

Bethe-Salpeter calculation of three-nucleon observables with multirank separable interactions

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(Received 17 December 1991)*

Bethe-Salpeter equations for two and three particles are employed to study relativistic effects in the three-nucleon bound-state system. The used covariant, multirank separable interactions are constructed from one phenomenological (Graz II) and two different meson-exchange potentials (Paris and Bonn). The obtained increase in the triton binding energy due to relativity ranges from 0.29 to 0.38 MeV. In an approximation where boost effects are neglected and an angular-averaging procedure is applied in the construction of the total trinucleon wave function, the elastic charge form factors are calculated.

PACS number(s): 21.45.+v, 21.30.+y, 11.10.St, 11.80.Jy

I. INTRODUCTION

In an article [1] hereafter referred to as RT88, we have employed Bethe-Salpeter-Faddeev (BSF) equations with phenomenological, separable interactions of rank one to study relativistic effects in the three-nucleon system. The results indicated the promising possibility of increasing the triton binding energy, thereby reducing the discrepancy with the experimental value, and also of making the falling-off of the ^3He charge form factor steeper. However, the not so realistic nature of the used forces was clearly revealed by the failure to reproduce the experimentally observed diffraction minimum in the latter observable.

In the present paper, several multirank interactions will be employed in order to account for a more realistic description of the on- and off-shell nucleon-nucleon (NN) amplitudes. These will be either purely phenomenological, namely a covariant generalization of the nonrelativistic (NR) Graz-II [2] potential, or derived from separable approximations of the Paris [3] and Bonn [4] NN potentials. Our strategy will be the same as in RT88, which amounts to refitting, in the BS framework, the coupling constants stemming from the underlying NR potentials to the NN scattering data. The thus-determined relativistic interactions are then used to calculate the trinucleon observables, which will be compared with the NR predictions, as well as with two quasipotential approaches. Throughout this investigation the Dirac spin structure of the nucleons will be neglected, so that they

are effectively treated as scalar fermions. This approximation, also employed in RT88, is not imposed by any basic difficulty other than a sizable increase of the number of coupled channels in the three-body equations. The inclusion of the Dirac spin structure will be the subject of future study.

In Sec. II the different separable interactions are presented and the results of fits to the NN data are given. Section III generalizes the BSF equations derived in RT88 to the multirank case and deals with the construction of the trinucleon charge form factors from the bound-state solutions to these equations. In Sec. IV the results for the triton binding energy and the charge form factors are presented and discussed. Section V contains some concluding remarks and prospects for future work.

II. TWO-NUCLEON SYSTEM

A. BS equation with separable interactions

In momentum space, the BS equation for the T matrix describing relativistic two-particle scattering reads, in terms of the relative four-momenta p , p' , k , and the total four-momentum P ,

$$T(p, p'; P) = V(p, p') + \frac{i}{(2\pi)^4} \int d^4k V(p, k) S(k; P) T(k, p'; P), \quad (2.1)$$

where $V(p, p')$, in principle, stands for the set of all two-particle-irreducible diagrams, and $S(k; P)$ is the free two-particle Green's function. If one chooses a separable interaction kernel, Eq. (2.1) can be solved in closed form. Treating the nucleons as scalar fermions, which amounts to describing their spin degrees of freedom essentially

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in a nonrelativistic way, we may take for the coupled-channel case a covariant, multirank separable potential of the form

$$V_{LL'}(p, p') = \sum_{i,j=1}^N \lambda_{ij} g_i^{(L)}(p^2) g_j^{(L')}(p'^2), \quad (2.2)$$

where the coupling matrix $[\lambda]$ is symmetric. For simplicity, we assume that the form factors do not depend

on the total two-nucleon c.m. momentum. For equal-mass particles and in the c.m. frame, the solution for the partial-wave T matrix reads [1, 5]

$$T_{LL'}(p_0, |\mathbf{p}|; p'_0, |\mathbf{p}'|; s) = \sum_{i,j=1}^N \tau_{ij}(s) g_i^{(L)}(p^2) g_j^{(L')}(p'^2), \quad (2.3)$$

with

$$[\tau^{-1}(s)]_{ij} = [\lambda^{-1}]_{ij} - \frac{i}{4\pi^3} \sum_{L'=0,2} \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} k^2 d|\mathbf{k}| g_i^{(L')}(k^2) g_j^{(L')}(k^2). \quad (2.4)$$

Here, $S(k_0, |\mathbf{k}|; s) = [(\frac{1}{4}s - k_0^2 + E_k^2)^2 - sE_k^2]^{-1}$ with $E_k = \sqrt{\mathbf{k}^2 + m^2} - i\epsilon$ is the free, scalar two-body Green's function. The fully on-shell T matrix, defined by $T_{LL'}(\bar{p}) \equiv T_{LL'}(\bar{p}_0, |\bar{\mathbf{p}}|; \bar{p}_0, |\bar{\mathbf{p}}|; s)$, with $\bar{p}_0 = 0$ and $|\bar{\mathbf{p}}| = \sqrt{\frac{1}{4}s - m^2} = \sqrt{\frac{1}{2}mE_{\text{lab}}}$, is related to the scattering phase shifts as

$$T_{LL}(\bar{p}) = -\frac{8\pi\sqrt{s}}{|\bar{\mathbf{p}}|} e^{i\delta_L(\bar{p})} \sin \delta_L(\bar{p}). \quad (2.5)$$

Notice that exact two-particle unitarity will be satisfied except for those energies at which a pinch can occur of a pole of the Green's function with a singularity of the interaction. The integral over k_0 in Eq. (2.4) can be calculated either by applying Cauchy's theorem and accounting for the residues of all Green's-function and form-factor poles lying within the closed contour or by carrying out a Wick rotation in this variable, thereby also taking into account the possibly encountered poles in the rotation process. In RT88, only the former method was used, while here both are employed whenever feasible, so as to have an additional numerical check. The remaining integration(s) can then be done numerically, with a standard subtraction at positive energies.

