PHYSICAL REVIEW C VOLUME 44, NUMBER 6 DECEMBER 1991

Deviation of the $SU_q(2)$ prediction from observations in even-even deformed nuclei

J. Meng, C. S. Wu, and J. Y. Zeng*
China Center of Advanced Science and Technology (World Laboratory), Center of Theoretical Physics, P.O. Box 8730, Beijing 100080, China
and Department of Physics, Peking University, Beijing 100871, China
and Institute of Theoretical Physics, Chinese Academy of Science, Beijing 100080, China
(Received 15 February 1991)

The validity of the quantum group $SU_q(2)$ expression for the nuclear rotational spectrum is investigated thoroughly. Analysis [including the Mallmann plots, the relations for the $I(I+1)$ expansion coefficients, energy spectra, etc.] definitely display a systematical deviation of the $Su_q(2)$ prediction from the experimental data available for the even-even rare-earth and actinide nuclei. Only within a limited range of angular momentum the $Su_q(2)$ expression is suitable for rotational spectra. A significant angular momentum dependence of the q-deformation parameter is found. The q deformation is directly related to the nuclear softness.

I. INTRODUCTION

Recently, the quantum group or the q deformation of the universal enveloping algebras [1–3] has been attracting much interest in physics [4,5]. In Refs. [6,7], it was suggested that the spectra of rotational nuclei can be described by a Hamiltonian proportional to the second-order Casimir operator of the quantum algebra $SU_q(2)$. Therefore, it is worth checking to what extent the experimental data available on the even-even deformed nuclei can be reproduced by the quantum group.

$SU_q(2)$ is a q deformation of the Lie algebra of SU(2) [1–6] and is generated by the Hermitian operators, J_-, J_0, and J_+, which obey the commutation relations

$$[J_0,J_\pm]=\pm J_\pm,\quad [J_+,J_-]=[2J_0]\ , \tag{1}$$

where $[x]$ is the q number defined as

$$[x]=\frac{q^x-q^{-x}}{q-q^{-1}} \tag{2}$$

or equivalently in terms of $\gamma=\ln q$,

$$[x]=\frac{e^{\gamma x}-e^{-\gamma x}}{e^{\gamma}-e^{-\gamma}}=\frac{\sinh\gamma x}{\sinh\gamma}\ . \tag{3}$$

In the limit $q\to 1$ (i.e., $\gamma\to 0$), $[x]\to x$ and the $SU_q(2)$ is reduced to the usual SU(2). The irreducible representation of $SU_q(2)$ may be determined by the highest weight state $|IM=I\rangle$ with $J_+|II\rangle=0$ and $\langle II|II\rangle=1$, and the basic states $|IM\rangle$ $(I\geq M\geq -I)$ are expressed as

$$|IM\rangle=\left[\frac{[I+M]!}{[2I]![I-M]!}\right]^{1/2}J_-^{I-M}|II\rangle\ . \tag{4}$$

The second-order Casimir operator of $SU_q(2)$ is

$$C_2^q=J_-J_++[J_0][J_0+1] \tag{5}$$

with eigenvalue $[I][I+1]$.

A q rotor is a system with Hamiltonian

$$H=\frac{\hbar^2}{2\mathcal{J}^{(0)}}C_2^q+E_0\ , \tag{6}$$

where $\mathcal{J}^{(0)}$ is the moment of inertia for $q=1$ $(\gamma=0)$ and E_0 the bandhead energy which is chosen as zero for the ground-state band of an even-even nucleus. $\mathcal{J}^{(0)}$ and E_0 are regarded as constants in the model. Therefore, the rotational energy spectrum can be expressed as [6]

$$E=\frac{\hbar^2}{2\mathcal{J}^{(0)}}\frac{\sin(I|\gamma|)\sin[(I+1)|\gamma|]}{\sin^2|\gamma|}+E_0\ , \tag{7}$$

where a pure imaginary $\gamma(\equiv\ln q)=i|\gamma|$ is assumed.

The purpose of this paper is to examine systematically the validity of the q-deformation expression (7) for nuclear rotational spectra. In Sec. II the q-deformation expression is analyzed by using the Mallmann plot [8]. In Sec. III the relations for the $I(I+1)$ expansion coefficients of Eq. (7) are investigated and compared with the results obtained by using the least-squares fitting for the rotational spectra observed in the rare-earth and actinide nuclei. The comparison of the calculated energy levels with the experimental data is given in Sec. IV. The results show that only in a limited range of angular momentum the $SU_q(2)$ expression is suitable for rotational spectrum. Also the variation of q-deformation parameter $|\gamma|$ with angular momentum is investigated. In Sec. V the connection of Eq. (7) with the variable moment of inertia (VMI) model [9] is addressed.

