Parametrization scheme for effective interactions

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An algorithm is developed by which two nucleon effective interactions are constructed to fit on- and off-shell t- and/or g-matrix elements. The effective interaction is defined as plane-wave matrix elements of local operators that may have explicit energy and medium dependences. It comprises central, tensor, spin-orbit, quadratic spin-orbit, and angular momentum square operators, all with Yukawa form factors. As examples, the Paris and Bonn potentials are used to construct t matrices for projection onto chosen forms of effective interactions.

I. INTRODUCTION

Microscopic analyses of few body and nucleon-nucleus scattering ultimately require a specification of the nucleon-nucleon (NN) t matrix and/or g matrix, both on and off the energy shell [1, 2]. These matrices are solutions of the Lippmann-Schwinger and Bethe-Goldstone equations respectively when a realistic NN potential is used as input. Many such solutions are required in practical applications, and as they are obtained numerically, an interpolation method or an effective operator specification is almost mandatory. Collectively, these schemes are called *effective interactions* about which there exists an extensive literature. But as it is not our aim to give an historical review of the concept of effective NN interactions, we note only some pertinent studies [3–5].

The specification of effective interactions is a part of a program in which the relationship between two nucleon observables and complex nuclear structure and reactions is the central issue. Herein we concern ourselves with part of this total program. This is illustrated in Fig. 1, which is a scheme of various routes that link two nucleon observables with the notions of interactions between two nucleons.

Boson-exchange models [6–8] form one route and solutions of their equations of motion lead directly to partial-wave two-nucleon t matrices $t^{JST}_{L'L}(k', k; E)$. *Ex aequo* the same rationale applies to the specification of partial wave g-matrix elements, and so no further general reference will be made to them. As it is important to have a nonrelativistic potential, those field-theoretic studies construct NN potentials [one-boson-exchange potential (OBEP)] of central, tensor, spin-orbit and quadratic spin-orbit components, which give proper descriptions of the long- and medium-range interactions. The short-range part is simply parametrized as, ultimately, it must reflect quark degrees of freedom. The other route shown in Fig. 1 makes use of inverse scattering theory [9] in which the Schrödinger equation is the equation of motion. Inversion is the most appealing and mathematically rigorous link between data and potentials since it is based solely upon the chosen equation of motion and the existence of a sensible potential. Not only does inversion show how that link is made, but also it facilitates a direct connection between data and off-shell t matrices [10, 11]. In the most widely used form of inversion of NN data, the underlying potentials are local and energy independent for each partial wave [9, 11, 12]. Such potentials will be appropriate and very convenient for nuclear structure and reaction applications.

We consider the effective interaction to be a local operator in relative coordinates whose plane-wave matrix elements optimally reproduce the t-matrices on and off shell. An often used ansatz is a mix of central, tensor, and spin-orbit operators [3–5]. We extend that ansatz herein by adding quadratic spin-orbit, $(L\cdot S)^2$, and angular momentum square, $(L)^2$, terms. The resultant form, which is very similar to those of the one-boson-exchange potentials, has enough degrees of freedom to fit on- and off-shell t matrices in many partial waves, but yet be convenient for evaluations of two-body matrix elements. Since the application programs do not include the second tensor term $S_{12}(\hat{Q})$ of Love and his collaborators [5], which depends upon the direction of the vector $\vec{Q} = \vec{k} + \vec{k}'$, it is not considered herein. As we shall show, it is fallacious to

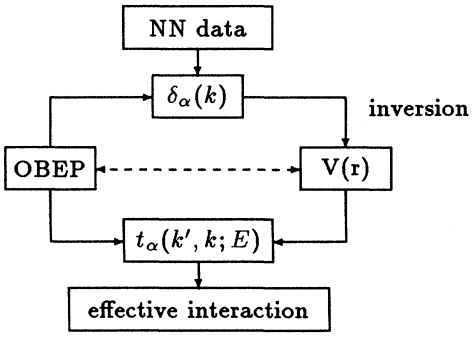

FIG. 1. A scheme which links NN data and effective interactions.

use only a part of an effective interaction in applications.

It remains to define the projection method by which the optimum set of parameters associated with the ansatz is determined. Details of the method are given in Sec. II and examples are presented in Sec. III.

