Strong absorption model and its associated potential

H. Leeb
Institut für Kernphysik, Technische Universität Wien, Wien, Austria

C. Steward and K. A. Amos
School of Physics, University of Melbourne, Parkville, Victoria 3052, Australia

L. J. Allen
School of Science and Mathematics Education, University of Melbourne, Parkville. Victoria 3052, Australia

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Using the example of ^{12}C-^{208}Pb elastic scattering at E_{lab} = 1449 MeV it is shown that, at sufficiently high energies, a full quantum mechanical inversion of heavy-ion scattering data can be performed. Furthermore it is pointed out that the strong absorption model as parametrized by McIntyre, Wang, and Becker is associated with a potential having strong repulsion at the origin. That repulsion is a remnant of the built-in point Coulomb S matrix. Finally, a systematic study is presented to define the necessary accuracy of experiment required to extend the knowledge of the nucleus-nucleus interaction to smaller radii.

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I. INTRODUCTION

Heavy-ion scattering has been a most active field of research in nuclear physics in the last decade. Central in analyses of most of the data from heavy-ion scattering experiments is the inter-ion interaction which may be deduced from elastic scattering cross sections (and vice versa). But the measured elastic scattering (differential) cross sections are quite diverse functions of momentum transfer. Depending upon the energies and masses of the colliding ions those cross sections vary from ones typical of Fraunhofer diffraction to ones characteristic of rainbow scattering. Analytically, such a variation in the shape of potential scattering cross sections reflects the interplay between the Coulomb interaction and a nuclear interaction; the latter often characterized by strong absorption. That interplay and the relative importance of each component is the basis of the success of algebraic models of the total scattering S matrices [1]. Also, the strong absorptive character of the nuclear interaction has been exploited to define the various strong absorption models (SAM) of the nuclear S matrices [2]. The SAM functional form given by McIntyre, Wang, and Becker [3] is a particularly successful, yet simple, parametrization of such nuclear S matrices for heavy-ion scattering. But physical descriptions of scattering processes generally are given in terms of forces and potentials, and, until recently, it was not clear just what specific forces and potentials underlay SAM analyses [4]. Finding the local potentials which, from direct solutions of the relevant Schroedinger equation give the SAM S matrices, corresponds to solving the inverse scattering problem at fixed energy [5]. Using different methods, several inversions of heavy-ion scattering data have been performed [6–11]. In some cases the inversion methods [6, 7] have led to strongly fluctuating potentials with doubtful physical relevance. The semiclassical WKB inversion [12], however, has been found [4, 10, 11] to be both stable and accurate in defining potentials to small radii.

In two recent papers [10, 11], we studied the energy and mass dependencies of the local nucleus-nucleus potential using a semiclassical inverse scattering procedure and starting from a parametrization of the S matrix of the form of McIntyre, Wang, and Becker [3]. In all cases considered, the process produced strong absorptive potentials with a short-range repulsive component in the real part of the potential. However, the onset of that repulsion is far inside of the radial region sensitive to the data, so that there is no implication that such repulsion is physical. Moreover those analyses aimed at a good reproduction of the cross section rather than an excellent description of the S matrix over the whole angular momentum range. Other SAM parametrizations also yielded repulsive cores. At present it is not clear whether the repulsion observed is a feature of the special parametrization [3] of the S matrix or an artifact of the inversion procedure. For further applications of the strong absorption model in reaction calculations, it is worthwhile to clarify this point.

The accuracy with which current data have been measured means that this repulsion cannot be attributed to a physical effect because its onset does not lie in the sensitive radial region. Therefore, the question arises whether this peculiar behavior is an intrinsic feature of the particular parametrization of the S matrix. In order to elucidate this point we consider ^{12}C-^{208}Pb elastic scattering at E_{lab} = 1449 MeV, a good example for the successful application of the strong absorption model. Unfortunately, to our knowledge, there is no way to determine analytically the potential associated with this S matrix. In our investigation we rely on the semiclassical WKB approximation [12]. At sufficiently high energies the WKB approximation has proved to be [10, 11] an excellent and stable method of solution and so should be an adequate

tool for the purposes of this investigation. We will check the quality of the WKB approximation by a comparison with the corresponding full quantal inversion, which has not been done in previous work [10,11]. The question as to the reality of any short-range repulsion in the SAM potentials in heavy-ion systems obtained by WKB inversion cannot be answered as yet because the strong absorption means that experiments where the radial sensitivity is not restricted to the peripheral region are difficult. For a better understanding of the interaction mechanism, especially for analyses and interpretation of reactions that originate within the region of overlap of colliding nuclei, it is desired to extend the sensitive radial range to the inner part. With this in mind, we study the ^{12}C-^{208}Pb elastic scattering, seeking the requirements of future experiments so that they yield information on the short-range interaction.

