-pion-exchange potential (OPEP) modified to produce a rudimentary pisobar [22] in a nucleus. Because of the lower energies involved in the subthreshold range, only π exchange is considered. However, ρ exchange will be included in subsequent work. Also, the formalism will be applied only to π^0 production so as not to unnecessarily complicate an already complicated formalism with Coulomb effects. Furthermore, pion absorption will be considered in later work. In this paper I will concentrate on finding out under what conditions will pion production be maximal in the subthreshold regime. The motion of the nuclei will be treated in the Born approximation, so that resulting calculations will represent an upper estimate of pion production. The reasons for starting with the Born approximation is that not only will the initial calculation be more tractable, but theoretical comparisons can be made between the tensor and central contributions in the interaction to the overall production amplitude and the importance of the shell-model signature on the outgoing pions can be determined without the entangling effects of distortion complicating the picture. The present formalism includes pions from both the valence and core of each nucleus, and it is expected that surface pions will only be minimally affected by absorption.

This work represents a completely new formulation compared to previous work [14,15] as it contains the all important tensor term in the nucleon-nucleon interaction and is an important step forward. This paper will be the first in a series of papers that describe solutions to the general problem. The latter part of this paper is a semiquantitative analysis of the formal results and will be useful in providing a feel for the various aspects of the solution before lengthy computer codes are written. The next paper in the series will describe the formal solutions to the Δ- and particle-hole coefficients where it is assumed that coherent spin-isospin modes are excited in both nuclei. This work is itself quite lengthy and is specific enough that it warrants a separate paper. The next paper will describe the cross sections obtained from the amplitudes, and the numerical results will be compared to existing data. The kinematics and energy dependences have been solved formally, but are quite involved in detail because of the various relativistic transformations needed to translate the results to the laboratory frame. Then a final paper will include distortion effects so that a theoretical comparison can be made with the Born-approximation results as well as to the data.

II. FORMALISM

The problem addressed is that of the collision of two nuclei producing a Δ-isobar-hole state in either nucleus

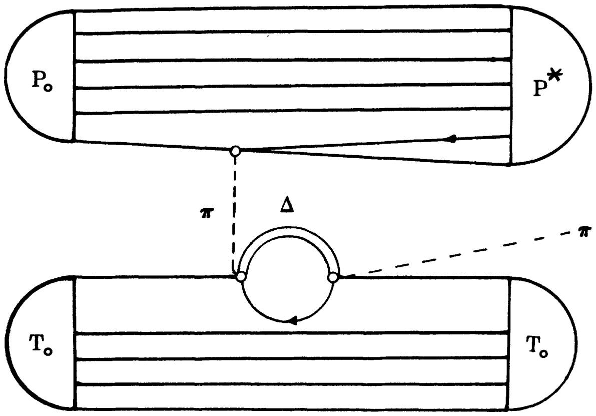

FIG. 1. Diagram showing a Δ-hole state excited in the target with simultaneous excitation of a particle-hole state in the projectile. The Δ then decays, producing a pion.

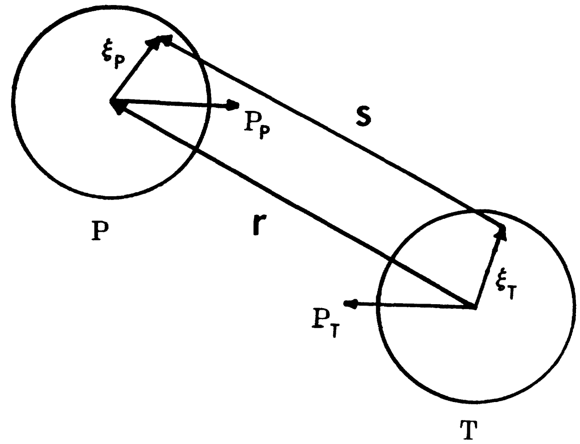

FIG. 2. Coordinates involved in the nucleus-nucleus collision viewed from the nucleus-nucleus rest frame.

where each nucleus is excited and the pions are then generated by the decay of the Δ isobar as shown in Fig. 1. For example, if coherent excited states are formed, then Δ-hole states may be created in the target while particle-hole states are created in the projectile. If we consider ^{12}C as the projectile, it might be excited to the $M1$ state at 15.11 MeV, whereas the target, which could also be ^{12}C, is excited to a ^{12}C(Δ) coherent state, which returns to its ground state after ejecting the pion. The diagram shows only the valence nucleons being promoted to excited states, but it is meant to be representative of excitation of any of the nucleons in either nucleus. Also, the labels P and T can be switched to generate the amplitudes for Δ-hole states excited in the projectile, and the total amplitude for excitation of Δ-hole states in either nucleus will then be the sum of the individual amplitudes.

