Relation between the bound state radius and the scattering length: Relativistic effects

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Relativistic effects on the relation between the root-mean-square radius r_D of the bound state and the scattering length a are examined for models which simulate the triplet S state of the two-nucleon system. Two models are considered; one is based on the one-body Dirac equation and the other on the two-body Dirac equation. The relation is expressed as $(r_D/a)^2=(c_0+c_1x^1+c_2x^2+\cdots)/8$, where $x\equiv\alpha r_0$; α is related to the binding energy and r_0 is the effective range. This expansion is compared with the corresponding expansion of the nonrelativistic Schrödinger case. The first three terms are shape independent. The c_0 $(=1)$ and c_1 $(=0)$ are the same as those of the nonrelativistic case but a relativistic correction appears in c_2.

I. INTRODUCTION

Recently several papers have appeared examining the relation between the root-mean-square (rms) radius of the deuteron r_D and the scattering length a (commonly denoted by a_t) of the triplet S state of the two-nucleon system [1–7]. The interest in this problem was inspired by the remark by Klarsfeld *et al.* [1] that the values of r_D and a, calculated on the basis of a number of realistic nucleon-nucleon potentials, exhibit a linear relation; when a is plotted against r_D one obtains a straight line. It is interesting that the experimental (r_D,a) point lies distinctly off the calculated line. The discrepancy, although only about 1%, is believed to be significant.

When the relativistic interaction consists of a strongly attractive Lorentz scalar and a strongly repulsive (zeroth component of) Lorentz vector, there could be significant relativistic effects even at low energies [8,9]. This observation motivated the analysis of relativistic effects on the r_D/a ratio [2]. Within limited one-dimensional model calculations, however, relativistic effects were found to be too small to remedy the discrepancy. Incidentally it was then conjectured that the discrepancy is a signature of nonlocality of the nucleon-nucleon interaction. Later analyses support this conjecture [3,4,6].

Through nonrelativistic (NR) model calculations Bhaduri *et al.* [4] found that, when r_D/a is expressed in the form of the expansion,

$$(r_D/a)^2=(c_0+c_1x^1+c_2x^2+\cdots)/8\ , \qquad (1.1)$$

where $x=r_0/a$ and r_0 is the effective range, the first three coefficients are shape independent:

$$c_0=1,\quad c_1=0,\quad c_2=\tfrac{1}{4}\ . \qquad (1.2)$$

The dependence on the shape of the potential begins to appear through c_3. Since $x\simeq 0.36$, Eq. (1.1) together with Eq. (1.2) explains why r_D/a is almost shape independent. The expansion has been further elucidated by Sprung *et al.* [5] and by Kermode *et al.* [6].

The purpose of this paper is to obtain relativistic versions of the expansion (1.1). Unlike Ref. [2] we consider three-dimensional models in this paper. However, we will point out an interesting difference between one and three dimensions. We consider two models, I and II. In model I we use the one-body Dirac equation with an external central potential. Model II is based on the two-body Dirac equation with an instantaneous interaction. Unlike in NR quantum mechanics, relativistic two-body problem cannot simply be reduced to a one-body problem; hence the one-body Dirac equation of model I can only simulate the two-nucleon system. The two-body Dirac equation of model II does describe the two-body system but the equation is not exactly covariant. Nevertheless, we hope to be able to get a feel for relativistic effects through these models.

II. THE rms RADIUS VERSUS SCATTERING LENGTH: NONRELATIVISTIC CASE

Before discussing the relativistic models, let us review the relation between r_D and a in the NR case. We consider an S state which contains only one bound state (which simulates the deuteron). The S matrix has a pole at $k=i\alpha$, which corresponds to the bound state. The binding energy is $\alpha^2/2\mu$, where $\mu=m/2$ is the reduced mass; m is the nucleon mass. The effective range expansion for the scattering phase shift δ can be done in two ways:

$$k\cot\delta=-\frac{1}{a}+\frac{1}{2}r_0k^2-Pr_0^3k^4+\cdots \qquad (2.1)$$

$$=-\alpha+\tfrac{1}{2}r_d(k^2+\alpha^2)-\cdots\ . \qquad (2.2)$$

The parameters in the two expansions are related by

$$\alpha=\frac{1}{a}+\frac{1}{2}\alpha^2r_0+P\alpha^4r_0^3+\cdots\ , \qquad (2.3)$$