In the following we address these questions in more detail. After a brief review of the inversion methods for heavy-ion potentials in Sec. II, we study in Sec. III the potentials associated with the special parametrization of the S matrix by McIntyre, Wang, and Becker [3]. Consideration of requirements on future experiments are discussed in Sec. IV.

II. INVERSION METHODS AT FIXED ENERGY

It is well known that heavy-ion scattering can be successfully described by the strong absorption model, central to which is a parametrized form of the S matrix, $S(l,k)$. In this paper we are dealing with the analysis of scattering data at a given energy (k=const). Consequently, the determination of a local central potential associated with the S matrix corresponds to an inverse scattering problem at fixed energy [5], and for which there exists several formulations [12–17]. Among these the exactly solvable models of Bargmann-type [15] are well suited for application. In particular, in the special case of the (so-called) rational inversion scheme [16], the S matrix is of very simple (rational) form,

$$S(\lambda) = S^{(0)}(\lambda) \prod_{n=1}^{N} \frac{\lambda^2 - \beta_n^2}{\lambda^2 - \alpha_n^2}, \tag{2.1}$$

where $\lambda = l + 1/2$ is the orbital angular momentum variable and α_n, β_n represent the poles and zeros of the S matrix, respectively. Furthermore $S^{(0)}(\lambda)$ is the S matrix of a possible reference potential $V_0(r)$. The potential associated with the S matrix Eq. (2.1) is given by [16]

$$V(r) = V_N(r), \tag{2.2}$$

$$V_n(r) = V_{n-1}(r) + V^{(n)}(r) \ , \quad n = 1, ..., N, \tag{2.3}$$

$$V^{(n)}(r) = \frac{2}{r}(\beta_n^2 - \alpha_n^2)\frac{d}{dr}\left[\frac{1}{r}\frac{1}{L_{\beta_n}^{(n-1)}(r) - L_{\alpha_n}^{(n-1)}(r)}\right]. \tag{2.4}$$

Here $L_\lambda^{(n)}(r)$ are the logarithmic derivatives of the Jost solutions $f_\lambda^{(n)}(r)$ to the potential $V_n(r)$. A given S matrix $S(\lambda)$ can be easily approximated by the rational form Eq. (2.1) and from the resulting $\{\alpha_n, \beta_n\}$ values, together with the reference potential, the associated potential $V(r)$ can be evaluated.

This procedure represents a full quantal inversion. However, as has been pointed out [16], only genuine Regge poles and zeros lead to reasonable potentials. False Regge poles and zeros are connected with singularities in the potential. As long as the false poles and zeros are far from the real axis one can overcome this difficulty by exploiting the similarity of the rational and a nonrational scheme [17] on the real axis. Usually the mixed procedure works rather well for higher energies. More details about this mixed method have been given in previous applications [18,19].

For many cases, especially at high energies, the inversion procedure can be facilitated by employing the WKB-approximation to the inverse scattering problem at fixed energy [12,20]. The key quantity in this approach is the quasipotential $Q(\sigma)$ which is determined from the S matrix as follows:

$$Q(\sigma) = -i\frac{2E}{\pi}\frac{1}{\sigma}\frac{d}{d\sigma}\left[\int_\sigma^\infty d\lambda \frac{\lambda}{\sqrt{\lambda^2 - \sigma^2}} \ln S(\lambda)\right]. \tag{2.5}$$

Subsequently, the evaluation of the potential in coordinate space is straightforward and one must solve the system of transcendental equations,

$$kr = \sigma \exp\left[\frac{Q(\sigma)}{2E}\right], \quad V(r) = E\left\{1 - \exp\left[-\frac{Q(\sigma)}{E}\right]\right\}. \tag{2.6}$$

In general, S matrices for optical model calculations are not unitary due to the complex nature of the optical potential. From the above formulas it is obvious that σ and the quasipotential $Q(\sigma)$ also become complex. In order to be able to perform an analytical continuation of $Q(\sigma)$ into the complex σ plane, an analytical solution of Eq. (2.5) is required. Therefore it is very important to know $Q(\sigma)$ in terms of analytical functions. In this respect many functional forms of the S matrix have been studied [21]. Of all, the rational form Eq. (2.1) leads to particularly simple relations.

