Force-range correction in the three-body problem: Application to three-nucleon systems

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A leading correction to the zero-range theory is considered for three-body systems. The correction is linear in the force range r_0. An explicit expression is obtained for this correction for the case of correlation between the three-body binding energy and the particle-pair scattering length, as well as for the practically interesting case of correlation between the triton binding energy and the neutron-deuteron doublet scattering length. The correction for the neutron-deuteron quartet scattering length is also found. Physics of the correction involves a modification of the effective long-range interaction $1/R^2$, which arises in the three-body systems under the conditions of the zero-range theory, by a singular correction r_0/R^3.

I. INTRODUCTION

During the early years of nuclear physics the zero-range theory played a useful role in understanding of the two-nucleon systems [1]. A similar approach for the three-nucleon systems was initiated in the fifties and sixties [2,3] but was utilized relatively rarely owing to insufficient understanding of its features (see Ref. [4] for the historical review). The unclear points were worked through with the time, and at present this approach seems to experience its revival. The new development has already led to a number of interesting results [4–7]. In particular, the explanation has been found for a once mysterious correlation, called the Phillips line, between the calculated values of the triton binding energy and the *nd* doublet scattering length.

The approach referred to shows in an explicit way that many properties of three-nucleon systems are actually determined by very few two-body and three-body parameters (the universality [4,5]). Therefore, the approach is a rather easy and concise way of qualitative understanding of both experimental data and results of numerical three-body calculations with model nuclear forces. Moreover, it provides a "background" that should be subtracted from experimental values of the three-body observables in order to get a better access to a really new and nontrivial information about the two- and three-nucleon systems.

The success of the zero-range theory warrants attempts to calculate a correction due to the finiteness of nuclear force range. For the two-nucleon system this correction, which is linear in the force range, is well known. The present paper concerns with the derivation of this correction for the case of three-nucleon systems.

This problem was also first posed in the sixties. Danilov [8] obtained an integral equation for the correction, which was numerically solved in Ref. [9] (see also Ref. [10]) for the relatively simple case of *nd* quartet scattering. The basic result of the present work is that the correction can in fact be found in an explicit analytical form.

This result seems to look rather natural. Indeed, as is known from the potential theory, the first perturbative correction to the binding energy and scattering length are both expressible in terms of unperturbed wave functions and perturbation potential. The situation in the present problem turns out the same, and we have derived the expressions of this sort. The peculiarity of our problem is that the unperturbed potential and perturbation are derived from the three-body theory and turn out singular at small distances (see below). Therefore, the derivation of final results is not just a trivial exercise in perturbation theory—it requires a clear understanding of the physics involved.

Sections II–IV deal with three spinless particles; Sec. V with three nucleons. The masses of particles are assumed to be unity.

II. THE STARTING EQUATION

We consider the symmetric *S*-wave state of three identical particles. This example enables us to demonstrate all principle steps of derivation of the linear correction. As was mentioned in the Introduction, the integral equation for the linear correction was first derived in Ref. [8]. On the basis of present-day experience, this equation can be understood in rather simple terms. Indeed, if we start with the Faddeev equation for our system and keep only first two terms of the expansion of the two-body t matrix in the force range [11], we will arrive at the equation of Ref. [8]. For the case of particle-pair scattering the equation reads (in comparison with Ref. [8] we slightly changed the notations and used a different grouping of terms)

$$\chi(p)=f\left[\left[E-\frac{3}{4}p^2\right]^{1/2}\right]\left[p_0^{-1}\ln\frac{p^2+p_0^2+pp_0-E}{p^2+p_0^2-pp_0-E}+\frac{2}{\pi}\int dp'\left[\ln\frac{p^2+p'^2+pp'-E}{p^2+p'^2-pp'-E}\right]\chi(p')\right] . \quad (1)$$

Here E is the energy of the three-body system; p_0 is the momentum of an incident particle in the center-of-mass system; $E=\frac{3}{4}p_0^2-\epsilon$, where ϵ is the absolute value of the pair binding energy. The function f is the two-body amplitude

$$f\left[\left(E-\frac{3}{4}p^2\right)^{1/2}\right]=[-\alpha+\gamma(p)]^{-1}+\frac{r_0}{2}\frac{\alpha+\gamma(p)}{-\alpha+\gamma(p)} , \tag{2}$$

where $\alpha=\sqrt{\epsilon}$, the r_0 is the two-body effective range and $\gamma(p)=(\frac{3}{4}p^2-E)^{1/2}$. The on-shell particle-pair scattering amplitude is related to $\chi(p)$ in the following way:

$$a(p_0)=\lim_{p\to p_0} p^{-1}(p^2-p_0^2)\chi(p) . \tag{3}$$

To describe the bound state of the three particles, one should omit the Born term on the right-hand side of Eq. (1).

Equation (2) is a leading piece of the standard low-energy expansion of the two-body amplitude. The function f depends on the relative two-body momentum $k=(E-\frac{3}{4}p^2)^{1/2}=i\gamma(p)$ expressed in terms of the energy E and the virtual momentum p of the third particle [12]. In the present paper we are dealing with the energy interval $E<0$, i.e., with the three-body bound states and the particle-pair scattering below the three-particle threshold. For these energies the momentum $\gamma(p)$ is real and positive.

