Boson mappings applied to the two-color delta model

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We study several aspects of a two-color delta model of interacting quarks, with the purpose of extracting information of relevance to real quark models of nuclei. We first describe a BCS treatment of the quark pair correlations that arise in the model at low densities and demonstrate that the rms radius of the dominant correlated pairs increases in the presence of the medium. We then consider the application of a Dyson boson mapping to the model and show that it incorporates, at the level of the Hartree-Bose approximation, the pair clustering dynamics of quark BCS. However, the collective bosons that emerge do not have the same structure as the correlated BCS pairs. The mapping also leads to a short-range repulsive interaction between the collective bosons, which we derive in both momentum and coordinate space. The additional correlations that result from this repulsive boson interaction can in principle be incorporated through a Brueckner treatment of the mapped boson Hamiltonian.

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I. INTRODUCTION

It is generally accepted that the strong interaction is responsible both for the clustering of quarks into nucleons and for the interactions between nucleons that lead to nuclear structure. The traditional philosophy of nuclear physics has been to view these two problems as decoupled. However, recent experiments [1] by the European Muon Collaboration (EMC) suggest that, at some level, such a simple picture may not be adequate. These experiments indicate that the structure of a nucleon in a nuclear medium is different from the structure of a nucleon in free space, suggesting that the interactions that build up the nucleon do not fully decouple from the interactions between nucleons. As a consequence, there is now great interest in trying to develop methods for deriving the properties of nuclei directly from the quark and gluon constituents of the nucleon, thereby avoiding the decoupling assumption. Such methods, once developed, would also provide a practical means of identifying where to look for explicit quark effects in nuclei.

Ideally, one would like to start from QCD which is generally believed to be the correct theory of the strong interaction. However, to date, little progress has been made along these lines. Where some progress has been made is in the use of constituent quark models, in which the antiquark and gluon degrees of freedom are suppressed. Such models have been used with impressive success to describe one- and two-baryon systems [2], but they have not yet been implemented for many-nucleon systems. Recently, a method was proposed [3] for treating multinucleon systems in a constituent quark framework using mapping techniques, but so far the method has only been tested in the context of a simple and not particularly realistic quark model of nuclei [4]. In particular, the model used to test the method does not produce spatially localized nucleons [5].

An important criterion of a realistic quark model of nuclei is that, at normal nuclear densities, it confines quarks in spatially localized three-quark clusters without yielding residual long-range forces between clusters. There have been several models [6–8] of interacting quarks that have been proposed and studied that satisfy this criterion.

In the present work, we focus on one such model [7,8], due to Koltun and collaborators, in which quarks (with color) move in one dimension (1D) and interact through a residual attractive delta-function potential. Despite the fact that nuclei live in a three-dimensional world, this model can, nevertheless, give important insight into how quarks cluster into nucleons within a nuclear environment. This model has, in fact, been developed both for quarks with three colors (as is clearly relevant for both hadronic physics and nuclear physics) and for quarks with only two colors. Both versions of the model share the following attractive features: (i) in the limit of infinite nuclear (or quark) matter, they can be solved exactly for the ground-state energy per particle using the Bethe ansatz, (ii) in the limit of very low density, they lead to spatially localized clusters of quarks (either pairs or triplets depending on the version) moving freely, and (iii) in the limit of very high density, they lead to a free quark gas. We will focus in this work on the two-color version of the model, since it is more directly amenable to many-body variational treatments while still providing many of the important features that would emerge from the more realistic three-color model.

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To date, the best approximate treatment of the two-color delta model has been based on the BCS approximation [9]. As emphasized by Tosa [8], the BCS equations of this model have a simple scaling property that permits a systematic solution of the BCS equations in powers of the density. Such an expansion is of particular interest in the low-density regime characteristic of real nuclei (or even real nuclear matter). From this analysis, it is clear that the BCS approximation, though a good starting point, does not include all of the relevant dynamics of the model. In particular, it does not incorporate the effects of the strong short-range repulsion between correlated BCS pairs that arises due to the quark Pauli principle. What is needed, therefore, is a method for going beyond mean field in a description of such systems.

We will consider in this paper the use of boson mapping techniques [10] to treat the low-density quark dynamics of this model. We will first consider the use of boson mappings coupled with a Hartree-Bose approximation and show that this leads to precisely the same physics as the earlier BCS approximation.

The Hartree approximation for bosons describes the ground state as a condensate of collective bosons in the lowest available boson state. Clearly, the lowest single-boson state involves momentum $K=0$. To improve on this description, it is necessary to include the excitation of bosons to states with $K\neq 0$. Such excitations will increase the kinetic energy of the system but at the same time will reduce the effect of the strong Pauli repulsion between bosons. The competition between these two effects dictates the dynamics of the system. The boson mapping method readily yields the interaction between all possible bosons, including those with the same internal structure as the collective $K=0$ boson but boosted in momentum. We will derive the form of the collective boson Hamiltonian both in momentum space and in coordinate space. The boson-boson interaction that emerges contains a strong short-range repulsion, reflecting the effect of the Pauli principle at the underlying quark level. An improved treatment of the dynamics of the collective bosons (beyond that of Hartree Bose or equivalently quark BCS) will thus require the development of methods akin to Brueckner theory [11], which is beyond the scope of the present work.

The structure of the paper is as follows: In Sec. II, we briefly review the two-color delta model and in Sec. III describe the BCS treatment of it carried out by Tosa. We also include a treatment of the structure of the correlated pairs that emerge from BCS as a function of the density of the system. In Sec. IV, we describe the application of boson mapping techniques to this model and demonstrate the equivalence of a variational Hartree treatment of collective bosons to the quark BCS treatment. Finally, in Sec. V, we obtain the collective boson Hamiltonian both in momentum space and in coordinate space and in Sec. VI make some concluding remarks.

