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Covariant Feynman rules at finite temperature: Time-path formulation

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A path-integral (time-path) formulation is used to derive Feynman rules for relativistic many-body systems at finite temperature and density. The generating functional of propagators is written in a form that involves evolution along contours in the complex time plane. Controversies regarding the factorization of this generating functional into separate contributions from real and imaginary times are resolved. The time paths are generalized to manifestly covariant form, and the distinction between evolution in the canonical and grand-canonical Heisenberg pictures is discussed. This unified path-integral approach produces manifestly covariant Feynman rules for both real and imaginary times, which were applied to hadronic field theories of hot, dense nuclear matter in an earlier paper.

I. INTRODUCTION

To interpret data from ultrarelativistic heavy-ion colliders, such as the existing Super Proton Synchrotron (SPS) Facility at CERN and the proposed Relativistic Heavy Ion Collider, and to study astrophysical systems, such as neutron stars, one needs an accurate description of hadronic matter under extreme conditions of temperature and density. This description requires a consistent microscopic treatment of strongly interacting, relativistic, quantum-mechanical systems, whether the degrees of freedom are hadrons or quarks and gluons.

In a previous paper [1], we studied hot, dense nuclear matter using a unified approach to relativistic many-body systems at finite temperature and density. This approach incorporated both real- and imaginary-time formalisms and emphasized Lorentz covariance. In particular, covariant real- and imaginary-time Feynman rules were presented and used to calculate nuclear matter properties at the one-baryon-loop level. We obtained manifestly covariant expressions for the energy density, baryon density, and other thermodynamic observables. In the present work, we complete the development by deriving the covariant Feynman rules using path integrals, which provide a natural framework for a unified treatment.

At zero temperature, there is little practical difference between the Feynman rules for Minkowski and Euclidean Green's functions. At finite temperature and density, however, the distinctions are significant. Thus there are two basic approaches to finite-temperature field theory: the imaginary-time (Euclidean) formalism and the real-time formalism (also known as the time-path method or thermofield dynamics). Practitioners of finite-temperature field theory (both relativistic and nonrelativistic) are often familiar with only one approach and apply it exclusively. In fact, there are advantages and disadvantages to each, and the choice between them should be based on the problem under consideration. (See Ref. [1] for further discussion on this issue.) In some cases, applying both real- and imaginary-time methods is appropriate, to check calculations or to resolve ambiguities that can arise at finite temperature and density [2]. The thermodynamic consistency of various approximations to the relativistic nuclear many-body problem can also be assessed. We therefore present a unified treatment.

We develop the finite-temperature Feynman rules using path integrals along contours in the complex time plane. A contour running from $t=0$ to $t=-i\beta$ (the so-called Matsubara contour) produces a generating functional for the familiar imaginary-time Green's functions. However, as Mills [3] and later Niemi and Semenoff [4] pointed out, a different contour with segments along and parallel to the real t axis leads to a generating functional for real-time Green's functions. This construction ultimately results in real-time propagators with a 2×2 matrix structure. Nevertheless, since the two approaches differ only in the choice of time contour, both real- and imaginary-time formalisms can be discussed within the same basic framework. Most of this paper will be devoted to constructing the generating functionals and the implied Feynman rules.

Our goal is to provide a self-contained introduction to these methods for nuclear physicists and to clarify some subtle aspects of the time-path approach (e.g., factorization, off-shell momenta) that have led to controversies in the literature. By concentrating on the basics, we hope to establish a framework in which consistent microscopic approximations can be developed for strongly interacting, relativistic, many-body systems. We follow the motivation in Ref. [1] and apply our methods to a hadronic field theory of nuclear matter; nevertheless, the formalism discussed here can also be applied to QCD and QED, as well as to more phenomenological models, and should therefore be useful to a more general audience. Since we derive the results in some detail, it should be possible to see how to modify our Feynman rules for more phenomenological approaches.

We emphasize in particular a covariant treatment of the finite-temperature formalism, which has been considered elsewhere only in a cursory manner and only for real-time Green's functions [2,5]. We discuss the implications of covariance for both real- and imaginary-time propagators and show how to generalize the path-integral constructions.

A covariant formulation allows calculations to be carried out directly in any convenient reference frame. This may be useful for describing the collision of two heavy ions, for which there is no frame in which all the matter is at rest. A covariant description also incorporates information that would be obscured by working in a fixed frame, such as the role of the fluid velocity and the momentum density as conjugate thermodynamic parameters [6]. Furthermore, by calculating covariant Green's functions, one can clearly identify the correct Lorentz structure and associated invariant functions. The importance of a covariant formalism in finite-temperature QED is stressed in Ref. [7].

As in Ref. [1], we specialize our discussion to the Walecka model [8] (also called QHD-I). This is a renormalizable relativistic quantum field theory that describes the nuclear system using a local Lagrangian density containing baryons and neutral scalar and vector mesons. It incorporates some basic elements of hadronic theories of nuclei and has been widely applied at the mean-field level with much phenomenological success [9]. This model provides a self-contained framework that can be used to study general features of the relativistic nuclear many-body problem at finite temperature and density. For example, one can investigate the importance of maintaining conservation laws and causality in various approximations to the underlying field theory. Moreover, this model illustrates the path-integral methods in sufficient detail that much of our discussion can be generalized or carried over directly to other theories, such as QED or QCD.

The outline of the paper is as follows. In Sec. II, we review some basic elements of covariant thermodynamics and introduce our model Lagrangian. Since most of the subtleties of the time-path formulation are distinct from the issue of covariance and are present with any Lagrangian, we first consider the simplest case: a self-interacting scalar field in the rest frame of the heat bath (the comoving frame). The generating functional for this problem is introduced in Sec. III, and the form of the noninteracting scalar propagator is derived. In Sec. IV, we show how to make the generating functional and propagator manifestly Lorentz covariant. The construction is generalized to include massive vector mesons and baryons in Sec. V, and the finite-temperature Feynman rules for QHD-I are derived. Section VI has some final comments and a summary.

II. PRELIMINARIES

We begin by briefly reviewing some concepts that will be central to our discussion of strongly interacting, relativistic, many-body systems: the covariant formulation of thermodynamics and statistical mechanics, the relativistic hadronic model of interest, and the definition of finite-temperature propagators for both real and imaginary times. Much of this material has been discussed more fully elsewhere [2,10]; in particular, covariant thermodynamics is considered in Refs. [6] and [11]. Our purpose is both to define the notation and to introduce various ingredients that will be referred to repeatedly in the subsequent development.

A. Covariant thermodynamics

In a covariant treatment of thermodynamics [11], the primary thermodynamic functions for a system in equilibrium are the energy-momentum tensor $T^{\mu\nu}$, the entropy flux vector S^{μ}, and the baryon current density vector B^{μ}, which involve no specification of a particular reference frame. These quantities are generally functions of six variables: the baryon thermal potential α, the inverse temperature β, the fluid four-velocity u^{μ}, and the volume $\mathcal{V}$. (The volume is taken to infinity at the end to define the "thermodynamic limit" and restore invariance under translations.) The variables α and β are Lorentz scalars defined by

$$\beta \equiv \frac{1}{T'}, \quad \alpha \equiv \frac{\mu'}{T'} , \tag{2.1}$$

where T' and μ' are the temperature and baryon chemical potential in the comoving frame, where the fluid three-velocity $\mathbf{v}$ is zero. (When we refer to a quantity that may be defined by an observer in any frame, the "proper" value taken in the comoving frame will be denoted with a prime.) We also define a timelike thermal four-vector

$$\beta^{\mu} \equiv \beta u^{\mu} \equiv \frac{1}{T'} u^{\mu} . \tag{2.2}$$

Our conventions are those of Ref. [10], with a metric $g^{\mu\nu} = \mathrm{diag}\,(+,-,-,-)$, and we use natural units with $\hbar = c = k_B = 1$.

The secondary thermodynamic functions are defined in the comoving frame and are thus Lorentz scalars. They are the pressure p, the proper energy density $\mathcal{E}'$, the proper entropy density σ', the proper baryon density ρ'_B, and the scalar density ρ_s. In the thermodynamic limit, these secondary quantities are functions of α and β (or μ' and T') only. The secondary thermodynamic functions can be used to construct the primary functions in any frame:

$$T^{\mu\nu} = (\mathcal{E}' + p) u^{\mu} u^{\nu} - p g^{\mu\nu} , \tag{2.3}$$

$$S^{\mu} = \sigma' u^{\mu} , \tag{2.4}$$

$$B^{\mu} = \rho'_B u^{\mu} . \tag{2.5}$$

In the thermodynamic limit, the primary quantities are functions of α, β, and u^{μ}, or equivalently α and β^{μ}.

To compute the thermodynamic functions in terms of ensemble averages of quantum-mechanical operators, a *grand partition function* Z and a four-vector thermodynamic potential $\Phi^{\mu}(\alpha, \beta^{\nu})$ are defined through

$$Z \equiv \exp\left[-\int d\Lambda_\mu \Phi^\mu(\alpha,\beta^\nu)\right]$$
$$\equiv \operatorname{Tr}\exp\left[-\int d\Lambda_\mu(\beta_\nu \hat{T}^{\nu\mu} - \alpha\hat{B}^\mu)\right], \tag{2.6}$$

which are manifestly Lorentz-invariant expressions. Here Λ is a spacelike hypersurface on which the fields and composite operators (e.g., $\hat{T}^{\mu\nu}$ and $\hat{B}^\mu$) are quantized. In the comoving frame, with $d\Lambda_\mu = d^3x'\delta_{\mu 0}$, Eq. (2.6) reduces to the familiar result

$$Z = \operatorname{Tr}\exp[-\beta(\hat{H} - \mu'\hat{B})], \tag{2.7}$$

where $\hat{H}$ is the Hamiltonian and $\hat{B}$ is the baryon number operator. Ensemble averages are computed from

$$A \equiv \langle\langle \hat{A} \rangle\rangle = Z^{-1}\operatorname{Tr}\left\{\hat{A}\exp\left[-\int d\Lambda_\mu(\beta_\nu \hat{T}^{\nu\mu} - \alpha\hat{B}^\mu)\right]\right\}, \tag{2.8}$$

where $\hat{A}$ is any operator built from the fields of the theory.

B. The model

The relativistic field theory QHD-I, including scalar meson self-couplings, is defined by the Lagrangian density [10]

$$\mathcal{L} = \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)]\psi + \tfrac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m_s^2\phi^2)$$
$$-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2}m_v^2 V_\mu V^\mu - V(\phi) + \delta\mathcal{L}. \tag{2.9}$$

Here $F_{\mu\nu} \equiv \partial_\mu V_\nu - \partial_\nu V_\mu$, $\delta\mathcal{L}$ contains renormalization counterterms that will be suppressed, and

$$V(\phi) = \frac{\kappa}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4. \tag{2.10}$$

In Ref. [6], the (renormalized) scalar self-couplings were chosen to be zero ($\kappa = \lambda = 0$), but here we will consider the more general case. The field equations from this Lagrangian are

$$(\partial_\mu\partial^\mu + m_s^2)\phi + \frac{\kappa}{2}\phi^2 + \frac{\lambda}{6}\phi^3 = g_s\bar{\psi}\psi, \tag{2.11}$$

$$(\partial_\nu\partial^\nu + m_v^2)V^\mu = g_v\bar{\psi}\gamma^\mu\psi, \tag{2.12}$$

$$[\gamma^\mu(i\partial_\mu - g_v V_\mu) - (M - g_s\phi)]\psi = 0. \tag{2.13}$$

The final equation implies that the baryon current $B^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved ($\partial_\mu B^\mu = 0$), which has been used to arrive at Eq. (2.12). At this stage, all these expressions involve c-number fields.

The energy-momentum tensor follows from the canonical definition [12] as

$$T^{\mu\nu} = \tfrac{1}{2}[-\partial_\lambda\phi\partial^\lambda\phi + m_s^2\phi^2 + \tfrac{1}{2}F_{\lambda\sigma}F^{\lambda\sigma} - m_v^2 V_\lambda V^\lambda]g^{\mu\nu}$$
$$+ V(\phi)g^{\mu\nu} + i\bar{\psi}\gamma^\mu\partial^\nu\psi + \partial^\mu\phi\partial^\nu\phi + \partial^\nu V_\lambda F^{\lambda\mu}, \tag{2.14}$$

and the field equations (2.11)–(2.13) imply that the four-momentum

$$P^\mu \equiv (H, \mathbf{P}) \equiv \int d^3x\, T^{0\mu} \tag{2.15}$$

is a constant of motion. [To work in a fully covariant manner, the integral in Eq. (2.15) is taken over the spacelike hypersurface Λ, so that $P^\mu \equiv \int d\Lambda_\nu T^{\nu\mu}$. By choosing $d\Lambda_\nu = d^3x\,\delta_{\nu 0}$, one reproduces the usual expression.] States of the system can therefore be labeled by their energy, three-momentum, and baryon number: E, $\mathbf{P}$, and B.

This theory can be quantized in the canonical fashion by imposing equal-time commutation relations (or, more generally, by imposing commutation relations on an arbitrary spacelike hypersurface Λ) [12]. For example, for the scalar field, we have

$$\begin{aligned}&[\hat{\Pi}(t,\mathbf{x}), \hat{\phi}(t,\mathbf{y})] = -i\delta^{(3)}(\mathbf{x}-\mathbf{y}),\\ &[\hat{\phi}(t,\mathbf{x}), \hat{\phi}(t,\mathbf{y})] = [\hat{\Pi}(t,\mathbf{x}), \hat{\Pi}(t,\mathbf{y})] = 0,\end{aligned} \tag{2.16}$$

where $\hat{\Pi}(t,\mathbf{x}) = \partial\hat{\phi}(t,\mathbf{x})/\partial t$ from the Heisenberg equations of motion.

The composite operators $\hat{B}$, $\hat{H}$, and $\hat{\mathbf{P}}$ must be defined by subtracting infinite constants to choose a particular operator ordering. Since $\hat{B}$ is conserved due to an internal symmetry, a single vacuum subtraction is sufficient, which is equivalent to normal ordering the operator. In contrast, $\hat{H}$ and $\hat{\mathbf{P}}$ are conserved because of spacetime symmetries, so a vacuum subtraction alone is insufficient [2], and various counterterm subtractions must be included [10]. An expression for $\hat{T}^{\mu\nu}$ appropriate for uniform systems is given in Ref. [1].

C. Finite-temperature propagators

Our goal is to construct Feynman rules for the interacting propagators,

$$i\Delta^{(c)}(x_1,\ldots,x_n) \equiv \langle\langle T_c\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)\rangle\rangle, \tag{2.17}$$

$$iG^{(c)}(x_1,\ldots,x_n,x_1',\ldots,x_n') \equiv \langle\langle T_c\hat{\psi}(x_1)\cdots\hat{\psi}(x_n)\hat{\bar{\psi}}(x_1')\cdots\hat{\bar{\psi}}(x_n')\rangle\rangle, \tag{2.18}$$

$$iD^{(c)}_{\mu\cdots\nu}(x_1,\ldots,x_n) \equiv \langle\langle T_c\hat{V}_\mu(x_1)\cdots\hat{V}_\nu(x_n)\rangle\rangle, \tag{2.19}$$

that generalize to an arbitrary frame the conventional finite-temperature propagators defined in the comoving frame. We apply these definitions to times $t_1,\ldots,t_n$ and $t_1',\ldots,t_n'$ that lie on contours in the complex plane; this permits Feynman rules to be derived for both real and imaginary times. (The thermal contour propagators will be denoted with superscripts "c", and T_c is a contour-ordering operator.) For real time arguments, we can interpret the $(x_i)^\mu = (t_i, \mathbf{x}_i)$ as Lorentz four-vectors [13], but it is not immediately obvious how to make the contour propagators (2.17)–(2.19) covariant for complex times. Furthermore, we must clarify the role of the hypersurface Λ in the covariant statistical operator [see Eq. (2.8)] and in the definition of Heisenberg field operators. Our approach will be to separate the details of the time-path construction from the issues of covariance. Thus, we restrict the discussion for now to the comoving frame and postpone to Sec. IV the extension to arbitrary frames.

The "contour-ordered product" $T_c(\cdots)$ is a straightforward generalization of the familiar time-ordered product and puts operators occurring nearest the start of the contour at the right, with those nearest the end of the contour at the left. T_c also involves the usual sign changes for permutations of fermion operators. To be more precise, allow t to be given parametrically by $t \equiv t(\zeta)$, where ζ is real and increases monotonically along the contour. We can then define [2]

$$\theta_c(t-t') \equiv \theta(\zeta-\zeta') , \tag{2.20}$$

$$\delta_c(t-t') \equiv \left[\frac{dt}{d\zeta}\right]^{-1} \delta(\zeta-\zeta') = \frac{d}{dt}\theta_c(t-t') , \tag{2.21}$$

which correctly incorporates the orientation of the contour.

The propagators of Eqs. (2.17)–(2.19) are defined using *grand* Heisenberg-picture operators. In the comoving frame, the *grand-canonical Hamiltonian* $\hat{K}$ is defined as

$$\hat{K} = \hat{H} - \mu' \hat{B} , \tag{2.22}$$

and the ensemble average of an operator $\hat{A}$ is given by

$$\langle\langle \hat{A} \rangle\rangle \equiv \frac{\mathrm{Tr}\{e^{-\beta\hat{K}}\hat{A}\}}{\mathrm{Tr}\{e^{-\beta\hat{K}}\}} \equiv Z^{-1}\mathrm{Tr}\{e^{-\beta\hat{K}}\hat{A}\} , \tag{2.23}$$

where the trace is over any complete set of states. Here $\beta = 1/T'$, so that the propagators are functions of μ' and T', or equivalently, α and β defined in Sec. II A. The volume $\mathcal{V}'$ of the system will be considered fixed throughout the calculation, with the thermodynamic limit $\mathcal{V}' \to \infty$ taken at the end. We will generally focus on intensive variables, which are independent of $\mathcal{V}'$.

The grand Heisenberg-picture operators are

$$\hat{\phi}(x) \equiv \hat{\phi}(t,\mathbf{x}) = e^{it\hat{K}}\hat{\phi}(\mathbf{x})e^{-it\hat{K}} , \tag{2.24}$$

$$\hat{\psi}(x) \equiv \hat{\psi}(t,\mathbf{x}) = e^{it\hat{K}}\hat{\psi}(\mathbf{x})e^{-it\hat{K}} , \tag{2.25}$$

$$\hat{\psi}^\dagger(x) \equiv \hat{\psi}^\dagger(t,\mathbf{x}) = e^{it\hat{K}}\hat{\psi}^\dagger(\mathbf{x})e^{-it\hat{K}} , \tag{2.26}$$

and similarly for the vector field. Since t may be complex, $\hat{\psi}$ and $\hat{\psi}^\dagger$ must be defined separately, and $[\hat{\psi}^\dagger]^\dagger \neq \hat{\psi}$ in general. The meson field operators commute with $\hat{B}$, so the baryon chemical potential μ' cancels out in Eq. (2.24) and can be set to zero when discussing the meson propagators.

The interacting propagators of Eqs. (2.17)–(2.19) can be used to evaluate ensemble averages of various operators and thus compute observables. The Feynman rules derived below allow one to express these propagators as order-by-order expansions in the coupling constants and noninteracting propagators. (Note that since QHD-I is a strong-coupling theory, practical calculations require infinite summations over various classes of Feynman diagrams.) We emphasize that the propagators above differ from those defined in the usual (canonical) Heisenberg picture, which involves only the Hamiltonian $\hat{H}$. For real-time calculations, it is convenient to extract a phase factor from the Fourier transform of the fermion two-point function, so that the momentum-space Green's functions correspond to those defined in the canonical Heisenberg picture. This simplifies the real-time rules and generates Green's functions with well-defined zero-temperature limits that are equivalent to the usual $T=0$ functions [10]. Since the canonical ($\hat{H}$) and grand-canonical ($\hat{K}$) Heisenberg-picture fields differ by a unitary transformation (for real times), observables are not affected.

III. THE GENERATING FUNCTIONAL

The most efficient derivation of Feynman rules for propagators proceeds through a generating functional. We will illustrate the formalism for a self-interacting scalar field and generalize later (in Sec. V) to QHD-I. We also work initially in the comoving frame and postpone the extension to arbitrary frames until the following section.

The first step is to introduce a generating functional defined on contours in the complex time plane. The allowed contours are determined by requiring that thermal traces converge. The generating functional is then expressed as a path integral, which can be manipulated to produce an exact expression containing the noninteracting scalar propagator defined on a time contour and functional derivatives with respect to external sources. We then discuss the properties of the scalar contour propagator and the existence of a well-defined generating functional in the limit that the (real) time variables are extended to $\pm\infty$. We prove that in this limit, it is possible and convenient to factorize the full generating functional into two separate contributions: one that generates real-time Feynman rules and another that determines the imaginary-time rules. Finally, we derive explicit expressions for the real-time noninteracting scalar propagator in the comoving frame. The development in this section is intended to be self-contained, but some of the material is also described in Refs. [2,4,14].

A. Contours in the complex time plane

As a candidate generating functional, consider the quantity

$$\mathrm{Tr}\left\{ e^{-\beta\hat{H}} T \exp\left[i\int_{-t_0}^{t_0} dt \int d^3x\, j(x)\hat{\phi}(x) \right] \right\} , \tag{3.1}$$

where t is real, $j(x)$ is a classical source, T denotes the usual time ordering, and we set $\mu'=0$ for scalars. Evidently, the scalar propagators of Eq. (2.17) are given by functional derivatives of this expression with respect to $j(x_i)$, if we demand that the t_i are real and that $-t_0 < t_i < t_0$ for all $1 \le i \le n$. The problem is that we have no systematic way to evaluate the trace in (3.1), since the time-ordered exponential involves evolution in real time through factors like $e^{i\hat{H}(t-t')}$, while the statistical operator contains $e^{-\beta\hat{H}}$, which produces a translation in imaginary time.

This problem can be solved by extending the time integral in (3.1) to a contour in the complex time plane, using the contour-ordering operator T_c defined in Sec. II. Thus we define a generating functional

$$Z[j]\equiv Z\left\langle\!\!\left\langle T_c\exp\left[i\int_c d^4x\, j(x)\hat{\phi}(x)\right]\right\rangle\!\!\right\rangle , \tag{3.2}$$

where $\int_c d^4x\equiv\int_c dt\int d^3x$, and the allowed contours c will be discussed shortly. The scalar propagators can now be evaluated for any time arguments on the contour using

$$i\Delta^{(c)}(x_1,\ldots,x_n)=\frac{1}{Z[0]}\left[\frac{\delta^n Z[j]}{i\delta j(x_1)\cdots i\delta j(x_n)}\right]_{j=0} , \tag{3.3}$$

if we extend the definition of functional differentiation to [see Eq. (2.21)]

$$\frac{\delta}{\delta j(x)}j(x')=\delta_c(t-t')\delta^{(3)}(\mathbf{x}-\mathbf{x}')$$
$$\equiv\delta_c^{(4)}(x-x') \tag{3.4}$$

for sources that live on the contour. The normalization in (3.2) is chosen so that

$$Z\equiv Z[0]=\mathrm{Tr}\{e^{-\beta\hat{H}}\} \tag{3.5}$$

is the partition function at inverse proper temperature β.