If we retain only the first term in expression (2) for f, we obtain the zero-range theory [2,3]. The second term proportional to r_0 presents a small perturbation. It gives rise to a corresponding perturbation of the wave function $\chi(p)$ and to corrections to both the scattering amplitude and the three-body binding energy. Our aim will be the derivation of these corrections.

To solve Eq. (1), various procedures can be used. In Ref. [8] a boundary condition on the wave function $\chi(p)$ was imposed at a certain large momentum $p\gg(\alpha,\sqrt{|E|})$ (yet $pr_0\ll1$). The solution of Eq. (1) turns out to be a function of an additional parameter that comes with this boundary condition. Another procedure, which we actually employ, consists of cutting the integral in (1) at a certain momentum p_c [13]. We choose $p_c\gg(\alpha,\sqrt{|E|})$ (yet $p_cr_0\ll1$) and find the solution as a function of p_c. As was discussed earlier in connection with the zero-range theory [14], both procedures are equivalent. The physics of this equivalence stems from the fact that at low energies $E\ll1/r_0^2$ the specific details of forces between particles are irrelevant. The interaction in the region where all relative distances r_{ik} are of the order of r_0 (this region corresponds to the momenta $p\sim r_0^{-1}$) can, as far as the rest of configuration space is concerned, be parametrized either by a boundary condition on the wave function at the region boundary or by employing a simple model for the forces inside this boundary [the scale R_0 of this boundary is usually selected such that $r_0\ll R_0\ll(\alpha^{-1},|E|^{-1/2})$]. In the momentum representation the first choice corresponds to the procedure of Ref. [8]; the second, our procedure. The simple model employed consists of putting the two-body t matrix equal to zero at $p>p_c$ [13].

The equivalence we are talking about is a characteristic feature of low-energy theories. For example, analyzing the two-body problem at $r_0\sqrt{|E|}\ll1$, one can either set a boundary condition on the wave function at a certain distance r such that $r_0\ll r\ll|E|^{-1/2}$ or solve the problem using a simple potential. Both procedures lead to the same answer.

Note that in our approximation neither the boundary-condition parameter nor the cutoff momentum depend on the energy E. This is due to the fact that both these quantities parametrize the interaction in the region $r_{ik}\sim r_0$ where the energy E can be safely neglected [7].

III. LINEAR CORRECTIONS AT A FIXED CUTOFF MOMENTUM

In this section we derive expressions for the linear corrections to the binding energy of three particles and to the particle-pair scattering length. The cutoff momentum p_c is assumed fixed.

To develop the perturbation theory for Eq. (1), we transform Eq. (1) to a form that looks like the Schrödinger equation for the wave function $\chi(p)$. To this end we multiply both sides of (1) by $(\gamma-\alpha)[1-\frac{1}{2}r_0(\gamma+\alpha)]$ and neglect terms quadratic in r_0. We obtain

$$(\gamma(p)-\alpha)\chi(p)-\frac{2}{\pi}\int dp'\left[\ln\frac{p^2+p'^2+pp'-E}{p^2+p'^2-pp'-E}\right]\left[\frac{\pi}{2p_0}\delta(p'-p_0)+\chi(p')\right]-\frac{r_0}{2}[\gamma^2(p)-\alpha^2]\chi(p)=0 .$$

Since $\gamma(p)-\alpha$ annihilates $\delta(p-p_0)$, we can add $(\pi/2)p_0^{-1}\delta(p-p_0)$ to $\chi(p)$ in the first and last terms of the left-hand side. As a result, we obtain the equation

$$(\gamma-\alpha+V_0+V_1)\psi=0 , \tag{4}$$

where

$$\psi(p)=\frac{\pi}{2p_0}\delta(p-p_0)+\chi(p) ,$$

and the operators V_0 and V_1 are

$$V_0(p,p')=-\frac{2}{\pi}\ln\frac{p^2+p'^2+pp'-E}{p^2+p'^2-pp'-E} ,$$
$$V_1(p)=-\frac{r_0}{2}[\gamma^2(p)-\alpha^2] . \tag{5}$$

This equation is the basis of our further consideration.

Let us discuss the physical meaning of Eq. (4). The equation describes our problem in terms of scattering of a particle on an effective potential. The term $(\pi/2)p_0^{-1}\delta(p-p_0)$ in $\psi(p)$ is the free wave, and the function $\chi(p)$ is the scattering wave. The operator $\gamma-\alpha$ plays the role of $T-E$, and the potential V consists of two parts—the main part V_0 and the perturbation V_1. The potential V_0 is the source of two well-known phenomena of the three-body problem: the Thomas collapse [15] at $p_c\to\infty$ and the accumulation of loosely bound three-body states at $\alpha\to 0$ [16]. In the configuration space its effect is equivalent to an attraction $1/R^2$ cut off at the distances p_c^{-1} and α^{-1} [14,16]. The potential V_1 is proportional to r_0, and its effect we are going to study in the present paper [17].