II. MALLMANN PLOTS

Many years ago Mallmann [8] pointed out that the plots of $R_I\equiv(E_I-E_0)/(E_2-E_0)$ (or $r_I\equiv R_I-R_{I-2}$) vs R_4 showed a remarkably smooth trend. The advantage of the Mallmann plot was emphasized in Ref. [10] and was used to test the applicability of each recipe of the two-parameter VMI model (including the Harris two-parameter formula as a special case). For each two-parameter formula of rotational band, the Mallmann plot is unique and does not depend on the parameter values

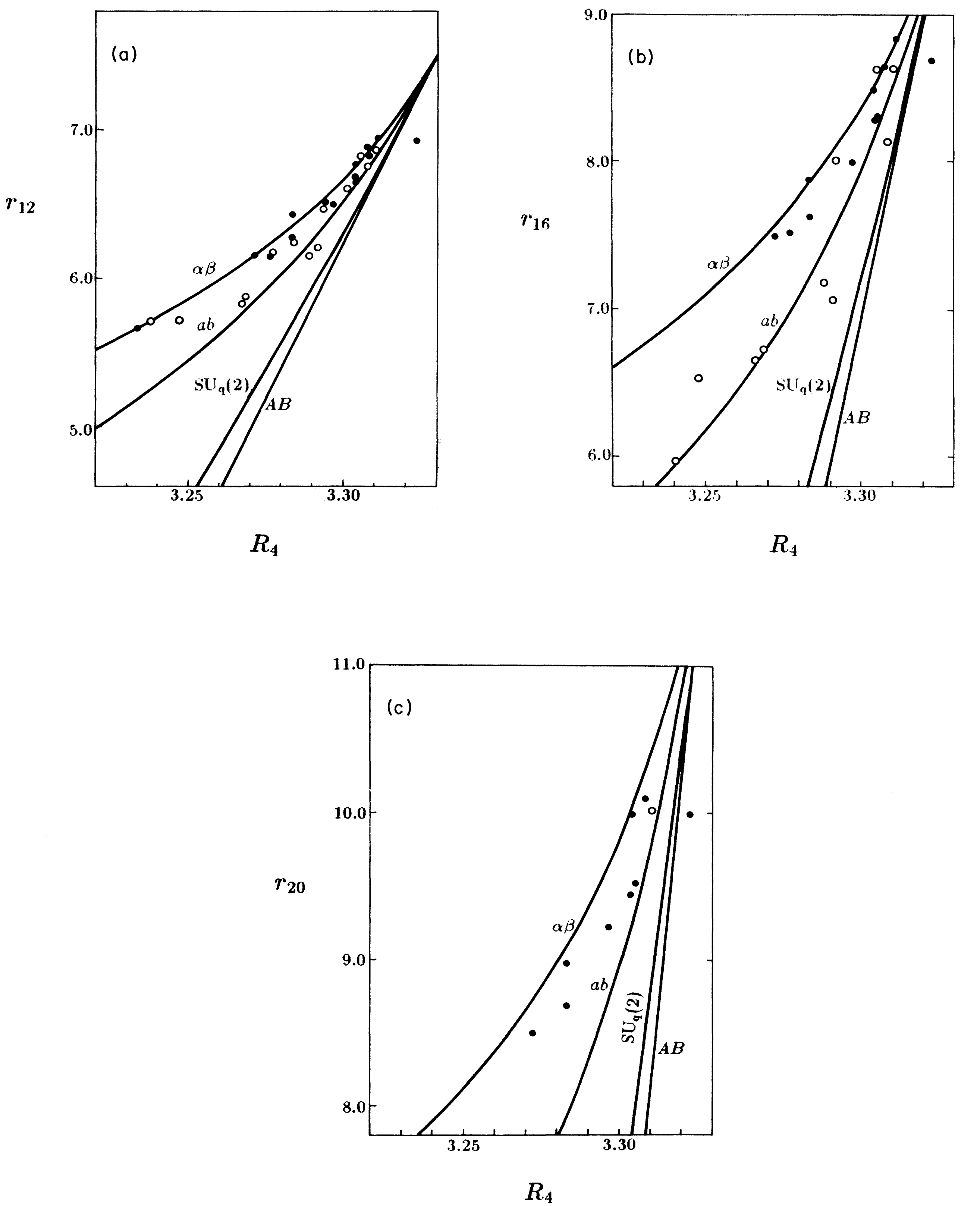

FIG. 1. The Mallmann plots for the $SU_q(2)$, AB, $\alpha\beta$, ab, expressions. The experimental values (taken from Ref. [14] and the references therein) are denoted by closed circles for the actinide nuclei and open circles for the rare-earth nuclei. (a) $I=12$. (b) $I=16$. (c) $I=20$.

involved. The Mallmann plot can be compared directly with the experimental data of rotational spectra over the entire range of nuclei. Thus, the Mallmann plot can give a clearcut picture of the relative success of each formula for rotational spectrum.

From Eq. (7) we have

$$R_I = \frac{\sin I|\gamma|\sin(I+1)|\gamma|}{\sin 2|\gamma|\sin 3|\gamma|} , \tag{8}$$

from which the Mallmann plot for the q rotor can be constructed. As illustrative examples the Mallmann plots for $I=12$, 16, and 20 are given in Figs. 1(a)–(c). For comparison, also given are the corresponding plots for the other two-parameter expressions of nuclear rotational spectra, namely, the two-parameter $I(I+1)$ expansion [11]

$$E = AI(I+1)+BI^2(I+1)^2 , \tag{9}$$

the Harris two-parameter ω^2 expansion [12]

$$E=\alpha\omega^2+\beta\omega^4 , \tag{10}$$

and the Wu-Zeng two-parameter closed expression [13]

$$E=a[\sqrt{1+bI(I+1)}-1] \tag{11}$$

derived from the Bohr Hamiltonian under certain approximation. These expressions are briefly referred to as the AB, $\alpha\beta$, and ab ones, respectively.