II. BRIEF REVIEW OF THE TWO-COLOR DELTA MODEL

In this model, a system of N nonrelativistic quarks with color c (which is allowed to take the two possible values $\pm\frac{1}{2}$) move in a one-dimensional box of size L subject to an attractive delta-function interaction. Letting $q_{kc}^{\dagger}$ (q_{kc}) denote the creation (annihilation) operator for a quark with momentum k and color c, we can express the model Hamiltonian as

$$H=\sum_{kc}\epsilon_k q_{kc}^{\dagger}q_{kc}-\frac{G}{2}\sum_{ijkl,c}q_{ic}^{\dagger}q_{j-c}^{\dagger}q_{l-c}q_{kc}\delta_{i+j,k+l}\ , \tag{1}$$

where $\epsilon_k=k^2/2m$ and m is the mass of the quark.

As noted by Tosa [8], we can go to the limit of an infinite box ($N\to\infty$, $L\to\infty$, $\rho=N/L$ finite) by replacing

$$\begin{aligned}&\sum_k\to\frac{L}{2\pi}\int dk,\\&\delta_{ij}\to\frac{2\pi}{L}\delta(i-j)\ ,\\&G\to g/L\ .\end{aligned} \tag{2}$$

An attractive feature of the infinite matter limit is that much of the relevant analysis can be carried out analytically.

III. BCS TREATMENT OF THE MODEL

For a Hamiltonian of the form (1), we expect strong correlations between pairs of quarks in a colorless state. A natural framework in which to describe such a correlation structure is the BCS approximation. The basic idea is to search for the optimum description of the system as a vacuum of quasiparticles. Introducing the quasiparticle transformation

$$a_{ck}^{\dagger}=u_k q_{ck}^{\dagger}-v_k q_{-c-k}\ , \tag{3}$$

we obtain, for the expectation value of the Hamiltonian (1) in the BCS quasiparticle vacuum,

$$\langle \mathrm{BCS}|H|\mathrm{BCS}\rangle=2\sum_k\epsilon_k v_k^2-G\sum_{kk'}(u_kv_ku_{k'}v_{k'}+v_k^2v_{k'}^2)\ . \tag{4}$$

Furthermore, the average number of particles in the quasiparticle vacuum is

$$\langle \mathrm{BCS}|\hat{N}|\mathrm{BCS}\rangle=2\sum_k v_k^2\ . \tag{5}$$

The BCS approximation involves minimizing $\langle \mathrm{BCS}|H-\lambda\hat{N}|\mathrm{BCS}\rangle$ (λ is the chemical potential and $\hat{N}$ is the number operator) subject to a constraint that $\langle \mathrm{BCS}|\hat{N}|\mathrm{BCS}\rangle=N$. This results in the well-known system of equations

$$\begin{aligned}&\rho=\frac{1}{L}\sum_k\left[1-\frac{\tilde{\epsilon}_k^2}{(\tilde{\epsilon}_k^2+\Delta^2)^{1/2}}\right]\ \text{(number equation)}\ ,\\&\frac{2}{G}=\sum_k\frac{1}{(\tilde{\epsilon}_k^2+\Delta^2)^{1/2}}\ \text{(gap equation)}\ ,\end{aligned} \tag{6}$$

where $\tilde{\epsilon}_k=\epsilon_k-\lambda-GN/2$, $\Delta=G\sum_k u_kv_k$, and $\rho=(2/L)\sum_k v_k^2$ is the density of the system.

Since the system of equations (6) scales [8] with ρ/mg, we introduce an expansion

$$v_k^2=b_k\rho+c_k\rho^2+\cdots\ ,$$
$$u_k^2=1-b_k\rho-c_k\rho^2+\cdots\ , \tag{7}$$

where (in the infinite matter limit) the density ρ is given by

$$\rho=\frac{1}{\pi}\int_{-\infty}^{+\infty}v_k^2dk$$
$$=\frac{1}{\pi}\left[\int_{-\infty}^{+\infty}b_k dk\,\rho+\int_{-\infty}^{+\infty}c_k dk\,\rho^2+\cdots\right]. \tag{8}$$

From (8), we see that the expansion coefficients b_k and c_k must satisfy

$$\int_{-\infty}^{+\infty}b_k dk=\pi,\quad \int_{-\infty}^{+\infty}c_k dk=0\ . \tag{9}$$

We also introduce density expansions for the pairing gap Δ and the chemical potential λ:

$$\Delta^2=\Delta_1^2\rho+\Delta_2^2\rho^2+\cdots\ ,$$
$$\lambda=\lambda_0+\lambda_1\rho+\cdots\ . \tag{10}$$

Inserting (7)–(10) into the number and gap equations (6) and equating powers of ρ leads to the following results:

$$b_k=\frac{m^3g^3}{4(k^2+m^2g^2/4)^2}\ ,$$
$$c_k=\frac{3m^4g^4}{4(k^2+m^2g^2/4)^3}-\frac{3m^2g^2}{8(k^2+m^2g^2/4)^2}-\frac{3m^6g^6}{16(k^2+m^2g^2/4)^4}\ ,$$
$$\lambda_0=-\frac{mg^2}{8}\ ,$$
$$\lambda_1=\frac{g}{4}\ , \tag{11}$$
$$\Delta_1^2=\frac{mg^3}{4}\ ,$$
$$\Delta_2^2=-\frac{3g^2}{8}\ .$$

As is well known, the BCS quasiparticle vacuum can be expressed as

$$|\mathrm{BCS}\rangle=c\exp(\Gamma^\dagger)|0\rangle\ , \tag{12}$$

where

$$\Gamma^\dagger=\sum_{kc}\frac{v_k}{u_k}q_{kc}^\dagger q_{-k-c}^\dagger \tag{13}$$

is the correlated pair that dominates in the description of the ground state of the system. Denoting $\Psi(k)=v_k/u_k$ and expanding

$$\Psi(k)=\Psi^0(k)+\Psi^1(k)\rho+\cdots\ , \tag{14}$$

we obtain

$$\Psi^0(k)=\frac{\sqrt{b_k}}{\sqrt{\pi}}\ ,$$
$$\Psi^1(k)=\frac{1}{2\sqrt{\pi}}\frac{b_k^2+c_k}{\sqrt{b_k}}\ . \tag{15}$$