Equation (4) is amenable to the standard methods of perturbation theory. Let us start with the correction to the binding energy. We write $\chi=\chi_0+\chi_1$, where χ_0 is the solution of (4) at $V_1=0$ and the function χ_1 is proportional to V_1. From Eq. (4) we find the following equation for χ_1:

$$(\gamma-\alpha+V_0)\chi_1+V_1\chi_0+\delta E(\gamma-\alpha+V_0)'\chi_0=0 .$$

Here δE is the energy shift due to V_1, and the prime denotes the derivative with respect to E. Multiplying this equation by χ_0 on the left, we obtain for δE

$$\delta E=\frac{\chi_0 V_1\chi_0}{-\chi_0(\gamma+V_0)'\chi_0} . \tag{6}$$

This expression turns into the standard perturbation-theory formula with the replacement of $\gamma-\alpha$ by $T-E$ and assuming $V_0'=0$. The derivatives of γ and V_0 can be easily calculated, and the explicit expressions will be given later. We have verified that the denominator in (6) is equal to the normalization of the full three-body wave function.

Let us turn to the correction to the scattering amplitude. Proceeding in a similar manner, we write the scattering wave χ in the form $\chi=\chi_0+\chi_1$. From (3) we have for the correction to the amplitude

$$\delta a(p_0)=\lim_{p\to p_0} p^{-1}(p^2-p_0^2)\chi_1(p)$$
$$=\frac{8\alpha}{3p_0}\lim_{p\to p_0}(\gamma-\alpha)\chi_1(p) . \tag{7}$$

The function $\chi_1(p)$ obeys the equation

$$(\gamma-\alpha+V_0)\chi_1+V_1\chi_0=0 ,$$

whose symbolic solution is

$$\chi_1=-(\gamma-\alpha+i0+V_0)^{-1}V_1\chi_0 .$$

Here $i0$ specifies the path of integration near the Green-function pole. To calculate the matrix element

$$\langle p_0|(\gamma-\alpha)(\gamma-\alpha+i0+V_0)^{-1}|\chi_0\rangle$$

needed in (7), we note that the function $(\gamma-\alpha-i0+V_0)^{-1}(\gamma-\alpha)|p_0\rangle$ differs from the unperturbed solution ψ_0 of Eq. (4) (i.e., the solution at $V_1=0$) only by the factor $(2/\pi)p_0$ and the sign in front of $i0$. We write this function in the form

$$(\gamma-\alpha-i0+V_0)^{-1}(\gamma-\alpha)|p_0\rangle=\frac{2p_0}{\pi}|\psi_0^-\rangle ,$$

where the superscript implies that the asymptotic form of χ_0 in ψ_0^- is an ingoing wave: $\psi_0^-=\psi_0^*$. Using this equality, we finally obtain for the $\delta a(p_0)$

$$\delta a(p_0)=-\frac{16\alpha}{3\pi}\langle\psi_0^-|V_1|\chi_0\rangle . \tag{8}$$

Expression (8) looks like the standard scattering-theory formula for the variation of scattering amplitude caused by a small perturbation of a potential. Indeed, we can replace χ_0 with ψ_0 in the ket vector because $V_1\delta(p-p_0)=0$. As a result, we obtain the matrix element $\langle\psi_0^-|V_1|\psi_0\rangle$, which is the standard matrix element for the variation of scattering amplitude. The factor in front of the matrix element is determined by the free-wave normalization and the reduced mass of incident particle.

Expression (8) can be cast in a more convenient form by isolating the contribution from the free wave in ψ_0^-. Using definition (5) of V_1 and the relation (3) between $a_0(p_0)$ and $\chi_0(p)$, we obtain

$$\delta a(p_0)=r_0\alpha a_0(p_0)-\frac{16\alpha}{3\pi}\langle\chi_0^-|V_1|\chi_0\rangle .$$

At $p_0=0$ this expression gives the correction to the particle-pair scattering length. Denoting the scattering length by A [A is equal to $-a(0)$] and writing it in the form $A=A_0+\delta A$, where A_0 is the scattering length corresponding to the case $V_1=0$, we find for the linear correction δA

$$\delta A=r_0\alpha A_0+\frac{16\alpha}{3\pi}(\chi_0V_1\chi_0) . \tag{9}$$

Here we have taken into account that $\chi_0^-=\chi_0$ at $p_0=0$.

On the basis of expressions (6) and (9) we can make two observations concerning the properties of the perturbation. First, it turns out attractive. Indeed, at $p_c\gg(\alpha,\sqrt{|E|})$ the main contribution to the matrix elements $\chi_0V_1\chi_0$ comes from the momenta $p\sim p_c$ because the function $\chi_0(p)$ decreases as p^{-1} at large p (this can be deduced from the equation for ψ_0) and V_1 grows as p^2 [see Eq. (5)]. Both matrix elements linearly increase with growth of p_c and are negative. Therefore, δE and δA are both negative.

Second, this attraction increases at small distances (large momenta). In addition to the fact that $V_1\sim r_0p^2$, one should take into account that at large p the phase space in the normalization integral [see the denominator in (6)] is proportional to $(\gamma+V_0)'\sim p^{-1}$. Therefore, the effect of V_1 is equivalent to the attraction r_0p^3. If estimated in the same manner, the effect of V_0 is equivalent to p^2. Correspondingly, since the potential V_0 is equivalent to the attraction $1/R^2$ in the configuration space, the potential V_1 appears to be equivalent to the attraction r_0/R^3 [18]. Note that despite the fact that V_1 grows faster than V_0 at large p, the effect of V_1 still remains small because the ratio $V_1/V_0 \sim pr_0$ is small at $p<p_c$.