Using the rational S-matrix parametrization Eq. (2.1), the WKB-inversion has already been applied successfully to heavy-ion systems [10,11]. Besides its simplicity it has the great advantage that one need not be concerned about the correct position of the poles and zeros of the S matrix in the complex plane. On the other hand, one should keep in mind that Eqs. (2.5) and (2.6) have been obtained by semiclassical considerations and they will fail at low energies where quantum mechanical effects (e.g., tunneling) become important.

In the presence of the long-range Coulomb interaction the behavior of the S matrix at large angular momentum cannot be represented readily by the rational form in Eq. (2.1). That behavior must be separated out by an adequate reference potential $V_0(r)$ with known S matrix, $S^{(0)}(\lambda)$. Following previous calculations we use the

reference S matrix,

$$S^{(0)}(\lambda) = e^{i\eta(\lambda^2+\lambda_C^2)}, \tag{2.7}$$

where η is the Sommerfeld parameter and λ_C is a smoothing parameter which shifts the singularity away from the origin. In the present calculations we choose $\lambda_C = 3\eta$. The reference S matrix has the same asymptotic λ dependence as the Coulomb S matrix. The corresponding potential $V_0(r)$ is found accurately by the semiclassical inversion Eqs. (2.5)–(2.7). It is smooth with a finite value at the origin and has the correct Coulombic tail asymptotically.

III. THE STRONG ABSORPTION MODEL

The elastic scattering of heavy ions is characterized by the strong absorption which they experience. In a semiclassical picture, all contributions with an impact parameter smaller than the so-called strong absorption radius (which in turn relates to a grazing angular momentum l_g) are completely absorbed. In terms of the S matrix, this behavior is described by

$$|S_l(k)| << 1 \quad \text{for} \quad l < l_g, \tag{3.1}$$

and is the basic content of the strong absorption model [2]. There exist several parametrizations of the S matrix [3, 22, 23] modeling this basic feature. In the following we deal only with the parametrization of McIntyre, Wang, and Becker [3] which has been applied successfully to a wide range of reaction systems. Therefore the following parametrization of the total S matrix is used:

$$S(\lambda) = \eta(\lambda)e^{2i\delta_{\rm nucl}(\lambda)}S_{\rm Coul}^{\rm point}(\lambda) \tag{3.2}$$

where $\eta(\lambda)$ and $\delta_{\rm nucl}(\lambda)$ describe mainly the effect of the nuclear interaction. Explicitly they are given by

$$\eta(\lambda) = \left[1 + \exp\left(\frac{l_g - l}{\Delta}\right)\right]^{-1}, \tag{3.3}$$

$$\delta_{\rm nucl}(\lambda) = \delta_0\left[1 + \exp\left(\frac{l - l_g'}{\Delta'}\right)\right]^{-1}. \tag{3.4}$$

Using this five-parameter model, excellent fits to differential cross sections have been obtained for a diversity of heavy-ion systems in a wide energy range.

As an example we consider ^{12}C-^{208}Pb elastic scattering [24] at $E_{\rm lab} = 1449$ MeV. Within the SAM parametrization a complete description of the elastic scattering cross sections can be given. The relevant parameter values are listed at the bottom of Table I and designated as the "plain SAM parametrization." Using them, the calculated differential cross section is that compared with the data in Fig. 1. However, because of the strong absorption for $l < 100$ the behavior of the S matrix, apart from its absolute value, is not determined from the experimental cross section. Rather the specific low l-value behavior is a direct consequence of the SAM parametrization. Therefore, we restrict our full quantal inversion of the S matrix to the range $l > 50$ which includes enough values below the grazing angular momentum to contain all the physically relevant information. The parameters of the best rational representation are given in Table II. Since all false Regge poles and zeros have rather large imaginary parts the associated potential can be determined by full quantal inversion in the mixed scheme [18]. We have also evaluated the corresponding potential in the WKB approach. Both calculations, the quantal and the semiclassical ones, lead essentially to the same potential and cannot be distinguished on the scale of Fig. 2.