A. Multipole decomposition of the interaction producing isobars

The spatial coordinates used in this calculation are shown in Fig. 2, where amplitudes for Δ production and decay will be calculated from the nucleus-nucleus rest frame. Then appropriate relativistic transformations are taken to relate all the input and output quantities to the laboratory frame. By using the Fourier integral theorem and generalized angular momentum expansions for the plane waves and products of plane waves and spin operators that appear in the Fourier transforms, it is possible to decompose the nn interaction into complicated sums of separate functions which are, individually, functions of the relative coordinate $\mathbf{r}$, and the projectile and target coordinates, ξ_p and ξ_t as [23]

$$g_{p\Delta}(\mathbf{s})=g^{\parallel}_{p\Delta}(\xi_p,\xi_\Delta,\mathbf{r})-g^{\perp}_{p\Delta}(\xi_p,\xi_t,\mathbf{r}) , \tag{1}$$

where

$$g^{\parallel}_{p\Delta}(\xi_p,\xi_\Delta,\mathbf{r})=\sqrt{4\pi}\sum_{nk_pk_\Delta L} i^{k_p-k_\Delta-L}\hat{k}_p\hat{k}_\Delta f^{\parallel}_n(k_n)\begin{pmatrix}k_p & k_\Delta & l\\ 0 & 0 & 0\end{pmatrix}\{[\mathcal{T}^{s\parallel}_{k_p}(k_n,\xi_p)\times\mathcal{T}^{s\parallel}_{k_\Delta}(k_n,\xi_\Delta)]\cdot\mathcal{T}_L(k_n,\mathbf{r})\}(\tau_p\cdot\mathbf{T}_\Delta) \tag{2a}$$

and

$$g^{\perp}_{p\Delta}(\xi_p,\xi_\Delta,\mathbf{r})=\sqrt{4\pi}\sum_{nk_pk_\Delta L} i^{k_p-k_\Delta-L}\hat{k}_p\hat{k}_\Delta f^{\perp}_n(k_n)\sum_{\kappa=\pm1}\begin{pmatrix}k_p & k_\Delta & L\\ \kappa & -\kappa & 0\end{pmatrix}\{[\mathcal{T}^{s+}_{k_p}(k_n,\xi_p)\times\mathcal{T}^{s-}_{k_\Delta}(k_n,\xi_\Delta)]\cdot\mathcal{T}_L(k_n,\mathbf{r})\}(\tau_p,\mathbf{T}_\Delta) . \tag{2b}$$

The longitudinal and transverse components of $f^{\parallel}_n(k_n)$ and $f^{\perp}_n(k_n)$ as well as $g^{\parallel}_{p\Delta}(k_n)$ and $g^{\perp}_{p\Delta}(k_n)$ are given in Ref. [23] with the addition of the nucleon form factor, which comes from the momentum-transfer dependence of the coupling constants, which leads to [24]

$$F_N(k_n)=\left[\frac{\Lambda_\pi^2-(m_\pi c^2)^2}{\Lambda_\pi^2+(\hbar c k_n)^2}\right] , \tag{3}$$

where $\Lambda_\pi=1300$ MeV. This factor is essentially constant over the subthreshold region and provides convergence for very large momentum transfers.

The other sums in (2a) and (2b) are sums of the projectile and target total angular momentum transfer and the relative orbital angular momentum transfer k_p, k_Δ, and L, which are coupled as $\mathbf{L}=\mathbf{k}_p+\mathbf{k}_\Delta$ via the 3-j symbols. Note that because of the 3-j symbol in (2a), the angular momentum sums are restricted to $k_p + k_\Delta + L =$even.

The tensors of rank k_i in (2a) and (2b), which separate out the relative motion space from the nuclear spaces, are also given in Ref. [23].

B. Second-order amplitude

The second-order amplitude C_{FI} for the process of Δn^{-1} excitation in the target while simultaneously exciting the projectile to nn^{-1} states and then the subsequent decay of the Δ to a produce a pion is

$$C_{FI} = \sum_N \frac{\langle F|V_2|N\rangle\langle N|V_1|I\rangle}{E_F - \tilde{E}_N} , \tag{4}$$

in which the initial, intermediate, and final states are given by

$$|I\rangle = |P_0 T_0 \Phi_{P_I}\Phi_{T_I} 0_\pi\rangle , \tag{5a}$$

$$|N\rangle = |N(P)N(T(\Delta))\Phi_{P_N}\Phi_{T_N} 0_\pi\rangle , \tag{5b}$$

$$|F\rangle = |F(P)T_0\Phi_{P_F}\Phi_{T_F} 1_\pi\rangle , \tag{5c}$$

where the first two states in the kets refer to the ground, intermediate, and final states of the projectile and target nucleus, the next two refer to the center-of-mass motion of each nucleus in the initial, intermediate, and final states as seen in the nucleus-nucleus rest frame, and the final ket refers to the pion space. In general, the nuclear motion can be described by distorted waves, but since the total energy of each nucleus is relatively high, plane-wave motion will be assumed where the distorted waves $|\Phi_{P_K}\Phi_{T_K}\rangle$ are replaced by the plane waves $|\mathbf{P}_{P_K}\mathbf{P}_{T_K}\rangle$ in the Born approximation for $K = I$, N, or F.