All the data now available for the ground rotational bands of even-even actinide and rare-earth nuclei (with band-crossing angular momentum $I_c \geq 16$) are displayed in the figures. The data are taken from Ref. [14] and the references therein. It can be seen that, as expected, the Mallmann plots for the $SU_q(2)$ expression (7) are better than those for the AB expression (10) because the higher power $I(I+1)$ terms have been included in Eq. (7) [see Eq. (14) below]. Physically, this fact perhaps implies that the effects of the vibration and other degrees of freedom may be accounted for, at least partly, by the q deformation. However, from the analyses of the abundant data on nuclear spectra, all the observed points (except one) lie on the left-hand side of the Mallmann plots for $SU_q(2)$, that is, for given R_4 (i.e., given $|\gamma|$) the calculated r_I's by using $SU_q(2)$ expression (8) are systematically smaller than the observed values. Therefore, the Mallmann plots for the nuclear rotational spectra definitely provide rather convincing evidence that, statistically speaking, the $SU_q(2)$ rotor prediction deviates from the abundant observed data and needs further improvement.

III. RELATIONS FOR THE $I(I+1)$ EXPANSION COEFFICIENTS

On the basis of symmetry consideration, Bohr and Mottelson [11] pointed out that the rotational energy of an even-even deformed nucleus can be expanded in powers of $I(I+1)$:

$$E = AI(I+1)+BI^2(I+1)^2+CI^3(I+1)^3 + DI^4(I+1)^4+ \cdots . \tag{12}$$

However, because only two parameters appears in Eqs. (7), (10), or (11), only two coefficients in their $I(I+1)$ expansion are independent and a series of relations among the coefficients are expected. Just as pointed out by Mottelson [15], the investigation of the relations for the $I(I+1)$ expansion coefficients is helpful for providing a test on the expression of rotational spectra. For example, if the Harris ($\alpha\beta$) expression (10) is expanded up to the $I^4(I+1)^4$ term, we may get

$$\frac{AC}{4B^2}=1, \quad \frac{A^2D}{24B^3}=1 . \tag{13}$$

For the $SU_q(2)$ expression, expanding the functions in the numerator of the right-hand side of Eq. (7) and collecting together the terms of the same power of $I(I+1)$, one may get

$$E=\frac{\hbar^2}{2\mathcal{J}^{(0)}}\frac{1}{[j_0(|\gamma|)]^2}[j_0(|\gamma|)I(I+1)-|\gamma|j_1(|\gamma|)I^2(I+1)^2+\tfrac{2}{3}|\gamma|^2J_2(|\gamma|)I^3(I+1)^3-\tfrac{1}{3}|\gamma|^3j_3(|\gamma|)I^4(I+1)^4+\cdots] , \tag{14}$$

TABLE I. The relations of the $I(I+1)$ expansion coefficients in various expressions for rotational spectra.

