Hyperspherical approach for the trinucleon system with hard-core potential

T. K. Das* and H. T. Coelho
Departamento de Física, Universidade Federal de Pernambuco, Recife, Pernambuco, Brasil

J. R. A. Torreão
Departamento de Informática, Universidade Federal de Pernambuco, Recife, Pernambuco, Brasil
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In this work we present a method for solving the hard-core (HC) three-body problem by the hyperspherical approach. We restrict ourselves to the totally symmetric S state of the dominant trinucleon system interacting via a central spin-dependent HC potential, but the method can be generalized to include other states.

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I. INTRODUCTION

The few-body problem has been an area of intense activity in nuclear physics for the last three decades. The interest is mainly due to the fact that the few-nucleon problem can be solved essentially exactly and vital information about nuclear interaction can be inferred from such calculations. The most widely used theoretical approaches to solve the few-body Schrödinger equation are the Faddeev [1] or Yakuboviskii [2] equation methods and the hyperspherical approach (HA) [3]. In most of these calculations, the nuclear potential chosen has a soft core (SC); i.e., it goes to infinity gradually as the interparticle separation goes to zero. However, only sporadic and feeble attempts have been reported so far in the literature to solve the three- or four-body problem interacting through two-body hard-core (HC) potentials. A hard-core potential is one in which the potential becomes $+\infty$, whenever the interparticle separation is less than the so-called hard-core radius r_c. The solution [4] of the two-body problem interacting via HC potentials is quite straightforward and is easily solved. However, the situation becomes much more complex even for the three-body system. In this case each of the three interparticle separations must be $\geq r_c$ for the total wave function to be nonvanishing. Imposition of a similar restriction for the two-body problem only restricts the scalar radial distance (r_{12}) independent of the orientation ($\mathbf{r}_{12}$). Hence the problem reduces to a differential equation in one variable where the r_{12} space is restricted to the interval $[r_c, \infty]$. However, for the three-body problem, the wave function vanishes whenever any of the pair separation distance $r_{ij} \leq r_c$ ($i,j=1,2,3$ cyclic). Naturally, then, the boundary conditions become very involved and difficult to handle.

In this paper we propose a method for solving the three-body Schrödinger equations by the HA method when the particles interact via HC potentials. The HA has been widely used [3,5] for the three- and four-body problems interacting via SC potentials. But, to our knowledge, little effort has been made to solve the three-body problem involving HC potentials, although the HC potential is of considerable interest in physics. In particular, it is well known that the two nucleon shows a hard-core behavior as demonstrated by standard potentials such as Hamada-Johnston [6] (HJ) or Reid hard-core [7] (RHC) potentials. Because of the fact that the inclusion of HC potentials for the three-body problem is very tricky, one encounters such calculations only by variational or other approximation methods [8]. It is important to note that the SC potentials have been formulated in such a way that they reproduce the two-nucleon data equally well. Thus, in the two-nucleon problem, there is no special choice between SC and HC potentials. But the situation is quite different in three-nucleon systems.

It has now been established that the three-nucleon force [9,10] (3NF) plays a significant role in the trinucleon system, accounting for about 15% of the ground-state binding energy. Now the form of the dominant two-pion-exchange three-nucleon force ($\pi\pi E$-3NF), for very short internucleon separation, is uncertain and a matter of great speculation and controversy. The simplest forms of $\pi\pi E$-3NF, like that of Fujita and Miyazawa [11], have a very strong singularity, which goes as r^{-6} for $r \to 0$, where r is the hyperradius [5], and is attractive for the most likely equilateral triangle configuration of the trinucleon system. This makes the Hamiltonian to be unbounded below, and there can be no stable bound state. But the fact of nature is that stable, bound trinucleon systems are well known. The reason for this apparent fallacy is the fact that the 3NF is not uniquely known for very short internucleon separations. Attempts have been made to include the pion-nucleon form factors in the derivation of the two-pion-exchange diagrams [9,10] or to consider heavier-meson-exchange diagrams to regularize the singular short-range behavior. Interestingly enough, the *ad hoc* cutoff parameter introduced to regularize the short-range behavior [5] agrees closely with the hard-core radius of HJ [6] or RHC [7]

*Permanent address: Physics Department, Calcutta University, Calcutta, India.