The final-state energy of the outgoing system in (4) is E_F and the intermediate energy $\tilde{E}_N$ is given by

$$\tilde{E}_N = E_N + \Delta E_N - i\Gamma_N/2 , \tag{6}$$

where, in the interest of obtaining a simplified expression for a preliminary calculation, the energy shift $\Delta E_N = \Delta(m_\Delta c^2)$ is approximately -30 MeV for the value of Δ-shifted mass in a nuclear medium [25]. Furthermore, if binding energies are neglected, then the energy denominator can be approximated by

$$E_F - \tilde{E}_N \cong \epsilon_\pi + m_n c^2 - m'_\Delta c^2 + i\Gamma'_\Delta(\epsilon_\pi)/2 , \tag{7}$$

where the kinetic energies of an individual nucleon or Δ particle from Fermi-gas considerations are small compared to their rest-mass energies. The pion energy dependence of the Δ width $\Gamma'_\Delta(\epsilon_\pi)$ will, however, be included where ϵ_π is the total energy of the emitted pion. The simplified second-order amplitude then becomes

$$C_{FI} = \frac{\sum_N \langle F|V_2|N\rangle\langle N|V_1|I\rangle}{\epsilon_\pi + m_n c^2 - m'_\Delta c^2 + i\Gamma'_\Delta(\epsilon_\pi)/2} , \tag{8}$$

where the primes on the Δ mass and width indicate values of the Δ isobar inside the nuclear medium.

C. Formation amplitude

In this section formal expressions will be obtained for the formation amplitude $\langle N|V_1|I\rangle$ assuming plane waves for the center-of-mass motion for each nucleus. Using a particle-hole model for the nuclear excited states,

$$|N(A)\rangle = \sum_{\text{ph}} x_{\text{ph}}(A)|\text{ph};J_A M_A T_A T_{ZA}\rangle , \tag{9}$$

where particle-hole (ph) coefficients x_{ph} have to be determined for each nucleus $A = P$ or T. The formal solutions obtained in this work will depend on the values of these coefficients, and so the solutions are quite general, involving excitations to any nuclear excited state described by the particle-hole model. If the coefficients are in phase, then the excitation is a coherent state. The ph states are angular momentum coupled and are given in Ref. [14].

After a great deal of Clebsch-Gordan algebra, substitution of Eqs. (1), (2), and (9) for the projectile and target nuclear states, the following is obtained:

$$\langle N|V_1|I\rangle = \langle N|V_1^{\parallel}|I\rangle - \langle N|V_1^{\perp}|I\rangle , \tag{10}$$

where

$$\langle N|V_1^{\parallel}|I\rangle = A_{J_T J_P}^{T_Z(T)} g^{\parallel}(K) \sum_L H_{J_T J_P}^{\parallel}(K)\Theta_{J_T J_P L}^{M_T M_P}(\hat{\mathbf{k}}) \times \delta_{T_T,1}^{T_P,1}\delta_{T_Z(T)}^{-T_Z(P)}\delta_{\mathbf{P}_{P_N}+\mathbf{P}_{T_N},\mathbf{P}_{P_I}+\mathbf{P}_{T_I}} , \tag{11}$$

with a similar expression for the transverse matrix element. The amplitude

$$A_{J_T J_P}^{T_Z(T)} = (-)^{T_Z(T)}\sqrt{8/3}(4\pi/V)\sqrt{4\pi}\, i^{J_T - J_P} , \tag{12}$$

while the angular function for the relative nuclear motion is

$$\Theta_{J_T J_P L}^{M_T M_P}(\hat{\mathbf{K}}) = \sum_M \begin{pmatrix} J_T & J_P & L \\ M_T & M_P & M \end{pmatrix} Y_L^{-M}(\hat{\mathbf{K}}) , \tag{13}$$

where the relative angular momentum and total target and projectile angular momentum in the intermediate states are coupled through the 3-j symbol in (13) as $\mathbf{L} = \mathbf{J}_T + \mathbf{J}_P$. The coupled longitudinal and transverse nuclear form factors are given by

$$H_{J_T J_P L}^{\parallel}(K) = G_{J_T}^{\parallel}(K)\hat{J}_T\hat{J}_P\hat{L}\begin{pmatrix} J_T & J_P & L \\ 0 & 0 & 0 \end{pmatrix} G_{J_P}^{\parallel}(K) , \tag{14a}$$

$$H_{J_T J_P L}^{\perp}(K) = \sum_{\kappa=\pm1} G_{J_T,\kappa}^{\perp}(K)\hat{J}_T\hat{J}_P\hat{L} \times \begin{pmatrix} J_T & J_P & L \\ \kappa & -\kappa & 0 \end{pmatrix} G_{J_P,-\kappa}^{\perp}(K) , \tag{14b}$$

where the form factors for the target and projectile nuclei are, respectively,

$$G_{J_A}^{\parallel}(K) = \sum_{k_i=\pm1} b_{J_A,k_i}^{\parallel} F_{J_A,k_i}(K) \tag{15}$$

and

$$G_{J_A,\pm\kappa}^{\perp}(K) = \frac{1}{\sqrt{2}} \sum_{k_i=-1}^{1} b_{J_A,k_i}^{\perp} F_{J_A,k_i}(K) , \tag{16}$$

in which the coefficients for $I=T$ or P and $i=t$ or p are defined as

$$b^{\parallel}_{J_A,k_i}=\frac{1}{\hat{J}_A}\sqrt{J_A+(1-k_i)/2},\quad k_i=\pm 1\ , \tag{17a}$$