We conclude this section with the detailed expressions for δE and δA. Substituting all quantities into (6) and (9), we obtain

$$\delta E=-\frac{r_0}{2}\int_0^{p_c} dp\left[\frac{3}{4}p^2-E_0-\alpha^2\right]\chi_T^2(p)\left[\frac{1}{2}\int_0^{p_c}\frac{dp\,\chi_T^2(p)}{(\frac{3}{4}p^2-E_0)^{1/2}}+\frac{4}{\pi}\int_0^{p_c}\frac{dp\,dp'pp'\chi_T(p)\chi_T(p')}{(p^2+p'^2-E_0)^2-(pp')^2}\right]^{-1}, \tag{10}$$

$$\delta A=r_0\alpha A_0-\frac{2}{\pi}r_0\alpha\int_0^{p_c}dp\,p^2\chi_0^2(p)\,. \tag{11}$$

In order to distinguish the functions χ_0 from (6) and (9), we introduced here the notation χ_T for the bound-state wave function χ_0. For the sake of reference it is also useful to rewrite Eq. (11) in terms of solution of the Skornyakov–Ter-Martirosyan equation [2]. The relation between this solution, which is denoted by $a_0(p)$, and $\chi_0(p)$ is

$$\chi_0(p)=\frac{pa_0(p)}{p^2-p_0^2}\,,$$

and at $p_0=0$ it turns to $\chi_0(p)=p^{-1}a_0(p)$. Thus, in terms of $a_0(p)$, the correction δA is

$$\delta A=r_0\alpha A_0-\frac{2}{\pi}r_0\alpha\int_0^{p_c}dp\,a_0^2(p)\,. \tag{12}$$

An example of application of this expression is given in the Appendix.

IV. LINEAR CORRECTION TO CORRELATION

Now we are going to exclude the cutoff momentum p_c from the expressions derived in the previous section. This momentum is a parameter of our model of interaction of three particles at small distances. Excluding p_c, we should arrive at a universal (i.e., model-independent) linear correction.

As was pointed in Sec. II, this procedure is similar to that used in the low-energy two-body problem. One would select there a simple attraction potential (for example, the square well) as a model of force, and find the two-body binding energy $-\epsilon$ and scattering length a as functions of a certain parameter of the model (for example, the well depth or the range). After excluding this parameter one would obtain a universal relation between these quantities, which in the zeroth approximation in r_0 has the form $\epsilon_0(a)=1/ma^2$; in the next approximation there is a correction $\Delta\epsilon(a)=r_0/ma^2$. In the same fashion, excluding p_c in our model, we first arrive at a universal correlation $E_0(A)$ of the zero-range theory; in the next approximation a correction $\Delta E(A)$ linear in r_0 arises. This correction is derived in this section.

To exclude p_c we pick up a certain value of A and find the corresponding value of E. When the perturbation V_1 is neglected, this value of A is obtained at a certain value of the cutoff momentum p_c [such that $A_0(p_c)=A$]; the corresponding value of energy is $E_0(p_c)$. With the account of V_1, this value of A will be obtained at a slightly different cutoff momentum $p_c+\delta p_c$. The shift δp_c is determined by the condition

$$A_0(p_c+\delta p_c)+\delta A(p_c)=A\,,$$

where δA is found in Sec. III. Therefore, the shift is

$$\delta p_c=-\frac{\delta A(p_c)}{dA/dp_c}\,. \tag{13}$$

To find the value of E, we take into account both the correction δE from (10) and the shift of the cutoff momentum

$$E=E_0(p_c+\delta p_c)+\delta E(p_c)$$
$$=E_0(p_c)+\delta E(p_c)-\frac{dE_0/dp_c}{dA_0/dp_c}\delta A(p_c)\,.$$

Thus, the linear correction $\Delta E(A)$ is

$$\Delta E(A)=\delta E(p_c)-\frac{dE_0}{dA}\delta A(p_c)\,, \tag{14}$$

where the derivative dE_0/dA is determined by the zeroth-approximation correlation $E_0(A)$ and is calculated at $A=A_0$. According to what was said above, correction (14) should be universal. We will explicitly show it later, after we calculate the derivative dE_0/dA.

To do it we slightly vary the cutoff momentum in the equation of the zeroth approximation [Eq. (4) with $V_1=0$], find corresponding small variations of the quantities E_0 and A_0, and then calculate their ratio.