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potentials. However, one must admit that these are all *ad hoc* in nature and the "correct" 3NF can never be known, since it is impossible to include all possible diagrams and all possible effects into account (such as quark structure of the hadrons involved, for instance). But a little thought will reveal that all these complications are almost totally redundant since the strong two-nucleon short-range repulsion will not permit the nucleons to come so close as to invoke more complicated diagrams and processes (all higher-order diagrams involve heavier-particle transfer and therefore shorter range). However, the use of a soft-core two-nucleon force, when the nucleons are treated as pointlike particles, will not forbid the nucleons from coming close enough and producing catastrophic effects due to the strong 3NF singularity. Inclusion of form factors or heavier-meson-exchange diagrams effectively introduces "cutoff parameters" [e.g., the Λ in Tucson-Melbourne (TM) [10] or Brazilian (BR) potentials [9]]. A simple phenomenological approach [5] is to cut off the singular 3NF below a "cutoff radius" r_0 and replace the 3NF for $r<r_0$ by its value at r_0. Although quite crude and *ad hoc* in its appearance, this form is, in practice, equivalent to the more sophisticated methods for recalculating 3NF [9,10]. One should further remember that the calculated ground-state properties [binding energy (BE), charge form factors, charge radius, etc.] are strongly dependent on the cutoff or smoothing parameters (for all the alternative forms of 3NF), thus making the calculations rather uncertain and inconclusive. A particular choice of the cutoff parameter (r_0 or Λ as the case may be) may reproduce the BE [5,9,10], but it is hardly justifiable or reliable, as is evident from the calculated numbers. We have to keep in mind that the introduction of a hard core in the forces is just a mathematical device, not a physical mechanism. If one introduces such a mathematical hard core, then the sensitivity of the cutoff dependence of the 3NF will shift to a sensitivity to the hardcore radius. However, the latter can be determined by fitting two-nucleon data.

Clearly, the singularity of the 3NF will not present any catastrophic situation when the HC potential is used, as the interacting particles simply do not come close enough. This is true for the simplest and highly singular Fujita-Miyazawa form of 3NF. Most of the additional effects of the sophisticated 3NF will, so to say, be hidden behind the "hard core" and will produce little detectable consequence. Thus a solution of the three-body Schrödinger equation including the HC potential is of great interest from the point of view of nuclear interactions in addition to the inherent interest in facing the challenge.

This is the motivation to propose a method for solving the HC three-body problem by the HA method. In this paper we restrict ourselves to the totally symmetric S state of the trinucleon system interacting via a central spin-dependent HC potential.

In Sec. II we present the theoretical method for incorporating the HC potential and the resulting equations to be solved. In Sec. III we present the numerical results. Conclusions are drawn in Sec. IV. An appendix is included to clarify the methods used in Sec. II.

II. HYPERSPHERICAL APPROACH FOR HARD-CORE POTENTIALS

In this work we adopt the HA for solving the Schrödinger equation for three nucleons of mass m (after c.m. motion is removed),

$$\left[-\frac{\hbar^2}{m}(\nabla_{\mathbf{x}}^2+\nabla_{\mathbf{y}}^2)+V\right]\Psi(\mathbf{x},\mathbf{y})=E\Psi(\mathbf{x},\mathbf{y}) , \tag{1}$$

which is written, in terms of Jacobi coordinates,

$$\begin{aligned}&\mathbf{x}=\mathbf{r}_2-\mathbf{r}_1 ,\\ &\mathbf{y}=(2/\sqrt{3})[\mathbf{r}_3-\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2)] ,\end{aligned} \tag{2}$$

when $\mathbf{r}_i$ $(i=1,2,3)$ are the particle coordinates, while $V=\sum_{i<j}V_{ij}+W$ is the sum of pairwise two-nucleon forces (2NF's), $\sum_{i<j}V_{ij}$, plus the three-nucleon force W. In the standard HA approach for SC potentials, Ψ is expanded in a complete orthonormal set of hyperspherical functions [3]:

$$\Psi(\mathbf{x},\mathbf{y})=\sum_{K\alpha}\left[\frac{u_{K\alpha}(r)}{r^{5/2}}\right]F_{K\alpha}(\Omega) , \tag{3}$$

where $r^2=(x^2+y^2)^{1/2}$, $x=r\sin\phi$, and $y=r\cos\phi$ $(0\le\phi\le\pi/2)$. The symbol Ω stands for five angles, namely, $\Omega\equiv\{\mathbf{x}\equiv(\theta_x,\phi_x),\mathbf{y}\equiv(\theta_y,\phi_y),\phi\}$, where $\mathbf{x}$ and $\mathbf{y}$ define the usual spherical polar angles of $\mathbf{x}$ and $\mathbf{y}$.

The complete orthonormal set $\{F_{K,\alpha}\}$ are given by

$$F_{K\alpha}=\left[\Gamma_{ST}^{m_S m_T}(R)\otimes\mathcal{Y}_{K,(l_x,l_y)L,M}^{(R^*)}\right]_{J=1/2}^{M_J} , \tag{4}$$