and

$$b^{\pm\kappa}_{J_A,k_i}=\begin{cases}\pm i\kappa, & k_i=0 \quad (17b)\\ \dfrac{(-1)^{(1-ki)/2}}{\hat{J}_A}\sqrt{J_A+(1+k_i)/2}, & k_i=\pm 1\ ,\end{cases} \tag{17c}$$

where $\hat{J}_A=\sqrt{2J_A+1}$ for $i=p$ or Δ. The individual target multipole form factor from the particle-hole model is

$$F_{J_T,k_t}(K)=\sum_{\Delta h}(-1)^{l_h}x^*_{\Delta h}\hat{l}_\Delta\hat{l}_h\hat{j}_\Delta\hat{j}_h R^{\Delta h}_{J_T-k_t}(K)\times\frac{(\widehat{J_T-k_T})}{\sqrt{4\pi}}\begin{pmatrix}l_h & l_\Delta & J_T-k_t\\ 0 & 0 & 0\end{pmatrix}\sqrt{4}\times\begin{Bmatrix}l_h & l_\Delta & J_T-k_t\\ \frac{1}{2} & \frac{3}{2} & 1\\ j_h & j_\Delta & J_T\end{Bmatrix}, \tag{18}$$

with a similar expression for $F_{J_p,k_p}(K)$, where $\Delta\rightarrow p$, $T\rightarrow P$, $t\rightarrow p$, and $\sqrt{4}$ is replaced by $\sqrt{6}$ and the center element in the 9-j symbol is $\frac{1}{2}$. The Δ-hole integral is given by

$$R^{\Delta h}_{J_T-k_t}(K)=\int_0^\infty d\xi\, u_{n_\Delta l_\Delta}(\xi)j_{J_T-k_t}(K\xi)u_{n_h l_h}(\xi)\ , \tag{19}$$

with a similar expression for $R^{\text{ph}}_{J_P-k_p}(K)$. The three-dimensional harmonic-oscillator function $u_{nl}(\xi)$ is corrected for rest mass. The nucleon mass m_n is replaced by the nuclear medium value m'_Δ in order to calculate the Δ-harmonic-oscillator length parameter [15]. The particle-hole integrals will be large whenever the overlap between the oscillator and spherical Bessel functions is maximal. The factors $\sqrt{4}$ and $\sqrt{6}$ in the form factors come from evaluations of the spin matrix elements for Δ formation. Because of these factors, it is slightly easier to excite the particle hole than the Δ hole in terms of spin considerations.

D. Decay amplitude

After the Δ-hole state is formed in the target, the pion is produced by the decay of $\Delta\rightarrow n\pi$, and it is assumed that this decay is independent of the projectile nucleus with the target returning to its ground state. Plane waves are again assumed for the intermediate and final nuclear motion, and the final state of the projectile may be excited to an isobar analog state that may decay via either the weak or electromagnetic interaction. The decay amplitude then reduces to

$$A^{M_T}_{J_T}(\mathbf{k}_\pi)=h(k_\pi)F_{J_T}(k_\pi)Y^{M_T}_{J_T}(\hat{\mathbf{k}}_\pi)\ , \tag{20}$$

where the angular distribution of the target state is picked up by the pion since the target returns to its ground state and

$$h(k_\pi)=\left[\frac{4}{3}\right]^{1/2}\frac{4\pi k_\pi f_n(k_\pi)}{\sqrt{2\epsilon_\pi V}}F_{\Delta n\pi}\ , \tag{21}$$

which contains the nucleon form factor

$$f_n(k_\pi)=\int d^3\xi\,\rho(\xi)\exp(-i\mathbf{k}_\pi\cdot\xi)\ . \tag{22}$$

The target form factor from the particle-hole model is given by by a multipole expansion as

$$F_{J_T}(k_\pi)=\sum_k(-i)^{k+1}\hat{k}\begin{pmatrix}k & 1 & J_T\\ 0 & 0 & 0\end{pmatrix}D^\Delta_{J_T,k}(k_\pi)\ , \tag{23}$$

where the total orbital angular momentum J_T delivered to the pion comes from the plane-wave orbital multiple angular momentum $\hbar k$ and the spin difference $\Delta S=1$ which comes from the conversion of the spin-$\frac{3}{2}$ Δ particle to a spin-$\frac{1}{2}$ nucleon such that $J_T=k$, $k\pm 1$, for all values of k. The Δ-decay multipole form factor is defined as a sum of 9-j symbols over Δ-hole states as

$$D^\Delta_{J_T,k}(k_\pi)=\sum_{\Delta h}(-1)^{l_h}x_{\Delta h}\hat{l}_\Delta\hat{l}_h\hat{j}_\Delta\hat{j}_h R^{\Delta h}_k(k_\pi)\times\frac{\hat{k}}{\sqrt{4\pi}}\begin{pmatrix}l_h & l_\Delta & k\\ 0 & 0 & 0\end{pmatrix}\sqrt{4}\begin{Bmatrix}l_h & l_\Delta & k\\ \frac{1}{2} & \frac{3}{2} & 1\\ j_h & j_\Delta & J_T\end{Bmatrix}, \tag{24}$$

where the multipole decay integral is

$$R^{\Delta h}_k(k_\pi)=\int_0^\infty d\xi\, u_{n_h l_h}(\xi)j_k(k_\pi\xi)u_{n_\Delta l_\Delta}(\xi)\ . \tag{25}$$