With the momentum p_c slightly varied, the equation takes the form

$$(\gamma-\alpha+V_0+\delta V_0)\psi=0\,, \tag{15}$$

where

$$(\delta V_0)\psi=-\frac{2}{\pi}\int_{p_c}^{p_c+\delta p_c}dp'\left[\ln\frac{p^2+p'^2+pp'-E_0}{p^2+p'^2-pp'-E_0}\right]\psi(p')$$
$$\approx\delta p_c V_0(p,p_c)\psi(p_c)\,.$$

Equation (15) has the same structure as Eq. (4), with V_1 replaced with δV_0. Therefore, to find the variations of E_0 and A_0 caused by the perturbation δV_0, we can use the expressions derived in the preceding section, just replacing V_1 with δV_0. From (6) we have for the variation of E_0

$$\delta E_0 = \frac{\chi_0 \delta V_0 \chi_0}{-\chi_0(\gamma + V_0)'\chi_0} .$$

Using here the expression for δV_0 and the equation $(\gamma - \alpha + V_0)\chi_0 = 0$ for χ_0, we obtain

$$\delta E_0 = \frac{[\chi_0(\gamma-\alpha)\chi_0]_{p=p_c}}{\chi_0(\gamma+V_0)'\chi_0}\delta p_c . \quad (16)$$

It is seen that the perturbation δV_0 is equivalent to the potential $-(\gamma-\alpha)\delta(p-p_c)\delta p_c$.

Using the same procedure we obtain from (8) the variation of the scattering length

$$\delta A_0 = -\frac{16\alpha}{3\pi}[\chi_0(\gamma-\alpha)\chi_0]_{p=p_c}\delta p_c \quad (17)$$

[cf. Eq. (9); the first term on the right-hand side of (9) does not appear in (17) because the free wave does not contribute at $p=p_c$]. Combining Eqs. (16) and (17) we arrive at the expression for the derivative dE_0/dA

$$\frac{dE_0}{dA} = -\frac{3\pi}{16\alpha}\frac{Z}{\chi_T(\gamma+V_0)'\chi_T} ,$$

where

$$Z = \left\{ \frac{\chi_T(\gamma-\alpha)\chi_T}{\chi_0(\gamma-\alpha)\chi_0} \right\}_{p=p_c} = \frac{(\frac{3}{4}p_c^2 - E_0)^{1/2} - \alpha}{(\frac{3}{4}p_c^2+\epsilon)^{1/2}-\alpha}\frac{\chi_T^2(p_c)}{\chi_0^2(p_c)} .$$

Here we again switched to the notation χ_T for the bound-state wave function.

Now we are ready to finish our derivation of the correction $\Delta E(A)$. Using in Eq. (14) the expressions for the derivative and for the corrections δE and δA from (6) and (9), we obtain

$$\Delta E(A) = \frac{(\chi_T V_1 \chi_T) - Z(\chi_0 V_1 \chi_0) - (3\pi/16) r_0 A Z}{-\chi_T(\gamma+V_0)'\chi_T} . \quad (18)$$

Let us check that this correction is indeed universal. In other words we should show that the parameter p_c disappears from (18) at $p_c \gg (\alpha, \sqrt{|E|})$. To do this we consider the matrix elements in (18), one by one. The normalization integral in the denominator is well convergent at large momenta. Therefore, only the low momenta $p \sim \sqrt{|E|}$ are significant in this integral, and we can safely extend the upper limit to infinity. Each of the matrix elements in the numerator is linearly divergent at large momenta, as was already mentioned in Sec. III. Nevertheless, their combination in (18) is convergent. Indeed, the difference $(\chi_T V_1 \chi_T) - Z(\chi_0 V_1 \chi_0)$ contains an integral over p which at large p is

$$\int dp\, p^2[\chi_T^2(p) - Z\chi_0^2(p)] . \quad (19)$$

At large p_c the factor Z is equal to $\chi_T^2(p_c)/\chi_0^2(p_c)$. Since at large momenta the energies E and α^2 can be neglected in the basic equation, the functions χ_T and χ_0 differ only by a factor at large p, and this factor is exactly $\sqrt{Z}$. Hence, the divergent contribution linear in p_c disappears from (19). We can further show that the difference between $\chi_T(p)$ and $\sqrt{Z}\,\chi_0(p)$ has the order $(E/p^2)\chi_T$ at large p. Therefore, the integral (19) converges as $\int dp\, p^{-2}$ [recall that $\chi_T(p)$ decreases as p^{-1}]. This means that only the low momenta $p \sim (\alpha, \sqrt{|E|})$ are significant in the difference $(\chi_T V_1 \chi_T) - Z(\chi_0 V_1 \chi_0)$, and hence the integration can also be safely extended to infinity.

Thus, the final expression for $\Delta E(A)$ is

$$\Delta E(A) = -\frac{W}{N} , \quad (20)$$

where

$$W = \frac{3\pi}{16} r_0 A Z + \frac{r_0}{2}\int_0^\infty dp[(\tfrac{3}{4}p^2 - E_0 - \alpha^2)\chi_T^2(p) - \tfrac{3}{4}p^2 Z \chi_0^2(p)] , \quad (21)$$

$$N = \frac{1}{2}\int_0^\infty \frac{dp\,\chi_T^2(p)}{(\frac{3}{4}p^2 - E_0)^{1/2}} + \frac{4}{\pi}\int_0^\infty \frac{dp\, dp' pp' \chi_T(p)\chi_T(p')}{(p^2+p'^2-E_0)^2 - (pp')^2} , \quad (22)$$

and Z is the asymptotic ratio $\chi_T^2(p)/\chi_0^2(p)$ at $p \to \infty$. Its value depends on the normalization of the wave function χ_T. Of course, the relation between Z and the normalization can be reversed: one can select some value of Z, say, $Z=1$, and this condition will determine the normalization of χ_T.