$$r_d=r_0+4P\alpha^2r_0^3+\cdots\ . \qquad (2.4)$$

Following Ref. [6], we put the rms radius of the bound state into the form

$$\left[\frac{r_D}{a}\right]^2=\frac{\int_0^\infty \rho(\alpha,r)(r/2)^2dr}{a^2\int_0^\infty \rho(\alpha,r)dr} =\frac{1-4\alpha^3 I_2(\alpha)}{8(\alpha a)^2[1-2\alpha I_0(\alpha)]}, \tag{2.5}$$

where

$$I_0(\alpha)=\int_0^\infty [e^{-2\alpha r}-\rho(\alpha,r)]dr , \tag{2.6}$$

$$I_2(\alpha)=\int_0^\infty [e^{-2\alpha r}-\rho(\alpha,r)]r^2dr . \tag{2.7}$$

The $\rho(\alpha,r)$ is related to the bound state wave function $u(\alpha,r)/r$ by

$$\rho(\alpha,r)=u^2(\alpha,r) , \tag{2.8}$$

where $u(\alpha,r)$ is such that $u(\alpha,r)\to e^{-\alpha r}$ as $r\to\infty$, and $\int_0^\infty \rho(\alpha,r)dr=1$.

Kermode *et al.* [6] derived the expansion

$$\left[\frac{r_D}{a}\right]^2=\tfrac{1}{8}[1+\tfrac{1}{4}(\alpha r_0)^2+c_3(\alpha r_0)^3+\cdots] , \tag{2.9}$$

where

$$c_3=(1-2J+8P)/4 , \tag{2.10}$$

$$J=I_2(0)/I_0(0)^3 . \tag{2.11}$$

Obviously c_3 is shape dependent. Actually, Kermode *et al.* [6] gave an expression for r_D/a rather than for $(r_D/a)^2$; moreover, their expansion is with respect to $\alpha r_0/2$. Their a_3 is related to our c_3 by $a_3=4c_3$. Note that Eq. (2.9) is an expansion in terms of αr_0 rather than r_0/a which Bhaduri *et al.* [4] adopted; the c_3's of Eqs. (1.1) and (2.9) are slightly different, but their relation can easily be obtained by using Eq. (2.4).

In arriving at Eq. (2.9) there were two important steps, namely,

$$I_0(\alpha)=r_d/2 , \tag{2.12}$$

and that $I_2(\alpha)$ is finite when $\alpha\to 0$ as implied by Eq. (2.10). Equation (2.12) arises in the derivation of expansion (2.2). For the behavior of $I_2(\alpha)$ when $\alpha\to 0$, we know of no example of the bound S state such that $I_2(\alpha)$ diverges. We point out in Sec. III that relativistic corrections appear in these two steps, resulting in a correction of order α^2 in Eq. (2.9).

III. ONE-BODY DIRAC EQUATION

We consider the Dirac equation (in natural units $c=\hbar=1$)

$$[\boldsymbol{\alpha}\cdot\mathbf{p}+\beta(\mu+S)+V]\psi=E\psi , \tag{3.1}$$

where S and V are a Lorentz scalar and the zeroth component of a Lorentz vector, respectively. The S and V are both central potentials, and can be local or nonlocal. For the mass we take $\mu=m/2$ so that the NR reduction of Eq. (3.1) becomes the two-body Schrödinger equation. The angular part of Eq. (3.1) can be separated, and the ψ is reduced to a two-component form [10]. We denote the radial part of ψ by $g(r)/r$ and $f(r)/r$, which are the upper and lower components, respectively.