A closer look at the S matrix reveals significant differences between the rational representation Eq. (2.1) and

TABLE I. Parameter and χ^2 values of the modified SAM parametrization for ^{12}C-^{208}Pb scattering at $E_{\rm lab} = 1449$ MeV.

a	b	l_g	Δ	l_g'	Δ'	δ_0	χ^2
10		267.2039	27.0625	213.6129	16.6635	99.5880	51.06
8		267.0769	27.2637	212.4376	16.7441	103.6670	49.51
6	100	266.7859	25.2614	212.4376	18.6213	113.7210	49.35
4		266.1583	23.9942	208.9814	20.0107	125.9740	49.78
2		266.2149	23.9961	208.2838	20.1087	128.3956	49.91
10		266.9289	26.1831	211.0531	17.8673	133.3280	49.60
8		266.6681	25.4496	210.4868	18.5851	117.5175	49.28
6	80	266.3914	24.5523	209.3650	19.4913	123.5435	49.51
4		266.3133	23.9574	208.3515	20.1373	127.7501	49.83
2		266.2305	23.8172	207.9617	20.3120	129.4946	49.94
10		266.3533	24.3870	208.8552	19.7056	125.6608	49.50
8		266.2800	24.0549	208.3651	20.0596	127.7739	49.60
6	60	266.1751	23.8451	207.9623	20.2741	130.0324	49.86
4		266.1780	23.8856	207.9572	20.2437	129.9181	49.93
2		266.1728	23.8587	208.0077	20.2576	129.9956	49.95
			Parameters for plain SAM parametrization				
0		266.1897	23.8822	207.9316	20.2484	130.0369	49.96

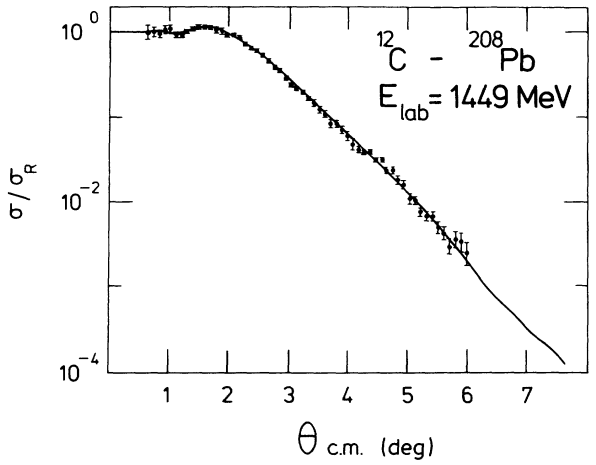

FIG. 1. Reproduction of the ^{12}C-^{208}Pb elastic scattering cross section by the McIntyre parametrization of the strong absorption model. The data are taken from Ref. [23].

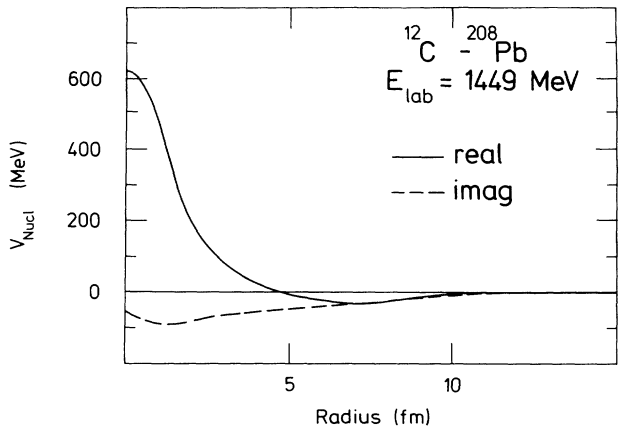

FIG. 2. ^{12}C-^{208}Pb optical potential at E_{lab} = 1449 MeV obtained by full quantal inversion in the mixed (rational and nonrational) scheme. The poles and zeros of the rational S matrix are given in Table II.

the SAM parametrization at small l values. These differences are displayed in Fig. 3. Similiar differences are also observed in comparisons with S matrices generated by best-fit conventional optical model potentials of Woods-Saxon or double-folding type. Since a local central potential is uniquely determined by the functional form of the S matrix on the positive real λ axis, those potentials, albeit reproducing the elastic cross sections, do not correspond to the SAM parametrization. Hence the question as to the precise form of the potential associated with a SAM S matrix.