II. THEORETICAL BACKGROUND

The problem of interest is to match on- and off-shell t-matrix elements to values of an effective interaction whose character and functional forms have been chosen for convenience of use in other studies. We seek a scheme that allows those t-matrix elements to be of varied origin. For example, the free on-shell NN t-matrix elements, the half off-shell free NN t-matrix elements, the fully off-shell free NN t-matrix elements, or the nuclear matter g-matrix elements may be selected as the input and they form, as such, the reference data. With reference data assumed to be in partial-wave form, we denote t-matrix elements by $t^{JST}_{L'L}(k',k;E)$. For the two-nucleon system k,k' are the relative momenta in the incident and outgoing channels whose orbital angular momenta are L and L' respectively, with E being the center-of-mass energy.

We seek a local energy-dependent, and possibly also density-dependent, operator whose plane-wave matrix elements reproduce, in a least-square sense, the reference data. This local operator is the *effective interaction* and it is written in the form

$$V(r) = \sum_{\beta} \mathcal{O}_{\beta}(\mu) V_{\beta}(E), \tag{1}$$

wherein the operator structure involves central, tensor, spin-orbit, quadratic spin-orbit and L^2 components that act upon relative two-nucleon states. We allow for a maximum of six Yukawa form factors to be associated with each operator whence the sum over β is at maximum 84. In first application we shall not use the full flexibility of this ansatz. In particular, we will constrain the $(L\cdot S)^2$ and L^2 form factors to have the same set of ranges as the $(L\cdot S)$ term. Such a choice facilitates tranformation of the ensuing effective interaction into forms using other operator specifications [13], which is also true for the central interactions if one wishes to transform to a form involving Wigner, Majorana, Bartlett, and Heisenberg potentials. With these reductions the effective interaction has a maximum of 18 ranges in its specification. Practical applications may reduce this even further. Typically, only four ranges are required for any component. On the other hand, all complex strengths are taken as independent. They are only functions of energy for the free NN interaction cases but, if the data to be matched are reaction matrix elements, then they will also be density dependent, viz. $V_{\beta}(E,k_F)$ where k_F designates any nuclear medium correction effects.

The complete t-matrix in partial-wave decomposition is

$$\langle \vec{k}'|T|\vec{k}\rangle = \frac{\hbar^2}{2m} \sum_{JSTL'LM} \mathcal{Y}^{JM}_{L'S}(\hat{\mathbf{k}}') t^{JST}_{L'L}(k',k;E) \mathcal{Y}^{JM\dagger}_{LS}(\hat{\mathbf{k}}). \tag{2}$$

Hereafter we shall use the collective channel quantum number $\alpha = (STL'LJ)$ whenever possible. Then, with the partial-wave expansion of the effective interaction

$$\langle \vec{k}'|V|\vec{k}\rangle = \sum_{\alpha\beta} \mathcal{Y}_{\alpha}(\hat{\mathbf{k}}') \langle k'\alpha|\mathcal{O}_{\beta}(\mu) V_{\beta}(E)|k\alpha\rangle \mathcal{Y}^{\dagger}_{\alpha}(\hat{\mathbf{k}}), \tag{3}$$

we can equate multipoles giving

$$t_{\alpha}(k',k;E) = \sum_{\beta} \langle k'\alpha|\mathcal{O}_{\beta}(\mu)|k\alpha\rangle V_{\beta}(E). \tag{4}$$

The matrix elements of $\mathcal{O}_{\beta}(\mu)$ are products of radial integrals $I^{\lambda}_{\alpha}(k',k,\mu)$ and matrix elements of the operators in the effective interaction. The integrals involve Bessel functions and are specified by

$$I^{\lambda}_{\alpha} = I^{\lambda}_{\alpha}(k',k,\mu) = \int_0^{\infty} j_{L'}(k'r) e^{-\mu r} r^{\lambda+1} j_L(kr)\, dr. \tag{5}$$

They are analytic in terms of Legendre functions of the second kind, with arguments

$$z = \frac{k'^2 + k^2 + \mu^2}{2kk'}. \tag{6}$$

Specifically

$$I^0_{L,L}(k',k,\mu) = \frac{1}{2kk'} Q_L(z) \tag{7}$$

and

$$I^2_{L,L}(k',k,\mu) = \frac{(L+1)}{(1-z^2)} \left(\frac{\mu}{k'k}\right)^2 \times \left[\left(-L + \frac{zkk'}{\mu^2} + \frac{2z^2}{1-z^2}\right) I^0_{L,L} - \left(\frac{2z}{1-z^2} + \frac{kk'}{\mu^2}\right) I^0_{L+1,L+1}\right]. \tag{8}$$