Let us examine the effective range expression for scattering. We introduce auxiliary functions $\mathscr{g}(r)$ and $\mathscr{f}(r)$ which are defined by

$$\mathscr{g}(r)=\frac{\sin(kr+\delta)}{\sin\delta},\quad \mathscr{f}(r)=\frac{k\cos(kr+\delta)}{(\mu+E)\sin\delta} , \tag{3.2}$$

where $E=(\mu^2+k^2)^{1/2}$. The $\mathscr{g}/r$ and $\mathscr{f}/r$ satisfy the Dirac equation in the absence of the interaction and the centrifugal terms. When $k\to 0$, $\mathscr{g}$ and $\mathscr{f}$, respectively, become

$$\mathscr{g}_0(r)=1-r/a,\quad \mathscr{f}_0(r)=-1/2\mu a . \tag{3.3}$$

We also choose the phase and normalization of $g(r)$ and $f(r)$ such that, when $r\to\infty$, they approach $\mathscr{g}(r)$ and $\mathscr{f}(r)$, respectively. Using a trick similar to that of the nonrelativistic case [11] we obtain

$$k\cot\delta=(\mu+E)\left[-\frac{1}{2\mu a}+(E-\mu)\int_0^\infty(\mathscr{g}\mathscr{g}_0+\mathscr{f}\mathscr{f}_0-gg_0-ff_0)dr\right] , \tag{3.4}$$

where g_0 and f_0 are for $k=0$. This leads to the expansion around $k=0$. However, note that, for r greater than the interaction range, $g=\mathscr{g}$ but $f=\mathscr{f}+O(1/r)$. Hence the integral in Eq. (3.4) gets a contribution from beyond the interaction range.

Next let us consider the expansion around $k=i\alpha$. To this end we introduce $\mathscr{g}_\alpha(r)$ and $\mathscr{f}_\alpha(r)$, defined by

$$\mathscr{g}_\alpha(r)=e^{-\alpha r},\quad \mathscr{f}_\alpha(r)=\frac{-\alpha}{\mu+E_\alpha}e^{-\alpha r} , \tag{3.5}$$

and obtain

$$k\cot\delta=(\mu+E)\left[-\frac{\alpha}{\mu+E_\alpha}+(E-E_\alpha)\int_0^\infty(\mathscr{g}\mathscr{g}_\alpha+\mathscr{f}\mathscr{f}_\alpha-gg_\alpha-ff_\alpha)dr\right] , \tag{3.6}$$

where $g_\alpha(r)$ and $f_\alpha(r)$ form the exact wave function for the bound state. When $k \simeq i\alpha$, the integral on the right-hand side of Eq. (3.6) can be approximated by

$$\int_0^\infty (\mathscr{g}_\alpha^2+\mathscr{f}_\alpha^2-g_\alpha^2-f_\alpha^2)dr$$
$$=\int_0^\infty \left[\frac{2\mu}{\mu+E_\alpha}e^{-2\alpha r}-(g_\alpha^2+f_\alpha^2)\right]dr$$
$$=\frac{2\mu}{\mu+E_\alpha}I_0(\alpha)\ , \tag{3.7}$$

where $I_0(\alpha)$ is defined by Eq. (2.6), together with

$$\rho(\alpha,r)=\frac{\mu+E_\alpha}{2\mu}(g_\alpha^2+f_\alpha^2)\ . \tag{3.8}$$

This $\rho(\alpha,r)$ replaces $u^2(\alpha,r)$ of the NR case. Note that $\rho(\alpha,r)\rightarrow e^{-2\alpha r}$ as $r\rightarrow\infty$; see Eq. (3.12). Putting Eq. (3.7) in Eq. (3.6), expanding around $k^2=-\alpha^2$, and comparing the result with Eq. (2.2), we obtain

$$r_d=-\frac{\alpha}{E_\alpha(\mu+E_\alpha)}+\frac{2\mu}{E_\alpha}I_0(\alpha)\ , \tag{3.9}$$

which leads to

$$I_0(\alpha)=\frac{1}{2}r_d+\frac{\alpha}{4\mu^2}-\frac{\alpha^2}{4\mu^2}r_d+O(\alpha^3)\ . \tag{3.10}$$

Next let us examine $I_2(\alpha)$ which is defined by Eq. (2.7) together with the $\rho(\alpha,r)$ of Eq. (3.8). This requires a closer look at the asymptotic form of g_α and f_α. Beyond the range of the interaction they are given by

$$g_\alpha\simeq e^{-\alpha r},\quad f_\alpha\simeq-\frac{\alpha}{\mu+E_\alpha}\left[1+\frac{1}{\alpha r}\right]e^{-\alpha r}\ , \tag{3.11}$$

and hence

$$e^{-2\alpha r}-\rho(\alpha,r)\simeq-\frac{\alpha^2}{2\mu(\mu+E_\alpha)}\left[\left[1+\frac{1}{\alpha r}\right]^2-1\right]e^{-2\alpha r}\ . \tag{3.12}$$