Equation (3.2) can be rewritten as

$$Z[j]=\mathrm{Tr}\left\{e^{-\beta\hat{H}}T_c\exp\left[i\int_c d^4x\, j(x)\hat{\phi}(x)\right]\right\}$$
$$=\sum_\phi\langle\phi(t_0)|e^{-\beta\hat{H}}T_c\exp\left[i\int_c d^4x\, j(x)\hat{\phi}(x)\right]|\phi(t_0)\rangle$$
$$=\sum_\phi\langle\phi(t_0-i\beta)|T_c\exp\left[i\int_c d^4x\, j(x)\hat{\phi}(x)\right]|\phi(t_0)\rangle , \tag{3.6}$$

where $|\phi(t)\rangle$ denotes a Heisenberg-picture state with field configuration $\phi(\mathbf{x})$ defined by eigenvalues of the Heisenberg-picture operators $\hat{\phi}(x)=\hat{\phi}(t,\mathbf{x})$ at complex time t. Field configurations at different times are related by [14]

$$\langle\phi(t)|=\langle\phi(0)|e^{-it\hat{H}},\quad |\phi(t)\rangle=e^{it\hat{H}}|\phi(0)\rangle , \tag{3.7}$$

which has been used to write the final equality in (3.6). The sum on ϕ runs over a complete set of states specified by their field configurations at an *arbitrary* time t_0. It follows that $Z[j]$ is independent of t_0 and that $Z=Z[0]$ is independent of the contour c. The preceding relations can be defined precisely using the continuum limit of a trace taken on a spacetime lattice, as illustrated in Ref. [14]. For the present discussion, it is more convenient to work with a continuum notation, but it should be understood that this notation is really just a shorthand for the corresponding expressions written on the lattice [15].

We now consider the possible contours. The matrix elements in (3.6) involve initial configurations at (complex) time t_0 and final configurations at time $t_0-i\beta$. To express the trace as a path integral, the contour should start at t_0, pass through all points of interest for the propagators, and terminate at $t_0-i\beta$. The contour should include only times where the propagators are well defined. To determine these values, examine the scalar two-point function $i\Delta^{(c)}(x-x')$ by writing

$$i\Delta^{(c)}(x-x')\equiv\langle\!\langle T_c\hat{\phi}(x)\hat{\phi}(x')\rangle\!\rangle \tag{3.8}$$
$$=\theta_c(t-t')\langle\!\langle\hat{\phi}(x)\hat{\phi}(x')\rangle\!\rangle+\theta_c(t'-t)\langle\!\langle\hat{\phi}(x')\hat{\phi}(x)\rangle\!\rangle$$
$$\equiv\theta_c(t-t')C^{>}(x-x')+\theta_c(t'-t)C^{<}(x-x') , \tag{3.9}$$

which defines the functions $C^{\gtrless}(x-x')$. These functions can be studied using a complete set of energy and momentum eigenstates $\hat{H}|n\rangle=E_n|n\rangle$, $\hat{\mathbf{P}}|n\rangle=\mathbf{P}_n|n\rangle$:

$$C^{>}(x-x')=C^{<}(x'-x)=Z^{-1}\mathrm{Tr}\{e^{-\beta\hat{H}}\hat{\phi}(x)\hat{\phi}(x')\}$$
$$=Z^{-1}\sum_{mn}\langle n|\hat{\phi}(\mathbf{x})|m\rangle\langle m|\hat{\phi}(\mathbf{x}')|n\rangle e^{-iE_m(t-t')}e^{iE_n(t-t'+i\beta)} , \tag{3.10}$$

where (2.24) has been used. The domain of analyticity of $\Delta^{(c)}$ is determined by the convergence of the sums over m and n. If we assume that the convergence is dominated by the exponentials (see, for example, Ref. [16]), and that the spectrum is bounded below, the propagator $\Delta^{(c)}(x-x')$ will be analytic if

$$-\beta<\mathrm{Im}(t-t')<0,\quad t \Longleftarrow t' ,$$
$$0<\mathrm{Im}(t-t')<\beta,\quad t' \Longleftarrow t , \tag{3.11}$$

where $t \Longleftarrow t'$ denotes that t follows t' on the contour.

This result has two consequences. First, the propagator functions $C^{\gtrless}(x-x')$ are defined only for relative times within two adjacent horizontal strips of width β in the complex time plane. To ensure the appropriate relative times, all contours must therefore start at some t_0, end at $t_0-i\beta$, and be wholly contained in the resulting horizontal strip of width β. Second, if all points on the contour are to be allowed as arguments of the propagators, the imaginary part of t must decrease monotonically along the contour. (The extension of the preceding analysis to higher n-point functions is discussed in Ref. [2].)

One acceptable contour is the Matsubara choice, which runs from $t=0$ to $t=-i\beta$ along the imaginary time axis. This contour is used to derive the conventional Euclidean Feynman rules at finite temperature [17]. Note that Eq. (3.11) gives the domain in which the propagator is analytic, since the sums in (3.10) converge absolutely. If one extends (3.11) to the boundaries of the allowed domain:

$$-\beta \le \mathrm{Im}(t-t') \le 0, \quad t \Leftarrow t' ,$$
$$0 \le \mathrm{Im}(t-t') \le \beta, \quad t' \Leftarrow t , \tag{3.12}$$

the resulting propagator will become a (singular) generalized function [18,19]. In the following discussion, we will see that special care is needed to construct Feynman rules using generalized functions.

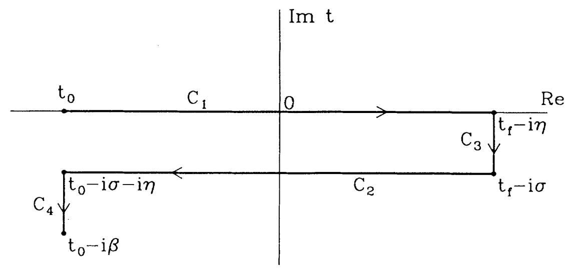

FIG. 1. An acceptable contour in the complex time plane. Here η is a positive infinitesimal, so that segments C_1 and C_2 have infinitesimal downward slopes, in agreement with Eq. (3.11).

B. Path-integral formulation

The most convenient way to derive Feynman rules from Eq. (3.6) is by rewriting it as a path integral. Consider the contour shown in Fig. 1, which satisfies the requirements discussed above. Equation (3.11) implies that the segments C_1 and C_2 must have infinitesimal downward slopes, as indicated in the figure. (We will suppress the infinitesimal arguments for brevity in the sequel.) Our ultimate goal is to generalize the discussion to allow for truly horizontal C_1 and C_2, and to extend these segments to $\pm\infty$, so that the propagators can be calculated for all real times and Fourier representations can be defined with continuous frequencies.

We return to Eq. (3.6) and rewrite it as

$$\begin{aligned} Z[j]= \sum_{\phi}\sum_{\phi'}\sum_{\phi''}\sum_{\phi'''} &\langle \phi(t_0-i\beta)|T_c \exp\left[i\int_{C_4} d^4x j(x)\hat{\phi}(x)\right]|\phi'(t_0-i\sigma)\rangle \\ &\times \langle \phi'(t_0-i\sigma)|T_c \exp\left[i\int_{C_2} d^4x j(x)\hat{\phi}(x)\right]|\phi''(t_f-i\sigma)\rangle \\ &\times \langle \phi''(t_f-i\sigma)|T_c \exp\left[i\int_{C_3} d^4x j(x)\hat{\phi}(x)\right]|\phi'''(t_f)\rangle\langle \phi'''(t_f)|T_c \exp\left[i\int_{C_1} d^4x j(x)\hat{\phi}(x)\right]|\phi(t_0)\rangle , \end{aligned} \tag{3.13}$$

by first decomposing the contour-ordered product into four pieces that are still in the proper order and then by inserting complete sets of states. Each term in the sum involves matrix elements of contour-ordered products of operators. These matrix elements can be written as path integrals over $\phi(x)$ and its associated momentum density $\Pi(x)$ using the Feynman-Mathews-Salam formula [6,20]. Since all the exponentials are correctly ordered, the path integrals can be recombined, leading to

$$Z[j]=\int_{\phi}^{\phi} D(\phi)D(\Pi)\exp\left\{i\int_c d^4x[\Pi(x)\dot{\phi}(x)-\mathcal{H}(\Pi,\phi) + j(x)\phi(x)]\right\} . \tag{3.14}$$

Here ϕ and Π are classical fields, $D(\phi)D(\Pi)$ denotes the measure for their integration, and $\dot{\phi}\equiv\partial\phi/\partial t$ is a directional derivative along the time contour. By returning to a discrete spacetime lattice, one can verify that there are the correct number of integrations to combine factors in Eq. (3.13) and arrive at a single path integral [14].

Due to the assumed infinitesimal downward slopes on C_1 and C_2, $i(dt/d\xi)$ has a positive real part everywhere on the contour. Thus, if the Hamiltonian is bounded below (which we will assume), the path integral in (3.14) exists, once the usual ultraviolet divergences are removed by renormalization. This implies that if the zero-temperature Feynman rules are *defined* by the limit of the finite-temperature results, there is no need to discuss "Wick rotations" to Euclidean time or the "euclidicity postulate."

Equation (3.14) is valid for any contour c consistent with Eq. (3.11), as can be proven by decomposing it into infinitesimal vertical and horizontal segments and using the same arguments as above. The periodic boundary condition on the scalar field,

$$\phi(t_0,\mathbf{x})=\phi(t_0-i\beta,\mathbf{x}) , \tag{3.15}$$

follows from the trace and is indicated symbolically by the limits on the path integral in (3.14). Integration over the momentum density $\Pi(x)$ is unrestricted. If we assume that the Hamiltonian density $\mathcal{H}(\Pi,\phi)$ is purely quadratic in $\Pi(x)$, we can perform the Gaussian functional integral over $\Pi(x)$ with the result

$$Z[j]=\mathcal{N}\int_{\phi}^{\phi} D(\phi)\exp\left\{i\int_c d^4x[\mathcal{L}(\phi,\partial^{\mu}\phi)+j(x)\phi(x)]\right\} . \tag{3.16}$$

The normalization factor $\mathcal{N}$ can be determined by writing (3.14) on the lattice, but we will see that this is not necessary.

The Lagrangian $\mathcal{L}$ can be decomposed into an interacting part $\mathcal{L}_{\rm int}(\phi)$ and a noninteracting part $\mathcal{L}_0(\phi,\partial^\mu\phi)$ in the usual fashion [2,4,14,20]. We can remove interaction terms from the path integral (3.16) by writing $\mathcal{L}_{\rm int}$ in terms of functional derivatives with respect to the source j:

$$Z[j]=\mathcal{N}\exp\left[i\int_c d^4x'\mathcal{L}_{\rm int}\left[\frac{\delta}{i\delta j(x')}\right]\right]\int_\phi^\phi D(\phi)\exp\left[i\int_c d^4x[\mathcal{L}_0(\phi,\partial^\mu\phi)+j(x)\phi(x)]\right] . \tag{3.17}$$

Since $\mathcal{L}_0$ is bilinear in ϕ by construction, it is possible to define a kernel for the Gaussian integration and shift variables to yield

$$Z[j]=\exp\left[i\int_c d^4x'\mathcal{L}_{\rm int}\left[\frac{\delta}{i\delta j(x')}\right]\right]\exp\left[-\frac{i}{2}\int_c d^4x\int_c d^4y\, j(x)\Delta_0^{(c)}(x-y)j(y)\right]\mathcal{N}\int_\phi^\phi D(\phi)\exp\left[i\int_c d^4x\,\mathcal{L}_0(\phi,\partial^\mu\phi)\right] . \tag{3.18}$$

Here $\Delta_0^{(c)}(x-y)$ is the noninteracting contour propagator (i.e., the inverse of the Gaussian kernel), which will be evaluated and discussed in the next subsection.

Our expression (3.18) for the generating functional is valid on any acceptable contour. If we turn off the interactions and sources, we find

$$Z_0[0]=\mathcal{N}\int_\phi^\phi D(\phi)\exp\left[i\int_c d^4x\,\mathcal{L}_0(\phi,\partial^\mu\phi)\right] . \tag{3.19}$$

But $Z[0]$ and likewise $Z_0[0]$ are *independent* of the contour, as noted above. Thus Eq. (3.19) is simply the noninteracting partition function Z_0, which is evaluated explicitly using a path integral in Ref. [14]. Finally, Eq. (3.18) becomes

$$Z[j]=Z_0\exp\left[i\int_c d^4x'\mathcal{L}_{\rm int}\left[\frac{\delta}{i\delta j(x')}\right]\right]\exp\left[-\frac{i}{2}\int_c d^4x\int_c d^4y\, j(x)\Delta_0^{(c)}(x-y)j(y)\right] . \tag{3.20}$$

This is an exact expression for the generating functional for an arbitrary acceptable contour, written in the rest frame of the thermal bath. The normalization factor $\mathcal{N}$, which can be determined by writing the path integral on a lattice, has been correctly included in the partition function Z_0, which provides the overall normalization for $Z[j]$. Renormalization can be performed by including the counterterms from Eq. (2.9) in $\mathcal{L}_{\rm int}$ and by subtracting the interacting zero-point energy [21]. This last quantity can be determined by evaluating the functional derivatives in (3.20), setting the sources to zero, and taking the zero-temperature limit.

C. The scalar contour propagator

Since the noninteracting Lagrangian density $\mathcal{L}_0$ is bilinear in ϕ, the path integral in Eq. (3.17) can be written as

$$\int_\phi^\phi D(\phi)\exp\left\{i\int_c d^4x\left[\tfrac{1}{2}\int_c d^4y\,\phi(x)A(x-y)\phi(y)+j(x)\phi(x)\right]\right\} . \tag{3.21}$$

One can now shift field variables to

$$\phi'(x)=\phi(x)+\int_c d^4y\,\Delta_0^{(c)}(x-y)j(y) \tag{3.22}$$

and rewrite Eq. (3.17) as in (3.18), where $\Delta_0^{(c)}(x-y)$ is the inverse of the kernel $A(x-y)$. The kernel and propagator are constructed explicitly on a lattice version of the Matsubara contour in Ref. [14]. In the continuum limit, which is of interest here, the contour propagator satisfies

$$(-\partial_t^2+\nabla_{\mathbf{x}}^2-m_s^2)\Delta_0^{(c)}(x-x')=\delta_c^{(4)}(x-x') , \tag{3.23}$$

where m_s is the scalar mass, ∂_t is a directional derivative along the contour, and the contour delta function $\delta_c^{(4)}(x-x')$ is defined by Eqs. (2.21) and (3.4). This differential equation also follows directly from the noninteracting versions of the definition (3.8) and the scalar field equation (2.11), when one remembers that ∂_t acts on the contour-ordering operator T_c.

To solve Eq. (3.23), we need boundary conditions on $\Delta_0^{(c)}(x-x')$. In analogy to Eq. (3.9), we define

$$\begin{aligned} i\Delta_0^{(c)}(x-x')&\equiv\langle\langle T_c\hat\phi(x)\hat\phi(x')\rangle\rangle_0=\theta_c(t-t')\langle\langle\hat\phi(x)\hat\phi(x')\rangle\rangle_0+\theta_c(t'-t)\langle\langle\hat\phi(x')\hat\phi(x)\rangle\rangle_0\\ &\equiv\theta_c(t-t')C_0^{>}(x-x')+\theta_c(t'-t)C_0^{<}(x-x') , \end{aligned} \tag{3.24}$$

where the subscripts "0" denote that the system is noninteracting. By writing out the interaction-picture operators with Eq. (2.24) and by using the cyclic properties of the trace, we immediately obtain the familiar Kubo-Martin-Schwinger (KMS) boundary condition

$$C_0^{>}(t-t',\mathbf{x}-\mathbf{x}')=C_0^{<}(t-t'+i\beta,\mathbf{x}-\mathbf{x}') \ . \tag{3.25}$$

If $-\beta<\mathrm{Im}(t-t')<0$, as in Eq. (3.11), both sides of Eq. (3.25) can be computed from sums over matrix elements, as in Eq. (3.10), and Eq. (3.25) relates two functions that are defined when $t-t'$ is in the original horizontal strip of width β.

Evidently, $C_0^{>}$ is defined for relative time arguments below the real axis, while $C_0^{<}$ is defined for relative time arguments above the axis, so these functions exist in two adjacent horizontal strips of width β in the complex time plane. As discussed in Ref. [16], microscopic causality ensures that $C_0^{>}$ and $C_0^{<}$ will agree on a finite segment of the real axis, if the domain is extended as in Eq. (3.12). Thus, by the edge-of-the-wedge theorem [22], we can define a *single* analytic function containing both $C_0^{>}$ and $C_0^{<}$, which is in fact the propagator of Eq. (3.24):

$$i\Delta_0^{(c)}(t-t',\mathbf{x}-\mathbf{x}') = \begin{cases} C_0^{>}(t-t',\mathbf{x}-\mathbf{x}'), & -\beta<\mathrm{Im}(t-t')<0 \ , \\ C_0^{<}(t-t',\mathbf{x}-\mathbf{x}'), & 0<\mathrm{Im}(t-t')<\beta \ . \end{cases} \tag{3.26}$$

Thus $\Delta_0^{(c)}(x-x')$ exists for $-\beta<\mathrm{Im}(t-t')<\beta$. Note that for $\mathrm{Im}(t-t')=0$, the propagator is defined by taking the limit from above or below the real $t-t'$ axis; these limiting values will be different except on the part of the real axis where $C_0^{<}$ and $C_0^{>}$ are equal. If desired, one can extend $\Delta_0^{(c)}$ to the entire complex plane by periodic repetition [16].

The KMS condition (3.25) was motivated above using the definition of $\Delta_0^{(c)}$ as a thermal average, as in Eq. (3.24). We emphasize, however, that due to the periodicity condition (3.15) on ϕ, the KMS condition arises *automatically* when one inverts the kernel A in the path integral (3.21). (This was demonstrated explicitly in Ref. [14] and is also discussed by Niégawa [23].) The inverse of the kernel is thus also periodic, in the sense of (3.25) and (3.26), which implies that the shifted field ϕ' in Eq. (3.22) satisfies Eq. (3.15). Moreover, $\Delta_0^{(c)}$ defined by inverting the kernel A is identical to that defined in Eq. (3.24).

Rather than solve Eq. (3.23) directly, it is easier to use the free-field expansion

$$\hat{\phi}(t,\mathbf{x})=\frac{1}{\sqrt{\mathcal{V}'}}\sum_{\mathbf{k}}\frac{1}{\sqrt{2\omega_k}}(c_{\mathbf{k}}e^{-ik\cdot x}+c_{\mathbf{k}}^{\dagger}e^{ik\cdot x})_{k_0=\omega_k} \ , \tag{3.27}$$

where $\omega_k\equiv(\mathbf{k}^2+m_s^2)^{1/2}$ and $\mathcal{V}'$ is the quantization volume in the comoving frame. For a noninteracting system,

$$\begin{aligned} \langle\langle c_{\mathbf{k}}^{\dagger}c_{\mathbf{k}'}\rangle\rangle_0&=n_b(\beta\omega_k)\delta_{\mathbf{k}\mathbf{k}'}, \\ \langle\langle c_{\mathbf{k}}c_{\mathbf{k}'}^{\dagger}\rangle\rangle_0&=[1+n_b(\beta\omega_k)]\delta_{\mathbf{k}\mathbf{k}'} \ , \end{aligned} \tag{3.28}$$

where the Bose distribution function is defined by

$$n_b(z)\equiv\frac{1}{e^z-1} \ , \tag{3.29}$$

and thermal averages of two creation or two annihilation operators vanish. These results follow from the free-field thermodynamic potential and the commutation relations (2.16) on $\hat{\phi}$.

A straightforward calculation produces

$$\begin{aligned} \langle\langle\hat{\phi}(x)\hat{\phi}(x')\rangle\rangle_0&=C_0^{>}(x-x')=C_0^{<}(x'-x) \\ &=\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\{e^{ik\cdot(x-x')}n_b(\beta\omega_k)+e^{-ik\cdot(x-x')}[1+n_b(\beta\omega_k)]\}_{k_0=\omega_k} \ , \end{aligned} \tag{3.30}$$

where we have taken the thermodynamic limit $\mathcal{V}'\to\infty$. For contours constrained by (3.11), the integrand always contains exponential damping factors that render the integral finite and ensure that $C_0^{\gtrless}(x-x')$ are analytic functions of $t-t'$. In contrast, if (3.12) is used instead, the integrands may oscillate at large $|\mathbf{k}|$ and $C_0^{\gtrless}(x-x')$ become generalized functions [18]. It can also be verified that (3.30) satisfies the periodicity condition (3.25). [The relation $e^{\beta\omega_k}n_b(\beta\omega_k)=1+n_b(\beta\omega_k)$ is useful for proving this result.]

The three-dimensional Fourier representation (3.30) can be used to construct a set of Feynman rules, as discussed by Iredale [24]. To generate covariant rules, however, it is convenient to introduce the four-dimensional representations

$$C_0^{\gtrless}(x-x')\equiv\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-x')}C_0^{\gtrless}(k) \ , \tag{3.31}$$

where $C_0^{\gtrless}(k)\equiv C_0^{\gtrless}(k_0,\mathbf{k})$ are functions of both k_0 and $\mathbf{k}$, which are *real* variables. After introducing a dummy integration over k_0 into Eq. (3.30), comparison with (3.31) yields

$$\begin{aligned} C_0^{>}(k)&=2\pi[1+n_b(\beta k_0)]\mathrm{sgn}(k_0)\delta(k^2-m_s^2) \\ &=e^{\beta k_0}C_0^{<}(k) \ , \end{aligned} \tag{3.32}$$

where $\mathrm{sgn}(k_0)\equiv\theta(k_0)-\theta(-k_0)$. A *spectral representation* of the contour propagator follows from Eqs. (3.24), (3.31), and (3.32):

$$i\Delta_0^{(c)}(x-x')=\int \frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-x')}\rho_0(k)[\theta_c(t-t')+n_b(\beta k_0)]\ , \tag{3.33}$$

where the noninteracting *spectral density* is

$$\rho_0(k)\equiv C_0^{>}(k)-C_0^{<}(k)=2\pi\,\mathrm{sgn}(k_0)\delta(k^2-m_s^2)\ . \tag{3.34}$$

These expressions are valid for any contour that satisfies Eq. (3.11), and it can be verified that (3.33) is consistent with the differential equation (3.23) and the periodicity condition (3.25). Thus the spectral representation (3.33) has all the required properties.

As discussed after Eq. (3.30), $\Delta_0^{(c)}(x-x')$ is an analytic function of $t-t'$ for contours consistent with (3.11). If the contours are extended, however, to those defined by Eq. (3.12), the propagator will become a generalized function. This result has little effect on the discussion of a single noninteracting propagator, but it causes problems for the derivation of Feynman rules, as these will involve convolutions of noninteracting propagators. It is well known that products of generalized functions do not always produce generalized functions [18], and typically, convolutions of such functions are not well defined [19]. The subtleties involved in constructing Feynman rules using a singular $\Delta_0^{(c)}$ will play an important role in the subsequent discussion.

D. Extension to infinite times

We now have an exact expression for the generating functional [Eq. (3.20)] and have determined a spectral representation for the propagator $\Delta_0^{(c)}(x-x')$ that is valid on contours of finite extent consistent with Eq. (3.11). In practice, however, it is a great advantage to consider contours that run along the whole real axis ($-\infty\le t\le+\infty$). This makes four-dimensional Fourier representations of contour-ordered propagators possible and is important for expressing results in covariant form.

To this end, consider the contour of Fig. 1 with $\sigma=\beta$ (see Fig. 2), so that we have only the segments C_1, C_2, and C_3. From Eq. (3.20), it follows that the functional derivatives will generate a set of diagrams (some with closed loops and some with sources) where vertices on each of the three segments are connected by various factors of $\Delta_0^{(c)}$. Contours C_1 and C_2 will allow us to compute the real-time propagators, which are useful for determining the dynamic properties of the system. Contour C_3 resembles the Matsubara contour, which allows us to define Euclidean propagators and compute equilibrium thermodynamic properties. These observations will be made more precise in the following discussion.