The nondiagonal integrals for the tensor force are more involved and we use the results (A.8) and (A.10) of Haftel and Tabakin [14] to find

$$I^2_{2,0}(k',k,\mu) = \frac{1}{4kk'^3}\left\{ 3\log\left(\frac{z+1}{z-1}\right) + \frac{1}{(1-z^2)kk'}\left[15\mu^2 + k'^2 - 3k^2 + \frac{2z}{(1-z^2)}\left(\frac{\mu^2}{kk'}\right)(3\mu^2 + k'^2 - 3k^2)\right] + 12k\frac{k+k'}{\mu^2+(k+k')^2}\left(1 - \frac{\mu^2}{\mu^2+(k+k')^2}\right) - 12k\frac{k-k'}{\mu^2+(k-k')^2}\left(1 - \frac{\mu^2}{\mu^2+(k-k')^2}\right)\right\}, \tag{9}$$

and

$$I^2_{L+3,L+1}(k',k,\mu) = \frac{2L+5}{2L+3}\left(\frac{k}{k'}\right)[I^2_{L+2,L} + I^2_{L+2,L+2}(k',k,\mu)] - I^2_{L+1,L+1}(k',k,\mu). \tag{10}$$

The central, $(L\cdot S)$, $(L\cdot S)^2$, and L^2 matrix elements involve $L' = L$ and $\lambda = 0$. The tensor matrix elements, however, involve $L' = L$ and $L' = L \pm 2$ with $\lambda = 2$.

It is convenient to use matrix notation, with which (4) becomes

$$T_\alpha(E) = \mathcal{O}_{\alpha\beta}(\mu)V_\beta(E). \tag{11}$$

Since we assume all ranges to be independent of k', k, and E, and all strengths $V_\beta(E)$ to depend only upon energy, this system of equations must be satisfied for any pair of momenta k' and k. But, with a finite number of potential strengths and ranges to match t matrices in a hyperplane of $\{k',k,E\}$, the problem is an ill-posed one. This situation is typical of inverse problems, for which there exists an extensive literature on methods of solution [15] and in which regularization techniques play a prominent role [16]. For the ill-posed problems we face, *a posteriori* regularization is most pertinent since then physics is the determining factor in selecting the optimal regularization procedure. As a consequence, we consider here only the solution of the normal equation which amounts to minimization of the Euclidean norm

$$\min := \|\mathcal{O}_{\alpha\beta}(\mu)V_\beta(E) - T_\alpha(E)\|. \tag{12}$$

This minimum must be found with respect to variation of the set of ranges $\{\mu\}$ and interaction strengths $\{V_\beta(E)\}$. With the conditions imposed, the problem is linear with respect to the strengths but nonlinear in the ranges. Nevertheless, that problem is separable so that the solution is found in two steps. In the first, we formally replace $V_\beta(E)$ by the normal solution of (11) and search for

$$\min := \|\mathcal{O}_{\alpha\beta}(\mu)\mathcal{O}^+_{\beta\gamma}(\mu)T_\gamma(E) - T_\alpha(E)\| \tag{13}$$

to determine optimal values of the ranges $\mu \rightarrow \mu_0$; a result independent of the values of the interaction strengths. The matrix $\mathcal{O}^+_{\beta\alpha}$ signifies the generalized inverse [17] of $\mathcal{O}_{\alpha\beta}$. Then, with that set of optimal ranges, the normal equation for the strengths has solutions

$$V_\beta(E) = \sum_\alpha \mathcal{O}^+_{\beta\alpha}(\mu_0)T_\alpha(E). \tag{14}$$

We recall that the matrix elements of $\mathcal{O}_{\alpha\beta}(\mu)$ and those of the generalized inverse $\mathcal{O}^+_{\beta\alpha}(\mu)$ are functions of k' and k while those of the t-matrix are functions of k', k, and E. To evaluate the norm in (12) and (13), it is customary to choose a diagonal weight function w_α and to discretize k', k, and E. For convenience, these weight factors are included in (11) and they shall be specified in Sec. III when the numerical results are given. Thus, we obtain a rectangular matrix, $\mathcal{O} \in R^{m,n}$ and $m \gg n$, where

$$m = d(\alpha)d(k')d(k)d(E) \tag{15}$$

and

$$n = d(\beta) \tag{16}$$

with $d(i)$ being the dimensionality of the components. The reference data, $T_\alpha(E) = T_\alpha(k',k;E)$, then form a column vector of length m.