For R greater than the interaction range we obtain

$$\int_R^\infty [e^{-2\alpha r}-\rho(\alpha,r)]r^2dr$$
$$\simeq-\frac{1}{2\mu\alpha(\mu+E_\alpha)}(1+\alpha R)e^{-2\alpha R}\ , \tag{3.13}$$

which diverges as $\alpha\rightarrow 0$. Consequently, $I_2(\alpha)$ behaves like $-1/(4\mu^2\alpha)$ as $\alpha\rightarrow 0$ [12]. We therefore define J by

$$I_2(\alpha)\equiv-\frac{1}{4\mu^2\alpha}+\frac{1}{8}r_d^3J+O(\alpha)\ . \tag{3.14}$$

Taking account of the relativistic corrections of Eqs. (3.10) and (3.14) we obtain the Dirac version of Eq. (2.8),

$$\left[\frac{r_D}{a}\right]^2=\frac{1}{8}\left[1+\frac{3\alpha^2}{2\mu^2}+\frac{1}{4}(\alpha r_0)^2+c_3(\alpha r_0)^3+\cdots\right]\ , \tag{3.15}$$

$$c_3=(1-2J+8P)/4\ . \tag{3.16}$$

The c_3 is shape dependent. The term $3\alpha^2/2\mu^2$ in Eq. (3.15) is a relativistic correction. For the deuteron, $(\alpha r_0)^2/4\simeq 0.041$ and $3\alpha^2/(2\mu^2)\simeq 0.014$. This relativistic correction on r_D is only 0.7%, but the discrepancy regarding r_D/a that we are concerned with is about 1%.

If one considers a fictitious one-dimensional model with the one-dimensional Dirac equation, the correction due to Eq. (3.14) does not appear; this is because the $1/\alpha r$ term does not appear in the one-dimensional counterpart of f_α. Then the J is defined by Eq. (2.11). The one-dimensional version of Eq. (3.15) is obtained by replacing $3\alpha^2/2\mu^2$ in Eq. (3.15) with $\alpha^2/(2\mu^2)$.

IV. TWO-BODY DIRAC EQUATION

By the "two-body Dirac equation" we mean

$$[(\boldsymbol{\alpha}_1-\boldsymbol{\alpha}_2)\cdot\mathbf{p}+(\beta_1+\beta_2)m+U]\psi=E\psi\ , \tag{4.1}$$

where subscripts 1 and 2 refer to the two nucleons [13, 14]. We have taken the center-of-mass system; $\mathbf{p}=\mathbf{p}_1-\mathbf{p}_2$ is the relative momentum. The potential U can be any linear combination of Lorentz scalar, vector, etc., but we need not specify it until we consider an explicit example later.

There are various configurations of the two-nucleon system, but let us consider the simplest one which corresponds to the ^1S state. Of course the deuteron is the ^3S state, but Eq. (4.1) is very complicated for the ^3S state. To use the ^1S state to simulate the ^3S state is like dropping the tensor force in the NR case. This is admittedly crude; nevertheless, let us try. We assume that the potential is such that there is a bound state (in the ^1S state) which simulates the deuteron.

According to Moseley and Rosen [14], the relevant components of the wave function for the ^1S state are those of the form

$$I=f_1(r),\quad A_4=f_2(r),\quad i\mathbf{F}=f_3(r)\mathbf{r}/r\ . \tag{4.2}$$

There is a departure from Moseley and Rosen's notation; our $i\mathbf{F}$ corresponds to their $\mathbf{F}$, i.e., our f_3 is equal to their if_3. In this way all the f_i's can be taken as real functions. Equation (4.1) becomes

$$(E-U_1)f_1+2\left[r\frac{d}{dr}+3\right]\frac{f_3}{r}+2mf_2=0\ , \tag{4.3}$$

$$(E-U_2)f_2+2mf_1=0\ , \tag{4.4}$$

$$(E-U_3)f_3-2\frac{df_1}{dr}=0\ , \tag{4.5}$$

where U_1, etc., are certain linear combinations of the Lorentz scalar part, vector part, etc., of U.