Next we consider the structure of the dominant correlated pairs in coordinate space. Carrying out a Fourier transformation of $\Psi(k)$ [(14) and (15)], we obtain

$$\Psi(x)=\Psi^0(x)+\Psi^1(x)\rho+\cdots\ , \tag{16}$$

where

$$\Psi^0(x)=\mu^{1/2}e^{-\mu|x|}\ ,$$
$$\Psi^1(x)=\mu^{1/2}e^{-\mu|x|}\left[\frac{3}{8\mu}+\frac{3}{4}|x|-\frac{1}{4}\mu x^2\right]\ , \tag{17}$$

and

$$\mu=mg/2\ .$$

The correlated-pair wave function [(16) and (17)] is only normalized at $\rho=0$. Its norm to first order in ρ is given by

$$\eta(\rho)=\int_{-\infty}^{+\infty}\Psi^*(x)\Psi(x)dx=1+\frac{5}{4\mu}\rho\ . \tag{18}$$

Finally, we can evaluate the rms radius of the dominant correlated pair as a function of density (still assuming that the density is low enough that it suffices to consider the linear correction to the collective wave function). The resulting expression is

$$\langle x^2\rangle^{1/2}=\frac{\sqrt{2}}{mg}\left[1+\frac{\rho}{4mg}\right]\ . \tag{19}$$

We see from (19) that the presence of the medium increases the rms radius of the dominant correlated pair. This occurs even though, as we will show in Sec. V, the residual interaction between correlated pairs is repulsive.

The rms radius only gives the average size of the correlated diquark pair. In Fig. 1, we plot the density distribution $\Psi^*(x)\Psi(x)/\int\Psi^*(x)\Psi(x)dx$ of the correlated pair for $mg=20$ and for $\rho=0$ and 2. From this figure, we see that there are several effects of the finite density medium on the structure of the correlated pair: (i) the wave function decreases very slightly in the region from roughly $x=0$ to 0.04; (ii) it then increases in the region from roughly $x=0.04$ to 0.25; (iii) beyond $x=0.25$, there is initially a decrease in the wave function and then the appearance of additional nodes at large values of x. The additional nodes that show up are a reflection of partial deconfinement of the correlated pairs due to the medium; however, they have no influence on the rms radius of the correlated pair since they are so small in magnitude. In Fig. 2, we show a blowup of Fig. 1 in the important region from $x=0$ to 0.4. The net increase in the rms radius can be traced to the fairly small rise that occurs in the window from $x=0.04$ to 0.25, which dominates over the decreases that occur for other x values. We conclude that the overall change in the size of the correlated BCS

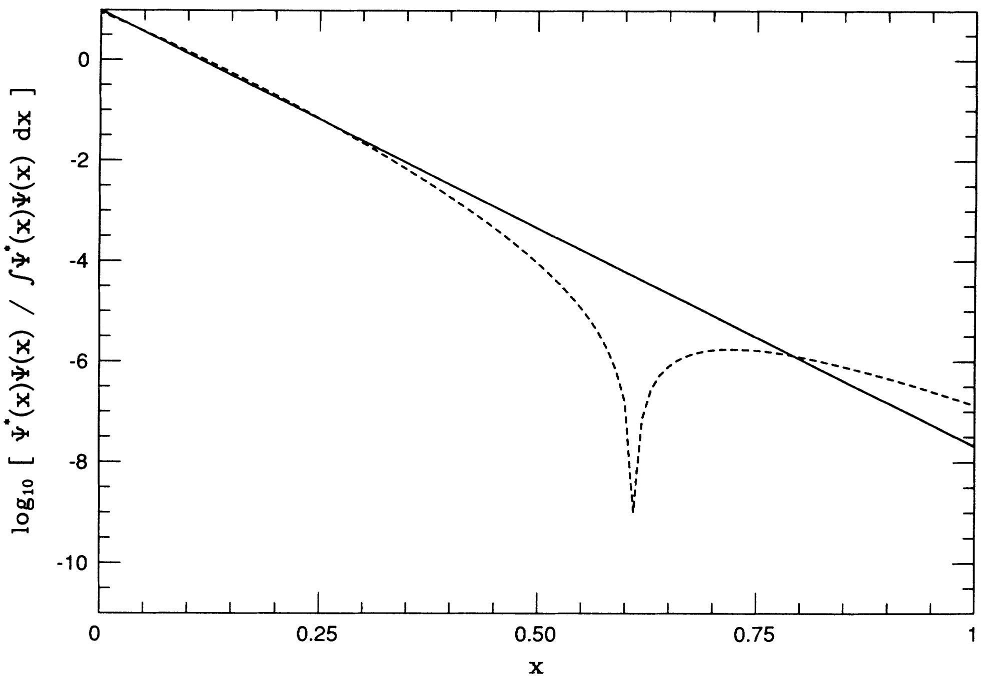

FIG. 1. The density distribution $\Psi^*(x)\Psi(x)/\int \Psi^*(x)\Psi(x)dx$ of correlated pairs that arise in a BCS treatment of the two-color delta model. Results are shown for $mg=20$ and for $\rho=0$ (solid curve) and $\rho=2$ (dashed curve).