When solving (13), we use the singular value decomposition (SVD) [18] of the matrix $\mathcal{O}$, namely,

$$\mathcal{O} = UDV^\dagger. \tag{17}$$

The details of this decomposition follow by considering the semipositive Hermitian matrices and their eigenvalue systems, viz.

$$\mathcal{O}\mathcal{O}^\dagger u_i = \lambda_i u_i \tag{18}$$

and

$$\mathcal{O}^\dagger\mathcal{O}v_i = \lambda_i v_i. \tag{19}$$

The same positive-definite eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$ and zero values $\lambda_{r+1} = \cdots = \lambda_n = \cdots = \lambda_m = 0$ occur in both equations, with the rank $r = r(\mathcal{O})$ being the same for both matrix products. Using $\sigma_i = +\sqrt{\lambda_i}$, $i = 1,\ldots,r$ the eigenvectors satisfy

$$\mathcal{O}v_i = \sigma_i u_i \tag{20}$$

and

$$\mathcal{O}^\dagger u_i = \sigma_i v_i \tag{21}$$

so that that triple set $\{u_i, v_i, \sigma_i\}$ form the singular value system of $\mathcal{O}$. Furthermore, the matrix U consists of the orthonormal eigenfunctions $\{u_i\}$ being r column vectors of length m. Likewise $\{v_i\}$ form the column vectors of V, and D is the diagonal matrix

$$D = \mathrm{diag}(\sigma_1,\ldots,\sigma_r). \tag{22}$$

The generalized inverse of $\mathcal{O}$ is defined thereby as

$$\mathcal{O}^+ = VD^+U^\dagger \tag{23}$$

wherein

$$D^+ = \mathrm{diag}(\sigma_1^{-1},\ldots,\sigma_r^{-1}). \tag{24}$$

With this representation, the minimization problem (13) reduces to

$$\min := \sum_{i=1}^m |T(i)|^2 - \sum_{j=1}^r \left|\sum_{i=1}^m u_j(i)^* T(i)\right|^2 \tag{25}$$

or equivalently

$$\max := \sum_{j=1}^r \left|\sum_{i=1}^m u_j^*(i)T(i)\right|^2. \tag{26}$$

Because the eigenvectors $\{u_j\}$ depend upon the ranges $\{\mu\}$ in a complex functional form, the minimum of Eq. (25) or the maximum of Eq. (26) must be found numerically.

With the optimal set of ranges $\{\mu_0\}$, the strengths $V_\beta(E)$ are computed from (14) using the data in subsets for every chosen value of E. The dimensions of the column vectors in $\mathcal{O}$ and T in (14) reduce thereby to

$$m_1 = d(\alpha)d(k')d(k), \tag{27}$$

for an interaction with energy-dependent strengths matching all chosen on- and off-shell t-matrix elements. A less general effective interaction results from matching on- and half off-shell t-matrix elements for which $E = \hbar^2k^2/2m$ and the dimension is reduced to

$$m_2 = d(\alpha)d(k'). \tag{28}$$

The least constrained effective interaction uses only the on-shell data, for which as $k' = k$,

$$m_3 = d(\alpha). \tag{29}$$

This case can have $m_3 < n$ and so be underdetermined. Even so, the method remains valid and gives a unique solution. Various other schemes can be designed. For example, if an energy-independent representation of the on-shell t matrices is desired, then the dimension is

$$m_4 = d(\alpha)d(E). \tag{30}$$

Finally, we note that the solution of (14), for $\{V_\beta(E)\}$, is the result of applying the optimal set of ranges in the computation of the generalized inverse of the matrix $\mathcal{O}$. It is during this last step of finding the strengths $\{V_\beta(E)\}$ that various options of regularization procedures would enter.