	$SU_q(2)$ expression	ab expression[a]	$\alpha\beta$ expression[b]
$\frac{AC}{4B^2}$	$\frac{1}{10}-\frac{2}{525}\|\gamma\|^2+\cdots$	$\frac{1}{2}$	1
$\frac{A^2D}{24B^3}$	$\frac{1}{280}-\frac{1}{3150}\|\gamma\|^2+\cdots$	$\frac{5}{24}$	1
$\frac{AC}{4B^2}-\frac{A^2D}{24B^3}$	$\frac{27}{280}-\frac{11}{3150}\|\gamma\|^2+\cdots$	$\frac{7}{24}$	0

[a]Reference 14.
[b]Reference 11.

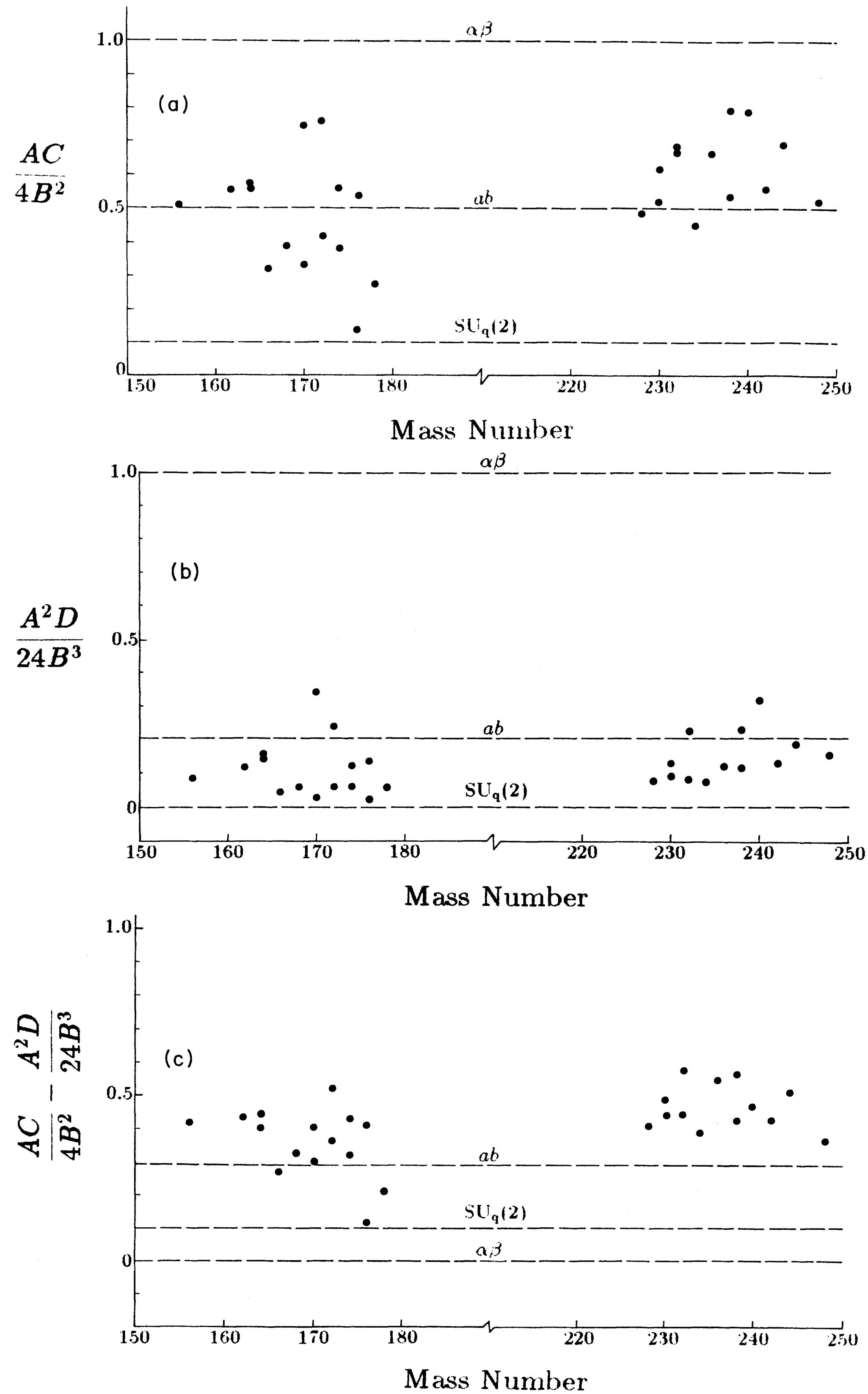

FIG. 2. The relations for the $I(I+1)$ expansion coefficients. The coefficients are determined by the least-squares fitting of the rotational energy levels (below $20\hbar$ for actinide nuclei and $16\hbar$ for rare-earth nuclei). (a) $AC/4B^2$, (b) $A^2D/24B^3$, (c) $AC/4B^2 - A^2D/24B^3$.

where $j_n(|\gamma|)$ is the n-order spherical Bessel function. The relations among the first four coefficients in the $I(I+1)$ expansion are given in Table I. For comparison, the similar relations for the $\alpha\beta$ expression (10) and the ab expression (11) are also shown there.

Now the data on the ground rotational bands of even-even deformed actinide and rare-earth nuclei ($I_c \geq 16$) are employed to determine the first four coefficients in the expansion (12) by using the least-squares fitting. Thus the "experimental" values of $AC/4B^2$ and $A^2D/24B^3$ and their differences can be obtained. The results are shown in Figs. 2(a), (b), and (c), respectively. It can be seen that the $SU_q(2)$ values of $AC/4B^2$ and $A^2D/24B^3$ are systematically smaller than the experimental values. To account for this fact, the values of higher coefficients in the $I(I+1)$ expansion, C and D, should be larger relative to those predicted by the $SU_q(2)$ theory [see Eq. (14)]. It is interesting to note that the observed results can be accounted for quite well by the ab expansion (11), which may provide some useful indication for further improvement of $SU_q(2)$ deformation theory.