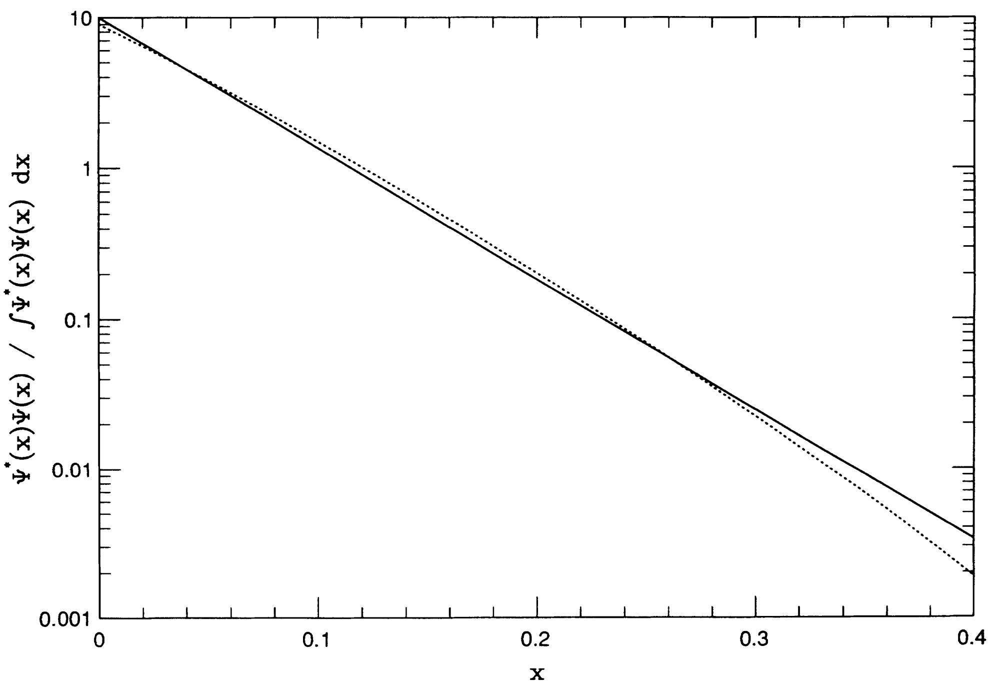

FIG. 2. A blowup of Fig. 1, focusing on the region of x values from $x=0$ to 0.4.

pair, as measured by its rms radius, involves a subtle competition between several different effects.

IV. DYSON BOSON MAPPING APPLIED TO THE MODEL

The BCS approximation incorporates the pairing correlations between quarks and includes the effects of the medium on these correlations. However, this is not the whole story. The exact Bethe ansatz results suggest that there is a residual short-range repulsion between the BCS pairs that arises from the Pauli principle at the quark level. Indeed, Pauli repulsion between composite objects built up out of fermions is quite common in physics; the interaction between hydrogen atoms in a hydrogen molecule is perhaps the best known example. The mean-field BCS approximation is unable to accommodate these further correlations. Clearly, what is needed is a scheme that, on the one hand, reduces to BCS while, on the other hand, permits additional correlations beyond mean field to be incorporated. In this section, we suggest that boson mapping techniques are naturally suited for this purpose. As mentioned in the Introduction, mapping techniques were recently proposed as a general framework in which to describe colorless three-quark clustering in real 3D nuclei.

We will consider the use of the Dyson boson mapping [10], which has the attractive feature that it maps finite quark Hamiltonians onto finite boson Hamiltonians. On the other hand, it leads to non-Hermitian boson Hamiltonians; we will see, however, that this is not a major complication.

A. Boson mapping of the Hamiltonian

The Dyson boson mapping is based on the requirement that the commutation algebra of operators in the original quark space should be preserved. Introducing the boson creation and annihilation operators $B^\dagger_{cic'j}$ and $B_{c'jci}$ ($c \neq c'$) which take the place of the quark pair operators $q^\dagger_{ci}q^\dagger_{c'j}$ and $q_{ci}q_{c'j}$, respectively, one can show that the commutation algebra is exactly preserved if operators are mapped according to

$$q^\dagger_{ci}q^\dagger_{c'j} \rightarrow B^\dagger_{cic'j} - \sum_{kl} B^\dagger_{cic'k}B^\dagger_{c'jcl}B_{c'kcl} , \tag{20}$$

$$q_{ci}q_{c'j} \rightarrow B_{c'jci} , \tag{21}$$

and

$$q^\dagger_{ci}q_{c'j} \rightarrow \delta_{cc'}\sum_k B^\dagger_{cic'k}B_{cjc'k} . \tag{22}$$

Applying the Dyson mapping (20)–(22) to the Hamiltonian (1) leads to the following boson Hamiltonian:

$$H_B = \sum_{ijc}\epsilon_i B^\dagger_{ci-cj}B_{ci-cj} - \frac{g}{2L}\sum_{ijklc} B^\dagger_{ci-cj}B_{ck-cl}\delta_{i+j,k+l} + \frac{g}{2L}\sum_{ijklmnc} B^\dagger_{ci-cm}B_{-cjcn}B_{-cmcn}B_{ck-cl}\delta_{i+j,k+l} . \tag{23}$$

Note that it contains only one- and two-boson terms and is inherently non-Hermitian.

It is useful to transform to a representation involving bosons with definite color. In this representation, only colorless ($C=0$) bosons contribute. We denote the colorless boson creation and annihilation operators by $\gamma^\dagger_{ij}$ and γ_{ij}, respectively. They are related to the earlier color-uncoupled operators by

$$\gamma^\dagger_{ij} = \sum_c (\tfrac{1}{2}c\tfrac{1}{2}-c|00)B^\dagger_{ci-cj} ,$$
$$\gamma_{ij} = \sum_c (\tfrac{1}{2}c\tfrac{1}{2}-c|00)B_{ci-cj} , \tag{24}$$

and satisfy the commutation relation

$$[\gamma_{ij},\gamma^\dagger_{kl}] = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} . \tag{25}$$