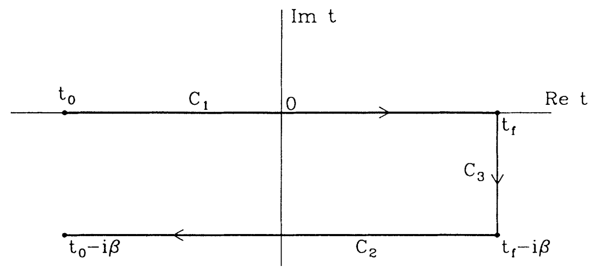

FIG. 2. Contour in the complex time plane used for discussing the $t_0\to-\infty$, $t_f\to+\infty$ limit. Note that here the segments C_1 and C_2 are strictly horizontal.

Consider the $t_0\to-\infty$ and $t_f\to+\infty$ limit of the generating functional $Z[j]$. Since the full contour must always lie within a strip of width β, the contour slopes must vanish in the limit, and the segments C_1 and C_2 behave as if they are truly horizontal for any finite times. Thus, in this limit, the contours C_1 and C_2 are in the class defined by Eq. (3.12), and $\Delta_0^{(c)}(x-x')$ will become a generalized function of $t-t'$. Moreover, we must ensure that the limit

$$\lim_{\substack{t_f\to+\infty\\ t_0\to-\infty}}\int d^3x\int d^3x'\int_{t_0}^{t_f}dt\int_{t_0}^{t_f}dt'\,j(x)\Delta_0^{(c)}(x-x')j(x') \tag{3.35}$$

exists, so that the final exponential in Eq. (3.20) (which is proportional to the noninteracting generating functional) is well defined. Let us concentrate first on contour C_1 and extend the arguments later to the other segments.

We begin by evaluating the k_0 integral in Eq. (3.33), which produces

$$i\Delta_0^{(c)}(x-x')=\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\{[\theta(t'-t)+n_b(\beta\omega_k)]e^{ik\cdot(x-x')}+[\theta(t-t')+n_b(\beta\omega_k)]e^{-ik\cdot(x-x')}\}_{k_0=\omega_k}\ . \tag{3.36}$$

A simple θ function can now be used since both times are assumed to be on contour C_1. The angular integrals are easily done, and we find

$$i\Delta_0^{(c)}(x-x')=\frac{1}{4\pi^2r}\int_0^\infty\frac{kdk}{\omega_k}e^{-i\omega_k|t-t'|}\sin kr+\frac{1}{2\pi^2r}\int_{m_s}^\infty d\omega\cos(\omega|t-t'|)n_b(\beta\omega)\sin(r\sqrt{\omega^2-m_s^2})\ , \tag{3.37}$$

where $r\equiv|\mathbf{x}-\mathbf{x}'|$. Consider this expression as $|t-t'|\to\infty$. The second integral is the cosine transform of an absolutely integrable function of ω. Thus the Riemann-Lebesgue lemma [18] can be applied, which implies that the second integral vanishes in the desired limit. In contrast, the Riemann-Lebesgue lemma *cannot* be applied to the first integral, since a change of integration variables from k to ω_k shows that the function being

Fourier transformed is not absolutely integrable. Nevertheless, the first integral can be evaluated explicitly in terms of Bessel functions as (see Appendix C of Ref. [12])

$$\frac{i}{8\pi r}\frac{\partial}{\partial r}\begin{cases} H_0^{(2)}\{m_s[|t-t'|^2-r^2]^{1/2}\}\ , & |t-t'|>r\ ,\\ (2i/\pi)K_0\{m_s[r^2-|t-t'|^2]^{1/2}\}, & |t-t'|<r\ ,\end{cases} \tag{3.38}$$

which vanishes as $|t-t'|^{-3/2}$ when $|t-t'|\to\infty$. Thus we conclude [25] that

$$\lim_{|t-t'|\to\infty}\Delta_0^{(c)}(x-x')=0\ , \tag{3.39}$$

at least when both times are on C_1.

These arguments also hold when t and t' are both on C_2, since the imaginary time shifts cancel, and for propagators connecting C_1 and C_2, since the required integrals either satisfy the Riemann-Lebesgue lemma or can be evaluated explicitly in terms of Bessel functions, as above. When both times are on C_3, the discussion is unnecessary since the integration limits are finite. Thus the only outstanding question concerns propagators connecting contours C_1 or C_2 with C_3, which we will examine shortly.

First, however, notice that the above considerations are inadequate to ensure the existence of (3.35), since they apply only when $|t-t'|$ becomes large. If both t and t' go to infinity *together*, $\Delta_0^{(c)}(x-x')$ does not vanish, and the time integrals will not converge for arbitrary sources $j(x)$. This problem can be solved by imposing *adiabatic switching,* which requires that the sources vanish in the remote past and distant future. As can be seen from Eq. (3.20), this is equivalent to making the system noninteracting in this limit. Adiabatic switching must also be imposed in the zero-temperature many-body formalism to arrive at meaningful results [17]. Similar restrictions are also required for the sources on C_2, so we must have

$$\lim_{t\to\pm\infty} j(t\in C_1,\mathbf{x})=0\ ,\quad \lim_{\mathrm{Re}t\to\pm\infty} j(t\in C_2,\mathbf{x})=0\ . \tag{3.40}$$

Adiabatic switching need not be imposed on the sources on C_3, since the integration region is finite.

Finally, we consider the contours C_1 and C_3 and prove the important result:

$$\lim_{t_f\to+\infty}\int d^3x\int d^3x'\int_{t_0}^{t_f}dt\int_{t_f}^{t_f-i\beta}dt'\,j(x)\Delta_0^{(c)}(x-x')j(x')=0\ . \tag{3.41}$$

(The integral involving C_2 and C_3 behaves analogously.) If we parametrize the integral on C_3 by letting $t'\equiv t_f-i\lambda$ and realize that t' always follows t on the contour, Eq. (3.41) becomes

$$\lim_{t_f\to+\infty}-\int d^3x\int d^3x'\int_{t_0}^{t_f}dt\int_0^{\beta}d\lambda\, j(t,\mathbf{x})C_0^{<}(t-t_f+i\lambda,\mathbf{x}-\mathbf{x}')j(t_f-i\lambda,\mathbf{x}')=0\ . \tag{3.42}$$

From Eqs. (3.31) and (3.32), we find

$$\begin{aligned} C_0^{<}(t-t_f+i\lambda,\mathbf{k}) &= \int_{-\infty}^{+\infty}\frac{dk_0}{2\pi}e^{-ik_0(t-t_f+i\lambda)}C_0^{<}(k_0,\mathbf{k})\\ &= \int_{-\infty}^{+\infty}\frac{dk_0}{2\pi}e^{ik_0(t_f-t)}e^{(\lambda-\beta)k_0}[1+n_b(\beta k_0)]\rho_0(k)\ , \end{aligned} \tag{3.43}$$

so that

$$C_0^{<}(t-t_f+i\lambda,r)=\frac{1}{4\pi^2 r}\int_{m_s}^{\infty}d\omega[e^{-i\omega(t-t_f)}e^{\lambda\omega}n_b(\beta\omega)-e^{i\omega(t-t_f)}e^{-\lambda\omega}n_b(-\beta\omega)]\sin(r\sqrt{\omega^2-m_s^2})\ . \tag{3.44}$$

For $0<\lambda<\beta$, the integrand is sufficiently damped that the Riemann-Lebesgue lemma can be applied, showing that Eq. (3.42) holds as long as t is kept finite. When t and t_f go to infinity together, the adiabatic switching of Eq. (3.40) ensures that (3.42) remains valid. Note that no adiabatic switching is needed for the source on C_3. It is also clear that the entire preceding discussion holds when $t_0\to-\infty$ on the other end of the contour.

Let us summarize the results of this subsection. To obtain a generating functional with contours that run along the whole real time axis, we must consider propagators that are generalized functions. We must also ensure that the generating functional remains well defined in the limits $t_0\to-\infty$ and $t_f\to+\infty$. For propagators linking the horizontal segments, the Riemann-Lebesgue lemma and the asymptotic time dependence of the propagators show that they vanish when their time arguments are large. This is insufficient, however, to guarantee the existence of the generating functional, and we must also impose adiabatic switching for the sources on these segments. For the propagators connecting the horizontal and vertical segments, we prove the important result (3.41), which shows that these contributions vanish in the desired limit. Thus the noninteracting generating functional *factorizes* into well-defined contributions from the horizontal and vertical contours in this limit.

E. Factorization of the generating functional

We have proven that in the limit of infinite (real) times, the *noninteracting* generating functional factorizes into separate contributions from the horizontal and vertical contours. Since the interaction terms in Eq. (3.20) also factorize, it is tempting to conclude [2,4] that the *exact* generating functional factorizes as well:

$$\lim_{t_f\to+\infty}\lim_{t_0\to-\infty} Z[j]=Z_{12}[j]Z_3[j] \; . \tag{3.45}$$

Unfortunately, this conclusion is wrong when propagators with the singular spectral density of Eq. (3.34) are used in the infinite-time limit.

As shown by Iredale [24], even though Eqs. (3.39) and (3.41) are valid for these propagators, some contributions involving two propagator functions do *not* vanish. For example, the convolution of a propagator connecting C_1 to C_3 with one going back from C_3 to C_1 [see Eq. (3.49)] remains finite in the infinite-time limit. Thus one finds different results for the exact generating functional if the infinite-time limit is taken before or after the functional derivatives are performed in Eq. (3.20). This disturbing result occurs because the propagator is a generalized function in the infinite-time limit, and the singular spectral density produces only a power-law decay at large times; thus, the limit of the convolution of propagators is unequal to the convolution of the limits of the propagators. Nevertheless, as demonstrated by Iredale, these singular propagators can be used to derive a consistent set of Feynman rules, if the infinite-time limit is taken after all convolutions are evaluated. Unfortunately, these convolutions of propagators are awkward and the Feynman rules are unwieldy. In particular, it is impossible to factorize the exact generating functional as in Eq. (3.45) into real-time and imaginary-time contributions.

To avoid this problem, we interpret convolutions of propagators differently in the infinite-time limit [2,4]. To be precise, we will define $\rho_0(k)$ in a slightly different way and modify the spectral density of Eq. (3.34) so that it is a meromorphic function of the frequency k_0. First rewrite the contour propagator using Eqs. (3.24) and (3.30), which produces

$$i\Delta_0^{(c)}(x-x')=\int\frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\{[\theta_c(t'-t)+n_b(\beta\omega_k)]e^{ik\cdot(x-x')}+[\theta_c(t-t')+n_b(\beta\omega_k)]e^{-ik\cdot(x-x')}\}_{k_0=\omega_k} \; . \tag{3.46}$$

Now define a spectral density

$$\begin{aligned}\rho_0(k)&=\frac{i}{2\omega_k}\left[\frac{1}{k_0-\omega_k+i\epsilon}-\frac{1}{k_0-\omega_k-i\epsilon}-\frac{1}{k_0+\omega_k+i\epsilon}+\frac{1}{k_0+\omega_k-i\epsilon}\right]\\&\equiv\frac{\pi}{\omega_k}[\delta_\epsilon(k_0-\omega_k)-\delta_\epsilon(k_0+\omega_k)]\\&\equiv 2\pi\,\mathrm{sgn}(k_0)\delta_\epsilon(k^2-m_s^2) \; .\end{aligned} \tag{3.47}$$

Here the *regularized* delta function, denoted δ_ϵ, has been introduced. The spectral representation (3.33) evaluated with (3.47) reproduces (3.46), if the k_0 integral is interpreted as a contour integral and $\epsilon\to 0$ [26]. Note that Eq. (3.47) agrees with Eq. (3.34) if one applies the usual symbolic identity

$$\lim_{\epsilon\to 0}\frac{1}{\omega\pm i\epsilon}=\mathcal{P}\frac{1}{\omega}\mp i\pi\delta(\omega) \; , \tag{3.48}$$

which is valid for real ω when used inside an integral. (The Cauchy principal value is denoted by $\mathcal{P}$.)

In contrast, if ϵ is kept finite, the spectral density remains a meromorphic function of k_0. This modifies the damping of the propagator at large times from a power-law to an exponential decay and produces different results for convolutions of propagators, as we will see shortly. Note that the Bose distribution function $n_b(\beta k_0)$ in Eq. (3.33) must retain the "off-shell" k_0 argument to ensure that the KMS condition (3.25) is satisfied.

We emphasize that if one considers only contours defined by Eq. (3.11), the distinction between (3.34) and (3.47) is immaterial, since the $\epsilon\to 0$ limit can be taken *either before or after* the k_0 integration in Eq. (3.33) is performed. Moreover, for discussing the properties of a single propagator, the distinction is unimportant, even for contours defined by (3.12), provided one is willing to live with generalized functions. When infinite contours defined by (3.12) are allowed and used to construct the Feynman rules, however, one encounters convolutions of generalized functions with support on the whole real time axis. Such convolutions are not always well defined [19], so the order of these limiting procedures becomes relevant, and we can choose it to make the Feynman rules simple.

We will interpret convolutions of propagators using the four-dimensional spectral representation (3.33) with the regularized spectral density in Eq. (3.47) and retain the infinitesimal ϵ until the end of all calculations. This procedure is commonly used in many-body calculations at zero temperature [10,17,20]; the advantages here are that the spectral density remains a meromorphic function of k_0 and the generating functional factorizes, as we show below. Later we will discuss why this method produces results that are identical to those computed with a singular spectral density and the nonfactorized generating functional.

To prove that the exact generating functional factorizes when $\rho_0(k)$ is regularized, consider the convolution

$$\lim_{t_f\to+\infty}\int_{C_3} d^4y\, i\Delta_0^{(13)}(x-y)i\Delta_0^{(31)}(y-x') = \lim_{t_f\to+\infty}\int_{C_3} d^4y\, C_0^{<}(x-y)C_0^{>}(y-x') \; . \qquad (3.49)$$

Here the superscripts on the propagators denote the contours labeled in Fig. 2. The integral can be evaluated using Eqs. (3.24) and (3.33), together with the regularized spectral density (3.47), with the result

$$\lim_{t_f\to+\infty} -i\int \frac{d^4k}{(2\pi)^4} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}e^{-ik_0t}\int_{-\infty}^{\infty} dp_0 2\pi n_b(\beta k_0)[1+n_b(\beta p_0)]\mathrm{sgn}(k_0)\mathrm{sgn}(p_0) \times\delta_\epsilon(k_0^2-\omega_k^2)\delta_\epsilon(p_0^2-\omega_k^2)e^{ip_0t'}e^{it_f(k_0-p_0)}\left[\frac{e^{\beta(k_0-p_0)}-1}{k_0-p_0}\right] . \qquad (3.50)$$

Since the δ_ϵ functions are regularized, the integrands of the k_0 and p_0 integrals are absolutely integrable, and $k_0 \neq p_0$. The Riemann-Lebesgue lemma can be applied to both frequency integrals, leading to the conclusion that this "cross-contour" convolution vanishes. (This can also be verified explicitly by performing the frequency integrals; the infinitesimal ϵ factors in the regularized δ_ϵ functions produce a result proportional to $e^{-2\epsilon t_f}$, which vanishes in the infinite-time limit.) This result would not obtain if the δ functions were singular, since then the oscillating factor involving it_f in (3.50) would disappear, leading to a nonzero result for the integral [24]. Higher-order convolutions connecting the vertical and horizontal contours behave similarly.

Thus, when the spectral densities in the propagators are regularized, *all* contributions to the generating functional that connect the horizontal and vertical contours vanish in the infinite-time limit [27], and the exact generating functional factorizes as in Eq. (3.45). It is clear that similar arguments will apply to segment C_4 in this limit, if we consider general contours ($\sigma\neq\beta$) of the type in Fig. 1. Thus we can state the factorization theorem in the general case as

$$\lim_{t_f\to+\infty}\lim_{t_0\to-\infty} Z[j]=Z_{12}[j]Z_{34}[j] \; . \qquad (3.51)$$

We repeat the basic arguments that lead to this result: In the limit of infinite times and truly horizontal contours, both the noninteracting and interacting propagators become generalized functions, which follows from the analysis of Eq. (3.10). The Feynman rules for the interacting propagator will involve convolutions of singular noninteracting propagators with support on the whole real time axis, and such convolutions are not well defined [19]. We will evaluate these convolutions by using regularized noninteracting propagators in the intermediate stages of any calculation and then let $\epsilon\to 0$ at the end to define the interacting propagator as a generalized function. This regularization modifies the asymptotic time dependence of the noninteracting propagator from a power-law to an exponential decay and leads to the factorization of the generating functional. We discuss below how this procedure gives the same results as those obtained by using singular spectral densities and taking the infinite-time limit at the end of the calculation.

First, however, we note several important corollaries of the factorization theorem (3.51).

(i) Consider again the case $\sigma=\beta$ of Fig. 2. Segment C_3 looks just like the Matsubara contour (which is defined by $0\le t\le -i\beta$ along the imaginary time axis) shifted horizontally along the real axis. As proved in Ref. [14], however, the Euclidean part of the generating functional is *independent of the origin of the contour.* Thus, for $\sigma=\beta$, Eq. (3.51) becomes

$$\lim_{t_f\to+\infty}\lim_{t_0\to-\infty} Z[j]=Z_{12}[j]Z_M[j] \; . \qquad (3.52)$$

Here M denotes the Matsubara contour, and

$$Z_M[j]=Z_0\exp\left[i\int_M d^4x'\mathcal{L}_{\rm int}\left[\frac{\delta}{i\delta j(x')}\right]\right] \times\exp\left[-\frac{i}{2}\int_M d^4x\int_M d^4y\, j(x)\Delta_0^{(c)}(x-y)j(y)\right] . \qquad (3.53)$$

Note that the normalization factor Z_0 is included with $Z_M[j]$, and this expression must still be renormalized by the addition of counterterms to $\mathcal{L}_{\rm int}$ and the removal of the zero-point energy. Moreover, it is important that no adiabatic switching is applied to sources on C_3, to ensure the independence with respect to the shift of origin.

(ii) Now take the $j\to 0$ limit of Eq. (3.52):

$$\lim_{t_f\to+\infty}\lim_{t_0\to-\infty} Z[0]=Z=Z_{12}[0]Z_M[0] \; . \qquad (3.54)$$

Here the notation $Z[0]$ implies that functional derivatives appearing in $\mathcal{L}_{\rm int}$ must be taken *first* and j set to zero at the end, leaving only closed 'bubble" diagrams. As shown in Ref. [14], however, the partition function Z is given entirely by $Z_M[0]$. Thus we obtain the remarkable result

$$Z_{12}[0]=1 \; . \qquad (3.55)$$

This implies that the remainder of the generating functional,

$$Z_{12}[j]=\exp\left[i\int_{C_1C_2}d^4x'\mathcal{L}_{\rm int}\left[\frac{\delta}{i\delta j(x')}\right]\right]\exp\left[-\frac{i}{2}\int_{C_1C_2}d^4x\int_{C_1C_2}d^4y\, j(x)\Delta_0^{(c)}(x-y)j(y)\right], \tag{3.56}$$

is correctly normalized as it stands.

It is worth emphasizing this point. In the zero-temperature formalism, which involves real times, adiabatic switching is introduced to allow for well-defined manipulations when the time integrals are extended to infinity. Using the theorems of Gell-Mann–Low and Goldstone, one learns that the diagrammatic expansion of the propagators contains only "connected diagrams" [17]. In the present case, all disconnected graphs from the horizontal contours ($\propto \ln Z_{12}[0]$) *cancel* in the infinite-time limit, implying that no overall normalization or phase factor is needed. The normalization resides in the Euclidean part of the generating functional, where it serves to correctly reproduce the partition function [14].

(iii) As we will show in Sec. III G, real-time propagators derived from the generating functional $Z_{12}[j]$ *do not depend on where C_2 is placed within the horizontal strip of width β.* Thus Eq. (3.56) yields identical results *for real-time propagators* for any value of σ in Fig. 1, and we are free to choose σ to make the Feynman rules as simple as possible.

Summarizing the important conclusions thus far: In the limit of infinite real times, the generating functional splits into two factors, as in Eqs. (3.52), (3.53), and (3.56), provided that we use regularized spectral densities in the noninteracting propagators and take $\epsilon\to 0$ at the end of all calculations. The horizontal contours produce the real-time propagators, which determine the dynamical properties of the system. These contours do not contribute to the equilibrium thermodynamics (the partition function) by Eq. (3.55). The vertical contour is equivalent to the Matsubara choice, which produces the Euclidean propagators and allows us to compute equilibrium properties.

We remark that there are some instances in which the factorization of the generating functional is not possible. (See Sec. 2.4.1 of Ref. [2] and references cited therein.) To our knowledge, these cases involve nonrelativistic theories in which it is impossible to impose the adiabatic-switching condition (3.40) due to the presence of discrete zero-energy modes. It is difficult to construct Feynman rules for these theories, although they have been studied using thermofield dynamics. As the model of interest here is relativistic and has no known discrete zero-energy modes, we will apply the factorization theorem with confidence.

F. Discussion of limiting procedures

At this stage, we have shown that by using regularized propagators, it is possible to factorize the generating functional, which will lead to a simpler set of Feynman rules. One may ask, however, whether these rules are correct; in particular, do they yield the same results that would be obtained by working with finite-length contours until the end of the calculation and then letting the real time interval become infinite?

The basic difference between the two approaches involves the order in which the $\epsilon\to 0$ limit and the $t_0\to -\infty$, $t_f\to +\infty$ limits are taken. As shown in Ref. [24], if the $\epsilon\to 0$ limit is taken *first*, leading to the three-dimensional representation in Eq. (3.46), then $Z[j]$ does not factorize, since convolutions such as

$$\lim_{t_f\to +\infty}\int_{C_3}d^4y\,\Delta_0^{(13)}(x-y)\Delta_0^{(31)}(y-x')\neq 0 \tag{3.57}$$

do not vanish. The integrals that survive, when written in momentum space, always contain products of propagators with the same frequency argument; thus, oscillating exponentials that would otherwise damp the integral cancel, as we demonstrated explicitly in (3.50). These surviving "cross-contour" contributions contain self-energy insertions on the propagator lines and are thus analogous to "anomalous graphs" in the conventional imaginary-time formalism [3,17]. The purpose of these cross-contour contributions is also analogous, namely, they shift the argument in the distribution function $n_b(\beta\omega_k)$ from the *noninteracting* spectrum ω_k to the interacting spectrum, order-by-order in perturbation theory [23].

In contrast, when the $t_0\to -\infty$, $t_f\to +\infty$ limits are taken first and ϵ is retained until the end of the calculation, the propagator has the four-dimensional representation of Eq. (3.33), where the spectral function takes the regularized form

$$\rho_0(k)=2\pi\,\mathrm{sgn}(k_0)\delta_\epsilon(k^2-m_s^2)\ . \tag{3.58}$$

As noted after Eq. (3.48), it is necessary to retain the "off-shell" βk_0 argument in the distribution function $n_b(\beta k_0)$ to reproduce the KMS periodicity condition on the propagator. Because the frequency variables in different propagators are off shell, the anomalous contributions vanish by the Riemann-Lebesgue lemma, and the generating functional factorizes, as in Eq. (3.45). Moreover, when the surviving integrals over the frequency variables are performed, the integrands contain poles of order $n>1$, and the resulting derivatives of the distribution functions produce the correct (interacting) spectrum [28]. In the end, the results agree with those computed from the singular representation [23].

Thus, while either limiting procedure can be used to construct Feynman rules, the method used here is the simplest, since graphs connecting C_1 or C_2 to C_3 need not be considered. Furthermore, by working with a momentum-space representation that is analytic in both k_0 and $\mathbf{k}$, the propagators can be written in manifestly covariant form, as we discuss below.