IV. ENERGY SPECTRA

Now the $SU_q(2)$ expression (7) is applied to fit the observed rotational spectra of well-deformed nuclei to see how well the experimental data can be reproduced. As two illustrative examples, the calculated results for ^{238}U and ^{174}Yb are displayed in Fig. 3, where for comparison, are also given the AB and ab fits by using Eqs. (10) and (11), respectively. The analyses for other deformed nuclei are similar but omitted here to save space. It is found that, as expected, the $SU_q(2)$ fit is better than the AB fit, because the higher power $I(I+1)$ terms have been involved in the $SU_q(2)$ model. The deviation, $\Delta E \equiv |E_I^{\rm expt} - E_I^{\rm calc}|$, is less than 22 keV for ^{238}U ($I \leq 26$) and 8 keV for ^{174}Yb($I \leq 20$). However, the $SU_q(2)$ fit is worse than the ab fit.

It is well-known experimentally that within a rotational band, not only the energy E_I, but also the γ transition energy between the adjacent levels, $E_\gamma(I) \equiv E_I - E_{I-2}$ is a monotonic increasing function of angular momentum I. From Eq. (7), we have

$$E_\gamma(I) = \frac{\hbar^2}{\mathcal{J}^{(0)}} \cot|\gamma| \sin(2I-1)|\gamma| \ . \tag{15}$$

Therefore, to account for the observed rotational spectra, the following requirement has to be imposed:

$$0 \leq (2I-1)|\gamma| < \pi/2 \ , \tag{16}$$

i.e., the $SU_q(2)$ expression (7) is suitable only for $I \leq I_{\rm max}$,

$$I_{\rm max} = \text{integer part of } \left[\frac{\pi}{4|\gamma|} + \frac{1}{2} \right] . \tag{17}$$

For the well-deformed even-even nuclei, $|\gamma| \sim 0.03-0.06$, hence $I_{\rm max} \sim 28-16$. When $\pi/2 \leq (2I-1)|\gamma| \leq \pi$, $E_\gamma(I)$ becomes decreasing with I. Furthermore, when $\pi \leq (2I-1)|\gamma| \leq 2\pi$, negative values of $E_\gamma(I)$ would be obtained. However, experimentally there seems no indication to reveal such a tendency.