The Hamiltonian expressed in terms of these colorless boson operators takes the form

$$H_B = \sum_{ij}\epsilon_i\gamma^\dagger_{ij}\gamma_{ij} - \frac{g}{2L}\sum_{ijkl}\gamma^\dagger_{ij}\gamma_{kl}\delta_{i+j,k+l} + \frac{g}{4L}\sum_{ijklmn}\gamma^\dagger_{im}\gamma^\dagger_{jn}\gamma_{mn}\gamma_{kl}\delta_{i+j,k+l} . \tag{26}$$

The structure of the *dominant* colorless bosons is governed by the relative motion only. Thus, we next introduce a transformation to relative and total momenta,

$$K = k_1 + k_2, \quad k = \tfrac{1}{2}(k_1 - k_2) . \tag{27}$$

Denoting the creation and annihilation operators for colorless bosons with relative momentum k and total momentum K by $\Gamma^\dagger_{kK}$ and Γ_{kK}, respectively, we obtain for the boson Hamiltonian

$$H_B = H_0 + V_B , \tag{28}$$

where

$$H_0 = \sum_{kk'K}\left[\left(\frac{k^2}{2m}+\frac{K^2}{8m}\right)\delta_{kk'} - \frac{g}{2L}\right]\Gamma^\dagger_{kK}\Gamma_{k'K} , \tag{29}$$

and

$$V_B = \frac{g}{4L}\sum_{kk'qKK'}\Gamma^\dagger_{q/2-k'-K'/2+K/2,q+K} \times \Gamma^\dagger_{-q/2+k',K'-q}\Gamma_{k'K'}\Gamma_{kK} . \tag{30}$$

The operators $\Gamma^\dagger_{kK}$ and Γ_{kK} satisfy the commutation relation

$$[\Gamma_{kK},\Gamma^\dagger_{k'K'}] = \delta_{KK'}(\delta_{kk'}+\delta_{k-k'}) . \tag{31}$$

We now introduce a (nonunitary) collective transformation [12],

$$\Lambda^\dagger_{pK} = \sum_k x^p_k\Gamma^\dagger_{kK} ,$$
$$\lambda_{pK} = \sum_k y^p_k\Gamma_{kK} . \tag{32}$$

As noted earlier, the coefficients of the collective transformation depend solely on the relative momentum.

To guarantee that the transformation is canonical, we impose the commutation relation

$$[\lambda_{pK},\Lambda^\dagger_{p'K'}]=\delta_{pp'}\delta_{KK'}\ , \tag{33}$$

whereby the coefficients of the collective transformation can be shown to satisfy

$$\sum_k x_k^p y_k^{p'}=\tfrac{1}{2}\delta_{pp'}\ , \qquad \sum_p x_{k'}^p y_k^p=\tfrac{1}{4}(\delta_{kk'}+\delta_{k-k'})\ . \tag{34}$$

This can be used to invert the collective transformation, yielding

$$\Gamma^\dagger_{kK}=2\sum_p y_k^p\Lambda^\dagger_{pK}\ , \qquad \Gamma_{kK}=2\sum_p x_k^p\lambda_{pK}\ . \tag{35}$$

The basic idea will be to substitute (35) into (28)–(30) and then to carry out a nonunitary Hartree-Bose variational treatment [12]. The lowest ($p=1$) solution will then define the collective bosons that dominate in the ground state of the system. For simplicity of notation, we replace

$$\Lambda^\dagger_{p=1K}\rightarrow\Lambda^\dagger_K\ ,\quad \lambda_{p=1K}\rightarrow\lambda_K\ ,\quad x_k^{p=1}\rightarrow x_k\ ,\quad y_k^{p=1}\rightarrow y_k\ . \tag{36}$$

The boson Hamiltonian, expressed in terms of the collective bosons and their structure coefficients, is given by

$$H_B=H_0+V_B\ , \tag{37}$$

where

$$H_0=\sum_K\left[\sum_k\frac{4k^2+K^2}{2m}y_kx_k-\frac{2g}{L}\sum_{kk'}y_kx_{k'}\right]\Lambda^\dagger_K\lambda_K\ , \tag{38}$$

$$V_B=\frac{4g}{L}\sum_{KK'q}\left[\sum_{kk'}y_{q/2-k'-K'/2+K/2}\,y_{-q/2+k'}x_{k'}x_k\right]\times\Lambda^\dagger_{q+k}\Lambda^\dagger_{K'-q}\lambda_{K'}\lambda_K\ . \tag{39}$$

B. The Hartree-Bose variational approximation

The Hartree-Bose variational approximation is based on a boson condensate trial state

$$|\Phi_0\rangle=\exp(\sqrt{N_B}\Lambda^\dagger_{K=0})|0\rangle\ , \qquad \langle\Psi_0|=\langle 0|\exp(\sqrt{N_B}\lambda^\dagger_{K=0})\ , \tag{40}$$

where $N_B=N/2$. The trial state does not contain a definite number of bosons. Thus, as in BCS, we must introduce a constraint that ensures that it has the correct number on the average. The variational condition is therefore to minimize

$$E_0(\lambda)=\langle\Psi_0|H_B-\lambda\hat N_B|\Phi_0\rangle/\langle\Psi_0|\Phi_0\rangle\ ,$$

subject to the constraint that $\langle\Psi_0|\hat N_B|\Phi_0\rangle/\langle\Psi_0|\Phi_0\rangle=N/2$ (one-half the total number of quarks). The energy functional $E_0(\lambda)$ is given by

$$E_0(\lambda)=N\sum_k\frac{k^2}{m}x_ky_k-2N\lambda\sum_k x_ky_k-\frac{gN}{L}\sum_k x_k\sum_{k'}y_{k'}+\frac{gN^2}{L}\sum_{kk'}y_{k'}^2x_{k'}x_k\ . \tag{41}$$

To establish contact with quark BCS, we introduce v_k and u_k coefficients according to

$$x_k=\sqrt{1/N}\,v_ku_k,\quad y_k=\sqrt{1/N}\,\frac{v_k}{u_k}\ . \tag{42}$$

There is still an overall undetermined constant for u_k, which can be fixed by imposing the condition that the boson images of $q^\dagger_{ci}q^\dagger_{c'j}$ and $q_{c'j}q_{ci}$ have the same expectation values in the trial state [12]. This leads to the following relation between the x_k and y_k coefficients:

$$x_k=y_k-Ny_k^2x_k\ . \tag{43}$$

Substituting (43) into (42), we find that v_k and u_k satisfy the relations

$$\sum_k v_k^2=N,\quad v_k^2+u_k^2=1\ , \tag{44}$$

exactly as in BCS.