where $\Gamma_{ST}^{m_S m_T}(R)$ are the orthonormal irreducible representations of spin-isospin states for trinucleon system have a total spin S and a total isospin T, their projection being m_S and m_T, respectively, and which correspond to a specific symmetry (R) under pairwise exchanges, and $\mathcal{Y}^{(R^*)}$ are the hyperangular functions of conjugate representation (R^*) and are the part of homogeneous harmonic polynomials of degree K $(K=0,1,2,\ldots,\infty)$ in six-dimensional space. The label α stands for all the other quantum numbers required to specify the system, involving five degrees of freedom contained in Ω, namely $\alpha=\{l_x,l_y,L,M\}$. For large values of K, the number of independent hyperspherical harmonics (HH's) is very large [3]. In order to handle numerically the three-body system, one usually reduces the number of partial waves involved in the expansion Ψ to the minimum number of significant terms by using the so-called optimal subset [5,12]. That formally means the substitution $\{\mathcal{Y}_{K,\alpha}^{(R^*)}\}\rightarrow\{P_{K,\alpha}^{(R^*)}\}$, where $\{P_{K,\alpha}^{(R^*)}\}$ stands for the optimal subset (for the trinucleon system this condition restricts K to even values). The wave function Ψ including various symmetry components becomes

$$\Psi(\mathbf{x},\mathbf{y}) \cong \sum_{K,\alpha,R} \left[\frac{u_{K,\alpha}^{(R)}}{r^{5/2}} \right] [\Gamma_{ST}^{m_S m_T}(R) \otimes P_{K,\alpha}^{(R^*)}(\Omega)]_{J=1/2}^{M} . \tag{5}$$

Substitution of Eq. (5) into Eq. (1) and projecting it on a particular HH, one obtains a system of coupled differential equations [3]:

$$\left[-\frac{d^2}{dr^2} + \frac{\Lambda_K(\Lambda_K+1)}{r^2} + \varepsilon \right] u_{K\alpha}^{(R)}(r) + \sum_{R'K'\alpha'} \langle K\alpha R | v | K'\alpha' R' \rangle u_{K'\alpha'}^{(R')}(r) = 0 , \tag{6}$$

where $\Lambda_K = 2K + L + \frac{3}{2}$, $\varepsilon = (-m/\hbar^2)|E|$, $|E|$ the trinucleon BE, and $v = (m/\hbar^2)[(V_{12}+V_{13}+V_{23})+W]$. The matrix elements $\langle v \rangle$ contain spin-isospin operations beside integrals over the five angles, resulting in a function of r.

When a HC-2NF, with a hard-core radius r_c, is employed, the wave function vanishes for $r_{ij} < r_c$ $(i,j=1,2,3$ cyclic). This can be incorporated in the structure of Eq. (5) in the way to be presented below.

The interparticle separations are given by

$$\begin{aligned} \mathbf{r}_{21} &= \mathbf{r}_2 - \mathbf{r}_1 = \mathbf{x} , \\ \mathbf{r}_{32} &= \mathbf{r}_3 - \mathbf{r}_2 = \frac{\sqrt{3}}{2}\mathbf{y} - \frac{1}{2}\mathbf{x} , \\ \mathbf{r}_{13} &= \mathbf{r}_3 - \mathbf{r}_1 = \frac{\sqrt{3}}{2}\mathbf{y} + \frac{1}{2}\mathbf{x} . \end{aligned} \tag{7}$$

As $r_{ij} \geq r_c$ $(i,j,=1,2,3)$, we have the following conditions:

$$\sin\phi \geq \frac{r_c}{r} , \tag{8}$$

$$\frac{1}{4} + \frac{1}{2}\cos^2\phi \pm \frac{\sqrt{3}}{4}\mu \sin(2\phi) \geq \left[\frac{r_c}{r} \right]^2 , \tag{9}$$

where $\mu \equiv \mathbf{x}\cdot\mathbf{y}$. We should note that for SC interactions, $r_c = 0$ and then $0 \leq \sin\phi \leq 1$, as expected.

For illustration of our approach, let us consider only the space totally symmetric $S(L=0)$ state of the trinucleon system, since its probability is about 90% of the total trinucleon wave-function probability.

Our ansatz for the HC triton wave function is

$$\Psi_S(r,\Omega) = \Gamma_{1/2,1/2}^{1/2,-1/2}(A) \otimes \sum_K \left[\frac{u_K(r)}{r^{5/2}} \right] P_K^{(0)}(\Omega) \Theta(r,\phi,\mu) , \tag{10}$$

where $P_K^{(0)}(\Omega)$ is the HH basis function for the totally symmetric S state (suppressing other conserved quantum numbers for notation simplicity), $\Gamma_{1/2,1/2}^{1/2,-1/2}(A)$ is the totally antisymmetric spin-isospin wave function, and

$$\Theta(r,\phi,\mu) = \Theta(r_{12}-r_c)\Theta(r_{23}-r_c)\Theta(r_{13}-r_c) . \tag{11}$$

The function $\Theta(x)$ is the usual theta function defined as

$$\Theta(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x < 0 . \end{cases} \tag{12}$$

We introduce the theta function in Eq. (10) in order that Ψ_S vanish whenever any pair separation is less than or equal to r_c, as required for the HC potential.