B. Covariantization

In RT88, phenomenological, separable NN interactions of rank one were chosen, with form factors only depending on the square of the relative four-momentum p in order to ensure Lorentz covariance. Of course, this is by no means a unique way to obtain covariant form factors, for an obvious alternative would be to also include a dependence on $p \cdot P$ with P the total c.m. momentum; however, it certainly is the simplest one. It allows a simple procedure to construct relativistic separable interactions to be used in the BS equation, from NR ones in a Lippmann-Schwinger (LS) approach being functions of the three-momentum squared \mathbf{p}^2 , namely through the substitution $\mathbf{p}^2 \rightarrow p^2 = p_0^2 - \mathbf{p}^2$. At first sight though, it is not at all clear how to apply this method to NR potentials based on meson exchange. However, separable approximations to the Paris [3, 6] and Bonn [4] potentials in momentum space have been constructed, with form factors consisting of rational functions of \mathbf{p}^2 only, making the above "covariantization" procedure also ap-

plicable to these interactions. Naturally, this does not in general lead to phase equivalence of the respective LS and BS two-nucleon amplitudes, a necessary requirement to study relativistic effects in the three-nucleon system. Therefore, some parameter refitting has to be carried out in the BS framework. Yet, only the coupling constants λ_{ij} are adjusted and not the range parameters, which are, albeit indirectly, related to the meson masses in the underlying potential model. It is hoped that in this manner the off-shell behavior of the NR potentials is not changed too much, and thus a relativistic meson-exchange interaction for scalar nucleons will be mimicked via the covariantization scheme. However, one should realize that the covariant interactions obtained via this hybrid construction are not expected to be genuine separable approximations of such an underlying meson-exchange force, contrary to their NR counterparts.

C. Graz-II, Paris, and Bonn interactions

In the present investigation, we will employ six multirank separable interactions, to be derived from the phenomenological Graz-II (with three different D -state probabilities), the Paris (in two different separable parametrizations), and the Bonn potentials. The covariant Graz-II form factors read (see also Ref. [5])

$$g_1(p^2) = \frac{1}{p^2 - \beta_{11}^2 + i\epsilon} + \frac{\gamma_1 p^2}{(p^2 - \beta_{12}^2 + i\epsilon)^2}, \quad (2.6)$$

$$g_2(p^2) = \frac{p^2}{(p^2 - \beta_{21}^2 + i\epsilon)^2} + \frac{\gamma_2 (p^2)^2}{(p^2 - \beta_{22}^2 + i\epsilon)^3},$$

for the 1S_0 partial wave, and

$$g_1^{(0)}(p^2) = \frac{1 - \gamma_1 p^2}{(p^2 - \beta_{11}^2 + i\epsilon)^2},$$

$$g_2^{(0)}(p^2) = -\frac{p^2}{(p^2 - \beta_{12}^2 + i\epsilon)^2}, \quad (2.7)$$

$$g_3^{(2)}(p^2) = \frac{p^2(1 - \gamma_2 p^2)}{(p^2 - \beta_{21}^2 + i\epsilon)(p^2 - \beta_{22}^2 + i\epsilon)^2},$$

$$g_1^{(2)}(p^2) = g_2^{(2)}(p^2) = g_3^{(0)}(p^2) \equiv 0,$$

for the coupled 3S_1 - 3D_1 . Note that the parameters β_{ab}

TABLE I. Adjusted 1S_0 parameters of Graz-II interactions.

	γ_1	λ_{11} (GeV ⁴)	λ_{22} (GeV ⁴)
NR	0.563 486	-0.725 034	666.063
BSLT	0.585 669	-0.717 523	151.859
BS	0.582 344	-0.747 549	130.422

and γ_a are different for the two partial waves. Their numerical values are given in Ref. [2]. Furthermore, the coupling matrix $[\lambda_{ij}]$ is diagonal and of rank 2 in the singlet case, while it is of rank 3 and nondiagonal for the coupled triplet. Now, we proceed to refit the couplings to the scattering length a , effective range r_0 , and, in the triplet case, also to the deuteron binding energy B_d , D -state probability P_D , and asymptotic D/S -state ratio $\rho_{D/S}$, while keeping the phase shifts as close as possible to the ones of the original potential. In the singlet case, where we have only two couplings at our disposal, also the parameter γ_1 is adjusted so as to obtain a satisfactory fit, but, as mentioned before, the range parameters β_{ij} are always left unaltered. In order to study the dependence of the trinucleon observables on P_D , three different values of this quantity are chosen, viz. 4%, 5%, and 6%, rather than keeping it fixed at its original value of 4.82%. Consequently, in the NR triplet case a refit has to be performed, too. Apart from the described LS and BS approaches, we also use the BSLT reduction of the BS equation (see, e.g., RT88) as an additional comparison. This leaves us with three sets of parameters for the 1S_0 partial wave, shown in Table I, and nine for the coupled triplet, given in Table II. Notice that the latter are different from those in Ref. [7], where the same type of interactions was used, but with Dirac instead of scalar propagators, in order to study relativistic contributions to the deuteron electromagnetic form factors. The resulting effective-range parameters and deuteron properties can be found in Tables III and IV. We see that very similar results are obtained for the three used equations with, on the whole, relatively small changes in the coupling constants. In particular, the BS and BSLT predictions are almost identical.

The second set of separable interactions are constructed from the Paris and Bonn potentials, based on meson exchange. For the purpose of exploring the reliability of the covariantization procedure, we use two distinct separable approximations of the Paris potential. These have been published in Ref. [6], to be denoted by Paris 1, and in Ref. [3], originally named PEST3 and PEST4 for the 1S_0 and 3S_1 - 3D_1 case, respectively, here denoted by Paris 2. Although both parametrizations are of rank 3 and 4, the used form factors are very different. As a matter of fact, in the former work covariant separable interactions for the BS equation were developed in an analogous way as in the present investigation. In order to facilitate the p_0 integration in Eq. (2.4) with Cauchy's method, form factors were chosen containing only poles of low order. In the latter paper on the other hand, merely an NR separable potential was constructed, so that no such restrictions were needed. Here, we evaluate the p_0 integrals for the corresponding Paris-2 interaction using the Wick-rotation procedure only. Surprisingly, it turns out that, already for poles of relatively low order, this method, involving a two-dimensional numerical integration, becomes more efficient in terms of precision/CPU time than Cauchy's, where it is just one-dimensional.