In fact, when I approaches $I_{\rm max}$, the calculated results using the $SU_q(2)$ expression (7) deviate farther and farther from the experimental values. As shown in Fig. 4(a), when $I > 20$, the calculated transition energies for ^{238}U decrease with increasing I, which is definitely opposite to the observed behavior. A similar discrepancy can be found in other well-deformed nuclei. For example, see Fig. 4(b) for ^{170}Hf.

Furthermore, for an ideal q rotor, the value of $|\gamma|$ should be a constant independent of angular momentum. The value of $|\gamma|$ can be derived directly from the observed R_I value by using Eq. (8). In Fig. 5, the $|\gamma|$ values for four typical actinide nuclei are plotted against the angular momentum. The situation is similar for other deformed nuclei. As shown in the figure, the q-deformation parameter $|\gamma|$ definitely is not a constant, but varies, sometimes sharply, with angular momentum. The I dependence of $|\gamma|$ implies that some other effects have not been considered in such a simple model Hamiltonian, (6). Therefore, to account for the observed data,

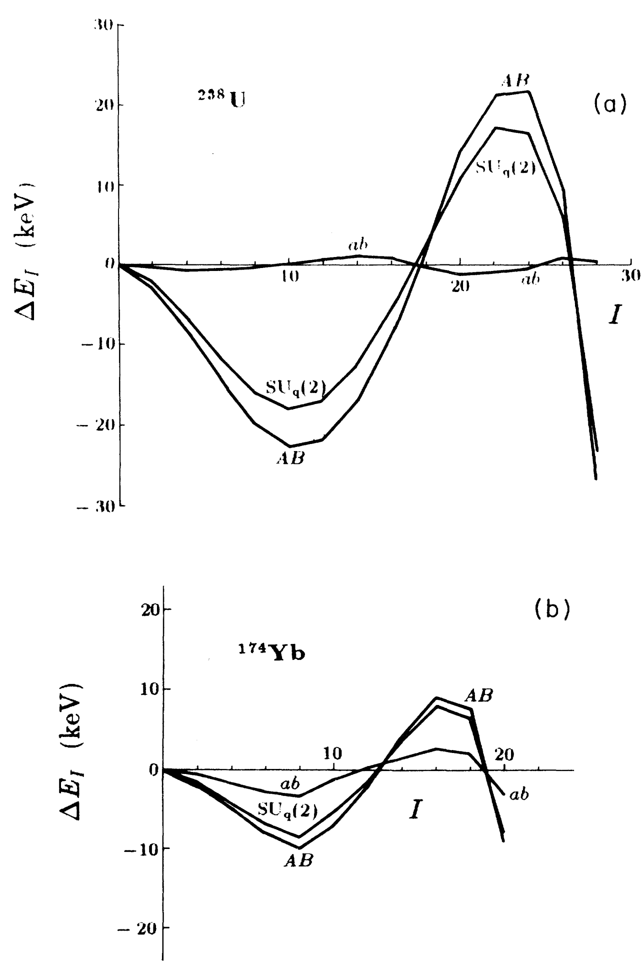

FIG. 3. $\Delta E_I = E_I^{\rm calc} - E_I^{\rm expt}$ vs I for two typical well-deformed nuclei. (a) ^{238}U, (b) ^{174}Yb.