We would like to point out, however, that Eq. (10) is more an ansatz for the wave function than a formal "expansion" of the wave function in a complete basis as the hyperangular space is restricted by the Θ function [Eqs. (11) and (12)]. However, the right-hand side of Eq. (10) should be a good representation of the HC wave function Ψ_S since this already satisfies the criterion that Ψ_S vanish whenever any pair separation goes to r_c. Substitution of Eq. (10) into Eq. (1) gives

$$\left[-\left[\frac{\partial^2}{\partial r^2} + \frac{5}{r}\frac{\partial}{\partial r} \right] - \frac{1}{r^2}\left[\frac{\partial^2}{\partial\phi^2} + 4\cot(2\phi)\frac{\partial}{\partial\phi} - \frac{\hat{l}^2}{\sin^2\phi} - \frac{\hat{l}_y^2}{\cos^2\phi} \right] + v + \varepsilon \right] \sum_{K'} \left[\frac{u_{K'}(r)}{r^{5/2}} \right] P_{K'}^{(0)}(\Omega)\Theta(r,\phi,\mu) = 0 , \tag{13}$$

where $P_{K'}^{(0)}$ can be further expanded as [5]

$$P_K^{(0)}(\Omega) = \sum_{\substack{l=0 \\ (\text{even})}}^{K} a_K^l \mathcal{Y}_{K,ll}^{00}(\Omega) , \tag{14}$$

where a_K^l are known coefficients obtained by symmetrization and are given by

$$a_K^l = \sqrt{2l+1}\, a_K^0 \left[\hat{\mathbf{P}}_{2K}^{ll}(0) + \hat{\mathbf{P}}_{2K}^{ll}\left[+\frac{2\pi}{3} \right] + \hat{\mathbf{P}}_{2K}^{ll}\left[-\frac{2\pi}{3} \right] \right] \Big/ \left[\hat{P}_{2K}^{00}(0) + \hat{P}_{2K}^{00}\left[+\frac{2\pi}{3} \right] + \hat{P}_{2K}^{00}\left[-\frac{2\pi}{3} \right] \right] , \tag{15}$$

where

$$a_K^0 = \frac{1}{\sqrt{3}} \frac{1}{\sqrt{K+1}} \times \begin{cases} \sqrt{K+3}, & K = 3n \\ \sqrt{K-1}, & K = 3n+1 \\ \sqrt{K+1}, & K = 3n+2, \end{cases} \tag{16}$$

and $n=0,1,2,\ldots$. Note that l is even since $P_K^{(0)}$ should be invariant under $1\leftrightarrow 2$.

For the S state, the $L=0$ hyperspherical harmonic [3] $\mathcal{Y}^{00}_{K,ll}(\Omega)$ is

$$\mathcal{Y}^{0,0}_{K,ll}(\Omega)=\sum_m \langle l,m,l,-m|00\rangle\, Y_{lm}(\mathbf{x})Y_{l,-m}(\mathbf{y})\hat{\mathbf{P}}^{ll}_{2K}(\phi)\ , \tag{17}$$

where

$$\hat{\mathbf{P}}^{ll}_{2K}(\phi)=\mathcal{N}_{Kl}\sin^l\phi\cos^l\phi P^{l+1/2,l+1/2}_{K-l}(\cos 2\phi)\ , \tag{18}$$

$P_n^{\alpha,\beta}(x)$ being a Jacobi polynomial and $\mathcal{N}_{Kl}$ the normalization constant [3] given by

$$\mathcal{N}_{Kl}=\left[\frac{4(K+1)(K-l)!(K+l+1)!}{\Gamma(K+\frac{3}{2})^2}\right]^{1/2}\ . \tag{19}$$

Substituting Eq. (14) into Eq. (13), multiplying the resulting equation from the left-hand side by $P_K^{(0)}(\Omega)$, and integrating over $d\Omega$, we obtain

$$\left[-\left[\frac{d^2}{dr^2}+\frac{5}{r}\frac{d}{dr}\right]+\varepsilon\right]\frac{1}{r^{5/2}}\left[\sum_{K'}N_{KK'}(r)u_{K'}(r)\right]+\frac{1}{r^{5/2}}\sum_{K'}[v_{KK'}(r)-M_{KK'}(r)]u_{K'}(r)=0\ , \tag{20}$$

where

$$N_{KK'}(r)=\tfrac{1}{2}\sum_{\substack{l=0\\(\text{even})}}^{[K,K']}a^l_K a^l_{K'}(2l+1)\int_{-1}^{1}d\mu\int_0^{\pi/2}\sin^2\phi\cos^2\phi\,d\phi[P_l(\mu)]^2\hat{\mathbf{P}}^{ll}_{2K}(\phi)\hat{\mathbf{P}}^{ll}_{2K'}(\phi)\Theta(r,\phi,\mu)\ , \tag{21}$$

$$M_{KK'}(r)=\frac{1}{2r^2}\sum_{\substack{l=0\\(\text{even})}}^{[K,K']}a^l_K a^l_{K'}(2l+1)\int_{-1}^{1}d\mu[P_l(\mu)]^2\int_0^{\pi/2}d\phi\,\hat{\mathbf{P}}^{ll}_{2K'}(\phi)\Theta(r,\phi,\mu)[f_1''-4f_2'-l(l+1)\hat{\mathbf{P}}^{ll}_{2K}(\phi)]\ ,$$