What is the physical reason that the divergence disappears when we turn from δE and δA to the correction $\Delta E(A)$? Let us use our earlier observation (see Sec. III) that V_0 and V_1 are equivalent to the attractions $1/R^2$ and r_0/R^3, respectively. When the perturbation r_0/R^3 is added to the potential $1/R^2$, the standard quantum-mechanical estimates show that the corrections to the binding energy and scattering length are both proportional to the integral $r_0\int dR/R^2$ which is singular at small R. This explains the singular structure of the corrections δE and δA. The important observation is that these two singular contributions relate to each other in such a way that they do not generate any deviation from the zeroth-approximation curve $E_0(A)$. Indeed, since the low-energy theory is not sensitive to the specific details of interaction at small distances $R \ll (\alpha^{-1}, |E|^{-1/2})$, any contribution from these R can be absorbed into the parameter of the boundary condition imposed on the wave function at small distances (we touched this issue in Sec. II). Remember now that the dependence $E_0(A)$ arises as a result of varying this parameter. Therefore, the effect of small distances on the correlation between E and A manifests itself as merely a slide along the curve $E_0(A)$. On the contrary, the correction $\Delta E(A)$ is defined to be a deviation from the curve $E_0(A)$. Therefore, the small distances (or large momenta) do not contribute to $\Delta E(A)$, in accordance with our result.

V. THREE-NUCLEON SYSTEMS

The generalization of the above derivation to the three-nucleon systems is straightforward. Therefore, we limit ourselves to the formulation of basic results.

For the nucleon-deuteron doublet S-wave scattering the basic equation has the matrix form [cf. Eq. (4)]

$$(\gamma-\hat{\alpha}+\hat{V}_0+\hat{V}_1)\psi=0 , \qquad (23)$$

where

$$\psi(p)=\frac{\pi}{2p_0}\delta(p-p_0)\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}u(p)\\v(p)\end{bmatrix}$$

and

$$\hat{V}_0(p,p')=-\frac{2}{\pi}\left[\ln\frac{p^2+p'^2+pp'-E}{p^2+p'^2-pp'-E}\right]\begin{bmatrix}\frac{1}{4} & \frac{3}{4}\\ \frac{3}{4} & \frac{1}{4}\end{bmatrix} ,$$

$$\hat{V}_1(p)=-\frac{\hat{r}_0}{2}[\gamma^2(p)-\hat{\alpha}^2] .$$

The upper element of the wave-function column ψ is the projection on the triplet state of a nucleon pair; the lower, on the singlet state. The free wave is present only in the triplet state; therefore, $E=\frac{3}{4}p_0^2-\epsilon_d$, where ϵ_d is the absolute value of the deuteron binding energy. The matrices $\hat{\alpha}$ and $\hat{r}_0$ are equal to

$$\hat{\alpha}=\begin{bmatrix}\alpha_t & 0\\ 0 & \alpha_s\end{bmatrix} , \quad \hat{r}_0=\begin{bmatrix}r_{0t} & 0\\ 0 & r_{0s}\end{bmatrix} ,$$

where $\alpha_t=\sqrt{\epsilon_d}$ and $\alpha_s=-\sqrt{\epsilon_s}$ (ϵ_s is the absolute value of energy of the singlet virtual state); r_{0t} and r_{0s} are the triplet and singlet effective ranges. The on-shell nucleon-deuteron scattering amplitude is related to $u(p)$ by the expression

$$a(p_0)=\lim_{p\to p_0} p^{-1}(p^2-p_0^2)u(p) ,$$

which is the analog of relation (3). The triton is described by Eq. (23) without the free wave in ψ.

We now formulate the results for the linear correction. We denote the *nd* doublet scattering length by a_2; the absolute value of the triton binding energy, by E_T. The linear correction to the correlation between E_T and a_2 is [cf. (18)]

$$\Delta E_T(a_2)=\frac{(\chi_T\hat{V}_1\chi_T)-Z(\chi_0\hat{V}_1\chi_0)-(3\pi/16)r_{0t}a_2Z}{\chi_T(\gamma+\hat{V}_0)'\chi_T} ,$$

where

$$Z=\left[\frac{\chi_T(\gamma-\hat{\alpha})\chi_T}{\chi_0(\gamma-\hat{\alpha})\chi_0}\right]_{p=p_c} .$$

Here $|\chi\rangle$ is the column $\binom{u}{v}$, and the summation over matrix indices is implied in the matrix elements. Using the explicit form of the operators we find [cf. (20)–(22)]

$$\Delta E_T(a_2)=\frac{W_t+W_s}{N} , \qquad (24)$$

where

$$W_t=\frac{3\pi}{16}r_{0t}a_2Z+\frac{r_{0t}}{2}\int dp[(\tfrac{3}{4}p^2+E_T-\epsilon_d)u_T^2(p)-\tfrac{3}{4}p^2Zu_0^2(p)] , \qquad (25)$$

$$W_s=\frac{r_{0s}}{2}\int dp[(\tfrac{3}{4}p^2+E_T-\epsilon_s)v_T^2(p)-(\tfrac{3}{4}p^2+\epsilon_d-\epsilon_s)Zv_0^2(p)] , \qquad (26)$$

$$N=\frac{1}{2}\int\frac{dp}{(\frac{3}{4}p^2+E_T)^{1/2}}[u_T^2(p)+v_T^2(p)]$$
$$+\frac{1}{\pi}\int\frac{dp\,dp'pp'}{(p^2+p'^2+E_T)^2-(pp')^2}[u_T(p)u_T(p')+v_T(p)v_T(p')+3u_T(p)v_T(p')+3v_T(p)u_T(p')] . \qquad (27)$$