For a full quantum inversion of the SAM S matrix, the simplest choice for $V_0(r)$ is that of a point Coulomb potential with, then, the nuclear part of the complete S matrix, Eq. (3.2) , being expressed by the rational form Eq. (2.1). But it has been shown [17] that a $1/r$ singularity taken in the background potential implies the same singularity in the total potential. The SAM S matrix therefore corresponds to a singular potential.

TABLE II. Poles α_n and zeros β_n of the rational representation of the SAM parametrization. A shift of $\lambda_c = 21.15$ has been chosen for the Coulomb background S matrix [Eq. (2.7)].

Reα_n	Imα_n	Reβ_n	Imβ_n
101.4711	288.8987	9.2802	277.8517
4.3742	-215.7872	87.4319	127.7401
257.1517	-186.1334	239.1705	90.0771
240.9615	-82.2528	175.8758	86.2074
265.8890	-76.1656	37.4563	84.2195
165.9116	-71.9888	22.4599	25.4414
26.3270	-15.8196	17.2048	14.2004
215.8013	45.4406	202.2733	-46.0926
200.9318	45.5382	215.6065	-46.7973
268.1469	75.2626	160.6596	-85.7738
224.1310	194.0603	64.3072	-90.0973
77.6289	314.0144	250.9952	-149.7245

Using the background S matrix Eq. (2.7) we cannot extend the full quantal inversion to the whole positive real λ axis because the rational approximation then contains false Regge poles and zeros with rather small imaginary parts, making a transfer to the mixed scheme impossible. The origin of these poles and zeros near the real axis in a rational approximation of SAM can be easily understood from the analytical form of $S^{\text{point}}_{\text{Coul}}(\lambda)$, viz.

$$S^{\text{point}}_{\text{Coul}}(\lambda) = \frac{\Gamma(\lambda + \frac{1}{2} + i\eta)}{\Gamma(\lambda + \frac{1}{2} - i\eta)}. \tag{3.5}$$

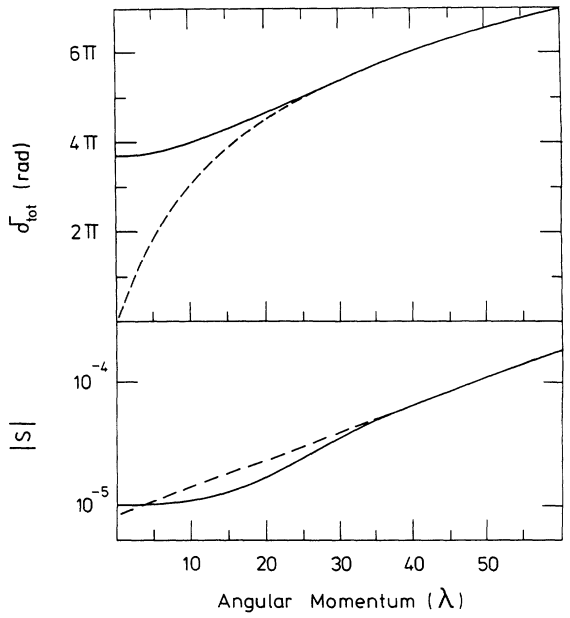

FIG. 3. Comparision of (a) the absolute values of the S matrix and (b) the total phase shifts of the SAM (dashed line) and of the rational representation (solid line) at small angular momenta.