E. Sum over intermediate states

After summing over intermediate states, the amplitude for excitation of the projectile to any excited state and Δ formation and decay in the target is

$$A^{P^*}_{T(\Delta)\rightarrow T\pi}=\sum_{J_T M_T}A^{M_T}_{J_T}(\mathbf{k}_\pi)A^{M_T M_P}_{J_T J_P}(\mathbf{K})\ , \tag{26}$$

such that the decay amplitude $A^{M_T}_{J_T}(\mathbf{k}_\pi)$ is given by Eqs. (20)–(25) and the formation amplitude $A^{M_T M_P}_{J_T J_P}(\mathbf{K})$ for the projectile momentum transfer $\mathbf{K}=\mathbf{k}_{P_I}-\mathbf{k}_{P_F}$ from the initial to final state is given by

$$A^{M_T M_P}_{J_T J_P}=A^{t_Z(\pi)}_{J_T J_P}\sum_L[g^{\parallel}(K)H^{\parallel}_{J_T J_P L}(K)-g^{\perp}(K)H^{\perp}_{J_T J_P L}(K)]\times\Theta^{M_T M_P}_{J_T J_P L}(\hat{\mathbf{K}})\ , \tag{27}$$

where the quantities in (27) are given in Eqs. (12)–(19). Since the amplitude for Δ-hole production in the projectile and excitation of the target is obtained simply by exchanging the target and projectile references, the total amplitude for exciting either nucleus to a Δ-hole state and the other nucleus to an ordinary particle-hole state is finally given by

$$c_{FI}=\frac{A^{P*}_{T(\Delta)\to T\pi}+A^{T*}_{P(\Delta)\to P\pi}}{\epsilon_\pi+m_nc^2-m'_\Delta c^2+i\Gamma'_\Delta(\epsilon_\pi)/2} . \tag{28}$$

This completes the presentation of the formal solutions of Δ-hole excitation and decay in one nucleus with simultaneous particle-hole excitation in the other nucleus. These results are general solutions to any particle-hole excitation since the results depend on the particle-hole coefficients found in the form factors of Eqs. (18) and (24). Solutions in which these coefficients are solved where the particle-hole states form a coherent excitation will be described in a subsequent publication.

III. SEMIQUANTITATIVE ANALYSIS

It is interesting, now that the formalism has been developed, to examine these expressions and approximately determine, to the extent possible, the conditions that would maximize the production of pions, before embarking upon the long road of writing all the programs that are needed to produce the final results. This analysis will then serve as a useful guide for the subsequent numerical calculations.

A. Breit-Wigner denominator

Beginning with the Breit-Wigner denominator in Eq. (28) and neglecting the energy dependence of the width for the moment, the pion kinetic energy at resonance is $t_\pi=124$ MeV in the nucleus-nucleus rest frame where the in-medium values of the Δ-isobar mass and width are 30 MeV smaller and 40 MeV wider than the free values. The interesting pion kinetic energies that lie within the resonance width roughly range from 50 to 200 MeV.

B. Decay amplitude

If the decay amplitude [Eq. (20)] is considered, it can be seen that the angular distribution is given by the spherical harmonic for a given value (J_T,M_T), which is maximized for pions in the forward direction. If we examine the factor $h(k_\pi)$ in Eq. (21) and assume for simplicity that the nucleon form factors take the approximate form from the cloudy-bag model [26],

$$f(k_\pi)\approx\exp[-(k_\pi R)^2/10] , \tag{29}$$

where the bag radius $R=1.0$ fm. The factor $h(k_\pi)$ is a broad function that starts linearly with k_π at small values, maximizes around 1.70 fm^{-1}, and tails off in a Gaussian shape for large values of k_π. The pion kinetic energy at maximum is about 223 MeV. This will provide a tendency to emphasize the larger values of pion kinetic energies so that the range of interesting pion values might go from 50 to 220 MeV.

Now examine the target form factor $F_{J_T}(k_\pi)$ as given in Eq. (23). The main convergence properties come from the radial form factor $R_k^{\Delta h}(k_\pi)$ as given in Eq. (25). This integral will be maximal when the maxima of the Δ-hole shell-model states match the maximum of the spherical Bessel function. An estimate has been made for these matching conditions. From the virial theorem, the rms value for a three-dimensional harmonic oscillator is $\text{rms}=\sqrt{N+\frac{3}{2}}b$, where the oscillator length parameter $b=\sqrt{\hbar/m_n\omega}$ and the energy quantum number $N=2(n-1)+1$, where n is the number of radial nodes in the radial wave function including the one at infinity, but excluding the one at the origin. The average rms value for a nucleus described by the three-dimensional harmonic oscillator is given by

$$\langle r^2\rangle_{\text{nuc}}=A^{-1}\sum_{i=1}^{A}\langle n_il_i|r^2|n_il_i\rangle , \tag{30}$$