G. The real-time scalar thermal propagator

We now concentrate on $Z_{12}[j]$ and derive explicit expressions for the scalar propagator that can be used to construct the Feynman rules. One of our goals is to show that real-time propagators derived from $Z_{12}[j]$ are independent of the value of σ in Fig. 1 (for $0<\sigma<\beta$). Begin by rewriting the first factor in Eq. (3.56) as

$$\exp\left[i\int_{C_1C_2} d^4x'\mathcal{L}_{\rm int}\left[\frac{\delta}{i\delta j(x')}\right]\right]=\exp\left[i\int_{-\infty}^{+\infty}dt'\int d^3x'\left\{\mathcal{L}_{\rm int}\left[\frac{\delta}{i\delta j_1(x')}\right]-\mathcal{L}_{\rm int}\left[\frac{\delta}{i\delta j_2(x')}\right]\right\}\right], \quad (3.59)$$

where we have defined the independent sources on each contour as

$$j_1(t,\mathbf{x})\equiv j(t,\mathbf{x}), \quad j_2(t,\mathbf{x})\equiv j(t-i\sigma,\mathbf{x}) . \quad (3.60)$$

Note that the time variables on the right-hand side of Eq. (3.59) are real. For the factor in Eq. (3.56) involving the propagator, define

$$\Delta_0^{(11)}(x-x')\equiv\Delta_0^{(c)}(t-t',\mathbf{x}-\mathbf{x}') , \quad (3.61)$$

$$\Delta_0^{(22)}(x-x')\equiv\Delta_0^{(c)}((t-i\sigma)-(t'-i\sigma),\mathbf{x}-\mathbf{x}') , \quad (3.62)$$

$$\Delta_0^{(12)}(x-x')\equiv\Delta_0^{(c)}(t-(t'-i\sigma),\mathbf{x}-\mathbf{x}')=-iC_0^{<}(t-t'+i\sigma,\mathbf{x}-\mathbf{x}') , \quad (3.63)$$

$$\Delta_0^{(21)}(x-x')\equiv\Delta_0^{(c)}((t-i\sigma)-t',\mathbf{x}-\mathbf{x}')=-iC_0^{>}(t-t'-i\sigma,\mathbf{x}-\mathbf{x}') , \quad (3.64)$$

where we have observed that all times on C_2 follow times on C_1. These definitions allow us to write

$$\exp\left[-\frac{i}{2}\int_{C_1C_2}d^4x\int_{C_1C_2}d^4x'j(x)\Delta_0^{(c)}(x-x')j(x')\right]$$

$$=\exp\left[-\frac{i}{2}\int_{-\infty}^{+\infty}dt\int d^3x\int_{-\infty}^{+\infty}dt'\int d^3x'j_r(x)\Delta_0^{(rs)}(x-x')j_s(x')\right] . \quad (3.65)$$

The repeated indices r and s are summed from 1 to 2 with the metric δ_{rs}, and functional differentiation with respect to the new sources is defined as

$$\frac{\delta}{\delta j_s(x')}j_r(x)=\delta_{rs}\delta^{(4)}(x-x')\equiv\delta_{rs}\delta(t-t')\delta^{(3)}(\mathbf{x}-\mathbf{x}') \quad (r,s=1,2) . \quad (3.66)$$

Note that $\delta(t-t')$ is the familiar Dirac δ function and *not* the contour function $\delta_c(t-t')$ defined in Eq. (2.21). The relative minus signs between Eqs. (2.21) and (3.66) that occur for sources on contour C_2 are incorporated in the definitions (3.63) and (3.64) and allow us to use the simple metric δ_{rs} when summing over contour labels.

We can therefore write the real-time generating functional as

$$Z_{12}[j]=\exp\left[i\int d^4y\left\{\mathcal{L}_{\rm int}\left[\frac{\delta}{i\delta j_1(y)}\right]-\mathcal{L}_{\rm int}\left[\frac{\delta}{i\delta j_2(y)}\right]\right\}\right]\exp\left[-\frac{i}{2}\int d^4x\int d^4x'j_r(x)\Delta_0^{(rs)}(x-x')j_s(x')\right] , \quad (3.67)$$

where all time integrals run from $-\infty$ to $+\infty$. The real-time scalar propagators follow from Eq. (3.3):

$$i\Delta(x_1,\ldots,x_n)=\left[\frac{\delta^n Z_{12}[j]}{i\delta j_1(x_1)\cdots i\delta j_1(x_n)}\right]_{j_1=j_2=0} . \quad (3.68)$$

All external functional derivatives involve sources on C_1, and no normalization factor is required because of Eq. (3.55).

We can now derive explicit forms for the propagators in Eqs. (3.61)–(3.64). The four-dimensional Fourier transform is defined as

$$i\Delta_0^{(rs)}(x-x')\equiv\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot(x-x')}i\Delta_0^{(rs)}(k) , \quad (3.69)$$

where all coordinate and momentum variables are real. This implies that the (constant) imaginary time shifts from contour C_2 are included in the Fourier coefficients $\Delta_0^{(rs)}(k)$. The spectral density $\rho_0(k)$ appearing in $\Delta_0^{(rs)}(k)$ is assumed regularized, as discussed previously.

Using Eqs. (3.33), (3.47), (3.61)–(3.64), and (3.69), it is a straightforward matter of algebra to arrive at the following expressions for the momentum-space propagator:

$$i\Delta_0^{(11)}(k)=i\theta(k_0)\Delta_{0F}(k)+i\theta(-k_0)\Delta_{0F}^{\dagger}(k)+\rho_0(k)n_b(\beta k_0) , \quad (3.70)$$

$$i\Delta_0^{(22)}(k)=-i\theta(k_0)\Delta_{0F}^{\dagger}(k)-i\theta(-k_0)\Delta_{0F}(k)+\rho_0(k)n_b(\beta k_0)=[i\Delta_0^{(11)}(k)]^{\dagger} , \quad (3.71)$$

$$i\Delta_0^{(12)}(k)=e^{\sigma k_0}\rho_0(k)n_b(\beta k_0) , \quad (3.72)$$

$$i\Delta_0^{(21)}(k)=e^{-\sigma k_0}\rho_0(k)[1+n_b(\beta k_0)]=e^{(\beta-2\sigma)k_0}i\Delta_0^{(12)}(k) , \quad (3.73)$$

where the scalar Feynman propagator is

$$\Delta_{0F}(k)=\frac{1}{k^2-m_s^2+i\epsilon}, \quad \Delta_{0F}^{\dagger}(k)=\frac{1}{k^2-m_s^2-i\epsilon} , \quad (3.74)$$

and the Bose distribution function $n_b(\beta k_0)$ is defined in Eq. (3.29).

We are now prepared to prove the independence of the real-time propagators with respect to the constant shift σ. (Our discussion follows Ref. [2], which in turn follows Ref. [29].) Note first that $\Delta_0^{(11)}(k)$ and $\Delta_0^{(22)}(k)$ are independent of σ, while $\Delta_0^{(12)}(k)\propto e^{\sigma k^0}$ and $\Delta_0^{(21)}(k)\propto e^{-\sigma k^0}$. Now consider any diagram contributing to the real-time propagator (3.68). [Only connected diagrams are relevant, because of Eq. (3.55).] Each diagram can be decomposed into subdiagrams that contain type-1 vertices only or type-2 vertices only, and since all external legs involve times on C_1, the subdiagrams are as shown in Fig. 3. As indicated, all "cross-contour" propagators $\Delta_0^{(12)}(k)$ or $\Delta_0^{(21)}(k)$ are associated with diagrams like Fig. 3(b); thus, some of the type-1 vertices contained in Fig. 3(a) have their legs amputated.

All diagrams like Fig. 3(a) are independent of σ, as are the shaded subdiagrams in Fig. 3(b). The only dependence on σ comes from the $\Delta_0^{(12)}(k)$ or $\Delta_0^{(21)}(k)$ propagators in the second class of diagrams. If all momenta k_i are labeled as outgoing, this σ dependence takes the form $\exp[\sigma(k_1^0+k_2^0+\cdots+k_n^0)]$, which assumes that the subdiagram has n propagator legs. Due to the Fourier representation (3.69), however, energy and momentum are conserved at each vertex, and $\sum_{i=1}^{n}k_i^0=0$ for every subdiagram like Fig. 3(b). Thus *there is no σ dependence in the real-time propagators.* If we restrict our attention to these propagators, Eq. (3.67) will yield identical results for any $0<\sigma<\beta$.

We are therefore free to choose σ as we like; not surprisingly, the most convenient results are obtained for $\sigma=\beta/2$, and we will adopt this choice for the remainder of this work. Equations (3.72) and (3.73) then imply

$$i\Delta_0^{(12)}(k)=e^{\beta k_0/2}\rho_0(k)n_b(\beta k_0)=i\Delta_0^{(21)}(k)\ , \tag{3.75}$$

so that $\Delta_0^{(rs)}(k)$ becomes a symmetric 2×2 matrix.

The scalar matrix propagator can now be rewritten using

$$\begin{aligned}\rho_0(k)&=\frac{i}{2\omega_k}\left[\frac{1}{k_0-\omega_k+i\epsilon}-\frac{1}{k_0-\omega_k-i\epsilon}-\frac{1}{k_0+\omega_k+i\epsilon}+\frac{1}{k_0+\omega_k-i\epsilon}\right]\\&=2\pi\,\mathrm{sgn}(k_0)\delta_\epsilon(k^2-m_s^2)\\&=i[\theta(k_0)-\theta(-k_0)][\Delta_{0F}(k)-\Delta_{0F}^\dagger(k)]\end{aligned} \tag{3.76}$$

and

$$\theta(k_0)n_b(\beta k_0)+\theta(-k_0)n_b(-\beta k_0)\equiv N_b(\beta k_0)=\frac{1}{\exp(\beta|k_0|)-1}\ . \tag{3.77}$$

If we define a unimodular "thermal" matrix $\underline{\mathbf{M}}_b$ as

$$\underline{\mathbf{M}}_b\equiv\begin{bmatrix}\cosh\Theta(k) & \sinh\Theta(k)\\ \sinh\Theta(k) & \cosh\Theta(k)\end{bmatrix}, \tag{3.78}$$

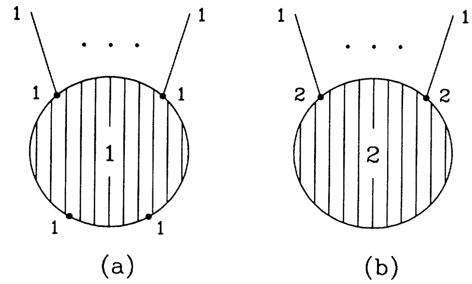

FIG. 3. Subdiagrams contained in real-time propagators. The shaded region in part (a) contains only type-1 vertices, while the shaded region in part (b) contains only type-2 vertices.

where

$$\sinh^2\Theta(k)\equiv N_b(\beta k_0),\quad \cosh^2\Theta(k)=1+N_b(\beta k_0)\ , \tag{3.79}$$

and $\Theta(k)\geq 0$, then

$$\Delta_0^{(rs)}(k)\equiv\underline{\Delta}_0(k)=\underline{\mathbf{M}}_b\begin{bmatrix}\Delta_{0F}(k) & 0\\ 0 & -\Delta_{0F}^\dagger(k)\end{bmatrix}\underline{\mathbf{M}}_b\ . \tag{3.80}$$

Note that the central matrix contains only causal and anticausal Feynman propagators; all the temperature dependence is in $\underline{\mathbf{M}}_b$.

We can also write $\underline{\Delta}_0(k)$ by observing that

$$\Delta_{0F}(k)-\Delta_{0F}^\dagger(k)=-2\pi i\delta_\epsilon(k^2-m_s^2)\ , \tag{3.81}$$

so that

$$\begin{aligned}\underline{\Delta}_0(k)&=\begin{bmatrix}\Delta_{0F}(k) & 0\\ 0 & -\Delta_{0F}^\dagger(k)\end{bmatrix}-2\pi i\delta_\epsilon(k^2-m_s^2)\sinh\Theta(k)\times\begin{bmatrix}\sinh\Theta(k) & \cosh\Theta(k)\\ \cosh\Theta(k) & \sinh\Theta(k)\end{bmatrix}\\&\equiv\underline{\Delta}_{0F}(k)+\underline{\Delta}_{0T}(k)\ .\end{aligned} \tag{3.82}$$

Thus the scalar matrix propagator also separates into the sum of a Feynman piece $\underline{\Delta}_{0F}(k)$ and a temperature-dependent piece $\underline{\Delta}_{0T}(k)$. This expression is more useful than Eq. (3.80) in certain applications.

Let us examine the scalar propagator in the zero-temperature limit ($\beta\to\infty$). Since $N_b(\beta k_0)\to 0$, $\sinh\Theta(k)\to 0$, and $\cosh\Theta(k)\to 1$ in this limit, the second term in Eq. (3.82) disappears, and $\underline{\Delta}_0(k)$ becomes *diagonal.* This also follows from Eq. (3.80), as $\underline{\mathbf{M}}_b$ becomes the unit matrix at zero temperature. Since $\underline{\Delta}_0(k)$ is diagonal, the contours C_1 and C_2 decouple:

$$\lim_{T\to 0}Z_{12}[j]=Z_1[j]Z_2[j]\ . \tag{3.83}$$

This shows immediately that all connected diagrams in the real-time propagators now involve only contour C_1 and will reproduce the familiar zero-temperature Feynman rules [20] in terms of $\Delta_{0F}(k)$. Note that disconnected diagrams involving C_1 will still be canceled by the disconnected diagrams from C_2, since Eq. (3.55) is valid at all temperatures. We thus have the interesting result that if one *defines* the Feynman rules for the zero-temperature propagators as the limit of the finite-temperature rules, the restriction to connected diagrams arises without the usual application of the Gell-Mann–Low theorem [17].

IV. COVARIANT TIME PATHS

Before we include vector mesons and baryons in the formalism, it is convenient to generalize the preceding results to manifestly covariant form. The generating functional of Eq. (3.2) is written for an observer who is comoving with the nuclear fluid, using operators that are quantized on his purely spacelike hyperplane ("equal times"), as in Eq. (2.16). We now want to consider "laboratory frame" observers for whom the fluid has a uniform, finite velocity $\mathbf{v}$ and a corresponding four-velocity

$$u^{\mu}=\eta(1,\mathbf{v}), \quad \eta\equiv(1-\mathbf{v}^2)^{-1/2} . \tag{4.1}$$

Note that $u_{\mu}u^{\mu}=1$, and in the comoving frame, $u^{\mu'}=(1,\mathbf{0})$. Any freely falling observer can quantize the system on his purely spacelike hyperplane, and taken together, these hyperplanes comprise all spacelike hyperplanes. Thus it is most equitable to allow the quantization to be performed on an arbitrary spacelike hyperplane by any observer [12].

The three-dimensional spacelike hyperplanes Σ_t are determined by four-vectors x^{μ} that satisfy

$$\Sigma_t:\ t-n^{\mu}x_{\mu}=0, \quad n^{\mu}n_{\mu}=1, \quad n^0>0 , \tag{4.2}$$

where t is a time parameter and n^{μ} is a unit normal vector [30]. Note that the time (or "instant") parameter t is not necessarily equal to the coordinate x^0, since the spacetime coordinate axes can be chosen in many different ways; however, t is a Lorentz scalar quantity that measures the time evolution for observers living on successive hyperplanes normal to n^{μ}. Although only real time variables enter in the preceding definition of Σ_t, we will generalize to complex times in order to study the partition function and generating functional on different time paths. When the time is complex, we interpret x^{μ} as the sum of two four-vectors: $x^{\mu}=tn^{\mu}+\tilde{x}^{\mu}$, where $\tilde{x}^{\mu}$ is spacelike and orthogonal to n^{μ}. The imaginary part of the time is related to the scalar quantity $n\cdot\beta=1/T$, where β^{μ} is defined in Eq. (2.2), and T is the temperature measured by observers living on the hyperplanes normal to n^{μ}. This generalizes the result of Ref. [6] that a lab-frame observer with $n^{\mu}=(1,0,0,0)$ measures a temperature $T=1/\beta^0=T'/\eta$, where T' is the proper temperature.

A. The generating functional

The first step in the construction of a covariant generating functional is to rewrite Eq. (2.6) in terms of a Euclidean time variable τ appropriate for the hyperplanes Σ_t. (The generalization to arbitrary spacelike *hypersurfaces* Λ is possible, but the notation is cumbersome at best.) From the definitions of β^{μ} and the thermal potential α, we have

$$\int d\Sigma_{\mu}(\beta_{\nu}\hat{T}^{\mu\nu}-\alpha\hat{B}^{\mu})=\int_0^{\beta}d\tau'\int d\Sigma_{\mu}(u_{\nu}\hat{T}^{\mu\nu}-\mu'\hat{B}^{\mu}) , \tag{4.3}$$

where $d\Sigma_{\mu}=n_{\mu}d\Sigma$. This result follows because $\hat{T}^{\mu\nu}$ and $\hat{B}^{\mu}$ are conserved, so their projections on n_{μ} integrated over Σ are constants of the motion [31]. The parameter τ' is the Euclidean counterpart of the *proper* time t' that determines the evolution for observers living on hyperplanes normal to u^{μ} (comoving observers). The relationship between τ and τ' can be deduced by observing that an infinitesimal timelike translation between two hyperplanes Σ_t and Σ_{t+dt} can be written as $dt=n\cdot u\,dt'$ (see Fig. 4), and thus $d\tau=n\cdot u\,d\tau'$. We can therefore write the partition function as

$$Z=\mathrm{Tr}\exp\left[-\int_0^{n\cdot\beta}d\tau\int d\Sigma_{\mu}[(n\cdot u)^{-1}u_{\nu}\hat{T}^{\mu\nu}-\mu\hat{B}^{\mu}]\right] , \tag{4.4}$$

where we have used $\mu'=\mu(n\cdot u)$. The chemical potential μ is that measured by an observer living on the hyperplanes normal to n^{μ}. Since we are considering here only scalar mesons, we will set $\mu=0$ for now and restore it later when the baryons are added.

Equation (4.4) expresses the partition function as the trace of an (imaginary-time) evolution operator with a grand-canonical Hamiltonian, where all quantities are defined for an observer living on hyperplanes normal to n^{μ}. The time path is the familiar Matsubara choice, which starts at the origin and runs along the imaginary time axis to $t=-i\tau=-in\cdot\beta$. Thus we can write the integral in the exponential using

$$\int_0^{n\cdot\beta}d\tau\int d\Sigma=i\int_0^{-in\cdot\beta}dt\int d\Sigma\equiv i\int_M d^4x , \tag{4.5}$$

where M denotes the Matsubara contour. The trace can be recast as a path integral over fields using the same techniques as in Sec. III B. The only subtlety arises because the quantization is carried out on an arbitrary spacelike hyperplane, which cannot be expressed in terms of purely spatial coordinates. We illustrate the procedure first for a free scalar field and include interactions at the end.

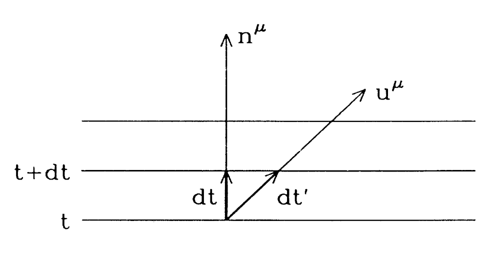

FIG. 4. Two-dimensional illustration of hyperplanes at successive times. Infinitesimal translations along the normal n^{μ} and fluid four-velocity u^{μ} are indicated by dt and dt', respectively.

Since the time evolution is along n^μ, we must use $\dot{\phi}=\partial\phi/\partial t \to n_\mu \partial^\mu \phi$ when computing the time derivative of the scalar field. Here the instant parameter t may be complex. The conjugate momentum density Π is defined by

$$\Pi \equiv n_\mu \Pi^\mu, \quad \Pi^\mu \equiv \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \ , \tag{4.6}$$

where $\mathcal{L}_0$ is the free scalar Lagrangian, and canonical quantization is performed by imposing the commutation relations [30]

$$\begin{aligned} &[\hat{\Pi}^\mu(x), \hat{\phi}(y)] = -i\delta^\mu(x,y) \ , \\ &[\hat{\phi}(x), \hat{\phi}(y)] = [\hat{\Pi}(x), \hat{\Pi}(y)] = 0 \quad (n \cdot x = n \cdot y) \ . \end{aligned} \tag{4.7}$$

Here $\int d\Sigma_\mu \delta^\mu(x,y) f(y) = f(x)$ for test functions $f(x)$. Note that $\delta^\mu(x,y)$ acts only in the hyperplane $\Sigma_{n\cdot x}$, and it is defined only when x and y have identical time parameters; thus, (4.7) is the appropriate generalization of the equal-time commutation relations (2.16) to an arbitrary spacelike hyperplane. Moreover, (4.7) implies that only the component of Π^μ normal to the hyperplane has a nonvanishing commutator with $\hat{\phi}$.

With these considerations in mind, one can divide the time contour into infinitesimal segments and rewrite the (noninteracting) scalar partition function as a path integral:

$$Z_0 = \int_\phi^\phi D(\phi) D(\Pi) \exp\left[i \int_M d^4x \left[\Pi n_\mu \partial^\mu \phi - \frac{1}{n \cdot u} n_\mu u_\nu T^{\mu\nu} \right] \right] . \tag{4.8}$$

Here $D(\phi)$ and $D(\Pi)$ denote the path-integral measures, and the limits on the path integral reflect the usual periodic boundary condition for scalar fields. Note that in general, n^μ and u^μ do not coincide, and thus the final term in the exponential contains both the Hamiltonian $H = \int d\Sigma\, n_\mu n_\nu T^{\mu\nu}$ and the total three-momentum $\mathbf{P}$, as well as the Lagrange multiplier $\mathbf{v}$ for the fluid velocity. To evaluate the path integral correctly, this final term must be expressed in terms of the field variables and their conjugate momenta, not the field variables and their time derivatives. In particular, since only the component of Π^μ along n^μ is a momentum density, one must take

$$\Pi^\mu = \Pi n^\mu + (\Pi^\mu - n^\mu \Pi) = \Pi n^\mu + \tilde{\partial}^\mu \phi \ , \tag{4.9}$$

where we have introduced the spacelike gradient in the hyperplane:

$$\tilde{\partial}^\mu \equiv \partial^\mu - n^\mu (n \cdot \partial) \ . \tag{4.10}$$

(We will consistently denote variables in the spacelike hyperplane Σ with a tilde.) By using the canonical definition of the energy-momentum tensor,

$$T^{\mu\nu} = -g^{\mu\nu} \mathcal{L}_0 + \Pi^\mu \partial^\nu \phi \ , \tag{4.11}$$

we arrive at

$$Z_0 = \int_\phi^\phi D(\phi) D(\Pi) \exp\left[i \int_M d^4x [\Pi n \cdot \partial\phi + \tfrac{1}{2}(\Pi^2 + \tilde{\partial}_\mu \phi \tilde{\partial}^\mu \phi - m_s^2 \phi^2) - \Pi^2 - \frac{1}{n \cdot u} \Pi u \cdot \tilde{\partial}\phi] \right] . \tag{4.12}$$

The path integral over the momentum density Π can be performed easily, leading to

$$\begin{aligned} Z_0 &= \mathcal{N} \int_\phi^\phi D(\phi) \exp\left[i \int_M d^4x \ \tfrac{1}{2}[(\mathcal{D}\phi)^2 + \tilde{\partial}_\mu \phi \tilde{\partial}^\mu \phi - m_s^2 \phi^2] \right] \\ &= \mathcal{N} \int_\phi^\phi D(\phi) \exp\left[i \int_M d^4x \ [\mathcal{L}_0]_{\partial \to d} \right] , \end{aligned} \tag{4.13}$$

where we have defined

$$d^\mu \equiv n^\mu \mathcal{D} + \tilde{\partial}^\mu, \quad \mathcal{D} \equiv n \cdot \partial - \frac{u \cdot \tilde{\partial}}{n \cdot u} \ . \tag{4.14}$$

The Lagrangian in this manifestly covariant partition function is to be evaluated by replacing the usual partial derivative ∂^μ with the new derivative d^μ, which contains a gradient $\tilde{\partial}^\mu$ in the hyperplane and a "convective" derivative $\mathcal{D}$ normal to the hyperplane. This is a natural result because the fluid is flowing past the observer when $n^\mu \neq u^\mu$, and the convective derivative automatically incorporates the Lagrange multiplier for the fluid velocity [6]. (Note that the convective derivative collapses to $\mathcal{D} = u \cdot \partial$ if $n^\mu = u^\mu$.) Renormalizable scalar self-interactions can be included easily, since they contain no derivatives and are carried along trivially through the manipulations discussed above; thus, the interacting Z also takes the form of Eq. (4.13).