III. EXAMPLES

By the very nature of its method of construction, an effective interaction will reflect not only the bias taken as to its form but also the selection of data from which it is specified. In our case, we have chosen the form of the effective interaction with applications in mind of defining microscopic model optical potentials and transition interactions of use in microscopic distorted-wave Born approximations (DWBA) of (p,p') and (p,n) reactions. As a consequence of the energy regime below 500 MeV that is pertinent to those applications, the data selected was from all coupled and uncoupled channels up to and including total angular momentum $J = 4$. Therewith this effective interaction development is an integral part of the latest version of Raynal's DWBA-90 program [13] as that permits use of the full complexity of these effective interactions. It can also use energy- and density-dependent g-matrices as well as microscopic optical potentials which may be generated consistently from the same effective interactions.

It was our aim also to define effective interactions that well represent both the on- and off-shell properties of the t matrices. This is achieved by explicitly using a com-

TABLE I. Effective interaction for $E = 172$ MeV, $k = 1.44$ fm^{-1} to the Paris t-matrix.

μ (fm^{-1})	Re V (MeV)	Im V (MeV)	Re V (MeV)	Im V (MeV)
	$S=0,\ T=0$		$S=0,\ T=1$	
Central interaction				
0.71000	18.024327	−1.0635907	−49.510696	34.575848
1.26374	148.69515	26.898631	236.41704	−237.31385
2.14196	−220.75975	−374.77139	−1146.2398	369.48829
4.00000	585.41253	834.47924	2042.9705	−134.57094
Volume integrals	1474.5	−185.95	−908.93	−99.071
	$S=1,\ T=0$		$S=1,\ T=1$	
Central interaction				
0.71000	−127.16301	42.080521	−10.110740	−0.25508064
1.26374	783.60799	−285.95988	89.933070	7.6999532
2.14196	−2539.0942	89.929157	−561.06459	−159.01109
4.00000	3497.4085	621.60107	1768.3668	209.08513
Volume integrals	−1211.7	−466.57	307.73	−217.08
Spin-orbit interaction				
0.94706	11.147494	35.375879	−13.271205	−11.864801
1.43559	−272.17358	−216.60942	178.43357	120.35606
1.95761	494.46009	468.71518	−489.34442	−266.55093
4.00000	−1877.2219	−1019.4288	49.537669	562.13762
Volume integrals	−1356.4	−88.816	−663.66	135.08
Tensor S_{12} interaction				
0.94706	−34.426498	4.1856042	3.3172104	0.31998681
1.43559	300.00814	−51.135144	−45.314688	−7.8849014
1.95761	−682.62652	126.32996	160.97252	32.279060
4.00000	1933.0394	−479.48753	−248.64338	−386.79969
Volume integrals	−139.37	−1.3474	43.282	−9.6968

plete table of on- and off-shell t matrices [19, 20] and they form a larger data base than used in other specifications. Previously only the on-shell values of the t-matrix amplitudes [3] or the integrands of the defining integrals of the partial-wave on-shell amplitudes [4] were fittted. Now we claim that it is not sufficient to have just the phase shifts from experiment, unless one makes use of quantum inversion [10] since the output kernels of Gelfand-Levitan or Marchenko fundamental inversion equations yield the correlated wave functions. From those functions half off-shell t matrices can be obtained and then the fully off-shell values follow from an integral equation [20]. At present our inversion studies are incomplete whence herein we make use of the Paris [6] and Bonn [7] potentials for computation of t matrices either in momentum [19, 21] or in coordinate [20] space.

TABLE II. Effective interaction for $E = 172$ MeV, $k = 1.44$ fm^{-1} to the Paris t matrix.