On the basis of Eqs. (4.3)–(4.5) we can derive effective range formulas similar to those for the one-body Dirac equation. Let us define $\mathscr{f}_i(r)$'s by

$$r\mathscr{f}_1=-\frac{2m}{E}r\mathscr{f}_2=\frac{\sin(kr+\delta)}{\sin\delta}\ ,$$
$$r\mathscr{f}_3=\frac{2k\cos(kr+\delta)}{E\sin\delta}\ , \tag{4.6}$$

where $E=2(m^2+k^2)^{1/2}$. When $k\to 0$, the $\not{f}_i$'s become

$$r\not{f}_{10}=-r\not{f}_{20}=1-\frac{r}{a},\quad r\not{f}_{30}=-\frac{1}{ma}\ . \tag{4.7}$$

We choose the phase and normalization of the f_i's such that they have the same asymptotic form as the $\not{f}_i$'s. The effective range expansion around $k=0$ can be obtained from

$$k\cot\delta=E\left[-\frac{1}{2ma}+\frac{1}{4}(E-2m)\sum_i\int_0^\infty(\not{f}_i\not{f}_{i0}-f_if_{i0})r^2dr\right], \tag{4.8}$$

where the f_{i0}'s are for $k=0$, and the $\sum_i$ is for $i=1$, 2, and 3.

For the expansion around $k=i\alpha$ we introduce

$$r\not{f}_{1\alpha}=-\frac{2m}{E_\alpha}r\not{f}_{2\alpha}=e^{-\alpha r},\quad r\not{f}_3(r)=\frac{-2\alpha}{E_\alpha}e^{-\alpha r}, \tag{4.9}$$

where $E_\alpha=2(m^2-\alpha^2)^{1/2}$, and arrive at

$$k\cot\delta=E\left[-\frac{\alpha}{E_\alpha}+\frac{1}{4}(E-E_\alpha)\sum_i\int_0^\infty(\not{f}_i\not{f}_{i\alpha}-f_if_{i\alpha})r^2dr\right], \tag{4.10}$$

where the $f_{i\alpha}$'s are the f_i's for the bound state.

We now define the two-body version of $\rho(\alpha,r)$ by

$$\rho(\alpha,r)=(E_\alpha^2/8m^2)r^2\sum_i f_{i\alpha}^2\ , \tag{4.11}$$

which behaves like $e^{-2\alpha r}$ as $r\to\infty$. This $\rho(\alpha,r)$ enters into $I_0(\alpha)$ and $I_2(\alpha)$. The two-body counterpart of Eq. (3.10) reads

$$I_0(\alpha)=\frac{1}{2}r_d+\frac{\alpha}{2m^2}-\frac{\alpha^2}{2m^2}r_d+O(\alpha^3)\ . \tag{4.12}$$

For r greater than the interaction range,

$$\begin{aligned} rf_{1\alpha}&\simeq-\frac{2m}{E_\alpha}rf_{2\alpha}\simeq e^{-\alpha r}\ ,\\ rf_{3\alpha}&\simeq-\frac{2\alpha}{E_\alpha}\left[1+\frac{1}{\alpha r}\right]e^{-\alpha r}\ , \end{aligned} \tag{4.13}$$

which leads to

$$I_2(\alpha)\equiv-\frac{1}{2m^2\alpha}+\frac{1}{8}r_d^3J+O(\alpha)\ . \tag{4.14}$$

Combining the above results we obtain

$$\left[\frac{r_D}{a}\right]^2=\frac{1}{8}\left[1+\frac{3\alpha^2}{m^2}+\frac{1}{4}(\alpha r_0)^2+c_3(\alpha r_0)^3+\cdots\right], \tag{4.15}$$

$$c_3=\tfrac{1}{4}(1-2J+8P)-\frac{1}{2(mr_0)^2}\ . \tag{4.16}$$

The correction $3\alpha^2/m^2$ in Eq. (4.15) is $\frac{1}{2}$ of the corresponding term of Eq. (3.15). The relativistic correction in $c_3(\alpha r_0)^3$ is $-\alpha^3r_0/(2m^2)\simeq-5\times10^{-5}$, which is negligible. For a fictitious one-dimensional model, the term $3\alpha^2/m^2$ of Eq. (4.15) is replaced by α^2/m^2.