Although the preceding derivation was carried out for the Matsubara contour, it can be generalized to an arbitrary time path by interpreting the time derivative $n \cdot \partial$ as a directional derivative along the contour (as well as a directional derivative in spacetime normal to the hyperplane) and by decomposing any contour into rectilinear segments as in Eq. (3.13). In other words, for the derivation of the path integral, it is immaterial whether the evolution parameter is real or complex. Thus, by adding interactions and a source term to Eq. (4.13), we can write a manifestly covariant expression for the generating functional that is analogous to Eq. (3.16):

$$Z[j] = \mathcal{N} \int_\phi^\phi D(\phi) \exp\left[i \int_c d^4x [\mathcal{L}(\phi, d^\mu \phi) + j(x)\phi(x)] \right] . \tag{4.15}$$

The integral $\int_c d^4x$ is over a spacelike hyperplane and an instant parameter, as in Eq. (4.5), and Eq. (4.15) is valid for any contour that starts at some t_0, ends at $t_0 - in\cdot\beta$, and has a monotonically decreasing $\mathrm{Im}\, t$. As before, the normalization factor $\mathcal{N}$ can be determined by rewriting (4.15) on a spacetime lattice, but we will see that this is not necessary. The generating functional depends on both the external source j and the thermodynamic parameters $T=(n\cdot\beta)^{-1}$, u^μ, and $\mathcal{V}\equiv\int d\Sigma$ (and μ for fermions), all defined appropriately for observers living on hyperplanes normal to n^μ.

B. Covariant contour propagators

We turn now to the covariant generalization of the contour propagators defined in Eqs. (2.17)–(2.19) and (3.24). These propagators are expressed as thermal averages of *grand* Heisenberg-picture fields (or grand interaction-picture fields) that are defined just as in Eqs. (2.24)–(2.26), except that $\mathbf{x}$ must now be interpreted as $\tilde{x}^\mu$, a four-vector in the spacelike hyperplane $\Sigma_{t=0}$, and the grand-canonical Hamiltonian $\hat{K}$ is determined from Eq. (4.4):

$$\hat{K}\equiv\int d\Sigma_\mu[(n\cdot u)^{-1}u_\nu\hat{T}^{\mu\nu}-\mu\hat{B}^\mu] . \tag{4.16}$$

If the time variable t is real, one can combine tn^μ and $\tilde{x}^\mu$ into a single Lorentz four-vector x^μ, which will allow for a covariant generalization of the real-time propagator of Sec. III G. In contrast, if t is complex, the two four-vectors tn^μ and $\tilde{x}^\mu$ must be kept separate in a covariant representation of the imaginary-time propagators.

For an arbitrary acceptable contour, the manipulations leading from Eq. (3.16) to Eq. (3.20) can also be carried out on the covariant expression (4.15), with the result

$$Z[j]=Z_0\exp\left[i\int_c d^4y\,\mathcal{L}_{\mathrm{int}}\left[\frac{\delta}{i\delta j(n\cdot y,\tilde{y})}\right]\right]\exp\left[-\frac{i}{2}\int_c d^4x\int_c d^4x'\,j(t,\tilde{x})\Delta_0^{(c)}(t-t',\tilde{x}-\tilde{x}')j(t',\tilde{x}')\right] . \tag{4.17}$$

Here the noninteracting contour propagator is defined by

$$\begin{aligned} i\Delta_0^{(c)}(t,\tilde{x})&\equiv\langle\langle T_c\hat{\phi}_K(t,\tilde{x})\hat{\phi}_K(0)\rangle\rangle_0\\ &=\theta_c(t)\langle\langle\hat{\phi}_K(t,\tilde{x})\hat{\phi}_K(0)\rangle\rangle_0+\theta_c(-t)\langle\langle\hat{\phi}_K(0)\hat{\phi}_K(t,\tilde{x})\rangle\rangle_0\\ &\equiv\theta_c(t)C_0^{>}(t,\tilde{x})+\theta_c(-t)C_0^{<}(t,\tilde{x}) , \end{aligned} \tag{4.18}$$

where the subscript K denotes the grand interaction-picture fields discussed above, and the trace involves the covariant statistical factor $\exp(-n\cdot\beta\hat{K}_0)$, with $\hat{K}_0$ the noninteracting version of (4.16) for $\mu=0$. Since the only difference between (3.16) and (4.15) is the replacement $\partial^\mu\to d^\mu$, the new propagator satisfies

$$\begin{aligned}(d_\mu d^\mu+m_s^2)\Delta_0^{(c)}(t,\tilde{x})&=-\delta_c^{(4)}(t,\tilde{x})\\ &\equiv-\delta_c(t)n_\mu\delta^\mu(\tilde{x},0) .\end{aligned} \tag{4.19}$$

To summarize the notation, $\tilde{x}^\mu$ is a four-vector lying in $\Sigma_{t=0}$, the scalar parameter t can be complex, d^μ is defined in Eq. (4.14), $\theta_c(t)$ and $\delta_c(t)$ are defined in Eqs. (2.20) and (2.21), and $\delta^\mu(x,y)$ is discussed after Eq. (4.7). The appropriate boundary conditions on the propagator follow just as in Sec. III C and are given by [compare Eq. (3.25)]

$$C_0^{>}(t,\tilde{x})=C_0^{<}(t+in\cdot\beta,\tilde{x}) . \tag{4.20}$$

The solution to Eq. (4.19) subject to (4.20) can be obtained through the following steps. First, Fourier transform the spatial variables by defining

$$\Delta_0^{(c)}(t,\tilde{x})=\int\frac{d\Sigma_k}{(2\pi)^3}e^{-i\tilde{k}\cdot\tilde{x}}\Delta_0^{(c)}(t,\tilde{k}) , \tag{4.21}$$

where Σ_k is the reciprocal spacelike hyperplane, and $\tilde{k}^\mu$ is a four-momentum lying in that hyperplane. Equation (4.19) then produces a differential equation

$$\left[\left[n\cdot\partial+i\frac{u\cdot\tilde{k}}{n\cdot u}\right]^2+\omega_k^2\right]\Delta_0^{(c)}(t,\tilde{k})=-\delta_c(t) , \tag{4.22}$$

with $\omega_k^2=-\tilde{k}^2+m_s^2$.

The two independent solutions to the homogeneous equation are easily found, and the appropriate linear combinations can be determined by enforcing the periodicity condition (4.20) and the proper jump discontinuity at $t=0$. One can then introduce a dummy frequency variable $n\cdot k$ just as in Sec. III C, where the frequency was denoted by k_0. Note that since the momentum and frequency variables are all taken to be real, they can be combined into a single four-vector $k^\mu=(n\cdot k)n^\mu+\tilde{k}^\mu$, and in particular, $d(n\cdot k)d\Sigma_k=d^4k$. Finally, the solution to Eq. (4.19) can be written as

$$i\Delta_0^{(c)}(t,\tilde{x})=\int\frac{d^4k}{(2\pi)^4}e^{-i\tilde{k}\cdot\tilde{x}}e^{-it(k\cdot u)/(n\cdot u)}\rho_0(k)\times\left[\frac{1}{1-e^{-\beta\cdot k}}\theta_c(t)+\frac{1}{e^{\beta\cdot k}-1}\theta_c(-t)\right] , \tag{4.23}$$

where the spectral density $\rho_0(k)$ is given covariantly by

$$\begin{aligned}\rho_0(k)&\equiv2\pi[\theta(n\cdot k)-\theta(-n\cdot k)]\delta(k^2-m_s^2)\\ &=2\pi[\theta(\beta\cdot k)-\theta(-\beta\cdot k)]\delta(k^2-m_s^2) .\end{aligned} \tag{4.24}$$

The final equality follows because n^μ, β^μ, and k^μ are all timelike here, so $n\cdot k$ and $\beta\cdot k$ have the same sign in all frames.

The covariant contour propagator (4.23) is an analytic function of t for contours consistent with Eq. (3.11), and it becomes a (singular) generalized function when the contours of Eq. (3.12) are allowed. One can also derive Eq. (4.23) directly from grand interaction-picture fields $\hat{\phi}(t,\vec{x})$ quantized on the hyperplane $\Sigma_{t=0}$.

The noninteracting Euclidean propagator follows by letting c be the Matsubara contour. With $t\equiv -i\tau$, we define

$$\Delta_0^{(E)}(\tau,\vec{x})\equiv\Delta_0^{(c)}(-i\tau,\vec{x}) \ , \tag{4.25}$$

and the covariant momentum-space propagator can be found easily from Eq. (4.23). We will present the results later, after the baryons and vector mesons have been included. Note that the propagators and Feynman rules generated by $\Delta_0^{(E)}$ are in the grand Heisenberg picture.

In contrast, real-time propagators are conveniently calculated in the *canonical* Heisenberg picture, which involves only the Hamiltonian $\hat{H}$ and not the momentum $\hat{\mathbf{P}}$ (nor the baryon number $\hat{B}$ for baryons). The canonical propagators simplify the real-time Feynman rules and have a zero-temperature limit that agrees with familiar real-time Green's functions. For scalars ($\mu=0$) in the comoving frame ($\mathbf{v}=0$), the grand and canonical Heisenberg pictures are identical, and that is why we have not considered the distinction until this point. However, as discussed in Ref. [1], time evolution in the canonical Heisenberg picture is performed along n^μ, with no reference to the moving fluid, whereas time evolution in the grand Heisenberg picture proceeds along u^μ [as is evident from Eq. (4.3)] and thus follows *fluid elements* along the hyperplanes Σ. It follows that real-time propagators in the grand Heisenberg picture depend explicitly on the fluid four-velocity u^μ and on $n\cdot u$ in particular, while propagators in the canonical Heisenberg picture are independent of n^μ.

In Sec. V, we will show how to extract Feynman rules for canonical real-time propagators from the generating functional, which is written in the grand Heisenberg picture. For now, we simply describe the relationship between real-time propagators in the two pictures and discuss the covariant generalization of the propagator in Sec. III G.

For real time arguments, the propagators in the two pictures differ only by a phase because the pictures are related by a unitary transformation. For a real time t, we can rewrite the scalar contour propagator (4.23) in terms of the Lorentz four-vector x^μ, with $t=x\cdot n$:

$$i\Delta_0^{(c)}(x^\mu)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}e^{i(x\cdot n)[k\cdot n-(k\cdot u)/(n\cdot u)]}\rho_0(k)\left[\frac{1}{1-e^{-\beta\cdot k}}\theta_c(x\cdot n)+\frac{1}{e^{\beta\cdot k}-1}\theta_c(-x\cdot n)\right] . \tag{4.26}$$

As noted above, this $\Delta_0^{(c)}$ is in the grand Heisenberg picture. However, since the canonical and grand-canonical evolution operators commute with each other, the phase factor between pictures is precisely the second exponential in Eq. (4.26). This follows by rewriting (4.18) in terms of the canonical Heisenberg-picture fields $\hat{\phi}_H(t,\vec{x})$ and using

$$[\hat{P}^\mu,\hat{\phi}_H(t,\vec{x})]=-i\partial^\mu\hat{\phi}_H(t,\vec{x}) \ . \tag{4.27}$$

[Note that thermal averages are always defined by the grand canonical *ensemble* with $\exp(-n\cdot\beta\hat{K})$.] If this phase factor is removed, what remains is an expression for the canonical Heisenberg-picture contour propagator, which is the desired covariant generalization of the real-time result that follows from Eq. (3.33). For example, the (1,1) component of the covariant real-time scalar propagator is given by

$$\begin{aligned} i\Delta_0^{(11)}(x^\mu)&=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\rho_0(k)\left[\frac{1}{1-e^{-\beta\cdot k}}\theta(x\cdot n)+\frac{1}{e^{\beta\cdot k}-1}\theta(-x\cdot n)\right]\\ &=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\rho_0(k)\left[\theta(x\cdot n)+\frac{1}{e^{\beta\cdot k}-1}\right]\\ &=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}\left[i\theta(\beta\cdot k)\Delta_{0F}(k)+i\theta(-\beta\cdot k)\Delta_{0F}^\dagger(k)+\rho_0(k)\frac{1}{e^{\beta\cdot k}-1}\right] , \end{aligned} \tag{4.28}$$

where the final equality follows by inserting the standard integral representation of the θ function, and the final term in brackets gives the covariant generalization of Eq. (3.70). Note that the dependence on n^μ has disappeared in the final expression.

Evidently, the covariant generalization of the real-time propagator of Sec. III G is obtained by replacing βk_0 with $\beta\cdot k$ in the thermal distribution functions and by changing the arguments of the θ functions from $k_0\to n\cdot k\to\beta\cdot k$. This final replacement is allowed because the θ functions survive only in combination with $\delta(k^2-m_s^2)$; since n^μ, β^μ, and k^μ are then all timelike, $n\cdot k$ and $u\cdot k$ (or $\beta\cdot k$) have the same sign in all frames. This fortunate result can be readily verified from Eq. (4.28), although it is already suggested by Eq. (4.24). Similar covariant expressions can be obtained for the other components of the real-time propagator [Eqs. (3.71)–(3.73)]; however, for propagators connecting C_1 with C_2, one must move the imaginary time shifts into the Fourier coefficients [as in Eqs. (3.72)–(3.73)], so that the real part of the time variable can be treated as above. In the end, only simple changes are needed to make the real-time propagators covariant, but these changes are intimately linked to the important distinction between the grand and canonical Heisenberg pictures.

The remaining analysis of Sec. III involving the extension to infinite times and the factorization of the generating functional proceeds as before, starting from Eqs. (4.17) and (4.23). The spatial variables are restricted to the quantization hyperplane, and the complex time variables produce the evolution normal to the hyperplane. One can verify that the asymptotic time dependence of the propagator $\Delta_0^{(c)}(t,\tilde{x})$ is also a power-law decay, as in the comoving frame. Moreover, all preceding arguments concerning factorization still apply; by replacing the singular spectral density of Eq. (4.24) with the regularized version [32]

$$\rho_0(k)=2\pi[\theta(\beta\cdot k)-\theta(-\beta\cdot k)]\delta_\epsilon(k^2-m_s^2)\ , \qquad (4.29)$$

and by keeping ϵ finite until the end of all calculations, one concludes that the factorization theorems of Eqs. (3.52) and (3.55) still hold. Thus the covariant Feynman rules can be determined from the generating functionals in Eqs. (3.53) and (3.56), once they are reexpressed in covariant form, as in Eq. (4.17), which contains the contour propagator $\Delta_0^{(c)}(t,\tilde{x})$. We will list the rules after adding the vector mesons and baryons, as discussed in the next section.

V. COVARIANT FEYNMAN RULES FOR QHD-I

In this section, we generalize the results of Secs. III and IV to the full quantum theory QHD-I, which is defined by the Lagrangian density in Eq. (2.9). First, we write the complete QHD-I generating functional as a covariant path integral, which is valid for an arbitrary contour (subject to the conditions discussed earlier and repeated below). We need only collect results derived elsewhere, since the covariant partition function for fermions is constructed in Ref. [1], and the important new features for vector mesons are discussed in the Appendix. Next, the Lagrangian is decomposed into interacting and noninteracting parts, and the interaction terms are removed from the path integral by writing $\mathcal{L}_{\rm int}$ in terms of functional derivatives with respect to the sources. After introducing noninteracting contour propagators for each of the fields, we obtain the generalization to QHD-I of the expression for $Z[j]$ in Eq. (4.17).

To derive Feynman rules, we specialize to particular contours of interest and note that the generating functional factorizes in the infinite-time limit, just as in Secs. III and IV, provided the spectral densities are regularized. Thus we consider separately the Matsubara contour and the conventional real-time contour (with $\sigma=n\cdot\beta/2$; see Fig. 5). On the Matsubara contour, we can directly extract Feynman rules for imaginary-time

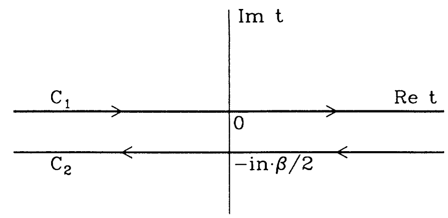

FIG. 5. Contour in the complex time plane used for deriving covariant real-time Feynman rules. The (constant) values of Im t on C_1 and C_2 are indicated.

propagators in the grand Heisenberg picture from the perturbative expansion of the generating functional. This expansion is in terms of noninteracting propagators and coupling constants. For the real-time contour, the 2×2 matrix structure of the propagators also arises as before. Although the generating functional produces propagators in the grand Heisenberg picture, we can define momentum-space Feynman rules for real-time propagators in the canonical Heisenberg picture. These propagators are independent of the quantization hyperplane defined by n^μ. We list in detail the momentum-space Feynman rules corresponding to both imaginary- and real-time contours.

A. The QHD-I generating functional

We start with the grand partition function Z of Eq. (2.6), write it as a path integral over the dynamical fields and conjugate momenta of the theory, and explicitly integrate over the momenta. As in Sec. IV, we specialize to the hyperplanes Σ_t defined by Eq. (4.2). Such path integrals were constructed for the scalar field in Secs. III and IV [Eq. (4.13)], for the fermion field in Sec. V of Ref. [1], and for the massive vector field in the Appendix. Although these derivations do not include all interaction terms, the QHD-I interactions may be added without further work, since the Lagrangian is renormalizable. Thus the interaction terms are Lorentz scalars with no derivatives, and they are carried along trivially through the same manipulations used in the noninteracting case [33].

After assembling these pieces, we obtain for the partition function the same expression as in the comoving frame, except that the Lagrangian density Eq. (2.9) is evaluated by replacing the partial derivative ∂^μ with d^μ, which is defined in Eqs. (4.10) and (4.14). We use the notation $[\mathcal{L}]_{\partial\to d}$ to indicate this replacement. We also write the chemical potential term in covariant form. Finally, to create the generating functional from the partition function, we add sources for each of the fields, including Grassmann sources $\bar{\xi}$ and ξ for the fermion fields. The result is

$$Z[\bar{\xi},\xi,j,J^\mu]=\mathcal{N}\int_{-\psi}^{\psi}D(\bar{\psi},\psi)\int_{\phi}^{\phi}D(\phi)\int_{V^\mu}^{V^\mu}D(V^\mu)\exp\left[i\int_c d^4x([\mathcal{L}]_{\partial\to d}+\mu n_\mu\bar{\psi}\gamma^\mu\psi+\bar{\xi}\psi+\bar{\psi}\xi+j\phi+J_\mu V^\mu)\right]\ , \qquad (5.1)$$

where the thermal boundary conditions on the fields are indicated symbolically as integration limits, and the coordinate dependence of the fields and sources has been suppressed. The integral $\int_c d^4x$ is to be interpreted in terms of a spacelike hyperplane and a complex instant parameter, as in Sec. IV. Thus we obtain a manifestly covariant expression for the generating functional of grand Heisenberg Green's functions in QHD-I. The functional is valid for any contour that starts at some t_0, ends at $t_0 - in\cdot\beta$, and has a monotonically decreasing Im t. It is normalized by $\mathcal{N}$, which can be determined by writing the path integral on a lattice. The explicit normalization factor does not occur in the Feynman rules for Green's functions and enters only in the partition function, so we do not consider it further in this discussion. As noted after Eq. (3.20), the generating functional will be well defined and finite when one includes counterterms and removes the interacting zero-point energy.

The generating functional depends on both the external sources, j, J^μ, $\bar{\xi}$ and ξ, and the thermodynamic parameters, $T=(n\cdot\beta)^{-1}$, μ, u^μ, and $\mathcal{V}\equiv\int d\Sigma$. As discussed in Sec. IV, the thermodynamic parameters are defined appropriately for observers living on hyperplanes normal to n^μ. We emphasize that $Z[\bar{\xi},\xi,j,J^\mu]$ depends on n^μ, but calculated observables must be independent of n^μ. An example of how this works in practice is given in Ref. [1].

The propagators defined in Eqs. (2.17)–(2.19) are generated for any time arguments on the contour using

$$i\Delta^{(c)}(x_1,\ldots,x_n)=\frac{1}{Z[0]}\left[\frac{\delta^n Z[\bar{\xi},\xi,j,J^\mu]}{i\delta j(x_1)\cdots i\delta j(x_n)}\right]_{\bar{\xi}=\xi=j=J^\mu=0}, \tag{5.2}$$

$$iD^{(c)}_{\mu\cdots\nu}(x_1,\ldots,x_n)=\frac{1}{Z[0]}\left[\frac{\delta^n Z[\bar{\xi},\xi,j,J^\mu]}{i\delta J^\mu(x_1)\cdots i\delta J^\nu(x_n)}\right]_{\bar{\xi}=\xi=j=J^\mu=0}, \tag{5.3}$$

$$iG^{(c)}(x_1,\ldots,x_n,x_1',\ldots,x_n')=\frac{1}{Z[0]}\left[\frac{(-1)^n\delta^{2n}Z[\bar{\xi},\xi,j,J^\mu]}{i\delta\bar{\xi}(x_1)\cdots i\delta\bar{\xi}(x_n)i\delta\xi(x_1')\cdots i\delta\xi(x_n')}\right]_{\bar{\xi}=\xi=j=J^\mu=0}. \tag{5.4}$$

In these equations and below, the coordinate arguments should be understood as a shorthand for the general form $x^\mu=tn^\mu+\tilde{x}^\mu$, where t is a complex scalar and $\tilde{x}^\mu$ is a four-vector in the quantization hyperplane. Functional derivatives for sources living on the contour are defined as in Eq. (3.4); the Grassmann derivatives on the contour are defined by

$$\frac{\delta}{\delta\xi(x')}\xi(x)=\frac{\delta}{\delta\bar{\xi}(x')}\bar{\xi}(x)=\delta_c^{(4)}(t-t',\tilde{x}-\tilde{x}'), \tag{5.5}$$

where the contour delta function is given in Eq. (4.19).

B. Contour propagators

The next step is to define the interaction Lagrangian and to remove interaction terms from the path integral by writing $\mathcal{L}_{\rm int}$ in terms of functional derivatives with respect to the sources. We decompose the Lagrangian according to $[\mathcal{L}]_{\partial\to d}=[\mathcal{L}_0]_{\partial\to d}+\mathcal{L}_{\rm int}$, with

$$[\mathcal{L}_0]_{\partial\to d}=\bar{\psi}(i\gamma_\mu d^\mu-M)\psi+\tfrac{1}{2}(d_\mu\phi\, d^\mu\phi-m_s^2\phi^2)-\tfrac{1}{4}(d_\mu V_\nu-d_\nu V_\mu)^2+\tfrac{1}{2}m_v^2V_\mu V^\mu, \tag{5.6}$$

$$\mathcal{L}_{\rm int}=\mathcal{L}_{\rm int}(\psi,\bar{\psi},\phi,V^\mu)=g_s\bar{\psi}\psi\phi-g_v\bar{\psi}\gamma_\mu\psi V^\mu-V(\phi)+\delta\mathcal{L}. \tag{5.7}$$

Note that the counterterm Lagrangian $\delta\mathcal{L}$ is simply carried along with $\mathcal{L}_{\rm int}$. Then we use the substitutions:

$$\psi(x)\to\frac{\delta}{i\delta\bar{\xi}(x)},\quad \bar{\psi}(x)\to\frac{\delta}{-i\delta\xi(x)},\quad \phi(x)\to\frac{\delta}{i\delta j(x)},\quad V_\mu\to\frac{\delta}{i\delta J^\mu(x)}, \tag{5.8}$$

to remove the interaction terms from the path integral (5.1).