and

$$v_{KK'}(r)=\tfrac{1}{2}\sum_{\substack{l=0\\(\text{even})}}^{[K,K']}a^l_K a^l_{K'}(2l+1)\int_{-1}^{1}d\mu[P_l(\mu)]^2\int_0^{\pi/2}\sin^2\phi\cos^2\phi\,d\phi\hat{\mathbf{P}}^{ll}_{2K}(\phi)v\hat{\mathbf{P}}^{ll}_{2K'}(\phi)\Theta(r,\phi,\mu)\ , \tag{23}$$

where f_1 and f_2 are given in the Appendix. The symbol $[K,K']$ means minimum of K and K'.

Calculating f_1' and f_2'' explicitly, Eq. (22) can be further simplified, resulting in a simpler expression for $M_{KK'}(r)$:

$$M_{KK'}(r)=-\frac{4K(K+2)}{r^2}N_{KK'}(r)\ . \tag{24}$$

Equation (20) is a matrix equation for a given r. If we define a column vector Φ as

$$Nu=\Phi\quad\text{or}\quad u=N^{-1}\Phi\ , \tag{25}$$

we obtain

$$u_{K'}(r)=\sum_{K''}[N^{-1}(r)]_{K'K''}\Phi_{K''}(r)\ . \tag{26}$$

Using Eq. (26) in Eq. (20), we obtain

$$\left[-\frac{d^2}{dr^2}+\frac{15}{4r^2}+\varepsilon\right]\Phi_K(r)+\sum_{K'}\left[[v_{KK'}(r)-M_{KK'}(r)]\sum_{K''}[N^{-1}(r)]_{K'K''}\right]\Phi_{K''}=0\ . \tag{27}$$

Using Eq. (24) and defining

$$\hat{\mathbf{v}}_{KK'}(r)=\sum_{K''}v_{KK''}(r)[N^{-1}(r)]_{K'',K'}+\frac{4K(K+2)}{r^2}\delta_{KK'}\ . \tag{28}$$

we finally obtain

$$\left[-\frac{d^2}{dr^2}+\frac{15}{4r^2}+\varepsilon\right]\Phi_K(r)+\sum_{K'}\hat{\mathbf{v}}_{KK'}(r)\Phi_{K'}(r)=0\ , \tag{29}$$

which has a form similar to Eq. (6).

III. NUMERICAL RESULTS

In order to solve Eq. (29), we need to calculate the $\hat{\mathbf{v}}$ matrix, which in turn is calculated in terms of the v and N matrices. The Θ function in Eqs. (21) and (23) can be used to change the limits of the double integration in $N_{KK'}(r)$, leading to

$$N_{KK'}(r)=\tfrac{1}{2}\sum_{\substack{l=0\\(\text{even})}}^{K,K'}a^l_K(2l+1)\int_{\phi_l}^{\phi_u}\sin^2\phi\cos^2\phi\,d\phi\times\hat{\mathbf{P}}^{ll}_{2K}(\phi)\hat{\mathbf{P}}^{ll}_{2K'}(\phi)\times\int_{\mu=-\mu_l}^{\mu=\mu_l}[P_l(\mu)]^2d\mu\ , \tag{30}$$

where

$$\phi_l = \sin^{-1}(r_c/r)\ ,$$
$$\phi_u = \sin^{-1}(s),\quad s = \min\left[\left\{\frac{3}{2} - 2\left[\frac{r_c}{r}\right]^2\right\}^{1/2}, 1\right]\ , \tag{31}$$
$$\mu_l = \min[z(\phi), 1],$$
$$z(\phi) = \frac{4}{\sqrt{3}\sin(2\phi)}\left[\frac{1}{4} + \frac{1}{2}\cos^2\phi - \left[\frac{r_c}{r}\right]^2\right]\ .$$

A similar expression is obtained for $v_{KK'}(r)$ by including $v = v_{12}(r_{12}) + v_{23}(r_{23}) + v_{13}(r_{13})$ in the integrand of the right-hand side of Eq. (30). One can immediately verify from Eq. (31) that for $r_c = 0$, $\phi_l = 0$, $\phi_u = \pi/2$, and $\mu_l = 1$, and using the orthonormality of $\hat{\mathbf{P}}^{ll}_{2K}(\phi)$ and $P_l(\mu)$, Eq. (30) gives

$$N_{KK'}(r) = \delta_{KK'} \quad \text{for } r_c = 0\ . \tag{32}$$

Furthermore, the expression for $v_{KK'}(r)$ coincides with the expression for the potential matrix for the SC case when $r_c = 0$. Thus the basic requirements in the limit $r_c = 0$ are satisfied by the set of Eqs. (21), (28), and (29). We have also checked our computer program with a suitable SC potential and $r_c = 0$; this reproduces the SC results obtained by a much simpler SC calculation.