The form factors of the Paris-1 interaction read [6, 5]

$$g_i(p^2) = \sum_{k=1}^5 \frac{C_{ik}}{-p^2 + \beta_k^2 - i\epsilon} \quad (2.8)$$

for the singlet (rank 3), and

$$g_i^{(L)}(p^2) = \sum_{k=1}^6 \frac{C_{ik}^{(L)} |p|^L}{(-p^2 + \beta_{ik}^{(L)^2} - i\epsilon)^{1+L/2}} \quad (2.9)$$

for the coupled triplet (rank 4). The parameters C_{ik} , β_k , and β_{ik} , which are kept fixed, can be found in Ref. [6] (Tables I and III), as well as the couplings λ_{ij} . Note that the value of the parameter β_{46} in the fourth column of Table III should be 3.009 451 6 instead of 2.009 451 6 [8]. Here, however, we repeat the fit of the couplings, for two reasons. In the first place, the singlet coupling matrix was originally taken to be diagonal, making a good reproduction of the effective-range parameters impossible,

TABLE II. 3S_1 couplings of Graz-II interactions.

P_D		λ_{11}^a	λ_{12}^b	λ_{13}^a	λ_{22}^c	λ_{23}^b	λ_{33}^a
4%	NR	$3.357\,24 \times 10^{-3}$	$-7.808\,89 \times 10^{-1}$	$-2.254\,84 \times 10^{-2}$	170.283	3.426 10	$4.052\,94 \times 10^{-2}$
	BSLT	$3.937\,60 \times 10^{-4}$	$-3.468\,36 \times 10^{-1}$	$-1.672\,42 \times 10^{-2}$	104.769	2.399 46	$2.491\,97 \times 10^{-2}$
	BS	$-6.180\,65 \times 10^{-4}$	$-2.330\,79 \times 10^{-1}$	$-1.546\,14 \times 10^{-2}$	90.4794	2.226 10	$1.987\,78 \times 10^{-2}$
5%	NR	$3.050\,65 \times 10^{-3}$	$-5.082\,87 \times 10^{-1}$	$-1.968\,67 \times 10^{-2}$	96.0983	2.592 44	$2.782\,35 \times 10^{-2}$
	BSLT	$5.873\,84 \times 10^{-4}$	$-1.953\,77 \times 10^{-1}$	$-1.601\,01 \times 10^{-2}$	57.9262	1.985 53	$1.834\,08 \times 10^{-2}$
	BS	$-3.634\,71 \times 10^{-4}$	$-1.123\,26 \times 10^{-1}$	$-1.510\,66 \times 10^{-2}$	51.0166	1.921 40	$1.436\,81 \times 10^{-2}$
6%	NR	$2.964\,42 \times 10^{-3}$	$-2.987\,33 \times 10^{-1}$	$-1.755\,66 \times 10^{-2}$	41.9506	2.006 21	$1.867\,86 \times 10^{-2}$
	BSLT	$9.374\,41 \times 10^{-4}$	$-8.327\,04 \times 10^{-2}$	$-1.561\,66 \times 10^{-2}$	22.5999	1.713 46	$1.353\,30 \times 10^{-2}$
	BS	$-5.654\,49 \times 10^{-5}$	$-4.293\,35 \times 10^{-3}$	$-1.478\,43 \times 10^{-2}$	17.0509	1.679 46	$9.814\,72 \times 10^{-3}$

^aIn GeV⁸.^bIn GeV⁶.^cIn GeV⁴.

TABLE III. 1S_0 effective-range parameters with Graz-II interactions.

	a (fm)	r_0 (fm)
NR	-23.77	2.683
BSLT	-23.77	2.683
BS	-23.77	2.683
Experiment (n - p)	-23.748	2.75

with eventual repercussions in the three-nucleon system [9]. We remedy this by relaxing the diagonality condition of $[\lambda]$, also for the LS equation. This is clearly legitimate, as the original Paris potential does not cover the neutron-proton (n - p) 1S_0 partial wave. Conversely, we will only be dealing with the n - p case for simplicity, since we are investigating the influence of relativity in the trinucleon and not possible charge-symmetry-breaking effects. Secondly, in Ref. [6] the accuracy of the computed NN effective-range parameters and deuteron properties is insufficient to warrant a reliable study of relativistic contributions in the three-nucleon case. Thus, we refit all the couplings for the three equations and the two partial waves.

The form factors of both the Paris-2 [3] and the Bonn [4] separable interactions have the structure

$$g_i(p^2) = \sum_{k=1}^4 \frac{C_{ik} (-p^2)^{k-1}}{(-p^2 + \beta_{ik}^2 - i\epsilon)^k} \quad (2.10)$$

for the rank-3 singlet, and

$$g_i^{(L)}(p^2) = \sum_{k=1}^4 \frac{C_{ik}^{(L)} (-p^2)^{k-1+L/2}}{(-p^2 + \beta_{ik}^{(L)2} - i\epsilon)^{k+L/2}} \quad (2.11)$$

for the rank-4 triplet. Notice that here the threshold behavior in the D state is described by $-p^2$, while for the Paris-1 interaction a factor \mathbf{p}^2 was taken as in Ref. [6]. There, magic vectors were used to guarantee covariance, also in P waves. However, the difference in the two prescriptions turns out to be immaterial. The parameters C_{ik} and β_{ik} can be found in Refs. [3] (Table VI of first article, Table I of second) and [4] (Tables II and III). Here, a comment is due with regard to two-particle unitarity. For all used separable interactions, this property will be exactly satisfied up to laboratory energies of at least 480 MeV, except for the Paris-2 case. There, one of the 32 triplet range parameters, viz. $\beta_{32}^{(0)}$, is slightly larger than twice the nucleon mass, leading, in principle,

TABLE IV. 3S_1 effective-range parameters and deuteron properties with Graz-II interactions.

P_D		a (fm)	r_0 (fm)	B_d (MeV)	$\rho_{D/S}$
4%	NR	5.418	1.780	2.2254	0.02502
	BSLT	5.419	1.779	2.2254	0.02592
	BS	5.419	1.780	2.2254	0.02591
5%	NR	5.419	1.780	2.2254	0.02796
	BSLT	5.420	1.779	2.2254	0.02896
	BS	5.420	1.779	2.2254	0.02896
6%	NR	5.420	1.779	2.2254	0.03061
	BSLT	5.421	1.778	2.2254	0.03171
	BS	5.421	1.778	2.2254	0.03170
4.82%	Graz-II	5.419	1.780	2.2254	0.02746
	Experiment	5.424	1.759	2.2246	0.0263

to unitarity violation at all scattering energies in the BS equation. This effect, however, is completely negligible, as has been checked numerically. The refitted coupling constants of the Paris-1, Paris-2, and Bonn interactions are given in Tables V and VI, and the resulting effective-range parameters and deuteron properties in Tables VII and VIII. Once more, one can observe that the couplings are quite similar for the LS, BSLT, and BS equations. We also remark that the predictions of the Paris-1 and Paris-2 interactions are not perfectly identical, which is hardly surprising, as they have been obtained from separable *approximations* to the original Paris potential.