Since $[\mathcal{L}_0]_{\partial\to d}+\mu n_\mu\bar{\psi}\gamma^\mu\psi$ is bilinear in the fields, we can introduce contour propagators, shift variables, and identify the noninteracting partition function Z_0 to obtain the generalization of Eq. (4.17). The intermediate steps parallel the scalar case discussed in Secs. III and IV, and the final result is

$$\begin{aligned}Z[\bar{\xi},\xi,j,J^\mu]=Z_0\exp&\left[i\int_c d^4y\,\mathcal{L}_{\rm int}\left[\frac{\delta}{i\delta\bar{\xi}(y)},\frac{\delta}{-i\delta\xi(y)},\frac{\delta}{i\delta j(y)},\frac{\delta}{i\delta J_\mu(y)}\right]\right]\\ &\times\exp\left[-\frac{i}{2}\int_c d^4x\int_c d^4x'\,j(t,\tilde{x})\Delta_0^{(c)}(t-t',\tilde{x}-\tilde{x}')j(t',\tilde{x}')\right]\\ &\times\exp\left[-\frac{i}{2}\int_c d^4x\int_c d^4x'\,J_\mu(t,\tilde{x})D_0^{(c)\mu\nu}(t-t',\tilde{x}-\tilde{x}')J_\nu(t',\tilde{x}')\right]\\ &\times\exp\left[-i\int_c d^4x\int_c d^4x'\,\bar{\xi}(t,\tilde{x})G_0^{(c)}(t-t',\tilde{x}-\tilde{x}')\xi(t',\tilde{x}')\right].\end{aligned} \tag{5.9}$$

The contour propagators satisfy the differential equations (remember that t is complex):

$$(d_\mu d^\mu + m_s^2)\Delta_0^{(c)}(t,\tilde{x}) = -\delta_c^{(4)}(t,\tilde{x}) \; , \tag{5.10}$$

$$[(d_\lambda d^\lambda + m_v^2)g^\mu{}_\rho - d^\mu d_\rho] D_0^{(c)\rho\nu}(t,\tilde{x}) = g^{\mu\nu}\delta_c^{(4)}(t,\tilde{x}) \; , \tag{5.11}$$

$$[\gamma_\mu(id^\mu + \mu n^\mu) - M]G_0^{(c)}(t,\tilde{x}) = \delta_c^{(4)}(t,\tilde{x}) \; , \tag{5.12}$$

subject to KMS boundary conditions on the corresponding retarded and advanced functions. For the bosons, these are given by Eqs. (4.18) and (4.20), while for the fermions, there is an additional minus sign:

$$\begin{aligned} iG_0^{(c)}(t,\tilde{x}) &\equiv \langle\langle T_c \hat{\psi}_K(t,\tilde{x})\hat{\bar{\psi}}_K(0)\rangle\rangle_0 \\ &= \theta_c(t)\langle\langle \hat{\psi}_K(t,\tilde{x})\hat{\bar{\psi}}_K(0)\rangle\rangle_0 \\ &\quad - \theta_c(-t)\langle\langle \hat{\bar{\psi}}_K(0)\hat{\psi}_K(t,\tilde{x})\rangle\rangle_0 \\ &\equiv \theta_c(t) g_0^>(t,\tilde{x}) + \theta_c(-t) g_0^<(t,\tilde{x}) \; , \end{aligned} \tag{5.13}$$

with

$$g_0^>(t,\tilde{x}) = -g_0^<(t + in\cdot\beta, \tilde{x}) \; . \tag{5.14}$$

Note that the KMS conditions have a particularly simple form because the propagators are defined using K-picture fields; if the propagators are defined with H-picture fields, the KMS conditions would contain additional multiplicative factors.

We can solve Eqs. (5.10)–(5.12) directly, as in Sec. IV B, by first constructing the solutions to the homogeneous equations and then determining the appropriate linear combinations by enforcing the KMS conditions and the correct jump discontinuities at $t=0$. [Note that the baryon propagator $G_0^{(c)}(t,\tilde{x})$ is itself discontinuous at $t=0$.] The solutions can be written in spectral form as

$$i\Delta_0^{(c)}(t,\tilde{x}) = \int \frac{d^4k}{(2\pi)^4} e^{-i\tilde{k}\cdot\tilde{x}} e^{-it(k\cdot u)/(n\cdot u)} \rho_0(k) \left[\frac{1}{1-e^{-\beta\cdot k}}\theta_c(t) + \frac{1}{e^{\beta\cdot k}-1}\theta_c(-t) \right] , \tag{5.15}$$

$$iD_0^{(c)\mu\nu}(t,\tilde{x}) = -\left[g^{\mu\nu} + \frac{d^\mu d^\nu}{m_v^2} \right] i\Delta_0^{(c)}(t,\tilde{x}) \; , \tag{5.16}$$

$$iG_0^{(c)}(t,\tilde{x})_{\xi\xi'} = e^{i\mu t}(i\gamma_\mu d^\mu + M)_{\xi\xi'} \int \frac{d^4k}{(2\pi)^4} e^{-i\tilde{k}\cdot\tilde{x}} e^{-it(k\cdot u)/(n\cdot u)} \rho_0(k) \left[\frac{1}{e^{-\beta\cdot k+\alpha}+1}\theta_c(t) - \frac{1}{e^{\beta\cdot k-\alpha}+1}\theta_c(-t) \right] , \tag{5.17}$$

where ξ and ξ' denote Dirac matrix indices. To permit the factorization of the generating functional (5.9), we use the regularized spectral density $\rho_0(k)$, which is defined as

$$\rho_0(k) \equiv 2\pi[\theta(\beta\cdot k) - \theta(-\beta\cdot k)]\delta_\epsilon(k^2 - m^2) \; , \tag{5.18}$$

with the appropriate mass (m_s, m_v, or M) in each case.

One can also construct the propagators directly from thermal averages, using grand interaction-picture fields quantized on the hyperplane $\sum_{t=0}$. For the scalar and baryon propagators, the procedures are straightforward, starting from Eqs. (4.18) and (5.13), respectively, but the construction is more subtle for the vector meson propagator. This is because the contour-ordering operator T_c must be defined appropriately for vector fields to ensure the correct discontinuity at $t=0$, as implied by Eq. (5.11). (This is related to the problem of "covariant time ordering" in the zero-temperature formalism, as discussed on pp. 135 and 223 of Ref. [20].)

At this point, we can derive perturbative expansions for n-point functions defined on a general contour in terms of coupling constants and the noninteracting contour propagators. The procedure is standard: Expand the first exponential in Eq. (5.9) in powers of the coupling constants, apply the functional derivatives, and set the sources to zero [10,20]. As usual, the Feynman rules can be deduced directly from low-order expansions. In the next two subsections, we specialize to two particular contours to obtain imaginary- and real-time Feynman rules.

C. Imaginary-time Feynman rules

The noninteracting Euclidean propagators follow from Eqs. (5.15)–(5.17) by letting c be the Matsubara contour. On this contour, t runs from 0 to $-in\cdot\beta$ on the imaginary-time axis, so the time parameter is purely imaginary. Thus the time arguments of the propagators (which are *relative* times) run from $-in\cdot\beta$ to $+in\cdot\beta$. However, the KMS conditions ensure that the propagators are periodic or antiperiodic with period $n\cdot\beta$ in this interval, so it is sufficient to consider only relative times between 0 and $-in\cdot\beta$.

With $t \equiv -i\tau$, we define

$$\Delta_0^{(E)}(\tau,\tilde{x}) \equiv \Delta_0^{(c)}(-i\tau,\tilde{x}) \tag{5.19}$$

and the Fourier transforms

$$\Delta_0^{(E)}(\nu_j,\tilde{p}^\mu) = \int_0^{n\cdot\beta} d\tau \int d\Sigma \, e^{i\tilde{p}\cdot\tilde{x}} e^{i\nu_j\tau} \Delta_0^{(E)}(\tau,\tilde{x}) \; , \tag{5.20}$$

$$\Delta_0^{(E)}(\tau,\tilde{x}) = \frac{1}{n\cdot\beta} \sum_j \int \frac{d\Sigma_p}{(2\pi)^3} e^{-i\tilde{p}\cdot\tilde{x}} e^{-i\nu_j\tau} \Delta_0^{(E)}(\nu_j,\tilde{p}^\mu) \; , \tag{5.21}$$

which involve discrete frequencies $\nu_j = 2j\pi/n\cdot\beta$ with integral j, because the time interval is finite and the propagator is periodic. Analogous definitions are made for $D_0^{(E)\mu\nu}$ and $G_0^{(E)}$. In the latter case, the discrete frequencies are $\omega_j=(2j+1)\pi/n\cdot\beta$, which reflect the antiperiodicity condition for a fermion propagator [2,17].

If one inserts the spectral form (5.15) into the Fourier transform (5.20), the thermal distribution functions disappear, leaving

$$i\Delta_0^{(E)}(\nu_j,\tilde{p}^\mu)=\int\frac{d(n\cdot p)}{2\pi}\frac{i\rho_0(p)}{\nu_j+i(p\cdot u)/(n\cdot u)}$$

$$=\frac{1}{-p_j^2+m_s^2}\ , \tag{5.22}$$

where p_j^μ is defined as

$$p_j^\mu\equiv\left[i\nu_j-\frac{\tilde{p}\cdot u}{n\cdot u}\right]n^\mu+\tilde{p}^\mu\ . \tag{5.23}$$

(It is permissible to set the regularization parameter $\epsilon\to 0$ in the Euclidean propagators, since all time integrals are of finite extent.) Similarly, the noninteracting vector meson propagator is given by

$$iD_0^{(E)\mu\nu}(\nu_j,\tilde{p}^\lambda)=\frac{-g^{\mu\nu}+p_j^\mu p_j^\nu/m_v^2}{-p_j^2+m_v^2}\ . \tag{5.24}$$

The numerator is simple because the d^μ operators in (5.16) merely bring down factors of $-ip_j^\mu$. The noninteracting fermion propagator is

$$iG_0^{(E)}(\omega_j,\tilde{k}^\mu)_{\xi\xi'}=\left[\frac{1}{-\not{k}_j+M}\right]_{\xi\xi'}=\frac{(\not{k}_j+M)_{\xi\xi'}}{-k_j^2+M^2} \tag{5.25}$$

$$=\frac{[(n\cdot\gamma)(i\omega_j-\tilde{k}\cdot u/n\cdot u+\mu)+\gamma\cdot\tilde{k}+M]_{\xi\xi'}}{(\omega_j+i\tilde{k}\cdot u/n\cdot u-i\mu)^2+E^2(\tilde{k})}\ , \tag{5.26}$$

with

$$k_j^\mu\equiv\left[i\omega_j-\frac{\tilde{k}\cdot u}{n\cdot u}+\mu\right]n^\mu+\tilde{k}^\mu \tag{5.27}$$

and

$$E^2(\tilde{k})\equiv-\tilde{k}^2+M^2\ . \tag{5.28}$$

We can construct a generating functional for Euclidean propagators by combining these definitions with Eq. (5.9) written on the Matsubara contour:

$$\begin{aligned}Z_M[\bar{\xi},\xi,j,J^\mu]=Z_0\exp&\left[\int d^4y_E\mathcal{L}_{\text{int}}\left[\frac{\delta}{\delta\bar{\xi}_E(y_E)},\frac{-\delta}{\delta\xi_E(y_E)},\frac{\delta}{\delta j_E(y_E)},\frac{\delta}{\delta J_{E\mu}(y_E)}\right]\right]\\ &\times\exp\left[\frac{i}{2}\int d^4x_E\int d^4x_E'\,j_E(\tau,\tilde{x})\Delta_0^{(E)}(\tau-\tau',\tilde{x}-\tilde{x}')j_E(\tau',\tilde{x}')\right]\\ &\times\exp\left[\frac{i}{2}\int d^4x_E\int d^4x_E'\,J_{E\mu}(\tau,\tilde{x})D_0^{(E)\mu\nu}(\tau-\tau',\tilde{x}-\tilde{x}')J_{E\nu}(\tau',\tilde{x}')\right]\\ &\times\exp\left[i\int d^4x_E\int d^4x_E'\,\bar{\xi}_E(\tau,\tilde{x})G_0^{(E)}(\tau-\tau',\tilde{x}-\tilde{x}')\xi_E(\tau',\tilde{x}')\right]\ ,\end{aligned} \tag{5.29}$$

where $\int d^4x_E\equiv\int_0^{n\cdot\beta}d\tau\int d\Sigma$, $j_E(x_E)\equiv j_E(\tau,\tilde{x})\equiv j(-i\tau,\tilde{x})$,

$$\frac{\delta}{\delta j_E(\tau',\tilde{x}')}j_E(\tau,\tilde{x})=\delta(\tau-\tau')n_\mu\delta^\mu(\tilde{x},\tilde{x}')\equiv\delta_E^{(4)}(\tau-\tau',\tilde{x}-\tilde{x}')\ , \tag{5.30}$$

and similarly for the other sources. The Feynman rules can be extracted by evaluating the functional derivatives to some low order in the coupling constants. We list below the imaginary-time, momentum-space Feynman rules in covariant form for the two-point functions of QHD-I. Other Green's functions (higher n-point functions) can be calculated using the same rules, by changing the number of external lines appropriately (generalizing rule 1 below) and by defining the overall phase consistently. These rules generate propagators in the covariant *grand* Heisenberg picture, and so the rules depend explicitly on the timelike unit four-vector n^μ. In general, the n^μ dependence does not disappear until observables are calculated; in Ref. [1], the one-loop approximation to QHD-I is used to illustrate how this works. The Feynman rules can also be applied to the calculation of the thermodynamic potential, but the combinatorial factors are rather intricate. (See Refs. [2] and [34] for a discussion.)

The rules for the mth order contribution to

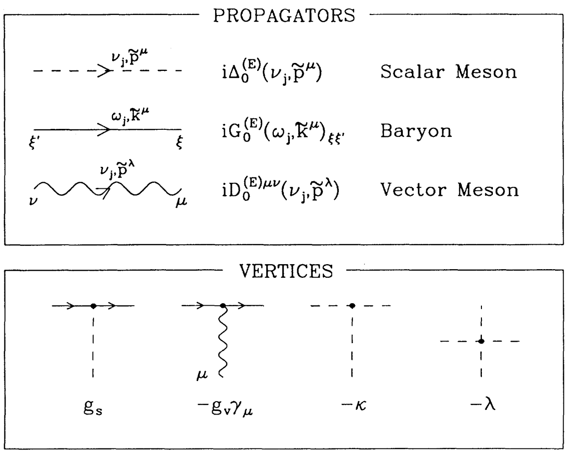

FIG. 6. Components of imaginary-time Feynman diagrams for QHD-I.

$(i)\times$(Euclidean propagator) in QHD-I are as follows (see Fig. 6):

(1) Draw all topologically distinct diagrams with two external propagator lines and m vertices connected by internal propagator lines. Do not include diagrams with completely disconnected pieces (that is, not connected to *any* external line).

(2) Assign a direction to each line. Associate a directed four-momentum on Σ_k (i.e., $\tilde{k}^\mu$ such that $n\cdot\tilde{k}=0$) and a discrete (Matsubara) frequency with each line. The frequencies obey

$$\nu_j=\frac{2j\pi}{n\cdot\beta}\quad\text{(bosons)}\,, \tag{5.31}$$

$$\omega_j=\frac{(2j+1)\pi}{n\cdot\beta}\quad\text{(fermions)}\,, \tag{5.32}$$

with integral j. Conserve four-momentum on Σ_k and Matsubara frequency at each vertex. This will leave only independent momenta and frequencies in the diagram.

(3) Each vertex is assigned a factor indicated in Fig. 6. Feynman rules for counterterm contributions follow directly.

(4) Each directed line gets a factor of (i) times the appropriate noninteracting propagator: $\Delta_0^{(E)}(\nu_j,\tilde{p}^{\,\mu})$, $D_0^{(E)\mu\nu}(\nu_j,\tilde{p}^{\,\lambda})$, or $G_0^{(E)}(\omega_j,\tilde{k}^{\,\mu})_{\xi\xi'}$, as indicated in Fig. 6.

(5) Sum over all repeated spacetime $(\mu,\nu,\dots)$ and Dirac $(\xi,\xi',\dots)$ indices.

(6) Integrate $\int d\Sigma_k/(2\pi)^3$ over all independent internal four-momenta on Σ_k and sum $(1/n\cdot\beta)\sum_j$ over all independent internal Matsubara frequencies.

(7) Include a factor of (-1) for each closed fermion loop.

(8) Any single-fermion line forming a tadpole loop must be multiplied by a convergence factor $e^{i\omega_j\eta}$, where $\eta\to 0^+$ at the end of the calculation, to regularize the sum over ω_j. Since the convergence factor defines an ordering that is not normal ordering, a subtraction is still needed to define the tadpole loop.

(9) A symmetry factor may be needed in diagrams with ϕ^3 and ϕ^4 vertices. The factor is the same as at $T=0$. (See pp. 63–65 of Ref. [35] for a discussion.)

If desired, one can replace $\int d\Sigma_k$ by $\int d^4k\,\delta(n\cdot k)$ in loop integrals and then let $\tilde{k}^{\,\mu}\to k^\mu$, but this is really just an alternative notation. However, it is possible in at least some circumstances to introduce an integral over $n\cdot k$ with an appropriate delta function that eliminates the dependence on n^μ. (See Ref. [1] for an example.)

We observe that the covariant imaginary-time rules are quite similar to those written in the comoving frame [36,37]. There are several new ingredients in the covariant formulation. First, two types of four-vectors are used: a real one in the quantization hyperplane and a complex one normal to the hyperplane. This enables us to define four-vectors with real spatial parts and imaginary temporal parts. Second, because we work in the grand-canonical Heisenberg picture, the Lagrange multipliers for the fluid velocity and chemical potential enter as imaginary shifts in the Matsubara frequency, leading to the complicated form for the complex frequencies. Finally, the Matsubara frequencies are defined using $T=1/n\cdot\beta$; thus, each observer defines these frequencies with the observed temperature in the laboratory frame, rather than the proper temperature $T'=1/\beta$.

D. Real-time Feynman rules

In this subsection, we follow the procedure used in Sec. IV B for a scalar theory to obtain covariant real-time Feynman rules for the full QHD-I theory. In particular, we start from Eq. (5.9) with the standard real-time contour (see Fig. 5) and generalize the analysis of Sec. III. The extension to infinite times and the factorization of the generating functional goes through unchanged, if we remember that the spatial variables are restricted to the quantization hyperplane, and the complex time variables evolve the system normal to the hyperplane.

After the factorization, we can separate the sources into distinct contributions from the two contour segments and then introduce a matrix structure for the propagators, as in Sec. III G. The generalization of Eq. (3.67) is

$$\begin{aligned}Z_{12}[\bar{\xi},\xi,j,J^\mu]=&\exp\left[i\int d^4y\left\{\mathcal{L}_{\rm int}\left[\frac{\delta}{i\delta\bar{\xi}_1(y)},\frac{\delta}{-i\delta\xi_1(y)},\frac{\delta}{i\delta j_1(y)},\frac{\delta}{i\delta J_{1\mu}(y)}\right]\right.\right.\\&\left.\left.-\mathcal{L}_{\rm int}\left[\frac{\delta}{i\delta\bar{\xi}_2(y)},\frac{\delta}{-i\delta\xi_2(y)},\frac{\delta}{i\delta j_2(y)},\frac{\delta}{i\delta J_{2\mu}(y)}\right]\right\}\right]\\&\times\exp\left[-\frac{i}{2}\int d^4x\int d^4x'\,j_r(x)\tilde{\Delta}_0^{(rs)}(x-x')j_s(x')\right]\exp\left[-\frac{i}{2}\int d^4x\int d^4x'\,J_{r\mu}(x)\tilde{D}_0^{(rs)\mu\nu}(x-x')J_{s\nu}(x')\right]\\&\times\exp\left[-i\int d^4x\int d^4x'\,\bar{\xi}_r(x)\tilde{G}_0^{(rs)}(x-x')\xi_s(x')\right]\,.\end{aligned}\tag{5.33}$$

In every integral, the time variable takes on real values from $-\infty$ to $+\infty$. We have written the noninteracting matrix propagators with tildes here to signify that they are still in the grand-canonical picture. The interacting real-time propagators can be determined from Eqs. (5.2)–(5.4), since all external sources are on segment C_1 and satisfy Eq. (3.66).

The noninteracting matrix propagators follow directly from the contour propagators in Eqs. (5.15)–(5.17) by applying the definitions in Eqs (3.61)–(3.64) (generalized appropriately for baryons and vector mesons). The relative time argument is either real (if the times are on the same segment) or has a constant imaginary part (if the times are on opposite segments). If the time parameter $t=x\cdot n$ is real, we can rewrite the scalar contour propagator in terms of the real four-vector x^μ:

$$i\tilde{\Delta}_0^{(c)}(x^\mu)=\int\frac{d^4k}{(2\pi)^4}e^{-ik\cdot x}e^{i(x\cdot n)[k\cdot n-(k\cdot u)/(n\cdot u)]}\rho_0(k)\times\left[\frac{1}{1-e^{-\beta\cdot k}}\theta_c(x\cdot n)+\frac{1}{e^{\beta\cdot k}-1}\theta_c(-x\cdot n)\right]. \tag{5.34}$$

For times on opposing segments, the constant imaginary time shift (which produces a real exponential) is grouped with the spectral density $\rho_0(k)$ and included in the Fourier coefficients $\Delta_0^{(rs)}(k)$.

As noted in Sec. IV, we can identify the second exponential in Eq. (5.34) as the phase factor between the grand Heisenberg- and canonical Heisenberg-picture scalar propagators. This phase originates from the $\mathbf{v}\cdot\mathbf{P}$ contribution to the grand-canonical Hamiltonian [6] in a frame where the fluid has velocity $\mathbf{v}$. Since the four-momentum is a constant of the motion, the same phase factor arises in the noninteracting and exact propagators. Moreover, since four-momentum is conserved at every vertex in every Feynman diagram, this factor depends only on the external time variables of the propagator, and it arises order-by-order in perturbation theory. Finally, by observing that the extra phase in Eq. (5.34) can be generated by placing

$$e^{(x\cdot n)\bar{u}\cdot\bar{\partial}/n\cdot u} \tag{5.35}$$

outside the integral, one concludes that the grand Heisenberg-picture propagators are related to the canonical Heisenberg-picture propagators by the derivative operator (5.35) in each order of perturbation theory. Thus, by omitting this factor, we can use the remaining momentum-space spectral representation to derive Feynman rules in the canonical Heisenberg picture.

Similar arguments hold for the $e^{i\mu t}$ factors in the baryon propagator, since the baryon number is a constant of the motion and is conserved at every vertex in every Feynman diagram. Moreover, it is easy to see that moving the derivative operator in (5.35) through a d^μ operator is equivalent to the replacement $d^\mu\to\partial^\mu$. Thus, to convert the vector and baryon propagators in Eqs. (5.16) and (5.17) to the canonical Heisenberg picture, we simply remove the extra phases and replace d^μ by ∂^μ.

Based on these observations, the (1,1) component of the canonical real-time scalar propagator is given by Eq. (4.28), and we emphasize that the dependence on n^μ has disappeared in the final expression. Similar expressions can be obtained for the other thermal matrix elements and the other propagators; we will simply compile the results here.