One can also verify that the minimum value of the hyperradius r is $\sqrt{2}r_c$, which corresponds to the smallest permissible configuration of an equilateral triangle of sides r_c, so that $x = y = r_c$ [see Eq. (2)]. We calculate the N, v, and $\hat{\mathbf{v}}$ matrices at each r-mesh point in the interval $\sqrt{2}r_c < r < r_\infty$, r_∞ being taken as $\sqrt{2}r_c + 15$ fm. The $N_{KK'}(r)$ matrix calculated from Eqs. (30) and (31) quickly approaches the unit matrix as r increases from $\sqrt{2}r_c$. The diagonal elements of $N_{KK'}(r)$ are appreciably different from 1 (by more than 1%) only in the range $\sqrt{2}r_c < r < \sqrt{2}r_c + 4.0$ fm, for which the maximum magnitude of the off-diagonal elements is about 0.2 (which occurs values of r slightly larger than $\sqrt{2}r_c$). In Fig. 1 we plot the diagonal elements of $N_{KK'}(r)$ as a function of r. Note that we plot this for only the narrow region of r, $\sqrt{2}r_c < r < \sqrt{2}r_c + 4$ fm, showing a magnified horizontal scale.

In Table I we present some representative diagonal elements $N_{KK}(r)$ for $\sqrt{2}r_c < r < \sqrt{2}r_c + 4$ fm, together with the magnitude of the largest off-diagonal element for several values of r.

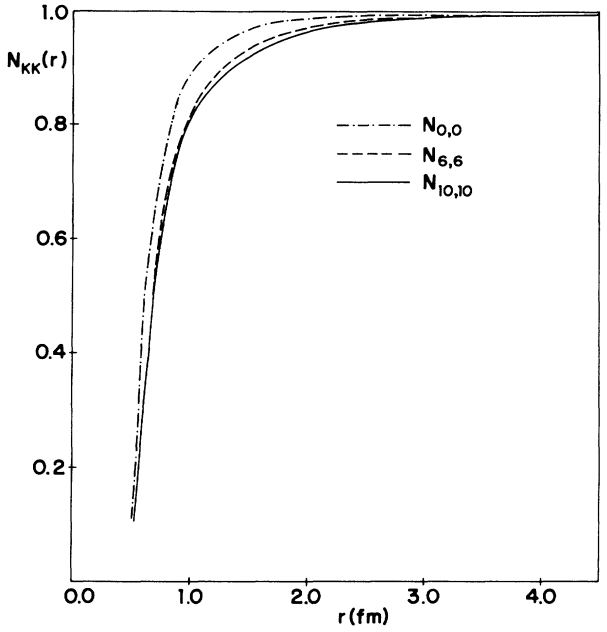

FIG. 1. Plot of $N_{KK}(r)$ for representative values of K, for $K_{\max} = 12$.

From these, it is clear that $N_{KK}(r)$ attains a rapid convergence as K increases and also becomes the unit matrix for $r > \sqrt{2}r_c + 4$ fm. Since the aim of this work is not to do a realistic calculation, but to investigate the feasibility of a solution with a hard-core potential, we choose our HC potential rather arbitrarily as that of a soft core [13] given by

$$V(r) = \tfrac{1}{2}[V_t(r) + V_s(r)]\ ,$$

where

$$\begin{aligned} V_t(r) &= 1000.0\exp[-(r/0.4303)^2] \\ &\quad -143.4\exp[-(r/1.105)^2] \\ &\quad -43.0\exp[-(r/1.291)^2]\ , \\ V_s(r) &= 880.0\exp[-(r/0.4385)^2] \\ &\quad -67.1\exp[-(r/1.270)^2] \\ &\quad -21.0\exp[-(r/1.620)^2]\ , \end{aligned} \tag{33}$$

TABLE I. Representative diagonal elements of $N_{KK}(r)$ for $\sqrt{2}r_c < r < \sqrt{2}r_c + 4$ fm, together with the magnitude of the largest off-diagonal element (LODE) for each value of r.

r (fm)	LODE	$N_{0,0}(r)$	$N_{3,3}(r)$	$N_{6,6}(r)$	$N_{8,8}(r)$	$N_{10,10}(r)$
0.524	0.10	0.227	0.117	0.120	0.128	0.134
0.624	0.14	0.537	0.418	0.386	0.395	0.445
0.724	0.17	0.723	0.587	0.568	0.620	0.630
0.824	0.20	0.829	0.667	0.696	0.734	0.750
0.924	0.18	0.892	0.730	0.780	0.805	0.807
1.474	0.061	0.986	0.931	0.934	0.961	0.958
2.474	0.036	0.997	0.980	0.964	0.969	0.975
3.474	0.021	0.999	0.992	0.983	0.982	0.982
4.744	0.012	0.999	0.996	0.991	0.990	0.989

with a suitable hard-core radius $r_c = 0.3$ fm.