and the empirical expression for $A^{1/3}>2$ is [27]

$$\text{rms}=(0.82A^{1/3}+0.58)\text{ fm} . \tag{31}$$

If we consider ^{12}C, $\text{rms}_{\text{nuc}}=2.46$ fm, and solving for the oscillator parameter, $b=1.67$ fm. From this value the rms value in the $1p$ state is 2.64 fm. Since rms values are weighted by an r^2 factor, the rms values are greater than the maximum value of the radial wave function, but they should give an approximate indication as to where the maxima are located. For the case where a Δ is produced in ^{12}C, the oscillator parameter b_Δ becomes slightly smaller [15] since $b_\Delta=(m_n/m'_\Delta)^{1/4}b$. Therefore, performing a similar calculation as above, the rms value for the Δ-particle state shifts downward by the same factor and for the $1p(\Delta)$ state is 2.48 fm. Averaging these values for the Δ- and nucleon-hole state, the folded maxima should occur at a radius value of $\xi_{\max}\lesssim 2.56$ fm. If the maxima of $j_k(x_{\max})$ are known, then a matching will occur when $k_\pi^{\max}=(x_{\max}/2.56)$. In Table I approximate values for the first maximum of the spherical Bessel func-

TABLE I. Estimated values of pion kinematics matching the first maximum of $j_k(x)$.

k	$x_{\max}$ (fm)	$k_\pi^{\max}$ (fm^{-1})	p_π (MeV/c)	t_π (MeV)	t_π^{lab} (MeV)
0	0	0	0	0	3
1	2.10	0.82	162	74	113
2	3.30	1.29	255	150	211
3	4.50	1.76	347	234	316
4	5.60	2.19	432	314	416
5	6.80	2.66	525	403	
6	7.90	3.09	610	486	
7	8.90	3.48	687	561	

tion are listed [28] for various orders k, along with the pion wave number at this maximum value $k_\pi^{\max}$, its momentum p_π, kinetic energy t_π, and its kinetic energy in the laboratory rest frame, $t_\pi^{\rm lab}$, where an incoming projectile energy of 85A MeV was chosen as an example, and relativistic transformations were used to determine the laboratory kinetic energies. Using conservation of energy and momentum in the forward direction, the maximum kinetic energy that can be attained is 370 MeV in the nucleus-nucleus frame, which corresponds to 486 MeV in the laboratory frame, so that the entries below 416 MeV in the last column are now allowed. The values in the other columns are allowed for a higher incident energy such as, for example, 120A MeV.

The third row in Table I is particularly interesting where $k=2$. In the case the first maximum of $j_k(x)$ matches the folded maximum of the Δ-hole states. This corresponds to a pion kinetic energy of 150 MeV, which for an 85A MeV incident energy transforms to 211 MeV in the laboratory frame. This matching would then maximize the decay radial form factor $R_k^{\Delta h}(k_\pi)$ of Eq. (25). Furthermore, it is also close to the resonant energy of 124 MeV, so that both the decay amplitude and Breit-Wigner factor are large.

The energy behavior of the radial form factor is expected to be a peaked function about the matching value of k_π. For low pion wave numbers, because of the low-energy behavior of $j_k(k_\pi\xi)$, the form factor is expected to go as k_π, whereas at high energies, the Bessel function oscillates so quickly that the integrand will have approximately equal and opposite contributions such that the integral will become very small. Also, the k dependence for a fixed argument strongly dies off for large k values, so that $R_k^{\Delta h}(k_\pi)$ will provide a convergence of the sum over those values.

C. Formation amplitude

The formation amplitude $A_{J_TJ_P}^{M_TM_P}(\mathbf{K})$ is given in Eq. (27). Again, because of the spherical harmonics present in Eq. (13) for the angular function $\Theta_{J_TJ_PL}^{M_TM_P}(\hat{\mathbf{K}})$, the maximum value will be obtained when the projectile scatters in the forward direction. This means that magnitude of the projectile momentum transfer in the forward direction, $\hbar K(0)$, will also attain its maximum value where

$$K(\theta_P)=k_I[1-2(P_{P_F}/P_{P_I})\cos\theta_P+(P_{P_F}/P_{P_I})^2]^{1/2}\ , \tag{32}$$

in which $P_{P_I}=\hbar k_I$ is the incident projectile momentum, P_{P_F} is the final projectile momentum, and θ_P is the projectile scattering angle from the incident direction. Since the nuclear form factors have properties very much like that of the decay form factor, similar arguments can be made concerning their convergence properties. The maximum values of these nuclear form factors will also depend on the matching conditions of each radial form factor. Since the particle-hole states set the scale for matching with the spherical Bessel function, a minimum value of $K(0)$ will be needed for that matching. Using nonrelativistic kinematics for the nuclei, which works well at subthreshold energies, but relativsitic kinematics for the pions, and solving the equations of conservation of energy and momentum in the forward direction, the following is obtained for equal-mass nuclei:

$$K(0)=k_I\left[1-\sqrt{T_{P_I}/T_{P_F}}\right]\ , \tag{33}$$

$$T_{P_F}=(T_{P_I}-\epsilon_\pi/2)-\sqrt{(p_\pi^2/2Am_n)(T_P-\epsilon_\pi/2-p_\pi^2/8Am_n)}\ , \tag{34}$$