μ (fm^{-1})	Re V (MeV)	Im V (MeV)	Re V (MeV)	Im V (MeV)
	$S = 0,\ T = 0$		$S = 0,\ T = 1$	
Central interaction				
0.71000	−18.330983	5.0464033	−55.963896	43.800022
1.26374	305.62993	−35.391743	345.85801	− 395.95911
2.14196	−509.36186	−182.13855	−1452.9948	813.88952
4.00000	842.81923	451.86596	2369.7517	− 606.72542
Volume integrals	1214.7	−296.66	−792.20	−271.06
L^2 interaction				
0.94706	22.195262	−1.5160382	13.100200	−12.667806
1.43559	−131.69523	5.5341767	−94.662342	87.859467
1.95761	210.41806	9.3558769	124.18207	−90.081534
4.00000	−252.03458	−54.982265	−17.251216	−80.021256
Volume integrals	0.0	0.0	0.0	0.0
	$S = 1,\ T = 0$		$S = 1,\ T = 1$	
Central interaction				
0.71000	−111.42369	62.779395	−56.400142	9.9482966
1.26374	873.08042	−496.07939	289.45052	−99.831498
2.14196	−2922.7137	625.43262	−979.25593	255.50527
4.00000	3956.5065	72.406612	2179.2348	−607.30475
Volume integrals	−805.52	−568.53	−98.992	−314.69
L^2 interaction				
0.94706	55.144410	− 22.932404	101.01598	−34.131979
1.43559	−428.39488	189.73695	−703.69514	289.14393
1.95761	345.37322	− 393.15008	1306.7686	−581.49844
4.00000	900.17687	577.50148	−1794.7433	791.91392
Volume integrals	0.0	0.0	0.0	0.0
Spin-orbit interaction				
0.94706	30.068101	−8.1954638	−86.701892	10.360891
1.43559	−198.39737	155.74963	690.26858	−72.034989
1.95761	417.53165	−270.46789	−1435.4625	134.04650
4.00000	−510.35050	1942.8008	1430.1383	−69.788931
Volume integrals	179.86	1473.8	−589.68	90.674
$(L \cdot S)^2$ interaction				
0.94706	−19.749942	19.295254	−100.24739	34.653355
1.43559	156.85739	−181.81620	719.63852	− 300.15510
1.95761	254.70007	479.65463	−1365.1688	623.75241
4.00000	−1928.8521	−935.27802	1901.0828	− 892.14394
Volume integrals	0.0	0.0	0.0	0.0
Tensor S_{12} interaction				
0.94706	−7.0996164	12.193188	0.70209051	1.2616654
1.43559	52.664619	− 126.78881	−31.818117	−15.882337
1.95761	−63.287466	314.31218	142.18029	46.445924
4.00000	−630.37120	− 1226.1995	−614.57626	−181.45882
Volume integrals	−40.183	24.103	8.3208	3.5533

TABLE III. Effective interaction for $E = 172$ MeV, $k = 1.44$ fm^{-1} to the Bonn-R t matrix.

μ (fm^{-1})	Re V (MeV)	Im V (MeV)	Re V (MeV)	Im V (MeV)
	$S=0,\ T=0$		$S=0,\ T=1$	
Central interaction				
0.71000	33.624745	−6.1094675	−53.013265	28.959560
1.26374	53.748090	58.821167	291.80127	−189.37474
2.14196	122.88682	−490.16148	−1261.3024	203.63369
4.00000	−609.34366	1318.4042	2075.3637	89.802735
Volume integrals	1119.1	3.4732	−850.16	−139.91
	$S=1,\ T=0$		$S=1,\ T=1$	
Central interaction				
0.71000	−96.154968	55.025157	−6.4485486	2.4880234
1.26374	575.69430	−425.43295	81.119633	−16.335390
2.14196	−1994.8973	524.57620	−578.53303	−97.894229
4.00000	2846.4309	71.881025	1858.1802	166.59942
Volume integrals	−1095.5	−482.60	352.37	−203.80
Spin-orbit interaction				
0.94706	59.186510	46.834783	−6.2174043	−9.2276092
1.43559	−667.07225	−328.65148	123.73341	96.315478
1.95761	1314.3374	741.17406	−368.46772	−206.29062
4.00000	−4328.2765	−2050.2645	−176.59858	401.44244
Volume integrals	−2327.8	−527.63	−679.60	96.836
Tensor S_{12} interaction				
0.94706	−37.911555	3.7295015	4.1370261	0.35353505
1.43559	339.37330	−49.102252	−57.125981	−8.7871237
1.95761	−803.71221	120.66045	200.96819	37.038233
4.00000	2851.6767	−445.34627	−444.55518	−418.06771
Volume integrals	−135.86	−5.6328	45.749	−9.3047