Although this potential is not a standard HC potential, we choose this mainly to test the convergence behavior of our numerical calculation. We prefer this potential over, for example, the Afnan-Tang HC potential [14] (which is purely attractive just outside the hard core) for the following reason. For a potential which is repulsive as $r_{ij} \to r_c + \delta$ [e.g., the potential of Eq. (33)], the change of potential at $r_{ij} = r_c$ is less drastic as compared with that for a potential which is purely attractive as r_{ij} approaches the hard-core radius. This leads to unstable results in the numerical algorithm adopted by us as r approaches $\sqrt{2}r_c$ for the Afnan-Tang (HC) potential, while for a potential which is repulsive just outside the hard core, the probability of finding the particles at such separations is small and the numerical algorithm presents no difficulty.

With the potential (33) we calculate the N and v matrices by numerical double integration and v matrix by Eq. (28) at each r-mesh point and then solve Eq. (29) by the hyperspherical adiabatic approach (HAA) [5]. The binding energies (BE's) calculated by both the extreme and uncoupled adiabatic approximation (denoted, respectively, by EAA and UAA have been presented in Table II for various $K_{\max}$ [equal to the truncated upper limit of the K' sum in Eq. (29)]. It is clear that the BE approaches a convergence reasonably fast as $K_{\max}$ increases. A convergence in BE to be better than half a percent is achieved with $K_{\max} = 12$. The exact BE is expected to be in between EAA and UAA values, since the energy (E) satisfies an important inequality [15] $E_{\rm EAA} \le E_{\rm exact} \le E_{\rm UAA}$. The smallness of $|E_{\rm UAA} - E_{\rm EAA}|$ lends credence to the HAA procedure.

In Fig. 2 we plot the lowest eigenpotential [5] $\omega_0(r)$, including the overbinding correction for $K_{\max} = 12$. Once again, a convergence of $\omega_0(r)$ as $K_{\max}$ increases is clearly demonstrated from our calculated numbers. The calculated $\omega_0(r)$ for $K_{\max} = 2$ to $K_{\max} = 12$ are so close to each other that they cannot be differentiated in Fig 2. Figure 3 is a plot of the ground-state hyperradial wave function (which coincides with Ψ_0) for representative values of $K_{\max}$. The convergence is clearly demonstrated as $K_{\max}$ increases. This wave function has the required behavior that it vanishes as $r \to \sqrt{2}r_c$. One can reconstruct the wave function Ψ for any configuration from Eq. (10) using the calculated hyperradial wave function and known hyperangular functions. Since the hyderradial wave function goes smoothly to zero as $r \to \sqrt{2}r_c$, the wave function Ψ will go smoothly to zero as $r_{ij} \to r_c$, and because of the presence of the theta function in Eq. (10), Ψ will remain zero for all $r_{ij} < r_c$. This wave function has the required behavior that it vanishes as $r \to \sqrt{2}r_c$.

TABLE II. Calculated binding energy.

$K_{\max}$	Binding energy (MeV) EAA	UAA
2	3.9738	3.9446
4	4.2302	4.1798
6	4.2784	4.2239
8	4.2949	4.2398
10	4.3143	4.2588
12	4.3298	4.2744

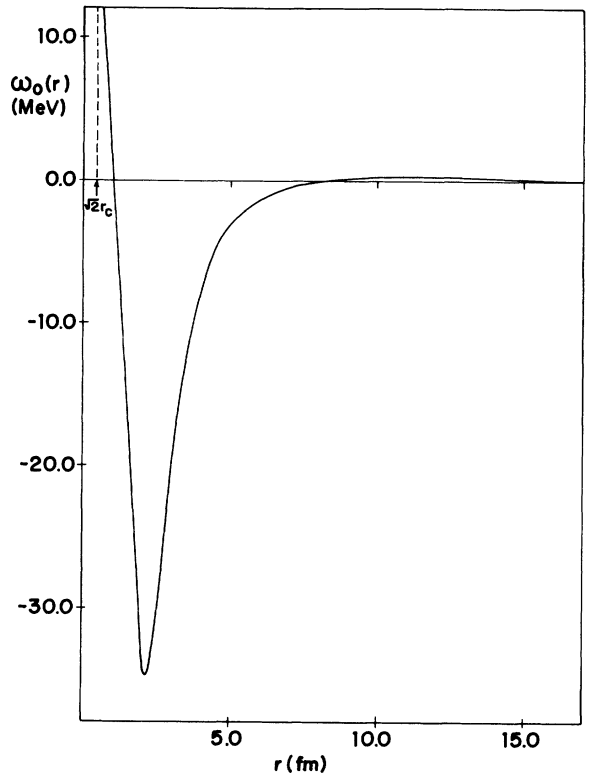

FIG. 2. Plot of $\omega_0(r)$ for $K_{\max} = 12$. The two-body force is indicated in the text. Curves for $K_{\max} < 12$ cannot be clearly distinguished in the scale used.