We define several thermal factors, which are Lorentz scalars. The Bose thermal distribution function $N_b(\beta\cdot k)$ is

$$N_b(\beta\cdot k)\equiv\theta(\beta\cdot k)n_b(\beta\cdot k)+\theta(-\beta\cdot k)n_b(-\beta\cdot k)=\frac{1}{e^{|\beta\cdot k|}-1}, \tag{5.36}$$

where $n_b(z)$ is given by (3.29), and the fermion counterpart $N_f(\beta\cdot k,\alpha)$ is

$$\begin{aligned}N_f(\beta\cdot k,\alpha)&\equiv\theta(\beta\cdot k)n_f(\beta\cdot k,\alpha)+\theta(-\beta\cdot k)\bar{n}_f(\beta\cdot k,\alpha)\\&\equiv\theta(\beta\cdot k)\frac{1}{e^{\beta\cdot k-\alpha}+1}+\theta(-\beta\cdot k)\frac{1}{e^{-\beta\cdot k+\alpha}+1}\\&\equiv\frac{1}{e^{|\beta\cdot k|-\alpha\,\mathrm{sgn}(\beta\cdot k)}+1}.\end{aligned} \tag{5.37}$$

These distribution functions depend on the baryon thermal potential $\alpha\equiv\beta\mu'$, a scalar, and the temperature four-vector $\beta^\mu\equiv\beta u^\mu$, where β is the inverse proper temperature and μ' is the proper baryon chemical potential [6]. When evaluated in the laboratory frame [where $u^\mu=(1-\mathbf{v}^2)^{-1/2}(1,\mathbf{v})$], these distribution functions have the same form as those given in Ref. [6]. Note, however, that the four-momenta appearing in the distribution functions in noninteracting propagators are "off shell," that is, they involve four independent components k^0 and $\mathbf{k}$. As discussed in Sec. III F, the appearance of off-shell arguments is crucial for producing the correct Feynman rules when combined with the regularized spectral functions and the factorized generating functional.

The boson matrix propagators can be written compactly by defining a unimodular "thermal" matrix $\underline{\mathbf{M}}_b$ as

$$\underline{\mathbf{M}}_b\equiv\begin{bmatrix}\cosh\Theta(k) & \sinh\Theta(k)\\ \sinh\Theta(k) & \cosh\Theta(k)\end{bmatrix}, \tag{5.38}$$

where

$$\begin{aligned}\sinh\Theta(k)&\equiv\sqrt{N_b(\beta\cdot k)}=\frac{1}{(e^{|\beta\cdot k|}-1)^{1/2}},\\ \cosh\Theta(k)&=e^{|\beta\cdot k|/2}\sqrt{N_b(\beta\cdot k)}=\frac{e^{|\beta\cdot k|/2}}{(e^{|\beta\cdot k|}-1)^{1/2}},\end{aligned} \tag{5.39}$$

so that $\Theta(k)\geq 0$. This is the covariant generalization of Eq. (3.78). The analogous matrix $\underline{\mathbf{M}}_f$ for fermions is

$$\underline{\mathbf{M}}_f\equiv\begin{bmatrix}\cos\Theta(k) & -\sin\Theta(k)\\ \sin\Theta(k) & \cos\Theta(k)\end{bmatrix}, \tag{5.40}$$

where

$$\sin\Theta(k)\equiv\sqrt{N_f(\beta\cdot k,\alpha)} =\frac{1}{(e^{|\beta\cdot k|-\alpha\,\mathrm{sgn}(\beta\cdot k)}+1)^{1/2}} ,$$
$$\cos\Theta(k)=[\theta(\beta\cdot k)-\theta(-\beta\cdot k)]\sqrt{1-N_f(\beta\cdot k,\alpha)} =\frac{\mathrm{sgn}(\beta\cdot k)e^{[|\beta\cdot k|-\alpha\,\mathrm{sgn}(\beta\cdot k)]/2}}{(e^{|\beta\cdot k|-\alpha\,\mathrm{sgn}(\beta\cdot k)}+1)^{1/2}} , \quad (5.41)$$

and $0\le\Theta(k)\le\pi$. The utility of these thermal matrices was first discovered in the context of thermofield dynamics, where they arise naturally as Bogoliubov transformation matrices.

With these definitions, the noninteracting scalar meson propagator can be written as

$$\Delta_0^{(rr')}(k)\equiv\underline{\boldsymbol{\Delta}}_0(k)=\underline{\mathbf{M}}_b\begin{bmatrix}\Delta_{0F}(k) & 0\\ 0 & -\Delta_{0F}^{\dagger}(k)\end{bmatrix}\underline{\mathbf{M}}_b , \quad (5.42)$$

where the scalar Feynman (F) propagator is

$$\Delta_{0F}(k)=\frac{1}{k^2-m_s^2+i\epsilon} ,\qquad \Delta_{0F}^{\dagger}(k)=\frac{1}{k^2-m_s^2-i\epsilon} . \quad (5.43)$$

Note that the central matrix in Eq. (5.42) contains only causal and anticausal Feynman propagators; all the temperature dependence is in $\underline{\mathbf{M}}_b$. The propagator can be decomposed into Feynman (F) and temperature-dependent (T) parts:

$$\underline{\boldsymbol{\Delta}}_0(k)=\begin{bmatrix}\Delta_{0F}(k) & 0\\ 0 & -\Delta_{0F}^{\dagger}(k)\end{bmatrix}-2\pi i\delta_\epsilon(k^2-m_s^2)\sinh\Theta(k)\begin{bmatrix}\sinh\Theta(k) & \cosh\Theta(k)\\ \cosh\Theta(k) & \sinh\Theta(k)\end{bmatrix}$$
$$=\begin{bmatrix}\Delta_{0F}(k) & 0\\ 0 & -\Delta_{0F}^{\dagger}(k)\end{bmatrix}-2\pi i\delta_\epsilon(k^2-m_s^2)N_b(\beta\cdot k)\begin{bmatrix}1 & e^{|\beta\cdot k|/2}\\ e^{|\beta\cdot k|/2} & 1\end{bmatrix} \quad (5.44)$$
$$\equiv\underline{\boldsymbol{\Delta}}_{0F}(k)+\underline{\boldsymbol{\Delta}}_{0T}(k) . \quad (5.45)$$

These expressions are more useful than Eq. (5.42) for some applications.

The noninteracting vector meson propagator is concisely expressed in terms of the scalar meson propagator,

$$D_0^{(rr')\mu\nu}(k)\equiv\underline{\mathbf{D}}_0^{\mu\nu}(k)=\left[-g^{\mu\nu}+\frac{k^\mu k^\nu}{m_v^2}\right]\Delta_0^{(rr')}(k)\Big|_{m_s^2\to m_v^2} , \quad (5.46)$$

where the prescription is to replace the scalar meson mass m_s by the vector meson mass m_v in $\Delta_0^{(rr')}(k)$ [Eqs. (5.42)–(5.45)]. It is reasonable to expect that the $k^\mu k^\nu$ factor can be dropped when calculating observables, as at $T=0$, because the baryon current is conserved. To our knowledge, however, there is no proof that this prescription is valid for $T>0$.

Finally, the noninteracting baryon propagator is given by

$$G_0^{(rr')}(k)_{\xi\xi'}\equiv\underline{\mathbf{G}}_0(k)_{\xi\xi'} =\underline{\mathbf{M}}_f\begin{bmatrix}G_{0F}(k)_{\xi\xi'} & 0\\ 0 & -G_{0F}^{\dagger}(k)_{\xi\xi'}\end{bmatrix}\underline{\mathbf{M}}_f , \quad (5.47)$$

where the fermion Feynman (F) propagator is

$$G_{0F}(k)_{\xi\xi'}=\frac{(\not k+M)_{\xi\xi'}}{k^2-M^2+i\epsilon} ,\qquad G_{0F}^{\dagger}(k)_{\xi\xi'}=\frac{(\not k+M)_{\xi\xi'}}{k^2-M^2-i\epsilon} , \quad (5.48)$$

and there is no complex conjugation of the Dirac matrices in the anticausal propagator. The Dirac matrix indices ξ and ξ' should not be confused with the thermal matrix indices r and r'. We can again decompose the propagator into Feynman (F) and temperature-dependent (T) parts:

$$\underline{\mathbf{G}}_0(k)_{\xi\xi'}=\begin{bmatrix}G_{0F}(k)_{\xi\xi'} & 0\\ 0 & -G_{0F}^{\dagger}(k)_{\xi\xi'}\end{bmatrix}+2\pi i(\not k+M)_{\xi\xi'}\delta_\epsilon(k^2-M^2)\sin\Theta(k)\begin{bmatrix}\sin\Theta(k) & \cos\Theta(k)\\ -\cos\Theta(k) & \sin\Theta(k)\end{bmatrix}$$
$$=\begin{bmatrix}G_{0F}(k)_{\xi\xi'} & 0\\ 0 & -G_{0F}^{\dagger}(k)_{\xi\xi'}\end{bmatrix}$$
$$+2\pi i(\not k+M)_{\xi\xi'}\delta_\epsilon(k^2-M^2)N_f(\beta\cdot k,\alpha)\begin{bmatrix}1 & \mathrm{sgn}(\beta\cdot k)e^{[|\beta\cdot k|-\alpha\,\mathrm{sgn}(\beta\cdot k)]/2}\\ -\mathrm{sgn}(\beta\cdot k)e^{[|\beta\cdot k|-\alpha\,\mathrm{sgn}(\beta\cdot k)]/2} & 1\end{bmatrix} \quad (5.49)$$
$$\equiv\underline{\mathbf{G}}_{0F}(k)_{\xi\xi'}+\underline{\mathbf{G}}_{0T}(k)_{\xi\xi'} . \quad (5.50)$$

In the zero-temperature limit ($\beta \to \infty$) with positive chemical potential ($\alpha > 0$),

$$\sin^2\Theta(k) = N_f(\beta\cdot k, \alpha) \to \theta(\beta\cdot k)\theta(\alpha - \beta\cdot k) , \quad (5.51)$$

$$\sin\Theta(k)\cos\Theta(k) \to 0 . \quad (5.52)$$

Thus, from Eq. (5.49), $\underline{\mathbf{G}}_0(k)_{\xi\xi'}$ becomes diagonal, and the (1,1) component gives a covariant generalization of the finite-density propagator in Eq. (5.5) of Ref. [10].

Given these expressions, we list the rules for the two-point functions in momentum space. These rules generate propagators in the *canonical* Heisenberg picture; they are generalizations of the $T=0$ rules for QHD-I in Ref. [10] to $T \neq 0$ and to an arbitrary reference frame, which is characterized by the fluid four-velocity u^μ. The new features at $T \neq 0$ are that each vertex now comes in two types, which will be labeled by $r=1,2$ and which differ only by a sign, and that each propagator has a corresponding 2×2 matrix structure (e.g., $\Delta^{(rr')} \equiv \underline{\mathbf{\Delta}}$). The rules of Ref. [10] can be recovered by taking the $T \to 0$ limit and specializing to the comoving frame with $u^\mu = (1,0,0,0)$. Other Green's functions (higher n-point functions) can be calculated using the same rules, by changing the number of external lines appropriately (generalizing rule 1 below) and by defining the overall phase consistently.

The rules for the nth order contribution to $(i)\times$(real-time propagator) in QHD-I are as follows (see Fig. 7):

(1) Draw all topologically distinct diagrams with two external propagator lines and n vertices connected by internal propagator lines. Do not include diagrams with completely disconnected pieces (that is, not connected to *any* external line).

(2) Assign a direction to each line. Associate a directed four-momentum with each line and conserve energy and momentum at each vertex. This will leave only independent four-momenta in the diagram.

(3) Each *internal* vertex is given a label r, which is summed over $r=1,2$. External points of physical real-time Green's functions have $r=1$ only. Each vertex is assigned a factor indicated in Fig. 7. These are the same factors as at $T=0$, except for the r-dependent sign. Feynman rules for counterterm contributions follow directly.

(4) Each directed line carrying momentum k gets a factor of (i) times the appropriate noninteracting propagator: $\Delta_0^{(rr')}(k)$, $D_0^{(rr')\mu\nu}(k)$, or $G_0^{(rr')}(k)_{\xi\xi'}$, as indicated in Fig. 7. The labels r and r' are those of the vertices at the ends of the propagator. Explicit expressions for these propagators are listed above. These expressions include regularized delta functions δ_ϵ, where $\epsilon \to 0$ eventually. In most cases, ϵ must be kept finite until the end of a calculation to ensure well-defined results; this is certainly true for diagrams involving products of propagators with the same momentum arguments.

(5) Sum over all repeated spacetime ($\mu, \nu, \ldots$), Dirac ($\xi, \xi', \ldots$), and thermal matrix ($r, r', \ldots$) indices.

(6) Integrate $\int d^4k/(2\pi)^4$ over all independent internal four-momenta.

(7) Include a factor of (-1) for each closed fermion loop.

(8) Any single-fermion line forming a tadpole loop *may* be interpreted as $e^{i\eta u\cdot k} i G_0^{(rr)}(k)_{\xi\xi}$, where $\eta \to 0^+$ at the end of the calculation. Such factors are unnecessary if integrals are regularized dimensionally.

(9) A symmetry factor may be needed in diagrams with ϕ^3 and ϕ^4 vertices. The factor is the same as at $T=0$.

It is customary to include a factor of $e^{i\eta u\cdot k}$ or $e^{i\eta k^0}$ (with $\eta \to 0^+$ at the end of the calculation) in closed tadpole loops. However, the operator ordering implied by this factor ($\hat{\bar{\psi}}$ appears to the left of $\hat{\psi}$) does not produce normal-ordered operators, so one must still define the tadpole integral by performing a subtraction. It is therefore equally acceptable to omit this factor and define the integral by subtraction anyway, which is the preferred procedure when dimensional regularization is used. An exception to this procedure involves the vertex for a conserved current; this can be normal ordered in the Lagrangian so that divergent (vacuum) tadpoles never appear.

Similar Feynman rules (for the comoving frame) have been given elsewhere [2,4,38], but several different phase conventions are used.

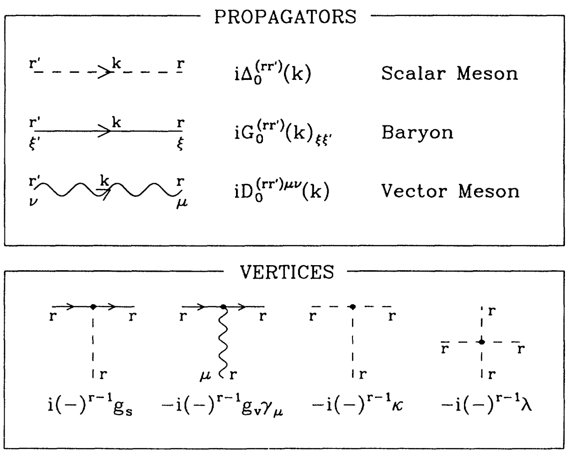

FIG. 7. Components of real-time Feynman diagrams for QHD-I.

E. Discussion

The Feynman rules presented above allow one to compute the interacting propagators as order-by-order expansions in the coupling constants and the noninteracting propagators. As we have emphasized, however, QHD-I is a strong-coupling theory, and such perturbative expansions must be summed to infinite order to produce meaningful results. The issue of which classes of diagrams are important is still an open question that is beyond the scope of this article and is currently under active investigation. Nevertheless, there are several points that arise in the application of these Feynman rules that are worthy of discussion.

Observable thermodynamic quantities and various response functions will typically be expressed as thermal averages of composite operators, which can be related to the interacting propagators, self-energies, and vertices. In general, the analytic expressions for these quantities will be divergent, and the physical observables must be defined by making subtractions to render these expressions finite. For some quantities, such as the baryon current, one need only normal order the current operator, which is equivalent to a vacuum-expectation-value subtraction. For most quantities, however, a vacuum subtraction is insufficient, and additional counterterms must be included. Various procedures for making these subtractions are discussed in the literature; see, for example, Refs. [1, 10, 21].

When calculating with the real-time Feynman rules, it is important to maintain the 2×2 matrix structure at intermediate stages of the calculation, even though the desired (physical) results depend only on the (1,1) components of the propagators. For example, the cancellation of disconnected diagrams resulting from Eq. (3.55) occurs only when contributions from both C_1 and C_2 are included [2]. This can be verified explicitly in the first few orders of perturbation theory, although it is usually easier to use the coordinate-space spectral representations [e.g., Eq. (5.15)] than to use the momentum-space rules given above. Moreover, as discussed in Sec. 3.2.2 of Ref. [2], the 2×2 matrix structure of the propagators removes pathologies (so-called "pinch singularities") that would arise if one kept only the (1,1) propagator components in the Feynman rules [39].

It is also important to remember the distinction between the canonical and grand-canonical Heisenberg pictures. As response functions and thermodynamic observables are most often expressed as thermal averages of canonical Heisenberg-picture fields, the latter must be rewritten in terms of grand-canonical fields before the imaginary-time propagators can be applied. The transformation is usually straightforward; for example, one has

$$\langle\langle \hat{\bar{\psi}}_H(-i\tau,\vec{x})\partial^\mu\hat{\psi}_H(-i\tau,\vec{x})\rangle\rangle = \langle\langle \hat{\bar{\psi}}_K(-i\tau,\vec{x})(d^\mu - i\mu n^\mu)\hat{\psi}_K(-i\tau,\vec{x})\rangle\rangle \qquad (5.53)$$

for baryon fields, with similar results for scalar and vector fields.

Finally, we consider how covariance is maintained in the preceding formalism. Our working definition of covariance is that well-defined quantities can be calculated by any freely falling observer with identical results. In the real-time formalism, the covariance is manifest, since all propagators are written in terms of quantities with definite Lorentz transformation properties, and there is no dependence on n^μ. For imaginary times, however, covariance has been imposed by requiring that all observers refer their calculations to a particular quantization hyperplane specified by n^μ. Although all calculated observables will be manifestly covariant, one must still show that they are independent of the choice of quantization hyperplane. At present we have no general proof that all observables will be independent of n^μ when computed from the imaginary-time propagators listed above; however, in Ref. [1], we showed how this works at the one-baryon-loop level. Since our Feynman rules maintain the distinction between n^μ and u^μ, one can explicitly check the independence of results with respect to n^μ. This is in contrast to an approach that is often advocated in the literature, in which quantization is always performed in hyperplanes normal to u^μ, and the independence of results on the choice of hyperplane is assumed.

VI. SUMMARY

The purpose of this paper is to present a framework for performing covariant finite-temperature calculations of strongly interacting, relativistic, many-body systems. This framework is based on finite-temperature Feynman rules in both real and imaginary time, which are derived using path integrals along contours in the complex time plane. These rules were illustrated in an earlier paper [1]. The advantage to covariant Feynman rules is that they incorporate information and constraints that would be obscured by working in a fixed frame [1,40]. Moreover, one can use them in situations where there is no frame in which all the matter is at rest, and one can directly identify the Lorentz structure and associated invariant functions that arise in the computation of observables.

We concentrate on the basic elements and procedures needed to progress from the generating functional to the Feynman rules, such as the factorization of the generating functional, the definition of noninteracting contour propagators, the implementation of covariance, and the distinction between the canonical and grand-canonical Heisenberg pictures. Although the methods are applied to quantum hadrodynamics, the derivation discussed here can be used for other relativistic field-theoretic Lagrangians.

The starting point of the development is a generating functional defined on time contours (paths) in the complex time plane. The allowed time paths are restricted to those for which the thermal traces converge, namely, paths that have a monotonically decreasing imaginary part, terminate at a point $-i\beta$ below the start of the contour, and lie within a horizontal strip of width β, the inverse temperature. The generating functional is independent of the origin of the contour. For contours of this type, the propagators derived from the generating functional are analytic functions of their time arguments. It is straightforward to rewrite the generating functional as a path integral over the dynamical fields and their conjugate momentum densities. This path integral can be manipulated in the usual fashion to produce an exact result containing the noninteracting propagators and interaction terms written as functional derivatives with respect to external sources [see Eqs. (3.20) and (5.9)]. We emphasize that the normalization of the path integral is well defined, and in the absence of external sources, the generating functional reduces to the (grand) partition function.

The noninteracting propagators appropriate for a particular contour can be determined from the form of the noninteracting Lagrangian. They are most conveniently written in a four-dimensional spectral representation, as in Eq. (3.33). Although the spectral density is singular in momentum space, one can verify that the propagators are analytic functions of the time for contours defined as noted above. Boson propagators satisfy the familiar KMS periodicity conditions, while the fermion propagators are antiperiodic.

It is possible to derive a consistent set of Feynman rules for this class of contours and then pass to the limit of infinite real time variables at the end of the calculation [24]. In particular, one can show that the noninteracting propagators have a power-law decay at large times, and the noninteracting generating functional is well defined in the infinite-time limit, if adiabatic switching is imposed on the external sources. Moreover, it follows that the noninteracting generating functional factorizes into real-time and imaginary-time contributions in the limit. However, the resulting Feynman rules are unwieldy, because the *exact* generating functional does *not* factorize; certain convolutions of propagators do not vanish because of the power-law time decay, and all convolutions must be evaluated before the infinite-time limit is taken. One also loses the advantage afforded by infinite real time intervals, which allow the propagators to be written as four-dimensional Fourier transforms and results to be expressed in manifestly covariant form.

We therefore define the Feynman rules differently, by taking the real time intervals to infinity before evaluating convolutions of propagators [1,2,4,23]. In this limit, it is necessary to allow contours that are strictly horizontal (that is, relative times can be real), and we must consider propagators that are (singular) generalized functions of the relative time variable. The exact generating functional then involves convolutions of generalized functions with support on the whole real time axis, which are not well defined [19]. We interpret these convolutions at intermediate stages of any calculation using the four-dimensional spectral representation with a regularized spectral density that is a meromorphic function of the frequency variable. The regularization parameter ϵ is retained until the end of the calculation. This modifies the asymptotic time dependence of the propagators from a power-law to an exponential decay, leading to different results for convolutions of propagators from those discussed above. Most important, the regularization procedure implies that the exact generating functional factorizes into real-time and imaginary-time contributions.

We stress that the regularization procedure has no real significance if one discusses the properties of a single noninteracting propagator. The only consequences occur in convolutions of propagators (which must be defined in the infinite-time limit) and in fact, only in convolutions that involve "anomalous diagrams." These diagrams contain self-energy insertions on internal propagator lines that are needed to shift the arguments of the thermal distribution functions from the noninteracting spectrum to the interacting spectrum, when one defines Feynman rules using the first approach mentioned above. In the second approach, however, the distribution functions must contain frequency variables that are *off the mass shell* for the propagators to satisfy the KMS conditions, and these off-shell variables are automatically replaced by the correct interacting spectrum when the integrals over internal momenta are performed. The anomalous diagrams are not needed, and they vanish due to the exponential time decay of the regularized noninteracting propagators. In the end, these two approaches yield identical results for the interacting propagators [23], but the second approach is more convenient, since the exact generating functional factorizes.

The factorization of the generating functional has several important consequences. First, it allows us to perform real-time and imaginary-time calculations separately. One can show that the vertical imaginary-time contour can be replaced by the familiar Matsubara choice; thus, these calculations allow us to study the equilibrium thermodynamics of the many-body system. The two horizontal contours produce a 2×2 matrix structure in the real-time propagators and allow us to study the dynamical response of the system to external perturbations. One also finds the important result that closed-loop "bubble" diagrams survive only in the imaginary-time part of the generating functional, where they produce the correctly normalized partition function. In contrast, closed-loop diagrams from the horizontal contours cancel [see Eq. (3.55)], so the real-time propagators do not contribute directly to the thermodynamics of the system, and the real-time part of the generating functional is correctly normalized without any additional phases or factors. Finally, by studying the analytic structure (in frequency space) of the regularized real-time propagators, one can prove the absence of various pathologies that would arise if one computed real-time results using a contour that runs only along the real time axis [2,39].