Finally, we point out that, with all interactions described in this section, the NN phase shifts for the three employed equations are very similar and practically indistinguishable graphically, so that we have not plotted them. Thus, relativistic and nonrelativistic NN interactions have been constructed which are to a high degree phase equivalent, making a comparison of the corresponding three-nucleon observables meaningful.

III. THREE-NUCLEON SYSTEM

A. BSF equations

In RT88, homogeneous, Faddeev-type (BSF) integral equations in two continuous variables were derived to describe the three-nucleon bound-state system, with separable, pairwise interactions of rank one. It was also shown that a Wick rotation could be carried out, leading to a

TABLE V. 1S_0 couplings of Paris and Bonn interactions; units: fm^{-4} .

		λ_{11}	λ_{12}	λ_{13}	λ_{22}	λ_{23}	λ_{33}
Paris-1	NR	-0.948 820	0.0396 328	-0.00587 557	-1.184 70	-0.0491 013	0.833 606
	BSLT	-1.020 93	-0.0127 224	0.0525 057	-1.447 85	0.336 379	0.570 440
	BS	-1.060 70	-0.0473 008	0.161 285	-1.512 88	0.198 404	0.578 036
Paris-2	NR	-0.00420 731	0.0888 985	0.0109 088	-2.178 09	-1.237 81	6.099 89
	BSLT	-0.00439 216	0.0973 919	0.0299 439	-2.722 19	-1.242 05	5.546 11
	BS	-0.00448 232	0.100 214	0.0239 941	-2.858 64	-0.997 939	4.844 84
Bonn	NR	-0.00400 648	0.106 667	0.556 320	28.9071	-75.7845	176.637
	BSLT	-0.00293 762	0.129 658	0.661 269	24.9786	-76.4435	207.104
	BS	-0.00257 235	0.137 886	0.710 601	23.3424	-76.7143	223.126

TABLE VI. 3S_1 couplings of Paris and Bonn interactions; units: fm^{-4} .

	NR	Paris-1		NR	Paris-2		NR	Bonn	
		BSLT	BS		BSLT	BS		BSLT	BS
λ_{11}	-0.205559	-0.231413	-0.238168	-2.30301	-2.28431	-2.34799	-1.93010	-1.91329	-1.99086
λ_{12}	-0.108958	-0.146062	-0.140148	-1.55623	-1.56471	-1.53749	-1.85908	-1.83626	-1.81676
λ_{13}	0.0625966	0.0636279	0.0624894	0.649750	0.640183	0.629960	0.600230	0.608954	0.591910
λ_{14}	0.0259473	0.00959093	0.00671305	0.0462466	0.0346558	0.0559728	0.843597	0.784281	0.781237
λ_{22}	0.246180	0.0783938	0.0821977	0.794700	0.668754	0.688313	0.708501	0.681793	0.623377
λ_{23}	0.0817942	0.0845026	0.0848610	0.856251	0.879083	0.882429	1.38688	1.45580	1.47213
λ_{24}	-0.127576	-0.0639813	-0.0750647	-0.717883	-0.692309	-0.727545	-0.723216	-0.758844	-0.772750
λ_{33}	-0.0295842	-0.0282376	-0.0295189	-0.277624	-0.282922	-0.284657	-0.253240	-0.274625	-0.286107
λ_{34}	-0.0471933	-0.0384208	-0.0373834	-0.317045	-0.344264	-0.346235	-1.25370	-1.30664	-1.29517
λ_{44}	0.227136	0.187132	0.167919	1.53599	1.57055	1.47533	1.92639	1.99451	1.96585

coupled set of integral equations with a compact kernel. As the generalization to the multirank case is straightforward, we give here the final expressions only (see also Ref. [5]). For a rank- N separable interaction, the Faddeev bound-state amplitude takes the form

$$\Psi^a(p, q) = \sum_{i,j=1}^N g_i^a(p) \tau_{ij}^a \left[\left(\frac{2}{3}P + q \right)^2 \right] \Phi_j^a(q), \quad (3.1)$$

$$\Phi_j^a(q_4, |\mathbf{q}|) = -\frac{1}{4\pi^3} \sum_{b=1}^2 \sum_{k,l=1}^{N_b} \int_{-\infty}^{\infty} dq'_4 \int_0^{\infty} q'^2 d|q'| Z_{jk}^{ab}(q_4, |\mathbf{q}|; q'_4, |q'|) \frac{\tau_{kl}^b \left[\left(\frac{2}{3}\sqrt{s} + iq'_4 \right)^2 - \mathbf{q}'^2 \right]}{\left(\frac{1}{3}\sqrt{s} - q'_4 \right)^2 - \mathbf{q}'^2 - m^2 + i\epsilon} \Phi_l^b(q'_4, |q'|), \quad (3.2)$$

with the driving term

$$Z_{jk}^{ab}(q_4, |\mathbf{q}|; q'_4, |q'|) = C^{ab} \int_{-1}^1 d \cos \theta_{\mathbf{q}, \mathbf{q}'} \frac{g_j^a \left(-\frac{1}{2}q_4 - q'_4, \left| \frac{1}{2}\mathbf{q} + \mathbf{q}' \right| \right) g_k^b \left(q_4 + \frac{1}{2}q'_4, \left| \mathbf{q} + \frac{1}{2}\mathbf{q}' \right| \right)}{\left(\frac{1}{3}\sqrt{s} + i(q_4 + q'_4) \right)^2 - (\mathbf{q} + \mathbf{q}')^2 - m^2 + i\epsilon}. \quad (3.3)$$

Here, N_b is the rank of the separable interaction in channel b , and s is related to the three-body binding energy by $\sqrt{s} = 3m - B_t$. The spin/isospin recoupling coefficients have the values $C^{11} = C^{22} = \frac{1}{4}$ and $C^{12} = C^{21} = -\frac{3}{4}$. Note that the restriction to S waves implies a truncation of the 3S_1 - 3D_1 two-body T matrix, which is, however, calculated in full. This so-called T_{00} approximation yields results quite close to a complete 5-channel calculation, at least in the NR case.