Here the integration is performed from 0 to p_c, and Z is

$$Z=\frac{[(\frac{3}{4}p_c^2+E_T)^{1/2}-\alpha_t]u_T^2(p_c)+[(\frac{3}{4}p_c^2+E_T)^{1/2}-\alpha_s]v_t^2(p_c)}{[(\frac{3}{4}p_c^2+\epsilon_d)^{1/2}-\alpha_t]u_0^2(p_c)+[(\frac{3}{4}p_c^2+\epsilon_d)^{1/2}-\alpha_s]v_0^2(p_c)} .$$

The integration can be safely extended to infinity, with Z equal to the asymptotic ratio $[u_T^2(p)+v_T^2(p)]/[u_0^2(p)+v_0^2(p)]$ at $p\to\infty$. As can be shown, $u_T\to v_T$ and $u_0\to v_0$ at large p. Therefore, Z is also equal $u_T^2(p)/u_0^2(p)$ or $v_T^2(p)/v_0^2(p)$.

Correction (24) contains two independent linear contributions proportional to r_{0t} and r_{0s}. One can observe from (25)–(27) that at large momenta $p\gg(\alpha,\sqrt{|E|})$ the effect of the potential $\hat{V}_1$ is equivalent to the attraction $(r_{0t}+r_{0s})p^3$. In the configuration space this corresponds to the attraction $(r_{0t}+r_{0s})/R^3$. The effective ranges enter this expression in a symmetric fashion because only the symmetric component of the three-nucleon wave function, which averages the effect of the triplet and singlet forces, survives at small distances.

VI. DISCUSSION

A few comments about our expressions for the corrections $\Delta E(A)$ and $\Delta E_T(a_2)$ are in order.

A. Conditions of applicability

The conditions of applicability of expressions (20) and (24) are given by the basic criteria of the present theory

$$r_0\alpha \ll 1 \ , \quad r_0\sqrt{|E|} \ll 1 \ . \tag{28}$$

In the course of derivation we also used the condition $r_0 p_c \ll 1$. The momentum p_c is an auxiliary intermediate quantity which disappears from our final expressions. Therefore, the condition $r_0 p_c \ll 1$ can be omitted.

B. Corrections for the series of levels

At $r_0\alpha \ll 1$ the three-body system has, in fact, more than one level. The number of levels is known to be $\approx (1/\pi)\ln(r_0\alpha)^{-1}$ [16]. Which of them was actually considered above? The answer is that we have calculated the corrections ΔE for any of these levels. To get the correction for a given level, one should use the wave function of this level. Note that the second condition (28) gradually gets weaker with increasing the binding. For the lowest level this condition may turn to $r_0\sqrt{|E|}\sim 1$. For such a level our consideration may not be applicable.

C. Physics of the correction

As far as physics is concerned, our problem is similar to the simple quantum-mechanical problem for a particle moving in the attraction potential $1/R^2$ cut at the distances r_0 and α^{-1}, and affected by the additional small perturbation r_0/R^3 cut at the same distances. Let us summarize physics of our problem in the light of this analogy.

If the perturbation is neglected, we have a series of $(1/\pi)\ln(r_0\alpha)^{-1}$ levels generated by the $1/R^2$ potential. At $r_0\sqrt{|E|} \ll 1$ the energy of each level can be expressed in terms of the scattering length A which accumulates all low-energy information about the region $R\sim r_0$. Under conditions (28) this dependence $E_0(A)$ does not contain the range r_0 and can be found either directly for the potential $1/R^2$ with $r_0=0$ or by using the limiting procedure $r_0\rightarrow 0$ [19]. If we turn the perturbation on and do it in such a way that the scattering length A remains constant (to achieve this, we should slightly vary the cutoff profile at $R\sim r_0$), we obtain the correction $\Delta E(A)$ for the energy of each level. It is just the correction we have derived in the present paper. The correction is proportional to r_0; therefore, we can put $r_0=0$ in the proportionality coefficient. This means that in calculating the effect of the perturbation we can again extend the potential $1/R^2$ to $R=0$, just as it was done in the calculation of $E_0(A)$. Although the energy level we are talking about will now have an infinite number and, correspondingly, its wave function will contain an infinite number of oscillations at the small distances $R<r_0$, this is not important in the calculation of ΔE because this correction is determined by the large distances $R\sim(\alpha^{-1},E^{-1/2})$ [20].

D. One-level case

The series referred to can contain only one level. This occurs, for example, in the case of three-nucleon systems. At first glance, criteria (28) are hard to satisfy in the one-level case. However, as follows from the formula $n\approx(1/\pi)\ln(r_0\alpha)^{-1}$ for the number of levels, the values of r_0 (or α) that correspond to the formation of adjacent levels of the series, differ from one another very substantially, by about $e^{\pi}\approx 23$ times. Therefore, a situation is possible when r_0 may be noticeably less than α^{-1} and $|E|^{-1/2}$, yet no second level is still formed. For this situation our consideration of the correlation $E_0(A)$ and the correction $\Delta E(A)$ will be correct.