If we now substitute v_k and u_k for x_k and y_k in (41), we obtain for $E_0(\lambda)$ the following result:

$$E_0(\lambda)=\sum_k(\epsilon_k-\lambda)v_k^2-\frac{g}{L}\sum_{kk'}u_kv_ku_{k'}v_{k'}\ , \tag{45}$$

which is precisely the BCS expression, except that it does not contain the Fock term $-2GN(=-2g\rho)$.

Even though the energy functional (45) does not contain the Fock term of BCS, a minimization of it with respect to the v_k coefficients nevertheless leads precisely to the gap and number equations (6). Thus, the occupation amplitudes v_k^2 and u_k^2 that emerge are identical to those of BCS.

C. The structure of the collective bosons

In quark BCS, the correlated pairs that dominate the ground state are defined by (13), and thus their structure coefficients are given by v_k/u_k. In the Hartree-Bose (HB) approximation, the collective boson is defined by (32), which for $p=1$ and $K=0$ reads

$$\Lambda^\dagger_0=\sum_k x_k\Gamma^\dagger_{k0},\quad \lambda_0=\sum_k y_k\Gamma_{k0}\ . \tag{46}$$

Thus, there are distinct structure coefficients for the operator that creates the collective boson and the operator that annihilates it. The operator that annihilates the collective boson has the same structure as the correlated BCS pair, namely, v_k/u_k. However, the operator that creates the collective boson has structure coefficients v_ku_k. Note that this difference reflects the manner in which quark Pauli effects are transmitted to the collective

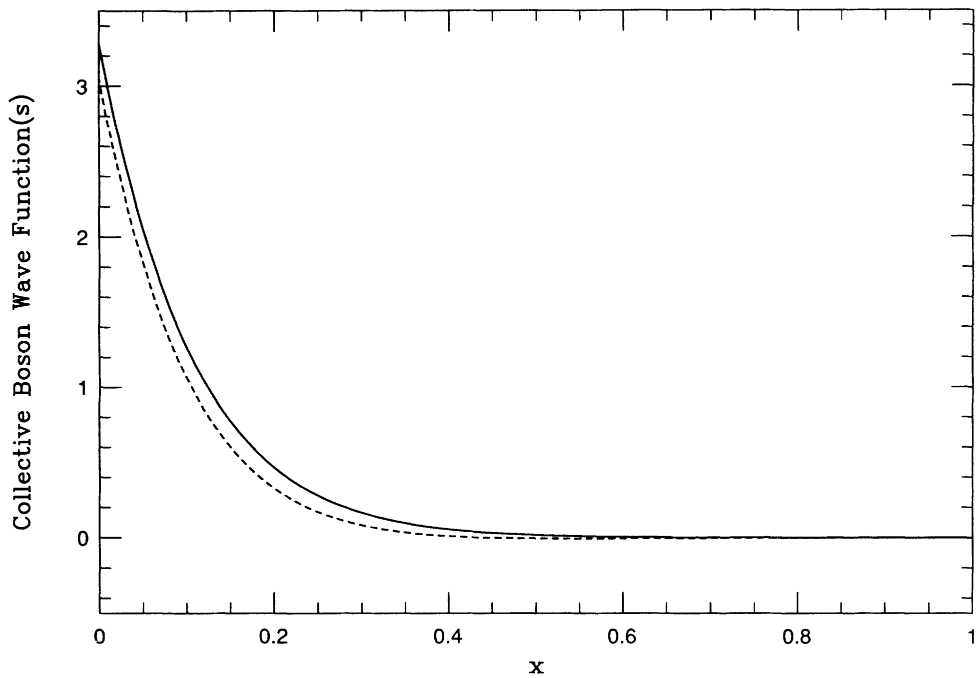

FIG. 3. The collective boson wave functions in coordinate space calculated for $\rho=2$ and $mg=20$. The solid curve gives the wave function $\Psi(x)$ corresponding to the annihilation of a collective boson and the dashed curve the wave function $\Phi(x)$ corresponding to the creation of a collective boson.

bosons by the Dyson mapping.

It is interesting to examine this difference in coordinate space. To do so, we Fourier transform the wave functions associated with the creation and annihilation of a collective boson. The structure of the collective boson annihilation operator is, as noted above, identical to that of the correlated BCS pair, and is thus given by the same $\Psi(x)$ as in Eqs. (16) and (17). The structure of the collective boson creation operator, on the other hand, is described by a relative wave function

$$\Phi(x)=\Phi^0(x)+\Phi^1(x)\rho+\cdots , \tag{47}$$

where $\Phi^0(x)=\Psi^0(x)$, as given by (17), and

$$\Phi^1(x)=-\mu^{1/2}\exp^{-\mu|x|}\left[\frac{3}{8\mu}+\frac{1}{2}\mu x^2\right] . \tag{48}$$

In Fig. 3, we show the wave functions $\Psi(x)$ and $\Phi(x)$, corresponding to the annihilation and creation of a collective boson, respectively, for $\rho=2$ and $mg=20$. These results illustrate the effect of the Pauli principle on the structure of the collective bosons in the presence of a medium.

The spatial wave functions describing the creation and annihilation of a collective boson satisfy

$$\int_{-\infty}^{+\infty}\Psi^*(x)\Phi(x)dx=1 , \tag{49}$$

and thus are binormalized for all ρ.