where p, q are the relative four-momenta with respect to the spectator particle, and the superscript a refers to a specific two-body channel. Then, restricting ourselves to S waves and equal masses just as in RT88, we get, in the three-particle c.m. frame, for the partial-wave-decomposed spectator function Φ the Wick-rotated integral equation

B. Trinucleon charge form factors

The trinucleon charge form factors can readily be found in the relativistic impulse approximation [10]. They are determined from the Faddeev amplitude given in Eq. (3.1) by taking the appropriate matrix elements of the single-nucleon current operator, accounting for all three-nucleon contributions. Keeping in mind that in this way pair currents will automatically be included, one can

TABLE VII. 1S_0 effective-range parameters with Paris and Bonn interactions.

		a (fm)	r_0 (fm)
Paris-1	NR	-23.67	2.805
	BSLT	-23.72	2.805
	BS	-23.72	2.810
Paris-2	NR	-23.72	2.817
	BSLT	-23.72	2.817
	BS	-23.72	2.817
Bonn	NR	-23.75	2.731
	BSLT	-23.75	2.727
	BS	-23.75	2.734
Experiment ($n-p$)		-23.748	2.75

TABLE VIII. 3S_1 effective-range parameters and deuteron properties with Paris and Bonn interactions.

		a (fm)	r_0 (fm)	B_d (MeV)	$\rho_{D/S}$
Paris-1	NR	5.428	1.780	2.2241	0.02611
$P_D =$	BSLT	5.426	1.774	2.2247	0.02612
5.77%	BS	5.426	1.775	2.2246	0.02610
Paris-2	NR	5.411	1.765	2.2250	0.02614
$P_D =$	BSLT	5.414	1.765	2.2250	0.02614
5.77%	BS	5.413	1.765	2.2250	0.02614
Bonn	NR	5.484	1.853	2.2250	0.02669
$P_D =$	BSLT	5.479	1.845	2.2249	0.02666
4.58%	BS	5.484	1.851	2.2250	0.02656
Experiment		5.424	1.759	2.2246	0.0271

essentially proceed similarly to the NR case, where such effects would have to be treated separately. Thus, we define the amplitudes

$$\Psi_{(i)}^S = \frac{1}{\sqrt{2}}(\Psi_{(i)}^s - \Psi_{(i)}^t), \quad (3.4)$$

$$\Psi_{(i)}^{S'} = \frac{1}{\sqrt{2}}(\Psi_{(i)}^s + \Psi_{(i)}^t), \quad (3.5)$$

where $\Psi_{(i)} \equiv \Psi(p_i, q_i)$, and the superscripts s and t stand for singlet and triplet, respectively. Next, we can construct the totally symmetric and mixed symmetric vertex functions

$$v_0 \equiv u = \Psi_{(1)}^S + \Psi_{(2)}^S + \Psi_{(3)}^S, \quad (3.6)$$

$$v_1 = \frac{1}{2}(\Psi_{(1)}^{S'} + \Psi_{(2)}^{S'} - 2\Psi_{(3)}^{S'}), \quad (3.7)$$

$$v_2 = \frac{\sqrt{3}}{2}(\Psi_{(1)}^{S'} - \Psi_{(2)}^{S'}). \quad (3.8)$$

$$\Psi((p_2)_4, |\mathbf{p}_2|; (q_2)_4, |\mathbf{q}_2|) \approx \frac{1}{2} \int_{-1}^{+1} d \cos \theta_{\mathbf{p}, \mathbf{q}} \Psi(-\frac{1}{2}p_4 - \frac{3}{4}q_4, |\frac{1}{2}\mathbf{p} + \frac{3}{4}\mathbf{q}|; p_4 - \frac{1}{2}q_4, |\mathbf{p} - \frac{1}{2}\mathbf{q}|), \quad (3.9)$$

and a similar expression for $\Psi_{(1)}$. It should be noted that, while this approximation is known to work well in the NR case, the same is not necessarily true in the BSF formalism, for in the latter case there is an additional dependence on $\theta_{\mathbf{p}, \mathbf{q}}$ in two of the three propagators. However, at zero momentum transfer, the exact result can be computed quite straightforwardly, and the angular-averaging approximation has been verified to be very reasonable in this limit. A possible improvement would be to consider the right-hand side of Eq. (3.9) the first term in a partial-wave expansion and to include some more terms. In fact, in the mentioned limit, this procedure turns out to converge very rapidly to the exact answer. For arbitrary momentum transfers, it might still be numerically feasible, which will be investigated in future work.

The three-nucleon charge form factors can be written in terms of the proton and neutron ones as

$$2F_{\text{ch}}(^3\text{He}) = (2F_{\text{ch}}^p + F_{\text{ch}}^n)F_1 - \frac{2}{3}(F_{\text{ch}}^p - F_{\text{ch}}^n)F_2, \quad (3.10)$$

$$F_{\text{ch}}(^3\text{H}) = (F_{\text{ch}}^p + 2F_{\text{ch}}^n)F_1 + \frac{2}{3}(F_{\text{ch}}^p - F_{\text{ch}}^n)F_2, \quad (3.11)$$

where F_1 and F_2 are the usual body form factors. With the vertex functions defined in Eq. (3.6), the former are then given by (see also RT88)

$$F_1(Q) = \int d^4p \int d^4q \sum_{n=0}^2 v_n^*(p, q') G_3(q') \Gamma_0(q, q') \times G(p, q) v_n(p, q), \quad (3.12)$$

$$F_2(Q) = -3 \int d^4p \int d^4q v_1^*(p, q') G_3(q') \Gamma_0(q, q') \times G(p, q) u(p, q),$$

where $q' = q - \frac{2}{3}Q$ with Q the photon momentum, $G(p, q) = G_1(p, q)G_2(p, q)G_3(p, q)$ is the free three-body Green's function, and $\Gamma_\mu(q, q') = \frac{2}{3}P_\mu + Q_\mu - 2q_\mu$ is the vertex operator for scalar electrodynamics. For simplicity, this operator was taken to be unity in RT88. Strictly speaking, the correct expression is needed to warrant