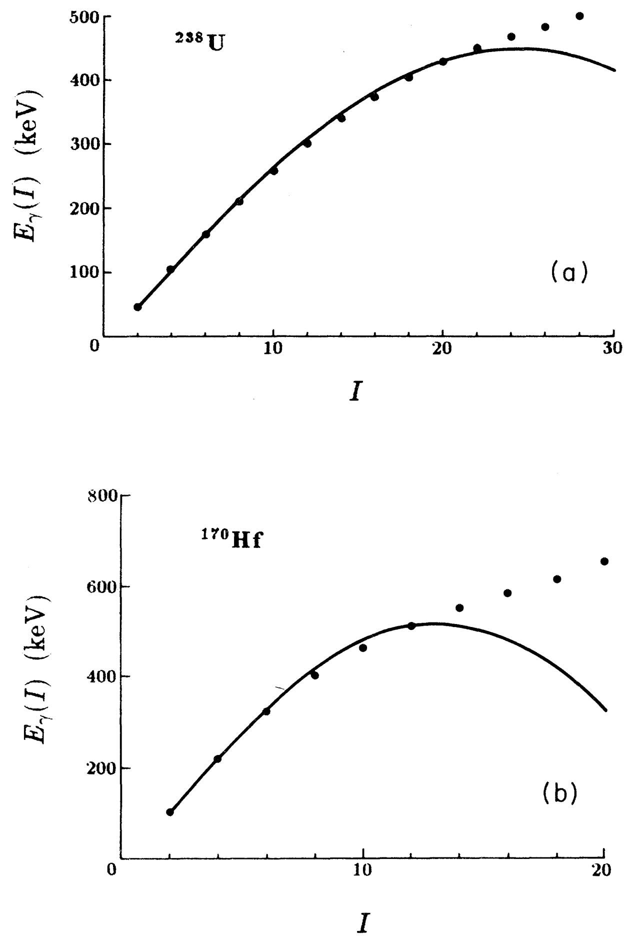

FIG. 4. The variation of transition energy, $E_\gamma(I)$, with I. The curves stand for the calculated $SU_q(2)$ results and the dots for the experimental data. (a) ^{238}U, (b) ^{170}Hf.

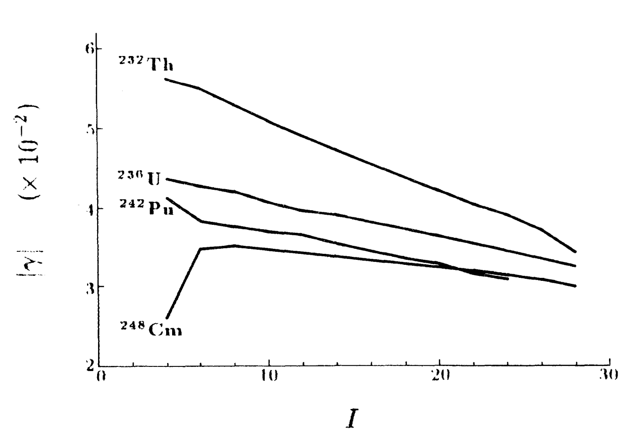

FIG. 5. The variation of the q-deformation parameter, $|\gamma|$, with the angular momentum.

it appears that an additional residual interaction needs to be introduced into Hamiltonian (6), or another kind of quantum algebra than $SU_q(2)$ should be considered.

V. MEANING OF PARAMETERS IN $SU_q(2)$ EXPRESSION

In this section we shall discuss the physical implication of the parameters in $SU_q(2)$ expression (7).

According to Eq. (7), the rotational energy deviates from the simple $I(I+1)$ rule valid for a rigid rotor. This deviation, as pointed out by Bohr and Mottelson [11], may be viewed as a dependence of the moment of inertia on the rotational angular momentum.

As usual in the cranked shell model, the nuclear moment of inertia is defined as [16]

$$\mathcal{J}_I=\frac{\hbar\sqrt{I(I+1)}}{\omega} \tag{18}$$

with

$$\omega=\frac{1}{\hbar}\frac{dE}{d\sqrt{I(I+1)}} \; . \tag{19}$$

From Eq. (7), one may get

$$\mathcal{J}_I=\mathcal{J}^{(0)}\frac{(2I+1)\sin^2|\gamma|}{|\gamma|\sin(2I+1)|\gamma|} \; , \tag{20}$$

which means that the nuclear moment of inertia is an increasing function of I

$$\mathcal{J}_I=\mathcal{J}_0\frac{(2I+1)\sin|\gamma|}{\sin(2I+1)|\gamma|} \; , \tag{21}$$

where $\mathcal{J}_0$ is the ground-state moment of inertia

$$\mathcal{J}_0=\mathcal{J}^{(0)}\frac{\sin|\gamma|}{|\gamma|} \; , \tag{22}$$

which differs from $\mathcal{J}^{(0)}$ by a reduction factor of $\sin|\gamma|/|\gamma|$ due to q-deformation. When $|\gamma|\to 0$, $\mathcal{J}_0\to\mathcal{J}^{(0)}$. The numerical calculation shows that when $I\sim I_{max}$ [i.e., $(2I+1)|\gamma|\sim\pi/2$], $(\mathcal{J}_I-\mathcal{J}_0)/\mathcal{J}_0\sim 0.7$, which seems to be smaller than the observed results [17] $(\mathcal{J}_I-\mathcal{J}_0)/\mathcal{J}_0\sim 1$ for $I\sim 20$–30.