For the nucleons we have $r_{0t}\alpha_t\approx 0.4$ and $r_{0s}\alpha_s\approx 0.1$. The energy of the only three-nucleon bound state is 8.5 MeV, which is noticeably less than the typical magnitude of nuclear force ($\approx$30 MeV). Therefore, criteria (28) are satisfied, though not in a strong fashion. This enables us to believe that the expressions derived can be applicable for this case. The calculations of Ref. [7] indeed show that the above theory is able to explain the correlation between E_T and a_2 existing in the three-nucleon systems.

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APPENDIX: NUCLEON-DEUTERON QUARTET SCATTERING LENGTH

The *nd* quartet scattering serves as an interesting example of application of expression (12). In this case the three-nucleon configuration-space wave function has the mixed permutation symmetry. In our formalism this can be taken into account by multiplying the right-hand side of Eq. (1) by the factor $\frac{1}{2}$. Correspondingly, the potential V_0 in Eq. (4) is replaced with $-\frac{1}{2}V_0$. As can be readily shown, this replacement does not affect the derivation of expression (12): the result for the correction to the scattering length remains the same.

The distinctive feature of the mixed-symmetry case is that the function $a_0(p)$ rapidly decreases with growth of p. This is a reflection of the fact that in the quartet state the Pauli principle prohibits the close approach of three nucleons. Therefore, the integral in (12) converges at large momenta, and we can set $p_c=\infty$ as its upper limit. Thus, we have for the quartet scattering length

$$\delta a_4 = r_{0t}\alpha_t a_4^{(0)} - \frac{2}{\pi} r_{0t}\alpha_t \int_0^\infty dp\, a_0^2(p) \ , \tag{A1}$$

where $a_4^{(0)}$ is its value in the zero-range theory and δa_4 is the linear correction.

The value of δa_4 was calculated in Ref. [9] by numerically solving the integral equation derived in Ref. [8] and was found to be 0.97 fm. Expression (A1) enables us to calculate δa_4 directly provided the function $a_0(p)$ is known. To check the usefulness of the expression (A1) we took $a_0(p)$ from Fig. 2 of Ref. [9], supplementing it with the known asymptote [3,16] $a_0(p) \propto p^{-s}$, $s=2.17$, at large p. Using the values $\alpha_t = 0.2316\ \text{fm}^{-1}$, $r_{0t}=1.75$ fm, and $a_4^{(0)}=5.09$ fm from Ref. [9], we found after an elementary calculation that $\delta a_4 = 0.99 \pm 0.03$ fm. The errors are due to the small scale of the figure. This simple calculation is seen to give a rather accurate result.

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[17] The idea of transforming the equation of the zero-range theory to an equivalent Schrödinger equation dates as early as to Skornyakov–Ter-Martirosyan's paper [2]. In this paper the Schrödinger equation was constructed for the wave function $p^{-1}(\gamma+\alpha)^{-1/2}\psi$ (times a certain normalization factor). The advantage of this choice is that $\gamma-\alpha$ in Eq. (4) turns into the standard Schrödinger-like expression $\gamma^2-\alpha^2=\frac{3}{4}(p^2-p_0^2)$. On the other hand, the expression for V_0 gets more complicated due to the inconvenient additional factors $(\gamma+\alpha)^{1/2}$. Such a factor also makes more complicated the relation between the above wave function and the full wave function of the three-body system.

[18] We will give another, more rigorous proof of this fact elsewhere by deriving the linear correction directly in the configuration space. We also note that the form $V_1 \sim r_0/R^3$ naturally arises from the dimensional analysis [7]. Indeed, at the distances $r_0 \ll R \ll \alpha^{-1}$ the linear correction to the interaction $1/R^2$ is expected to have the structure $(1/R^2)(r_0/R)$, the first power of r_0 being compensated by an extra power of R to retain the dimension of the interaction intact. We just remind the reader that the form $V_0 \sim 1/R^2$ itself arises as a result of a similar dimensional analysis [16].

[19] If the limiting procedure $r_0 \to 0$ is not accomplished, the correlation $E(A)$ will still be contaminated by the contributions of higher terms of the expansion in parameters (28). This is what occurs in a three-body model studied in Refs. [21] and [22]. In this model the cutoff momentum, which plays the role of inverse force range, is not tended to infinity but kept on the order of the inverse range of nuclear force. Therefore, the genuine correlation $E_0(A)$ of the zero-range theory cannot be calculated within framework of this model. By the same reason, and because of the fact that the solution of the equations of the model is not expanded in r_0, the model cannot give the genuine linear correction $\Delta E(A)$.

[20] The inadequacy of the wave function of the zero-range theory at the distances of the order of r_0 is related to the violation of the condition $pr_0 \ll 1$. In the two-body problem, for example, the deuteron wave function of the zero-range theory, $e^{-\alpha r}/r$, is obviously not good at the distances $r < r_0$. However, this inadequacy is not important if one calculates a matrix element in which the main contribution originates from the distances $r \sim \alpha^{-1}$.

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