Note that v_1 is symmetric under the interchange of particles one and two, while v_2 is antisymmetric. Now, if one of Faddeev amplitudes has been obtained by solving the BSF equations (3.2,3.3), say $\Psi_{(3)}$, then the other two can be constructed from the former by interpolation of the respective Jacobi variables. After all, they refer to the same function, since the particles are identical, but with different arguments. At this point, a difficulty arises because even if $\Psi_{(3)}$ is restricted to S waves only, as is the case, the other two will automatically get higher partial-wave components in the interpolation process. In other words, the latter will depend also on the angle between the relative momenta \mathbf{p} and \mathbf{q} , which poses an extremely difficult numerical problem when computing the charge form factors. A remedy is to apply an angular-averaging approximation, also used in RT88, which amounts to writing

gauge invariance of the associated electromagnetic (EM) three-nucleon current; hence, we include it in the present investigation. Nevertheless, its effect on the form factors turns out to be negligible, at least when recoil is treated in an NR manner. This static approximation, which amounts to disregarding boost effects, has been amply discussed in RT88, and will also be applied here. Boost corrections to the deuteron EM form factors have been studied in Ref. [7], leading to the conclusion that they are small for the *charge* form factor. However, the same is not necessarily true in the trinucleon case, in view of the very peaked behavior of the spectator function Φ in Eq. (3.2), as a function of q_4 (see RT88). Despite the numerical difficulties, at least part of the boost effects, e.g., the boost on the final-state one-particle propagator $G_3(q')$ in Eq. (3.12), which turned out to give by far the largest contribution in the deuteron, will be included in forthcoming studies.

IV. RESULTS AND DISCUSSION

Equations (3.2) and (3.3) are solved in essentially the same way as in RT88, namely by discretizing the two integrations and iterating the resulting matrix eigenvalue problem. It turns out that for all considered interactions this procedure rapidly converges to the sought solution. In other words, all other eigenvalues of the discretized kernel are smaller than unity in absolute value. This feature, in combination with a careful choice of mappings for the integration points, allows a very accurate determination of the binding energy and spectator function.

In Tables IX and X, the triton binding energy B_t is given for all six interactions defined in Sec. II, using four different Faddeev-type approaches with respect to relativity just as in RT88. These are denoted by NRF, QP₁, QP₂, and BSF, standing for nonrelativistic Faddeev, type-1 quasipotential, type-2 quasipotential, and Bethe-Salpeter-Faddeev equations, respectively. The QP₁ and QP₂ equations are derived from dispersion integrals in the three- and two-body invariant energy, respectively

TABLE IX. Triton binding energies (MeV) for Graz-II interactions with equations described in text.

	NRF	QP ₁	QP ₂	BSF
$P_D^d = 4\%$	8.369	8.539	8.606	8.701
$P_D^d = 5\%$	7.961	8.129	8.188	8.290
$P_D^d = 6\%$	7.566	7.735	7.787	7.892

(see RT88 for the definitions). The numerical errors are less than 0.005 MeV in the BSF case and less than 0.001 MeV in the others. From Table IX we see that, for the Graz-II interactions, the BS formalism gives rise to an extra binding of about 0.33 MeV, practically independent of the deuteron D -state probability P_D^d . Note that the BSF results are slightly different from those given in Ref. [5], where a minor error occurred in the evaluation of the off-shell two-body T -matrices for complex arguments by using Cauchy's theorem. This inaccuracy has now been eliminated and verified with the Wick-rotation method discussed in Sec. II A. Furthermore, the binding energy turns out to scale almost linearly with P_D^d , increasing by about 0.4 MeV/%. Table X contains the binding energies for the interactions derived from meson-exchange potentials. In the case of the Paris interactions, the extra binding with the BSF approach is 0.29 MeV for both parametrizations, despite the 0.06 MeV binding-energy difference of one with respect to the other. This may indicate that the covariantization procedure does not alter the off-shell behavior of the two-body amplitudes substantially, thereby lending extra support to the whole hybrid scheme of constructing relativistic interactions. After all, the two employed EST parametrizations of the Paris potential are very different in the used basis functions, and nothing guarantees *a priori* that, after covariantizing these functions and refitting the couplings, the resulting off-shell T matrices will still be comparable. For the Bonn interaction, the increase in binding is 0.38 MeV, which is very significant, since the NR versions of this potential already produce results much closer to the experimental value of 8.48 MeV than any other realistic potential, basically due its low P_D^d value. In fact, it seems safe to conclude that in the BSF framework some version of Bonn will indeed be capable of reproducing this number, when accounting for higher partial waves. That does not necessarily mean the Bonn potential provides a correct description of the NN forces, for there are serious problems with the trinucleon charge form factors, as will be shown below. Finally, concerning all quasi-potential results in Tables IX and X, the same pattern

TABLE X. Triton binding energies (MeV) for Paris and Bonn interactions with equations described in text.

	NRF	QP ₁	QP ₂	BSF
Paris-1	7.245	7.389	7.435	7.535
Paris-2	7.183	7.363	7.408	7.474
Bonn	7.822	8.083	8.140	8.201

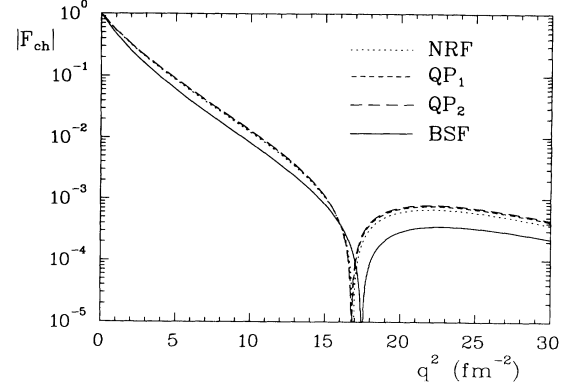


FIG. 1. ^3He charge form factor with Paris-1 interaction and equations described in text.

for B_t is obtained as in RT88, which can be symbolically summarized by $\text{NRF} < \text{QP}_1 < \text{QP}_2 < \text{BSF}$.