V. MODEL CALCULATIONS

Let us consider two models. The first one is the nonlocal separable potential model of Ref. [12]. This model uses the one-body Dirac equation. The relation between r_D and a of this model has been examined recently [7]. In order to see the relativistic effects, the results are compared with those of a phase-equivalent NR model. The NR model is constructed by means of the inverse scattering method. Also the NR model has a bound state of the same binding energy as that of the relativistic model.

When a is plotted against r_D, the (r_D,a) lines of the relativistic and NR models lie close to each other, but the relativistic one is slightly shifted to the right (such that r_D is larger for the same value of a); see Fig. 1 of Ref. [7]. The increase in r_D (for the same value of a) is (0.2–0.3)%. The term $3\alpha^2/2\mu^2$ of Eq. (3.15) increases r_D by 0.7%. On the other hand, the J of the relativistic models is larger than the J of the phase-equivalent NR models. This tends to compensate for the $3\alpha^2/2\mu^2$ term. In fact, $I_2(\alpha)$ for the relativistic and NR models are nearly equal. This is because the density distribution $\rho(r,\alpha)$ differs very little between the two models. We should add that, the phase shift in these models does not become negative at high energies; so it does not simulate the empirical phase shift very well. The NR separable potential has no short-range repulsion.

The second model is based on the two-body Dirac equation of Sec. IV. For the potential U we assume a combination of Lorentz scalar $S(r)$, vector $V(r)$, and pseudovector $V_\pi(r)$;

$$\begin{aligned} U=&\beta_1\beta_2S(r)+(1-\boldsymbol{\alpha}_1\cdot\boldsymbol{\alpha}_2)V(r)\\ &-\tfrac{1}{3}(\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2-\Gamma_1\Gamma_2)V_\pi(r)\ , \end{aligned} \tag{5.1}$$

where $r=|\mathbf{r}_1-\mathbf{r}_2|$ and $\Gamma=-i\alpha_x\alpha_y\alpha_z(=\gamma^5)$ [14]. Then U_1, U_2, and U_3 of Eqs. (4.3)–(4.5) are given by

$$U_1=S+4V+\tfrac{4}{3}V_\pi\ , \tag{5.2}$$

$$U_2=S-2V+\tfrac{2}{3}V_\pi\ , \tag{5.3}$$

$$U_3=-S\ . \tag{5.4}$$

For the radial dependence of the potential we assume [15]

$$S(r)=-g_s\exp[-(r/a_s)^2]\ , \tag{5.5}$$

$$V(r)=g_v\exp[-(r/a_v)^2]\ , \tag{5.6}$$

$$V_\pi(r)=-g^2[\exp(-m_\pi r)-\exp(-\Lambda r)]/r\ . \tag{5.7}$$

The V_π with $g^2=0.08$ and $m_\pi=138$ MeV is a cutoff one-pion-exchange potential. For the cutoff parameter Λ we arbitrarily assume $\Lambda = m/2$.

We solve the two-body Dirac equation without any approximation. The parameters in S and V are fixed such that there is a bound state of the deuteron binding energy 2.225 MeV ($\alpha = 0.04866m$), and that the ^3S scattering phase shift is well fitted as shown in Fig. 1. The values of the parameters of S and V, i.e., a_s, a_v, g_s, g_v, together with the calculated values of a, r_D, r_0, P, J, and c_3 are listed in Table I. The potential is similar to that of model $R1$ of Ref. [2] (which is the same as model A of Ref. [8]) in which $a_s = 5/m$, $a_v = 3/m$, $g_s = 0.37m$, $g_v = 0.46487$ [16]. The two-body Dirac equation can be rewritten into the form of a Schrödinger-like equation [15]. The effective potential W that appears in the Schrödinger-like equation of the present model is similar to the W of the one-dimensional model shown in Fig. 2 of Ref. [8]. The W is energy dependent. At low energies, W is almost entirely attractive, and is much softer than the nucleon-nucleon potential of the usual NR models.