To implement covariance in the formalism, we allow the theory to be quantized on any spacelike hyperplane, as in Eq. (4.2), rather than on the hyperplane that is comoving with the fluid. For real times, the generalization is straightforward, and temporal and spatial variables can be incorporated into a single Lorentz four-vector with familiar transformation properties. For complex times, however, it is necessary to keep the temporal and spatial variables separate and to Lorentz transform them separately. The imaginary part of the time is related to the inverse temperature observed on the chosen quantization hyperplane. One can construct a covariant generating functional by again appealing to contours in the complex time plane; the result is similar to that derived in the comoving frame, except that derivative operators in the Lagrangian now have an explicit spacelike part that acts in the quantization hyperplane and a "convective" temporal part that acts normal to the hyperplane.

By implementing covariance in this fashion, we introduce an auxiliary timelike vector n^{μ} that defines the direction of time evolution of the system. One must still prove that computed observables are independent of the choice of hyperplane. Here it is relevant that propagators derived from the generating functional are in the grand-canonical Heisenberg picture, which is defined using a Hamiltonian that contains Lagrange multipliers for the fluid velocity and baryon chemical potential. This picture corresponds to the Lagrangian description of fluid dynamics, in which one follows the evolution of fluid elements on successive hyperplanes. For the real-time rules, it is easy to relate these propagators to those defined in the canonical Heisenberg picture (which contains no Lagrange multipliers and corresponds to the Eulerian description of fluid dynamics), and to show that the difference between propagators is only a phase. Because four-momentum and baryon number are conserved at every vertex in every Feynman diagram, one can remove this phase and write the real-time rules solely in terms of canonical propagators that are independent of the choice of quantization hyperplane. In the end, the generalization of canonical propagators from the comoving frame to an arbitrary frame is simple [see the discussion after Eq. (4.28)]. The covariant real-time Feynman rules are listed in Sec. V D.

In contrast, imaginary-time propagators are computed in the grand-canonical Heisenberg picture, and thus contain an explicit dependence on the quantization hyperplane. Although we do not have a general proof that calculated observables will be independent of the hyperplane, we showed in Ref. [1] how this works at one-baryon-loop order, and that demonstration makes the independence to all orders plausible. In any event, the imaginary-time rules given in Sec. V C are written explicitly in terms of n^μ, so by retaining this variable throughout the intermediate stages of a calculation, one can see whether it disappears in the final results (much as one uses a gauge parameter to determine the gauge invariance of calculated results in QED or QCD).

Finally, we consider the application of our formalism to other relativistic models. Our motivation here was to derive covariant Feynman rules in a simple and widely used hadronic model with several different degrees of freedom and to concentrate on basic elements of the finite-temperature procedures. Within the context of QHD, these rules provide a framework for performing covariant calculations beyond the one-loop level, which may be useful in finding a suitable expansion procedure for reliable finite-density calculations; this is still an unsolved problem [21]. Many of the steps in our derivation are independent of the form of the Lagrangian and can thus be carried over to QED and QCD, when various (familiar) modifications are made to incorporate local gauge invariance in the path integrals [20,41,42]. Nevertheless, we emphasize that certain steps depend on the use of a local, renormalizable, Lorentz-invariant Lagrangian density; for example, the analytic structure of the finite-temperature propagators depends on microscopic causality [16]. Thus, while we expect that our formalism can be extended to more phenomenological approaches used to study hadronic and nuclear systems, it may be incorrect to simply apply the Feynman rules listed here without due regard for how these rules were obtained.

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APPENDIX: GENERATING FUNCTIONAL FOR NEUTRAL, MASSIVE VECTOR MESONS

This appendix has two purposes: to derive the generating functional for neutral, massive vector mesons from the appropriate path integral over field variables and conjugate momenta, and to generalize the result to an arbitrary quantization hyperplane, as in Eq. (4.15). Although the vector meson generating functional is usually written in the literature in a familiar form [10], which involves path integrals over all four field components V^μ, this result is nontrivial, since one must start with a path integral over the dynamical fields and conjugate momenta and then integrate out the momentum variables [41]. Moreover, while it is often tacitly assumed that the familiar form for the path integral is correct by analogy to QED, the massive vector field is qualitatively different from the photon field, since the Lagrangian for the former has no local gauge invariance. Thus the usual QED gauge freedom does not exist, and no "choice of gauge" is needed (or allowed) in the quantization of the massive vector field. As a consequence, all four components V^μ are operators, even for free fields, in contrast to the case of QED in the Coulomb gauge [43]. In fact, the massive vector field resembles a collection of four constrained scalar fields more closely than it does the photon field.

Rather than consider noninteracting vector mesons, we will study the more interesting case where the mesons interact with baryons in the usual fashion. The Lagrangian is given by (2.9), with ϕ suppressed, and the field equations are (2.12) and (2.13). To simplify the notation, we begin in the comoving frame with a purely spacelike quantization hyperplane, so that the instant parameter is $t=x^0$. At the end, we will generalize to a quantization hyperplane with an arbitrary timelike normal vector n^μ.

Begin by defining electric and magnetic fields:

$$E^i \equiv -F^{0i} = -\dot{V}^i - \nabla^i V_0 \ , \tag{A1}$$

$$B^k \equiv -\tfrac{1}{2}\epsilon_{ijk}F^{ij} = (\nabla\times\mathbf{V})^k \ , \tag{A2}$$

which imply that the two Maxwell equations $\nabla\times\mathbf{E}=-\dot{\mathbf{B}}$ and $\nabla\cdot\mathbf{B}=0$ are automatically satisfied. (We conventionally take the three-dimensional antisymmetric symbol to be independent of the vertical position of the indices: $\epsilon_{ijk}=\epsilon^{ijk}$.) Note that the first of these Maxwell equations is an equation of motion, while the second is an equation of constraint. The remaining two Maxwell equations follow from

$$\partial_\mu F^{\mu\nu} + m_{\rm v}^2 V^\nu = g_{\rm v}\bar{\psi}\gamma^\nu\psi \ , \tag{A3}$$

namely,

$$\dot{\mathbf{E}} = \nabla\times\mathbf{B} + m_{\rm v}^2\mathbf{V} - g_{\rm v}\mathbf{j} \ , \tag{A4}$$

$$m_{\rm v}^2 V_0 = -\nabla\cdot\mathbf{E} + g_{\rm v}\rho_B \ , \tag{A5}$$

where we use the shorthands $\rho_B \equiv \psi^\dagger\psi$ and $\mathbf{j}=\psi^\dagger\boldsymbol{\alpha}\psi$. Note that the time derivative of V_0 does not appear in these equations; thus, the constraint equation (A5) is analogous to (A2). Nevertheless, since the baryon current is conserved, we find $\partial_\mu V^\mu = 0$, so that in a canonical approach to quantization, V_0 becomes an operator

[20,44]. Thus the definition of the propagator in Eq. (2.19) makes sense for all spacetime indices.

To construct the Hamiltonian, we first compute the conjugate momentum densities

$$\Pi_\psi \equiv \frac{\partial \mathcal{L}}{\partial \dot\psi} = i\psi^\dagger \ , \tag{A6}$$

$$\Pi^0 \equiv \frac{\partial \mathcal{L}}{\partial \dot V_0} = 0, \quad \Pi_i \equiv \frac{\partial \mathcal{L}}{\partial \dot V^i} = F^{0i} = -E^i \ . \tag{A7}$$

As expected, there is no momentum conjugate to the V_0 field. The Hamiltonian density follows from

$$\mathcal{H} = i\psi^\dagger \frac{\partial \psi}{\partial t} + \Pi_\lambda \dot V^\lambda - \mathcal{L} = i\psi^\dagger \frac{\partial \psi}{\partial t} - \mathbf{E}\cdot\dot{\mathbf{V}} - \mathcal{L} \ , \tag{A8}$$

where the right-hand side is to be expressed in terms of the dynamical variables ψ, $\psi^\dagger$, $\mathbf{V}$, and $\mathbf{E}$. If we assume that the field variables obey periodic boundary conditions on the surfaces of our large volume $\mathcal{V}$, which eliminates surface terms arising from partial integrations, it is easy to arrive at the Hamiltonian

$$H = \int d^3x\, \mathcal{H} = \int d^3x \left\{ \psi^\dagger(-i\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta M)\psi + \tfrac{1}{2}[\mathbf{E}^2 + (\boldsymbol{\nabla}\times\mathbf{V})^2] + \tfrac{1}{2}m_v^2\mathbf{V}^2 + \frac{1}{2m_v^2}(\boldsymbol{\nabla}\cdot\mathbf{E})^2 - \frac{g_v}{m_v^2}\rho_B\boldsymbol{\nabla}\cdot\mathbf{E} - g_v\mathbf{j}\cdot\mathbf{V} + \frac{g_v^2}{2m_v^2}\rho_B^2 \right\} . \tag{A9}$$

The relevant part of the partition function contains the vector fields and their couplings to the fermion sources, so we suppress the noninteracting fermion terms (including the chemical potential) in the sequel. Various normalization constants that arise in the ensuing manipulations will also be suppressed. These can be determined by writing the path integrals on a spacetime lattice [14], but this is not really necessary, as they will all be canceled when similar manipulations are performed on the path integral that defines the zero-point-energy subtraction. Thus we consider

$$Z \propto \int_{\mathbf{V}}^{\mathbf{V}} \prod_i D(V^i)D(E^i)\exp\left\{ i\int_c d^4x \left[-\mathbf{E}\cdot\dot{\mathbf{V}} - \tfrac{1}{2}\mathbf{E}^2 - \tfrac{1}{2}(\boldsymbol{\nabla}\times\mathbf{V})^2 - \tfrac{1}{2}m_v^2\mathbf{V}^2 + g_v\mathbf{j}\cdot\mathbf{V} - \frac{1}{2m_v^2}(\boldsymbol{\nabla}\cdot\mathbf{E} - g_v\rho_B)^2 \right] \right\} . \tag{A10}$$

Note that $\mathbf{V}$ and $\mathbf{E}$ are independent integration variables, and $\mathbf{V}$ satisfies the usual thermal periodicity condition [see Eq. (3.15)] at the ends of the contour c; moreover, we cannot use the classical equations of motion or constraint, as we must integrate over all possible field configurations.

Following standard procedure, we would like to integrate out the conjugate momenta ($\mathbf{E}$) to arrive at a path integral containing only the fields ($\mathbf{V}$). This is not straightforward due to the $(\boldsymbol{\nabla}\cdot\mathbf{E})^2$ term. Consider, however, the Gaussian path integral

$$\int D(V_0)\exp\left[i\int_c d^4x(\tfrac{1}{2}m_v^2V_0^2 + V_0\boldsymbol{\nabla}\cdot\mathbf{E}) \right] = \exp\left[-i\int_c d^4x \frac{1}{2m_v^2}(\boldsymbol{\nabla}\cdot\mathbf{E})^2 \right] \int D(V_0)\exp\left[i\int_c d^4x\tfrac{1}{2}\left[m_vV_0 + \frac{1}{m_v}\boldsymbol{\nabla}\cdot\mathbf{E} \right]^2 \right] . \tag{A11}$$

If inserted inside the path integral of Eq. (A10), the final Gaussian integral in (A11) gives a purely numerical factor (since $\boldsymbol{\nabla}\cdot\mathbf{E}$ is fixed) that will be canceled along with various normalization factors, as noted above. Oscillatory Gaussian path integrals can be defined in the usual way by rotating the field variable through $\pi/4$ in the complex plane. In contrast, if the time is complex, one can analytically continue to a real time variable for the subsequent manipulations and then continue back in Eq. (A16).

Clearly, we also have

$$\int D(V_0)\exp\left\{ i\int_c d^4x[\tfrac{1}{2}m_v^2V_0^2 + V_0(\boldsymbol{\nabla}\cdot\mathbf{E} - g_v\rho_B)] \right\} \propto \exp\left\{ -i\int_c d^4x\frac{1}{2m_v^2}(\boldsymbol{\nabla}\cdot\mathbf{E} - g_v\rho_B)^2 \right\} , \tag{A12}$$

which allows us to rewrite (A10) as

$$Z \propto \int_{\mathbf{V}}^{\mathbf{V}} \prod_i D(V^i)D(E^i)D(V_0)\exp\left\{ i\int_c d^4x \left[-\mathbf{E}\cdot\dot{\mathbf{V}} - \tfrac{1}{2}\mathbf{E}^2 - \tfrac{1}{2}(\boldsymbol{\nabla}\times\mathbf{V})^2 - \tfrac{1}{2}m_v^2\mathbf{V}^2 + g_v\mathbf{j}\cdot\mathbf{V} + \tfrac{1}{2}m_v^2V_0^2 + V_0(\boldsymbol{\nabla}\cdot\mathbf{E} - g_v\rho_B) \right] \right\} . \tag{A13}$$

We remark that if one desires to write the partition function in covariant form as a path integral over both V^μ and $F^{\mu\nu}$, one can introduce $\mathbf{B}$ as an auxiliary variable by using a similar Gaussian integral to replace $-\tfrac{1}{2}(\boldsymbol{\nabla}\times\mathbf{V})^2$ with $\tfrac{1}{2}\mathbf{B}^2 - \mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{V})$ in Eq. (A13).

It is now straightforward to perform the Gaussian path integrals over the electric field by defining longitudinal and transverse parts:

$$\mathbf{E}\equiv\mathbf{E}_L+\mathbf{E}_T,\quad \nabla\cdot\mathbf{E}_T=0,\quad \nabla\times\mathbf{E}_L=0\ , \tag{A14}$$

and similarly for $\mathbf{V}$. When integrated over $\mathcal{V}$, all longitudinal-transverse cross terms vanish by the assumed periodic boundary conditions:

$$\int d^3x\ \mathbf{a}_L\cdot\mathbf{b}_T\equiv\int d^3x(\nabla f_a)\cdot\mathbf{b}_T=\int d^3x[\nabla\cdot(f_a\mathbf{b}_T)-f_a(\nabla\cdot\mathbf{b}_T)]=0\ . \tag{A15}$$

Thus, up to overall normalization factors,

$$Z\propto\int_{\mathbf{V}}^{\mathbf{V}}D(V^\mu)\exp\left\{i\int_c d^4x[\tfrac{1}{2}\dot{\mathbf{V}}_T^2+\tfrac{1}{2}(\dot{\mathbf{V}}_L+\nabla V_0)^2-\tfrac{1}{2}(\nabla\times\mathbf{V}_T)^2+\tfrac{1}{2}m_v^2V_\mu V^\mu-g_v\rho_B V_0+g_v\mathbf{j}\cdot\mathbf{V}]\right\} \tag{A16}$$

$$=\int_{V^\mu}^{V^\mu}D(V^\mu)\exp\left\{i\int_c d^4x[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}+\tfrac{1}{2}m_v^2V_\mu V^\mu-g_vV_\mu\bar{\psi}\gamma^\mu\psi]\right\}\ . \tag{A17}$$

To summarize, the correct evaluation of the vector meson path integral leads to an expression that involves integration over all four field variables V^μ and an exponential that takes the familiar covariant form. One can now restore the fermion terms and add sources in the usual way (that is, $J_\mu V^\mu$ for the vector field) to define the generating functional.

Several comments are in order regarding the periodicity condition on the V_0 field, as indicated in Eq. (A17). The dynamical $\mathbf{V}$ fields must be periodic, as required by the thermal trace, but no such requirement holds on V_0, and (A16) is correct as it stands. However, to define a vector boson propagator, it is necessary to partially integrate some time derivatives in (A16) on to the V_0 field. Thus it is convenient to impose periodicity conditions also for this field, as in (A17), to avoid unwieldy surface terms. This implies that retarded and advanced functions corresponding to all spacetime components of the vector propagator $D_0^{\mu\nu}$ satisfy the KMS periodicity condition [analogous to Eq. (3.25)].

To generalize the preceding derivation to quantization on an arbitrary hyperplane (or equivalently, to time evolution along an arbitrary timelike normal n^μ), proceed as follows. (We will illustrate the required modifications for *free* vector fields, since the vector-baryon interactions contain no derivatives and can be added easily.) The conjugate momenta are now defined by

$$\Pi^\lambda\equiv n_\mu\Pi^{\mu\lambda}\equiv n_\mu\frac{\partial\mathcal{L}_0}{\partial(\partial_\mu V_\lambda)}=n_\mu F^{\lambda\mu}\ , \tag{A18}$$

which generalizes Eq. (A7). Note that there is no momentum along n^λ, since $n_\lambda\Pi^\lambda$ vanishes by the antisymmetry of $F^{\lambda\mu}$, and thus Π^λ has only three independent components. It is therefore convenient to write the vector field as

$$V^\mu=n^\mu(n\cdot V)+\tilde{V}^\mu\ , \tag{A19}$$

where the projection of V^μ along n^μ is nondynamical, and the field $\tilde{V}^\mu$ in the spacelike quantization hyperplane contains three independent variables.

Equation (A18) and the free-field equation $\partial_\mu F^{\mu\nu}+m_v^2V^\nu=0$ imply $n\cdot V=-\partial_\mu\Pi^\mu/m_v^2$, and we can write the field-strength tensor in terms of dynamical variables as

$$F^{\lambda\mu}=\Pi^\lambda n^\mu-\Pi^\mu n^\lambda+\tilde{\partial}^\lambda\tilde{V}^\mu-\tilde{\partial}^\mu\tilde{V}^\lambda\ , \tag{A20}$$

where $\tilde{\partial}^\mu$ is defined in Eq. (4.10). The first two terms in (A20) give the electric fields ($\mathbf{E}$) in covariant notation, while the final two give the magnetic fields ($\mathbf{B}$).

Now use

$$T^{\mu\nu}=-g^{\mu\nu}\mathcal{L}_0+\Pi^{\mu\lambda}\partial^\nu V_\lambda=-g^{\mu\nu}\mathcal{L}_0+F^{\lambda\mu}\partial^\nu V_\lambda\ , \tag{A21}$$

so that the partition function becomes [compare Eqs. (4.4) and (4.8)]

$$Z_0=\mathrm{Tr}\exp\left[i\int_c d^4x\left[-\frac{1}{n\cdot u}n_\mu u_\nu\hat{T}^{\mu\nu}\right]\right]$$

$$=\int_{\tilde{V}}^{\tilde{V}}D(\tilde{V}^\mu)D(\Pi^\lambda)\exp\left[i\int_c d^4x\left[\Pi^\lambda n\cdot\partial\tilde{V}_\lambda+\mathcal{L}_0-\frac{1}{n\cdot u}n_\mu u_\nu F^{\lambda\mu}\partial^\nu V_\lambda\right]\right]\ . \tag{A22}$$

Here the path integral is to be taken over the three dynamical $\tilde{V}^\lambda$ fields (with periodic boundary conditions indicated symbolically) and the three components of the momentum density Π^λ, which are unconstrained at the end points. The final two terms in the exponential must be written in terms of these dynamical variables; we want no "time derivatives" $n\cdot\partial\tilde{V}^\lambda$ or nondynamical fields $n\cdot V$. The desired results follow straightforwardly and are given by

$$-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}=-\tfrac{1}{2}[\Pi_\mu\Pi^\mu+\tfrac{1}{2}(\tilde{\partial}_\mu\tilde{V}_\nu-\tilde{\partial}_\nu\tilde{V}_\mu)^2]\ , \tag{A23}$$

$$\tfrac{1}{2}m_v^2 V^\mu V_\mu = \frac{1}{2m_v^2}(\partial_\mu \Pi^\mu)^2 + \tfrac{1}{2}m_v^2 \tilde{V}_\mu \tilde{V}^\mu , \tag{A24}$$

$$-\frac{1}{n\cdot u} n_\mu u_\nu F^{\lambda\mu}\partial^\nu V_\lambda = -\frac{1}{n\cdot u}\Pi^\lambda u\cdot\partial \tilde{V}_\lambda$$
$$= \Pi_\lambda \Pi^\lambda - \frac{1}{n\cdot u}\Pi^\lambda u\cdot\tilde{\partial}\tilde{V}_\lambda - \frac{1}{m_v^2}(\partial_\mu \Pi^\mu)^2 . \tag{A25}$$

Here we have anticipated that these expressions will be inserted into the path integral, so a spatial partial integration has been performed to arrive at (A25).

With the preceding results, the covariant analog of Eq. (A10) is given by

$$Z_0 \propto \int_{\tilde{V}}^{\tilde{V}} D(\tilde{V}^\mu) D(\Pi^\lambda) \exp\left\{ i\int_c d^4x \left[\Pi^\lambda n\cdot\partial\tilde{V}_\lambda + \tfrac{1}{2}\Pi^2 - \tfrac{1}{4}(\tilde{\partial}_\mu \tilde{V}_\nu - \tilde{\partial}_\nu \tilde{V}_\mu)^2 - \frac{1}{2m_v^2}(\partial^\lambda \Pi_\lambda)^2 + \tfrac{1}{2}m_v^2 \tilde{V}^2 - \frac{1}{n\cdot u}\Pi^\lambda u\cdot\tilde{\partial}\,\tilde{V}_\lambda \right]\right\} . \tag{A26}$$

As above, the integrations over the momentum variables Π^λ are complicated by the $(\partial^\lambda \Pi_\lambda)^2$ term. However, we can introduce a Gaussian path integral over an auxiliary integration variable $n\cdot V$ to recast (A26) in a form analogous to (A13):

$$Z_0 \propto \int_{\tilde{V}}^{\tilde{V}} D(V^\mu) D(\Pi^\lambda) \exp\left\{ i\int_c d^4x \left[\Pi^\lambda n\cdot\partial\tilde{V}_\lambda + \tfrac{1}{2}\Pi^2 - \tfrac{1}{4}(\tilde{\partial}_\mu \tilde{V}_\nu - \tilde{\partial}_\nu \tilde{V}_\mu)^2 + \tfrac{1}{2}m_v^2[\tilde{V}^2 + (n\cdot V)^2] - \frac{1}{n\cdot u}\Pi^\lambda u\cdot\tilde{\partial}\tilde{V}_\lambda + n\cdot V\partial_\lambda \Pi^\lambda \right]\right\} . \tag{A27}$$

It is now a simple exercise to perform the Gaussian path integrals over Π^λ and to show that the result can be written as in Eq. (A17), with all derivatives ∂^μ replaced by d^μ defined in Eq. (4.14). Since the interaction term has no derivatives, it also has the form in Eq. (A17). Finally, one can impose the periodicity condition on $n\cdot V$ to allow for the construction of a covariant vector propagator, as discussed after Eq. (A17).

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in (3.37) behave as $|t-t'|^{-3/2}$ as $|t-t'|\to\infty$, as do the integrals in (3.44).

[26] The astute reader will realize that the integrand of the k_0 integral contains poles both near the real axis and also along the imaginary axis, when k_0 is a multiple of $2\pi i/\beta$. These imaginary poles arise from $n_b(\beta k_0)$. However, when $\rho_0(k)$ of (3.47) is evaluated at these poles, it is of order ϵ, and all of the resulting contributions to the propagator and Feynman rules vanish when ϵ is taken to zero. We will therefore simply omit the contributions from these poles. Note that $n_b(\beta k_0)\rho_0(k)$ has no pole at $k_0=0$.

[27] The arguments used to eliminate the "cross-contour" convolutions fail for some *closed-loop* diagrams. This can be seen by setting $t'=t$ in Eq. (3.50) and integrating $\int_{C_1} dt$. Fortunately, the closed-loop contributions that connect segments C_1 and C_3 exactly cancel those that connect C_2 and C_3. Thus the conclusion that the exact generating functional factorizes remains valid.

[28] Although this argument appears to rely on perturbation theory, it remains valid when nonperturbative summations of Feynman diagrams are performed. With these summations, the single-particle self-energy appears in the denominators in the frequency integrals, thereby automatically shifting the poles to the appropriate interacting spectrum.

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