Another interesting quantity is the softness of nucleus, which is defined as [9]

$$\sigma=\frac{1}{\mathcal{J}_I}\frac{d\mathcal{J}_I}{dI}\bigg|_{I=0} \; . \tag{23}$$

From Eq. (20), we have

$$\begin{aligned}\sigma&=2(1-|\gamma|\cot|\gamma|)\\&=\tfrac{2}{3}|\gamma|^2(1+\tfrac{1}{15}|\gamma|^2+\tfrac{2}{315}|\gamma|^4+\cdots)\\&\approx\tfrac{2}{3}|\gamma|^2 \; .\end{aligned} \tag{24}$$

Therefore, the q-deformation is directly related to the softness of the nucleus. For the usual value of $|\gamma|\sim 0.03$–0.06 σ ranges from 6×10^{-4} to 2.4×10^{-3}. These values seem rather smaller than those adopted in the usual VMI model [9].

ACKNOWLEDGMENTS

The authors are grateful to Professor X. C. Song for informing them of the recent progress of the quantum group theory and for valuable discussions. This work was supported by the Doctoral Program Foundation of Institution of Higher Education of China, the National Natural Science Foundation of China, and the Science Foundation of Peking University.

---

*Mailing address: Department of Physics, Peking University, Beijing 100871, China.

[1] L. D. Faddeev, in *Integrable Models,* Les Houches Lectures, 1982, edited by J. B. Zuber and R. Stora (Elsevier, Amsterdam, 1984).
[2] W. G. Drinfeld, Quantum Groups, Proc. ICM (Berkeley, 1986), pp. 798-820.
[3] M. Jimbo, Lett. Math. Phys. **10**, 63 (1985).
[4] L. G. Biedenharn, J. Phys. A **22**, L873 (1989); A. J. Macfarlane, *ibid.* **22**, 4581 (1989); C. P. Sun and H. C. Hu, *ibid.* **22**, L983 (1989); Y. J. Ng, *ibid.* **23**, 1023, (1990).
[5] X. C. Song, J. Phys. A **23**, L821 (1990); B. Y. Hou, B. Y. Hou, and Z. Q. Ma, Commun. Theor. Phys. (Beijing) **13**, 181 (1990).
[6] P. P. Raychev, R. P. Roussev, and Yu. F. Smirnov, J. Phys. G **16**, L137 (1990).
[7] Dennis Bonatsos, E. N. Argyres, S. B. Drenska, P. P. Raychev, R. P. Rousev, and Yu. F. Smirnov, Phys. Lett. B **251**, 477 (1990).
[8] C. L. Mallmann, Phys. Rev. Lett. **2**, 507 (1959).
[9] M. A. J. Mariscotti, G. Scharff-Goldhaber, and B. Buck, Phys. Rev. **178**, 1864 (1969); G. Scharff-Goldhaber, C. Dover, and A. L. Goodman, Annu. Rev. Nucl. Sci. **26**, 239 (1976).
[10] J. L. Wood and R. W. Fink, Nucl. Phys. **A224**, 589 (1974).
[11] A. Bohr and B. R. Mottelson, *Nuclear Deformations, Vol. II of Nuclear Structure* (Benjamin, New York, 1975).
[12] S. M. Harris, Phys. Rev. Lett. **13**, 663 (1964); Phys. Rev. B **138**, 589 (1965).
[13] C. S. Wu and J. Y. Zeng, Commun. Theor. Phys. (Beijing) **8**, 51 (1987); H. X. Huang, C. S. Wu, and J. Y. Zeng, Phys. Rev. C **39**, 1617 (1989).
[14] F. X. Xu, C. S. Wu, and J. Y. Zeng, Phys. Rev. C **40**, 2337 (1989).
[15] B. R. Mottelson, Proceedings of the Nuclear Structure Symposium of Thousand Lakes, Joutsa, 1970. [Nordisk Institut for Theoretisk Atomfysik, Nordita, Report No. 417, 1971 (unpublished)].
[16] R. Bengtsson and S. Frauendorf, Nucl. Phys. **A327**, 139 (1979).
[17] M. J. de Voigt, J. Dudek, and Z. Szymański, Rev. Mod. Phys. **55**, 949 (1983).