IV. CONCLUSIONS

Although the solution of the two-body problem interacting via hard-core potentials is quite straightforward [4,16], the corresponding problem for more than two interacting particles is considerably more complex. In the present work, we attempt to solve the three-body problem, interacting via pairwise hard-core interactions, within the framework of the hyperspherical harmonics expansion method by including a product of three theta functions in the ansatz for the three-body wave function. By an algebraic manipulation, the resulting system of differential equations can be put in the standard coupled differential equation form of the HAA method. For a suitably chosen potential, we demonstrate that this calculation procedure converges fairly well and the calculated wave function satisfies the physical requirements. This shows that the calculation process is reliable and convergent, giving credence to the ansatz proposed here. Although the numerical algorithm is rather slow because of the numerical double integration at each mesh point, this method is a fairly straightforward and feasible one. The method can be generalized to include the mixed-

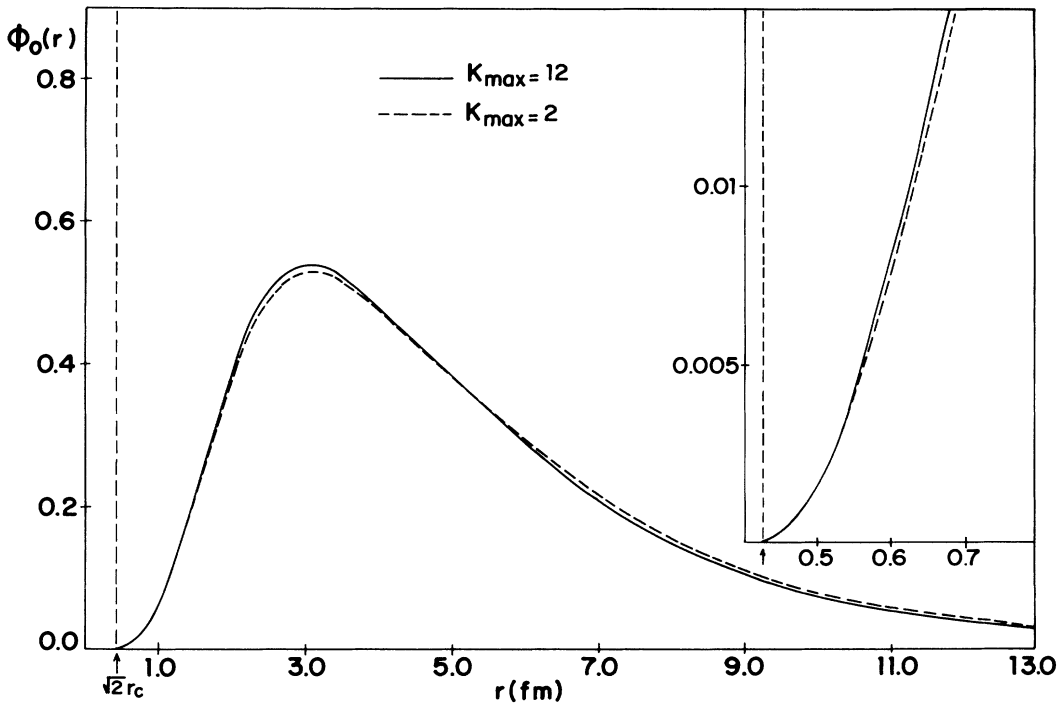

FIG. 3. Plot of the ground-state hyperradial wave function for $K_{max}=2$ and 12 showing the convergence of the wave function (region near $\sqrt{2}r_c$ is shown in an expanded scale in the inset).

symmetry and D states of the trinucleon, even though we restricted ourselves to the space totally symmetric S state.

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APPENDIX

To obtain Eq. (22), we define

$$I_i=\int_0^{\pi/2} f_1 \frac{\partial^2}{\partial\phi^2} X\, d\phi \ , \tag{A1}$$

$$I_2=4\int_0^{\pi/2} f_2 \frac{\partial}{\partial\phi} X\, d\phi \ , \tag{A2}$$

where

$$f_1=\hat{\mathbf{P}}^{ll}_{2K}(\phi)\sin^2\phi\cos^2\phi \ , \tag{A3}$$

$$f_2=\hat{\mathbf{P}}^{ll}_{2K}(\phi)\cot(2\phi)\sin^2\phi\cos^2\phi \ , \tag{A4}$$

$$X=\hat{\mathbf{P}}^{ll}_{2K'}(\phi)\Theta(r,\phi,\mu). \tag{A5}$$

Integrating by parts Eqs. (A1) and (A2), we obtain

$$I_1=\int_0^{\pi/2} f_1'' X\, d\phi \ , \tag{A6}$$

$$I_2=-4\int_0^{\pi/2} f_2' X\, d\phi \ . \tag{A7}$$

The evaluation of f_1'' and f_2' is straightforward.

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