As indicated in Sec. II, our procedure permits a weighting of the minimizations [(12), (13)] in select regions of momentum space. Specifically, we use

$$w(k',k,E_0) = w_0 \exp\Big(-\frac{(k'-k)^2}{\alpha^2} - \frac{(k-k_0)^2}{\beta^2}\Big), \quad (31)$$

where α and β are Gaussian widths, $\hbar^2 k_0^2/2m = E_0$ is a pertinent energy, and w_0 is the normalization. Additionally, in the search of ranges via (13) we add to the norm a regularization, namely,

$$\gamma \sum \frac{(\mu_i \mu_j)^\varepsilon}{|\mu_i - \mu_j|}, \quad (32)$$

to ensure a spread of the values of those ranges. Therein, γ and ε are two control parameters.

Tables I and II give two effective interactions that were obtained by a mapping onto the half off-shell Paris t matrices with parameter values α=5 fm^{-1}, β=0.4 fm^{-1}, and k_0=1.44 fm^{-1}. The dimensionality is given by (28). The first interaction comprises four Yukawa potentials $e^{-\mu r}/r$ each in both of the central and the spin-orbit components, and, multiplied by r^2, in the tensor components. Real and imaginary terms have the same ranges. The second interaction is an enlarged version of the first; enlarged by the inclusion of L^2 and $(L \cdot S)^2$ terms whose volume integrals were constrained to be zero. Thereby we can compare like components of the two interactions. The inclusion of the additional operators does improve the fit to the input t matrices but changes the strength parameters from those of the first interaction. Omitting the L^2 and $(L \cdot S)^2$ terms from Table II does not equate to Table I. Hence one cannot simply compare common components of any two effective interactions, e.g., the central singlet even part of the interactions [3–5], if either their

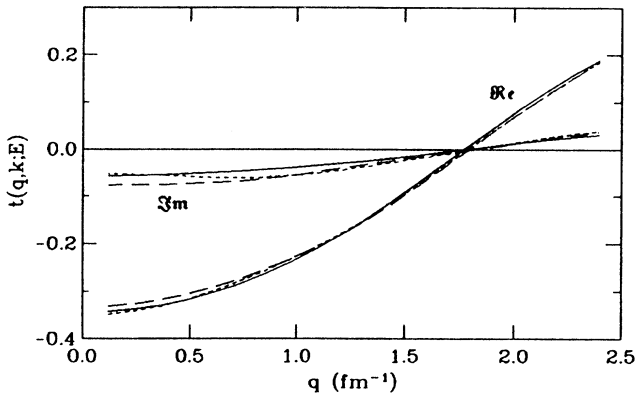

FIG. 2. Comparisons of $^1S_0$ half-off shell t matrices at E=172 MeV, equivalent to an on-shell momentum k=1.44 fm^{-1}. Their real (Re) and imaginary (Im) parts are shown for the Paris t matrix (solid line) and the effective interactions of Table I (dashed line) and Table II (dotted line).

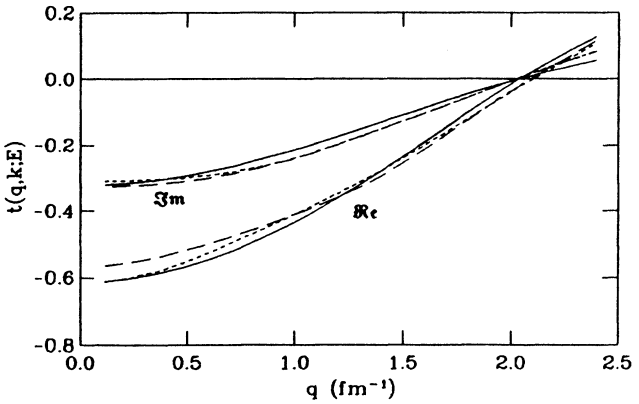

FIG. 3. As for Fig. 2 but for the $^3S_1$ channel.