and

$$T_{T_F}=\left[\sqrt{T_{P_F}}+\sqrt{p_\pi^2/2Am_n}\right]^2\ , \tag{35}$$

where the initial and final projectile kinetic energies are T_{P_I} and T_{P_F}, the total pion energy is ϵ_π of momentum p_π, and the baryon number of each nucleus is A. If a nonrelativistic assumption is made for the nuclei, then the incident projectile energy in the nucleus-nucleus frame is $T_{P_I}=\frac{1}{4}At_p^{\rm lab}$, where $t_p^{\rm lab}$ is the incident energy per nucleon in the laboratory frame. If the third term is neglected under the square root in Eq. (34), then because the terms under the square root must be positive, the maximum value for the pion kinetic energy is approximately

$$t_\pi^{\max}\approx\tfrac{1}{2}At_p^{\rm lab}-m_\pi c^2\ . \tag{36}$$

For 85 MeV/nucleon, $t_\pi\approx 370$ MeV, but for 120 MeV/nucleon it is approximately 580 MeV. This is the reason for the cutoff of the laboratory values of t_π in Table I.

If it is desired to keep $K(0)$ minimal in order to maximize the matching conditions, then from Eq. (34), assuming that the square-root term can be neglected, substituting into Eq. (33), and expanding for small argument with $\epsilon_\pi \ll 2T_{P_I}$,

$$\hbar K(0)\approx\sqrt{Am_n/8T_{P_I}}\epsilon_\pi\ . \tag{37}$$

This means that in order to minimize $K(0)$, either look for lower pion energies and/or increase the incident projectile energy in the attempt to maximize the nuclear form factors. Calculations with various values of pion kinetic energies, incident energies per nucleon, final-state energies of the projectile and target, and the momentum transfer in the forward direction have been done for ^{12}C on ^{12}C with only the mild assumption that $p_\pi^2/8Am_n \ll T_{P_I}-\epsilon_\pi/2$ in Eq. (34). These calculations show that $K(0)$ becomes smaller for lower pion energy and higher incident energy, and the results are given in Table II, where the last column is arranged with increasing momentum transfer starting from absolute threshold, reaching a maximum, and then decreasing with higher incident energies. If a similar argument is made for the radial form factor $R_{J_T-k_t}^{\Delta h}(K)$ in Eq. (19) as was done for the decay radial form factor, then for a average rms value of 2.56 fm for the product of the $1p$-Δ and $1p$-hole states along with a matching of the first maximum of $j_2(K\xi)$,

TABLE II. Energies and momenta transfer in the forward direction.

t_π (MeV)	t_p^{lab} (MeV/nucleon)	T_{P_F} (MeV)	T_{T_F} (MeV)	$K(0)$ (fm^{-1})
0	85	185	185	1.81
10	85	175	184	2.09
74	85	135	161	3.32
124	85	106	139	4.33
150	85	92	127	4.87
8	25	0.68	1.31	5.97
150	170	333	398	3.30
150	340	825	925	2.44
74	270	674	731	1.91
124	800	2200	2340	1.61
124	1600	4570	4770	1.31
124	2100	6050	6890	1.20

which occurs at $K\xi=3.30$ fm, then the value of the momentum transfer at matching is $K(0)=1.29$ fm^{-1}. It can be seen that this value is smaller than most of the values in Table II except for the highest incident energies which occur above threshold, and it appears to match at approximately 1600 MeV/nucleon. For subthreshold values the momentum transfer needed for a strong matching is not attained, so that the radial form factors will not be maximized in the subthreshold region. This mismatch will reduce the magnitude of the formation amplitude. The physical picture that emerges from this energy analysis for the case of a Δ produced in the target is roughly the following: The energy gained by the pion comes from the energy lost by both target and projectile; however, the projectile loses more energy than the target, since it has to provide energy in the forward direction to the pion that subsequently emerges in the forward direction. This asymmetry of energy loss tends to increase the momentum transfer. At higher projectile incident energies, the projectile plows its way forward more forcefully because of its greater momentum and is associated with more peripheral collisions. This leads to smaller energy and momentum transfer. The smaller energy transfer then gives less energy to the pion. So higher incident energies and smaller pion energies are associated with smaller momentum transfer.

IV. CONCLUSIONS

Summarizing the semiquantitative results, it appears that the maximum number of pions will be produced in the forward direction and for peripheral collisions, which is consistent with forward scattering of the projectile. For a pion kinetic energy of, say, 150 MeV, which is near the resonance value (~ 124 MeV), and at an incident energy of approximately 1600 MeV/nucleon, the formation and decay amplitudes may be near their greatest values. As one proceeds into the subthreshold region, the formation form factor is reduced as more mismatching occurs, but with a judicious choice of a range of pion energies, the decay form factors and resonance can still be maximized. The interesting range of pion energies might be from, say, 50 to 220 MeV. Since these results are applicable to any excited state well described by the particle-hole model, the effects of coherence has yet to be examined; however, each of the nuclear form factors in the formation and decay matrix elements depends on the sum of particle-hole coefficients over the shell-model states. It is plausible that large effects for the formation and decay amplitudes can be obtained, depending on the degree of coherence between the various coefficients. This provides the motivation for examining coherent, giant resonances, especially if produced in both nuclei.