In expansion (4.15), if we retain the terms up to $c_3(\alpha r_0)^3$, we find $8(r_D/a)^2 = 1.0300$, which can be compared with the exact calculated value of $8(r_D/a)^2 = 1.0305$. Hence the truncation error is 0.0005. Let us look into some details of the term in the square brackets of expansion (4.15). The relativistic terms of order α^2 is $3\alpha^2/m^2 = 0.0071$. The shape-dependent part of $c_3(\alpha r_0)^3$ is $-(J/2)(\alpha r_0)^3 = -0.0287$, which is not much smaller than the shape-independent term of order α^2, i.e., $(\alpha r_0)^2/4 = 0.0381$.

In Ref. [2] a variety of one-dimensional models, relativistic as well as NR, were examined. It was found that relativistic corrections on r_D/a were altogether negligible. This was surprising in the following sense. In contrast to the usual NR models in which the potential has a strong short-range repulsion, the effective potential W of the Schrödinger-like equation of the relativistic models of Ref. [2] is very soft. It was thought that the absence of the strong repulsion at short distances would result in a smaller size of the bound state as compared with the NR models. What actually happens is this. The value of J is larger in relativistic models than in the NR models; this indeed results in a smaller radius of the bound state. For r_D/a, however, the relativistic correction α^2/m^2 (in one dimension) nearly compensates for the increase in J. As we noted earlier, the relativistic correction of order α^2 is $3\alpha^2/m^2$ in three dimensions. This correction in three dimensions tends to overcompensate for the change in J, and shifts the (r_D, a) slightly to the right.

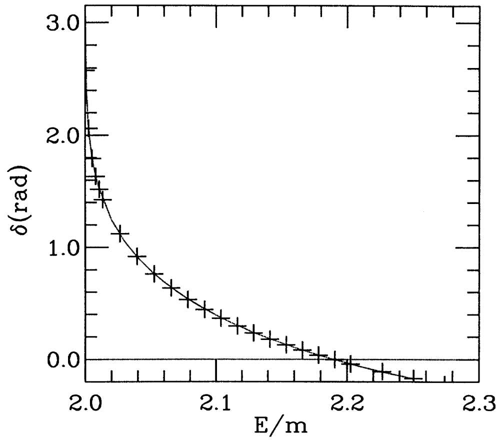

FIG. 1. The triplet S phase shift δ in radians versus the center-of-mass energy (including the rest mass) in units of $m = 939$ MeV. The experimental phase shift is indicated by the crosses. The line represents the calculated phase shift.

VI. SUMMARY

We examined the ratio $(r_D/a)^2$ for two relativistic models, I and II. Model I is based on the one-body Dirac equation and model II on the two-body Dirac equation. We obtained Eqs. (3.15) and (4.15) for models I and II, respectively. In both cases relativistic corrections begin to appear in order α^2; $3\alpha^2/2\mu^2$ and $3\alpha^2/m^2$ in the square brackets of Eqs. (3.15) and (4.15), respectively Since $\mu = m/2$, this correction term for I is twice as large as that of II. We also pointed out that, if one uses one-dimensional models as those of Ref. [2], the correction terms of order α^2 are reduced by a factor of 3 in both models.

The expansions we obtained are relativistic generalizations of Eq. (1.1). These formulas explain why the ratio r_D/a is not very sensitive to the details of the potential. In Sec. V we examined an explicit model based on the three-dimensional two-body Dirac equation. We discussed the mechanism which underlies the negligible relativistic effects on r_D/a found in the one-dimensional models of Ref [2]. In the three-dimensional model, the relativistic effects on r_D/a are slightly larger. We should

TABLE I. The parameters of the potential a_s, a_v, g_s, g_v, and various calculated quantities of the two-body model of Sec.. V. The values of a_s, a_v, a, r_D, and r_0 are in units of the nucleon Compton wavelength $1/m = 0.2101$ fm, and g_s, g_v, and α are in units of the nucleon mass $m = 939$ MeV. The parameters P, J, and c_3 are dimensionless.

a_s	a_v	g_s	g_v	$10^2\alpha$	a
4.6	3.0	0.356 62	0.439 02	4.866	25.54

r_D	r_0	$10^3 P$	J	c_3
9.168	8.024	−7.600	0.9626	−0.2543

add that the two-body calculation of Sec. V is only a simulation of the deuteron; we used the singlet state for simplicity. It would be interesting to do calculations for the triplet state.

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