At the energy considered the Sommerfeld parameter η is very large because of the big charges of the heavy ions. Consequently, there exist several integer l values with $l+1 < |\eta|$, which are connected with changes of the phase shifts between 45° and 90° from partial wave to partial wave, since

$$S_{\text{Coul}}^{\text{point}}(\lambda+1) = \exp\left[2i \arctan\left(\frac{\eta}{\lambda+\frac{1}{2}}\right)\right] S_{\text{Coul}}^{\text{point}}(\lambda). \tag{3.6}$$

Such equivalent "rotations" of the S matrix at low l values reflect the singularity of the point Coulomb potential at the origin. In a rational approximation this can only be simulated by additional pairs $\{\alpha_i, \beta_i\}$ having small imaginary parts.

In order to reduce the number of rotations of the SAM parametrization due to $S_{\text{Coul}}^{\text{point}}(\lambda)$, we propose a slightly modified parametrization of the SAM S matrix, namely,

$$S(\lambda) = \eta(\lambda) e^{2i\delta_{\text{nucl}}(\lambda)} S_{\text{Coul}}^{\text{SM}}(\lambda) \tag{3.7}$$

with

$$S_{\text{Coul}}^{\text{SM}}(\lambda) = \frac{\Gamma(\lambda+\frac{1}{2}+a^2 e^{-(\lambda-\frac{1}{2})^2/b^2}+i\eta)}{\Gamma(\lambda+\frac{1}{2}+a^2 e^{-(\lambda-\frac{1}{2})^2/b^2}-i\eta)}. \tag{3.8}$$

With appropriately chosen parameters a, b we obtain essentially the same fit to the cross-section data. Those parameter values are also given in Table I. With this model there is clearly a smoother low-l behavior of the S matrix. Taking into account the S matrix at all partial waves ($l = 0$ to 600) we have determined the corresponding potentials for different a and b values using the semiclassical WKB inversion technique. These potentials are displayed in Fig. 4. For decreasing values of a and b the total S-matrix Eq. (3.7) approaches the SAM S-matrix Eq. (3.2) and it is obvious from Fig. 4 that the corresponding potentials tend towards one which has a repulsive singularity at the origin. As expected, the singularity at the origin exhibits a $\frac{1}{r}$ dependence and is caused by the low-l behavior of the point Coulomb S matrix included in the SAM parametrization Eq. (3.2). The very noticeable changes in potential shapes with variation of the parameter values of a and b, while giving equivalent results for cross sections, clearly show the limitations of heavy-ion cross-section data fits in determining the potential.

IV. HEAVY-ION POTENTIALS AND THE ACCURACY OF MEASUREMENTS

The success of the SAM parametrization in describing the scattering of heavy-ion systems is certainly based on the fact that the inner part of the potential is hidden by the strong absorption of partial waves below the grazing angular momentum. Consequently, modifications of the potential at small radii cause only minor changes of the corresponding cross sections. As long as these changes are smaller than the accuracy of the measurements, no information about the potential at the corresponding radii can be extracted from the data.

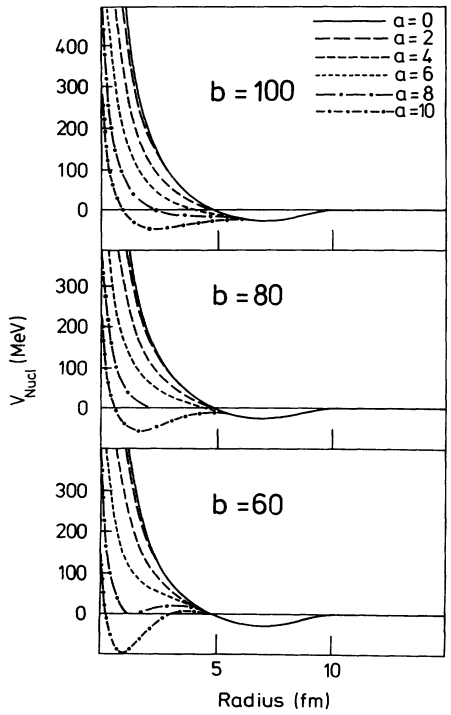

FIG. 4. Potentials obtained by semiclassical WKB inversion using different parameters a and b in the modified Coulomb potential.