A careful balancing may have to be performed if one considers thermal "hot spots" which produce low-energy pions [16] and the fact that higher incident energies open up many more channels, including fragmentation. This implies that the higher pion energies should be examined in the range suggested and that incident energies below, but near the threshold value, may be of significance.

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[1] G. F. Bertsch, Phys. Rev. C **15**, 713 (1977).
[2] C. Guet and M. Prakash, Nucl. Phys. **A428**, 119c (1984).
[3] M. Blann, Phys. Rev. C **32**, 1231 (1985).
[4] W. Cassing, Z. Phys. A **329**, 487 (1988).
[5] R. Shyam and J. Knoll, Phys. Lett. **136B**, 221 (1984); Nucl. Phys. **A426**, 606 (1984).
[6] J. Aichelin and G. Bertsch, Phys. Lett. **138B**, 350 (1984); J. Aichelin, Phys. Rev. Lett. **52**, 2340 (1984).
[7] C. Gale and S. Das Gupta, Phys. Rev. C **30**, 414 (1984).
[8] M. Prakash, P. Braun-Munzinger, and J. Stachel, Phys.

Rev. C **33**, 937 (1986).

[9] D. Vasak, B. Müller, and W. Greiner, Phys. Scr. **22**, 25 (1980); D. Vasak, H. Stöcker, B. Müller, and W. Greiner, Phys. Lett. **93B**, 243 (1980); M. Uhlig, A. Schäfer, and D. Vasak, Z. Phys. A **319**, 319 (1984); D. Vasak, W. Greiner, B. Müller, T. Stahl, and M. Uhlig, Nucl. Phys. **A428**, 291c (1984); D. Vasak, B. Müller, and W. Greiner, J. Phys. G **11**, 1309 (1985).

[10] M. Tohyama, R. Kaps, D. Masak, and V. Mosel, Phys. Lett. **136B**, 226 (1984); Nucl. Phys. **A437**, 739 (1985).

[11] A. Blin, C. Guet, and B. Hiller, Ganil Report No. P.85/11, 1985.

[12] M. Prakash, C. Guet, and G. E. Brown, Nucl. Phys. **A447**, 625c (1985).

[13] C. Guet, M. Soyeur, J. Bowlin, and G. E. Brown, Nucl. Phys. **A494**, 558 (1989).

[14] J. W. Norbury, F. A. Cucinotta, P. A. Deutchman, and L. W. Townsend, Phys. Rev. Lett. **55**, 681 (1985); P. A. Deutchman, J. W. Norbury, and L. W. Townsend, Nucl. Phys. **A454**, 733 (1986); P. A. Deutchman, Khin Maung Maung, J. W. Norbury, J. O. Rasmussen, and L. W. Townsend, Phys. Rev. C **34**, 2377 (1986); J. W. Norbury, *ibid.* **37**, 407 (1988).

[15] P. A. Deutchman, R. L. Buvel, Khin Maung Maung, J. W. Norbury, and L. W. Townsend, Phys. Rev. C **33**, 396 (1986); P. A. Deutchman and Khin Maung Maung, *ibid.* **41**, R423 (1990).

[16] P. Braun-Munzinger and J. Stachel, Annu. Rev. Nucl. Part. Sci. **37**, 97 (1987).

[17] J. Julien *et al.*, Z. Phys. A **330**, 83 (1988).

[18] R. Fonte, A. Insolia, G. Palamá, and G. V. Russo, Nucl. Phys. **A495**, 43c (1989).

[19] W. Benenson, Nucl. Phys. **A495**, 21c (1989).

[20] A Badalà *et al.*, Nucl. Phys. **A482**, 511c (1988).

[21] B. Erazmus *et al.*, Phys. Rev. C **44**, 1212 (1991).

[22] G. E. Brown, E. Oset, M. Vincent Vacas, and W. Weise, Nucl. Phys. **A505**, 823 (1989).

[23] F. Petrovich, Nucl. Phys. **A251**, 143 (1975); F. Petrovich, R. J. Philpott, A. W. Carpenter, and J. A. Carr, *ibid.* **A425**, 609 (1984).

[24] R. Machleidt, Adv. Nucl. Phys. **19**, 189 (1989).

[25] E. J. Moniz, Nucl. Phys. **A374**, 557c (1982); H. Esbensen and T. S. Y. Lee, Phys. Rev. C **12**, 1966 (1985); M. Roy-Stephan, Nucl. Phys. **A482**, 373c (1988).

[26] G. A. Miller, in *Quarks and Nuclei,* Vol. 1 of *International Review of Nuclear Physics,* edited by W. Weise (World Scientific, Philadelphia, 1984), p. 230.

[27] W. F. Hornyak, *Nuclear Structure* (Academic, New York, 1975), p. 235.

[28] *Handbook of Mathematical Functions,* Natl. Bur. Stand. Appl. Math. Ser. No. 55, edited by M. Abramowitz and I. A. Stegun (U.S. GPO, Washington, D.C., 1968), p. 457.