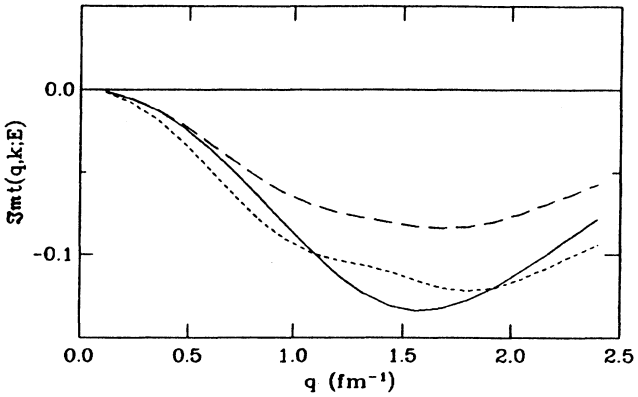

FIG. 5. As for Fig. 2 but for the imaginary part of the $^3D_2$ channel t matrix.

modes of construction vary or one involves more linear independent operators than the others. It is even less sensible to compare effective interactions, as derived herein, with any obtained from fits to select nuclear reaction data [22, 23] even if the latter were to use the same set of operators.

We show in Table III an effective interaction of precisely the same form including ranges and weighting parameters as for Table I. The input t matrices were based upon the Bonn-R potential [7]. As the Paris and Bonn-R potentials differ, so also do the strengths of their associated effective interactions.

Given the diversity between the parametrizations and numbers of Tables I and II, one might expect quite significant variations in the t matrices they produce. But in fact, for low partial waves, both give comparable and good fits to the input data. Those of the $^1P_1$ and $^3P_{0,1,2}$ channels fit both the real and imaginary parts of the Paris half off-shell t matrices to better than a few percent over an off-shell range of ± 1 fm^{-1} about the on-shell value, 1.44 fm^{-1} at an energy of 172 MeV. The $^1S_0$ and $^3S_1$ channel values are shown in Figs. 2 and 3. Therein the results obtained from Table I are presented as dashed lines, those of Table II are given by dotted lines, and the input data are shown as solid lines. Both are good representations of these t matrices on and off shell, with unitarity preserved to within 10%. From these results of low partial-wave channels, there is no obvious need for the introduction of L^2 and $(L \cdot S)^2$ operators. It is the higher partial-wave data that necessitate their inclusion. Consequently, analyses of data below, say, 100 MeV need only parametrizations as given by Table I. We show in Figs. 4 and 5 results for the real and imaginary parts, respectively, of the $^3D_2$ channel at 172 MeV wherein is used the same nomenclature as in Fig. 2. In this instance the improvement in fit that is obtained by using Table II rather than Table I is dramatic. Overall we have, with Table I, a fit that is better than 10%.

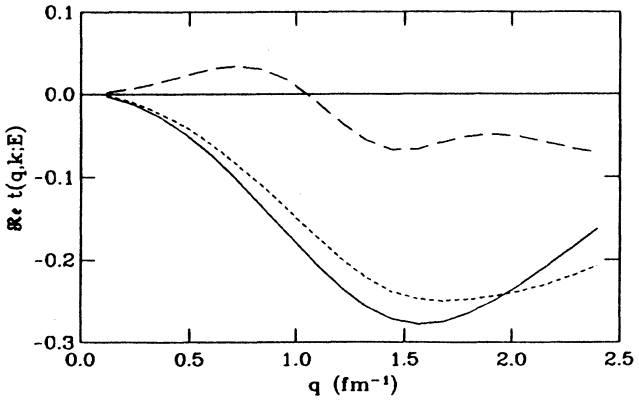

FIG. 4. As for Fig. 2 but for the real part of the $^3D_2$ channel t matrix.

IV. SUMMARY AND CONCLUSIONS

A method has been developed with which both on- and off-shell NN t or g matrices for many energies can be used to specify effective interactions. With the assumption that characteristic ranges of the effective interaction are energy independent, the procedure separates allowing determination of the optimal set of ranges which can then be used to solve the normal equation for the strengths. The use of the singular value decomposition of a matrix facilitates solution of the first step.

The method was applied to free NN t-matrix data obtained from the Paris and Bonn-R potentials, and mappings were achieved to local effective interaction in relative coordinates. Those interactions have the form of central, r^2S_{12}, $(L \cdot S)$, $(L \cdot S)^2$, and L^2 operators each with a combination of Yukawa form factors. Such an effective interaction form is used in the latest nuclear reaction analyses program DWBA-90. Herein we have found that the $(L \cdot S)^2$ and L^2 terms are important in fitting the total t matrix whenever partial waves with $L > 1$ are of significance. Such is the case for energies in excess of 100 MeV.

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