Turning to the trinucleon charge form factor, we first focus our attention on the ^3He case. In Figs. 1–7, $F_{\text{ch}}(^3\text{He})$ is plotted for the various interactions and equations. Since this investigation is primarily of a comparative nature, no experimental data points are displayed in the graphs. Suffice it to say that realistic NN potentials in an NR framework like the ones used here predict the first diffraction minimum and the secondary maximum at too high momentum transfers. Figure 1 shows the Paris-1 results with the four described equations. While the NRF, QP₁, and QP₂ curves are very similar, the BSF one has a somewhat different behavior. At low and intermediate momentum transfers, there is a slightly steeper falling-off (see also RT88), which however does not persist up to the diffraction minimum, so that the dip is moved outwards and the secondary maximum reduced, contrary to what is needed to get closer to the experimental data. In Fig. 2 the same curves are given for the Paris-2 interaction, showing only minor deviations from those in Fig. 1 as expected, although the NRF and BSF results depend noticeably more on the specific separable parametrization than the QP ones. Figure 3 displays the NRF and BSF form factors for both Paris interactions,

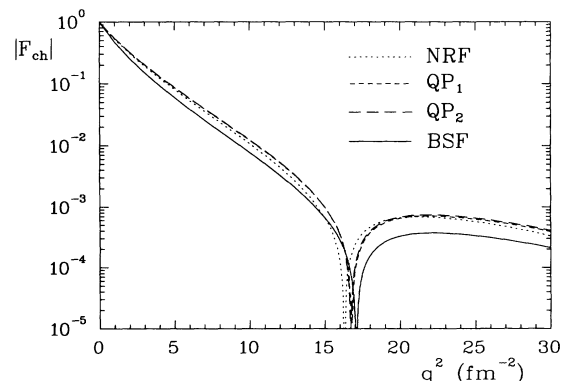


FIG. 2. Same as Fig. 1, with Paris-2 interaction.

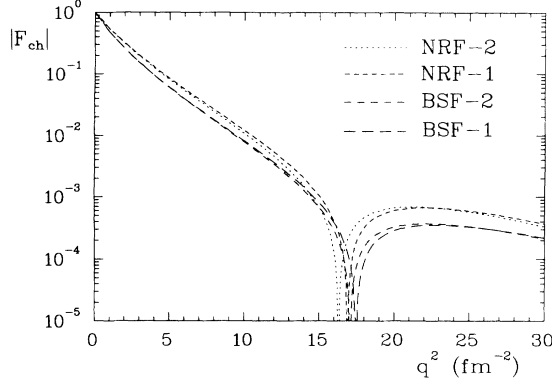


FIG. 3. Same as Fig. 1, with Paris-1 and Paris-2 interactions.

in order to focus on possible deviations due to the covariantization procedure, which are not observed, however. The Bonn results are plotted in Fig. 4, showing the same qualitative behavior as in the Paris case, but much more pronounced. We also see that the NR Bonn potential, in spite of yielding a triton binding energy much closer to the experimental value than the Paris potential, gives rise to a considerably worse charge form factor as compared to experiment.

Regarding the Graz-II interactions, the results are given in Figs. 5, 6, and 7, with P_D^d values of 4, 5, and 6%, respectively. First of all, we see that these interactions produce the first diffraction minimum at much too high values of q^2 , a deficiency already observed in the deuteron case [7]. Furthermore, increasing P_D^d makes the dip move inwards, also in accordance with Ref. [7]. Concerning the relativistic approaches, the same pattern emerges as for the Bonn interaction, namely with the two nearly coinciding QP curves lying between the NRF and BSF ones.

In Fig. 8, finally, we show the tritium charge form factor for the Paris-1 interaction. The same type of behavior can be seen as in the ^3He case, though the dip of the BSF curve is now only marginally shifted. Also, the falling off of all four curves at low momentum transfer is less

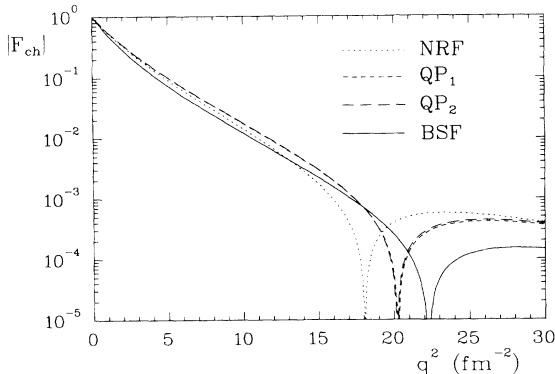


FIG. 4. Same as Fig. 1, with Bonn interaction.

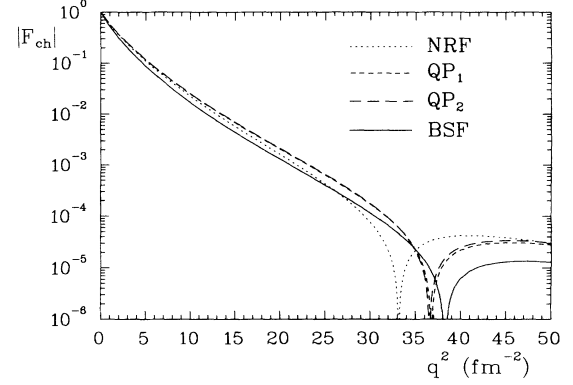


FIG. 5. Same as Fig. 1, with Graz-II interaction and $P_D^d=4\%$.

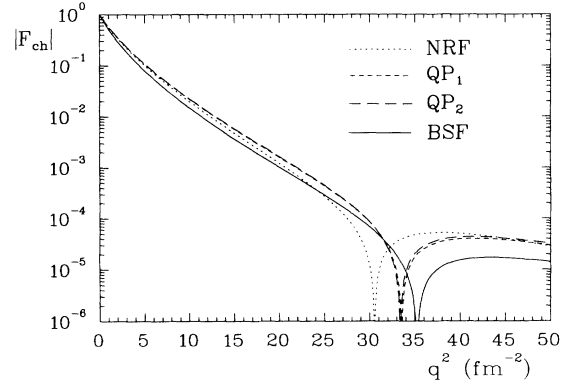


FIG. 6. Same as Fig. 1, with Graz-II interaction and $P_D^d=5\%$.

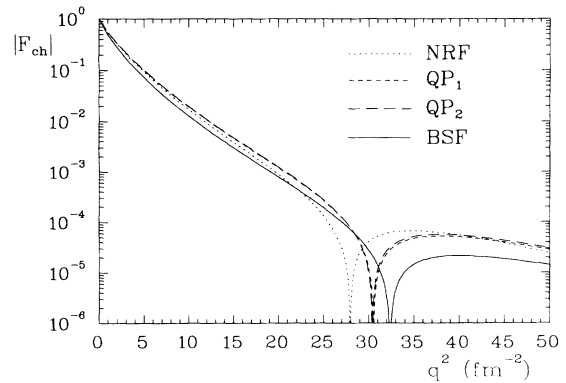


FIG. 7. Same as Fig. 1, with Graz-II interaction and $P_D^d=6\%$.