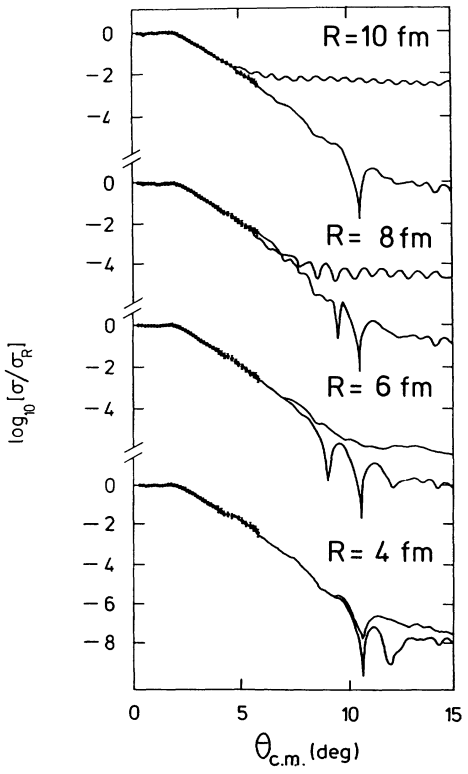

FIG. 5. Envelopes of the ^{12}C-^{208}Pb elastic scattering cross sections at $E_{\text{lab}} = 1449$ MeV generated using the modified potentials of Eq. (4.1).

For a better understanding of the scattering processes in heavy-ion systems, an extension of the radial sensitive region to smaller radii is necessary. Such an extension implies an improvement of the experiments with respect to their accuracy and of their scope. A reliable estimate of the requirements on a future experiment can easily be given by systematic variations of a potential extracted from the best experimental data.

In the following we give the example of such an estimate for ^{12}C-^{208}Pb elastic scattering at $E_{\rm lab} = 1449$ MeV. We modify the potential of the full quantal inversion (Fig. 2 and Tables I and II) by adding a contribution of the form

$$\Delta V(r) = \Delta V_0 \left[1 + \exp\left(\frac{r-R}{a}\right)\right]^{-1}. \quad (4.1)$$

We choose a rather small diffuseness $a = 0.1$ fm in order to get a clear falloff of the additional contribution Eq. (4.1). For a fixed radius R we calculate the corresponding cross sections assuming different ΔV_0 values covering uniformly the interval -10 MeV $\leq \Delta V_0 \leq 10$ MeV. The envelopes of the corresponding calculated differential cross sections are displayed in Fig. 5. It is obvious that information about the potential at a given radius can only be extracted from the experimental data when there is an angular range where uncertainties of the experimental data are smaller than the bandwidth defined by the envelopes. In particular one has to measure the ^{12}C-^{208}Pb differential cross sections at $E_{\rm lab} = 1449$ MeV for angles above 8° with a relative accuracy better than 10% in order to extend the sensitive radial region to $R = 6$ fm.

V. SUMMARY AND CONCLUSIONS

In this paper, we have presented a study of the potential asssociated with the SAM parametrization of McIntyre, Wang, and Becker. Based upon the physically determined SAM S matrix and using the example of ^{12}C-^{208}Pb elastic scattering at $E_{\rm lab} = 1449$ MeV, we have shown that a full quantum mechanical inversion of heavy-ion cross sections can be performed within the mixed rational and nonrational inversion scheme. The subsequent study using the S-matrix values on the whole positive real λ axis revealed the peculiar structure of the strong absorption model. Because of the inclusion of the point Coulomb S matrix in the chosen SAM parametrization, the associated potential exhibited a repulsive singularity at the origin. However, the onset of this repulsion is well below the sensitive radial region. Therefore, this repulsive contribution is not physically significant and is rather an artifact of the SAM parametrization. From this point of view the short distance repulsions found in previous analyses [10, 11] seem to be only a remnant of the same inbuilt point Coulomb S matrix.

The confirmation or otherwise of the simple SAM parametrization can only be given by more accurate experiments extending the radial sensitive region. In this respect the validity of the SAM parametrization might be only a question of the required quality of description of the scattering cross sections. In our final investigation we have tried to give a reliable estimate of the requirements on future experiments. The accuracies of the measurements required to resolve the heavy-ion potential at smaller radii provide a challenge for experimentalists. However, such accurate experiments will lead to valuable information about the interaction of heavy-ion systems.

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