Next, we evaluate the rms radius of the collective boson which, because of the condition (49), is given by

$$\langle x^2\rangle^{1/2}=\left[\int_{-\infty}^{+\infty}\Psi^*(x)x^2\Phi(x)dx\right]^{1/2} . \tag{50}$$

The resulting expression through first order in ρ is

$$\langle x^2\rangle^{1/2}=\frac{\sqrt{2}}{mg}\left[1-\frac{9\rho}{8mg}\right] . \tag{51}$$

Thus, in contrast to the correlated BCS pair, the rms radius of the collective HB boson decreases in the presence of the medium.

V. THE COLLECTIVE BOSON HAMILTONIAN INCLUDING $K\neq0$ BOSONS

In this section, we derive the structure of the collective boson Hamiltonian that would apply in an extension beyond mean field where collective bosons with momenta other than $K=0$ contribute. Our basic assumption will be that the collective bosons retain their internal structure when they are boosted in momentum. We first consider the structure of the collective boson Hamiltonian in momentum space; this can be obtained directly from the preceding analysis. We then derive the collective boson Hamiltonian in coordinate space.

A. The collective boson Hamiltonian in momentum space

The general form of the collective boson Hamiltonian was given in (37)–(39). The Hartree-Bose approximation yields, for x_k and y_k, the following expressions:

$$x_k=\sqrt{b_k/L}\left[1+\frac{1}{2}\left[\frac{c_k}{b_k}-b_k\right]\rho\right] ,$$
$$y_k=\sqrt{b_k/L}\left[1+\frac{1}{2}\left[\frac{c_k}{b_k}+b_k\right]\rho\right] , \tag{52}$$

where b_k and c_k are given in (11). Inserting these results into (37)–(39) and carrying out the integrations over relative momenta (k and k') yields the following expression for the collective boson Hamiltonian:

$$H_B=H_0+V_B , \tag{53}$$

where

$$H_0=\sum_K\left[-\frac{mg^2}{4}+\frac{K^2}{4m}\right]\Lambda_K^\dagger\lambda_K , \tag{54}$$

$$V_B=\sum_{q,\Delta,K} f(q,\Delta)\Lambda_{q+K}^\dagger\Lambda_{K+\Delta-q}^\dagger\lambda_{K+\Delta}\lambda_K , \tag{55}$$

and

$$f(q,\Delta)=\frac{4m^4g^5}{L}\frac{(q^2+\Delta^2-q\Delta+12m^2g^2)(1+\delta_{q0}\delta_{\Delta0}\delta_{q\Delta})}{[(q-\Delta)^2+4m^2g^2](q^2+4m^2g^2)(\Delta^2+4m^2g^2)} . \tag{56}$$

Note that the interaction strength depends on Δ and on q but not on K.

We have only presented in (53)–(56) those terms in the collective boson Hamiltonian that contribute through $O(\rho)$ to the energy per particle. The terms linear in ρ in the collective boson wave functions (52) also give rise to a density-dependent contribution to the two-boson interaction, which would contribute in a calculation of the energy per particle at $O(\rho^2)$.

The residual interaction between the collective bosons given in (55) and (56) is repulsive, as it arises solely from quark exchange. Ultimately, it would be interesting to consider the effect of $K\neq0$ bosons through an appropriate dynamical treatment that goes beyond the Hartree-Bose mean-field approximation. The strongly repulsive nature of the interaction has the physical effect of keeping the collective bosons apart, so that the quarks that constitute two different bosons can never be at the same point in space. The situation is very similar to that occurring in nuclear matter studies based on nucleon-nucleon interactions with a strongly repulsive core. Obviously, a basis of product wave function obtained, e.g., in a Hartree-Bose treatment, is not appropriate, since the strongly repulsive short-range interaction would lead to very large matrix elements between product states. In the nuclear matter problem, we know the correct solution; it is to use Brueckner theory [11] to extract an effective G matrix that can be used with product wave functions. We expect that the same procedure, albeit modified to composite bosons and a non-Hermitian Hamiltonian, should be applicable here as well. In fact, a Brueckner treatment for bosons should be simpler than for fermions since no Pauli exclusion of intermediate states is required.

B. The collective boson Hamiltonian in coordinate space

The strong repulsive interaction between bosons will induce short-range spatial correlations. It is natural to try to describe such spatial correlations directly in coordinate space. Thus, in this section we translate the above analysis (which was carried out in momentum space) to a coordinate-space framework.

We first introduce operators $q_c^\dagger(x)$ and $q_c(x)$ that, respectively, create and annihilate a quark with color c at the point x. The operators $q_c^\dagger(x)$ and $q_c(x)$ are related to the corresponding momentum-space operators $q_{ck}^\dagger$ and q_{ck} by

$$q_{ck}^\dagger=\frac{1}{\sqrt{L}}\int_{-\infty}^{+\infty}e^{-ikx}q_c^\dagger(x)dx ,$$
$$q_{ck}=\frac{1}{\sqrt{L}}\int_{-\infty}^{+\infty}e^{ikx}q_c(x)dx , \tag{57}$$

and satisfy the anticommutation relation

$$\{q_c(x),q_{c'}^\dagger(y)\}=\delta_{cc'}\delta(x-y) . \tag{58}$$

The Hamiltonian (1) can be expressed in terms of these coordinate-space operators as

$$H=-\frac{1}{2m}\sum_c\int q_c^\dagger(y)\frac{d^2}{dy^2}q_c(y)dy$$
$$-\frac{g}{2}\sum_c\int dy\, q_c^\dagger(y)q_{-c}^\dagger(y)q_{-c}(y)q_c(y) . \tag{59}$$