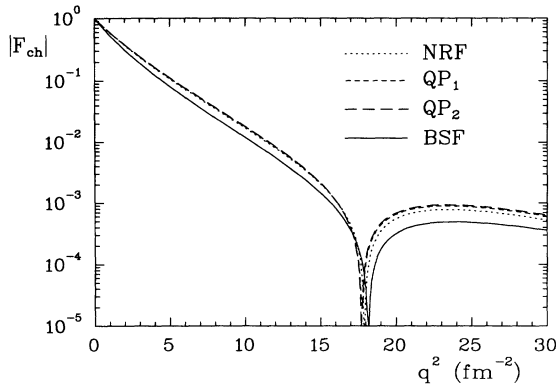


FIG. 8. ${}^3\text{H}$ charge form factor; description as in Fig. 1.

steep, reflecting a smaller charge radius of the triton as compared to ${}^3\text{He}$. We may conclude that, with the used approximations, the BSF approach does not improve the trinucleon charge form factors. Yet, a certain tendency can be observed for the shift of the diffraction minimum becoming larger as the dip moves outwards.

V. CONCLUSIONS AND OUTLOOK

In order to put the results obtained above in a correct perspective, we emphasize that the state-of-the-art of relativistic three-nucleon calculations is by no means comparable to the NR case. There, virtually converged results for nearly all popular NN potentials with a variety of very different methods have been produced. In the relativistic case, virtually *nothing* has been done in terms of corroborated predictions. This is clearly an unsatisfactory state of affairs, becoming all the more incomprehensible for quasipotential approaches, which do not require much more numerical effort than NR ones, especially when neglecting the Dirac spin structure of the nucleons. On the other hand, the solution of the BSF problem is numerically hard. Therefore, the present calculations have as yet been restricted effectively to two three-body channels, so that the results cannot be assumed to have converged. However, we do not expect that the higher partial waves, which give rise to an extra binding of about 0.2 (0.1) MeV for the NR Paris (Bonn) potential [11], contribute very differently in the relativistic case.

A picture seems to emerge from the studies of the relativistic effects on the trinucleon observables in the present framework. As for the triton binding energy B_t , an additional contribution varying from 0.29 to 0.38 MeV is obtained, displaying only a mild dependence on the global characteristics of the used interaction. This parameter-free prediction, which includes effective three-body con-

tributions due to retardation, is to be contrasted with the highly cutoff-dependent results of NR models with explicit three-body forces [12, 13]. However, whether the found increase in binding is sufficient to obtain agreement with experiment is still uncertain, in view of the strong dependence of B_t on the D -state probability of the underlying NN force.

With regard to the three-nucleon charge form factors, the situation seems to be less favorable. If the presented BSF results are taken at face value, the disagreement with experiment is even worsened, though this might not be the case for interactions (e.g., the Reid soft-core potential) that produce a diffraction minimum at lower momentum transfers than the ones employed here. Nevertheless, more definite conclusions are somewhat premature because of the applied approximations. Especially, the neglect of boost effects could very well give rise to substantial deviations at the intermediate and high momentum transfers considered. Besides, as suggested by nonrelativistic studies, meson-exchange currents should play an important role at high momentum transfer.

In conclusion, we wish to mention possible improvements and extensions of the present work. In the first place, of course, one should attempt to go beyond the static and angular-averaging approximations used in calculating the charge form factors, along the lines discussed in Sec. III B. Then, the T_{00} approximation can be improved upon by also including D waves in the three-body equations, despite a further increase in the size of the already very large kernel. Finally, some aspects of the nucleon Dirac structure may be incorporated. However, even if this is restricted to using Dirac instead of scalar propagators and neglecting the negative-energy components, the Lorentz-transformation properties of the off-shell two-body T matrices in the BSF equations will have to be reexamined. Such an analysis would also allow inclusion of the relativistic form of the EM current operator.

ACKNOWLEDGMENTS

We wish to thank E. F. Redish and S. J. Wallace for very useful discussions. Part of the work on the Graz-II interactions was done while one of the authors (G.R.) was at the Centre de Physique Théorique, CNRS, Marseille-Luminy, France. Part of the corresponding numerical computations was performed on the Cyber-205 of the Bochum University Computing Centre, Germany. The figures in this paper have been produced with the programs PLOTDATA and EDGR developed at TRIUMF. This work was realized with partial financial support from the Junta Nacional de Investigação Científica e Tecnológica, Lisbon, Portugal, through Grant No. PMCT/BPD/5/90.

- [1] G. Rupp and J. A. Tjon, Phys. Rev. C **37**, 1729 (1988).
- [2] L. Mathelitsch, W. Plessas, and W. Schweiger, Phys. Rev. C **26**, 65 (1982).
- [3] J. Haidenbauer and W. Plessas, Phys. Rev. C **30**, 1822

- (1984); **32**, 1424 (1985).
- [4] J. Haidenbauer, Y. Koike, and W. Plessas, Phys. Rev. C **33**, 439 (1986).
- [5] G. Rupp, Nucl. Phys. A **508**, 131c (1990).

- [6] K. Schwarz, J. Haidenbauer, and J. Fröhlich, Phys. Rev. C **33**, 456 (1986).
- [7] G. Rupp and J. A. Tjon, Phys. Rev. C **41**, 472 (1990).
- [8] K. Schwarz, private communication.
- [9] G. Rupp and J. A. Tjon, contributed paper to the *Twelfth International Conference on Few-Body Problems in Physics*, Vancouver, Canada, 1989 (TRIUMF Report No. TRI-89-2), p. D39; note that in the numerator of the rhs of Eq. (3) a factor $|\mathbf{p}|^L$ was erroneously omitted, but not in the calculation.
- [10] M. J. Zuilhof and J. A. Tjon, Phys. Rev. C **22**, 2369 (1980).
- [11] J. L. Friar, B. F. Gibson, and G. L. Payne, Phys. Rev. C **37**, 2869 (1988).
- [12] C. R. Chen, G. L. Payne, J. L. Friar, and B. F. Gibson, Phys. Rev. C **33**, 1740 (1986).
- [13] T. Sasakawa and S. Ishikawa, Few-Body Systems **1**, 3 (1986); S. Ishikawa and T. Sasakawa, *ibid.* **1**, 143 (1986).