Carrying out a Dyson mapping of the coordinate-space operators and truncating to colorless operators $\gamma^\dagger(x,y)$ and $\gamma(x,y)$ leads to the Hamiltonian

$$H_B=-\frac{1}{2m}\int dx\, dy\gamma^\dagger(x,y)\frac{\partial^2}{\partial^2x}\gamma(x,y)$$
$$-\frac{g}{2}\int dx\,\gamma^\dagger(x,x)\gamma(x,x)$$
$$+\frac{g}{4}\int dx\,dy\,dz\,\gamma^\dagger(x,y)\gamma^\dagger(x,z)\gamma(y,z)\gamma(x,x) . \tag{60}$$

In analogy with the momentum-space treatment, we now transform to relative and center-of-mass coordinates, $r=x-y$ and $R=\frac{1}{2}(x+y)$, for which the colorless boson creation and annihilation operators will be denoted $\Gamma^\dagger(r,R)$ and $\Gamma(r,R)$, respectively. We then truncate to collective bosons

$$\Lambda^\dagger(R)=\int dr\, x(r)\Gamma^\dagger(r,R) ,$$
$$\lambda(R)=\int dr\, y(r)\Gamma(r,R) . \tag{61}$$

Note that the structure of the collective bosons, which, as before, is obtained in the Hartree-Bose approximation, depends only on the relative coordinate. The structure coefficients $x(r)$ and $y(r)$ of the collective bosons can be obtained by Fourier transforming the x_k and y_k of Sec. IV. They satisfy the binormalization condition

$$\int_{-\infty}^{+\infty} y^*(r)x(r)dr=\tfrac{1}{2} . \tag{62}$$

The collective boson Hamiltonian, expressed in terms of the coordinate-space creation and annihilation operators, is given by

$$H_B=H_0+V_B , \tag{63}$$

where

$$H_0=-\frac{mg^2}{4}\hat{N}_B-\frac{1}{4m}\int dR\ \Lambda^\dagger(R)\frac{\partial^2}{\partial^2 R}\lambda(R) , \tag{64}$$

and

$$\begin{aligned}V_B=m^2g^3\int dR_1dR_2dR_3 &\\ \times e^{-mg\{|R_3|+|R_2-R_1+R_3|+|R_2-R_1+2R_3|\}} &\\ \times\Lambda^\dagger(R_2+R_3)\Lambda^\dagger(R_1-R_3)\lambda(R_1)\lambda(R_2). &\end{aligned}$$

The residual interaction between collective bosons is repulsive (as before) and inherently nonlocal. Having the collective boson Hamiltonian in this form, it may be possible to carry out the Brueckner treatment directly in coordinate space.

VI. CONCLUDING REMARKS

In this paper, we have studied several aspects of a two-color delta model, which exhibits many of the features characteristic of quark models of nuclei. Despite the fact that it is a one-dimensional model (whereas nuclei are three dimensional) and that it only permits two colors (whereas quarks can have three), it is, nevertheless, a good testing ground in which to explore issues such as (i) the manner in which quarks cluster together to form nucleons in a nuclear medium, and (ii) the possibility of developing meaningful approximate techniques to use in the problem of ultimate interest, the derivation of nuclei from quarks. Working in the infinite matter limit has the further attractive feature that much of the analysis can be carried out analytically.

The first feature of the model that we studied was the effect of the medium on the structure of the correlated pairs that dominate the ground-state dynamics of the system. We addressed this issue at the level of the BCS approximation. The BCS approximation was already developed earlier for this model; however, the effect of the medium on the structure of the collective pairs was not considered in those studies. We have shown that the correlated pairs that emerge in the presence of a low-density medium have a larger rms radius than those that emerge for $\rho=0$, even though the residual interaction between the correlated pairs is repulsive.

We then developed the method of Dyson boson mappings for this model. We showed that the Dyson method very naturally reproduces the physics of the BCS approximation, while, at the same time, providing a framework with which to incorporate additional correlations. An interesting outcome of the analysis was that the collective bosons that emerge at the same level of approximation as BCS have a different internal structure than the correlated BCS pairs; the rms radius of the collective bosons decreases in the presence of a low-density medium.

Finally, we derived a Hamiltonian for collective bosons whose internal structure is governed by BCS-like correlations but which can be boosted to nonzero total momenta. The collective Hamiltonian was obtained both in momentum space and in coordinate space. The additional correlations that result from the strongly repulsive interaction between collective bosons can, in principle, be treated by Brueckner theory.

We have focused on the Dyson boson mapping since it reproduced the dynamics of the BCS approximation with only two-boson interactions. However, it leads to non-Hermitian Hamiltonians and thus to different internal structures for the operators that create and annihilate collective bosons. In the generalized Holstein-Primakoff mapping [10], this would not be the case. Such mappings are inherently Hermitian; however, to carry out an analysis in the low-density regime with such a mapping it would be necessary to truncate the infinite series expansions that arise. When such truncations are imposed, the connection with standard fermion approximations, such as BCS, is lost. Nevertheless, such mappings should also be considered in the future.

As noted in the Introduction, our eventual goal is to develop a practical method to treat the dynamics of real nuclei, starting from constituent quarks. In an earlier paper [3], we suggested that mapping techniques might prove useful in this context and showed how to map from a problem of interacting quarks to a problem of interacting triplet fermions. A Hartree-Fock analysis in the triplet fermion space, which is analogous to the Hartree-Bose analysis of the present work, would then provide the structure of the collective fermions that dominate in the low-lying properties of nuclei, i.e., the nucleons. And, indeed, such a variational procedure naturally incorporates effects of the nuclear medium on the collective structure of the nucleons. For realistic quark Hamiltonians (assuming that they can be found), the interaction between collective triplet fermions (nucleons) will most likely contain a strong repulsive core, and thus a Brueckner treatment will be required. We believe that the insight gained by treating relatively simple models, such as the two-color delta model, can be very useful in determining the feasibility of implementing mapping methods in more realistic descriptions of nuclei in terms of quarks.

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