y-scaling analysis for inelastic scattering from relativistic targets

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We discuss the calculation of the hadronic tensor, which plays an essential role in the description of the interaction of a nucleus with an electromagnetic probe, in the kinematic regime where the concepts underlying the theory of y scaling may be used. In our previous work we determined the kinematic factors that relate the cross section to the scaling function, $F(y)$, defined in the theory of y scaling. That study was carried out for a deuteronlike bound state of two scalar particles. In this work we extend our analysis to the case where the target is composed of spin one-half nucleons. The origin of various kinematic factors arising from the off-shell characterization of the struck nucleon and its motion in the nucleus is clarified. The target nucleus is described as a relativistic system and, in the scaling limit, we discuss how the terms which parametrize the relativistic density matrix determine the value of the cross section.

I. INTRODUCTION

We are interested in calculating the cross section for quasielastic electron-nucleus scattering on rather general grounds, using as few approximations as possible. The first approximation we will have to make is to assume that the inclusive cross section is dominated by the ejection of a single nucleon from the target. We are aware of that, in general, various corrections to this simple picture are required. One may study the role of meson-exchange currents or the excitation of the delta resonance in modifying the picture developed here; however, when restricting ourselves to the special kinematic regime appropriate to y scaling [1] we may assume that these corrections are relatively unimportant [2].

If the struck nucleon has a large momentum, say $k > 400$ MeV, it is likely that this nucleon is one member of a strongly interacting pair of nucleons. If we neglect final-state interactions, we can argue that information concerning the dynamics of the other member of the pair is ultimately contained in the characterization of the density matrix of the target. (In this work we will discuss various assumptions which may be made concerning the structure of the density matrix). A novel feature of our analysis is that the density matrix here is that appropriate to a target that is described relativistically. (In this case, the density matrix, $\rho_{\alpha\beta}$, is a 4×4 matrix.)

Furthermore, in this work we will only consider the plane-wave impulse approximation. That is, we neglect the final-state interaction between the struck nucleon and the spectator nucleus. As was shown in a previous publication [3], these interactions modify the response of the nucleus to the electromagnetic probe, and only when the momentum transfer is large are final-state interactions negligible.

In Ref. [4] we described how the kinematic factors which relate the cross section to the scaling function of the y-scaling analysis should be chosen. It was shown that if these kinematic factors are not chosen properly, the momentum distribution of the target particles determined in a y-scaling study will contain errors [4]. In this work we consider a more realistic model in which the target is a nucleus composed of spin one-half nucleons. The analysis is somewhat more complicated than that of Ref. [4], since we here have to make use of the Rosenbluth formula to describe the interaction of the electrons with the target nucleon. However, we find that the kinematic factors are similar to those determined in our earlier work [4], where the electron-"nucleon" cross section was given by the Mott formula.

We are also interested in providing a general analysis of the scattering from a *relativistic* target in the kinematic regime where one may study the phenomenon of y scaling. If we describe the target as a relativistic system, a quantity which enters the analysis is the relativistic density matrix. There have been a number of self-consistent calculations for the ground state of nuclei using the Dirac equation to describe the motion of the nucleons. From these results, one may obtain an expression for the relativistic density matrix of the nucleus. (Actually, what is needed is the matrix which describes the probability of finding a nucleon of momentum k in the target; see Fig. 1. That quantity has a much simpler structure than the density matrix itself which, in general, depends on two momenta, k and k'.)

Almost all investigations of y scaling are based on the assumption that the target can be described in a nonrelativistic framework [5]. One usually starts with "smearing" the electron-nucleon cross section (which eventually contains corrections due to the fact that the nucleon one scatters from is off its mass shell and is moving) with the momentum distribution of the nucleons in the nucleus.

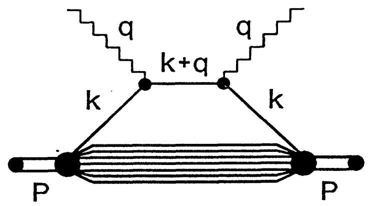

FIG. 1. The hadronic tensor may be obtained by evaluating the imaginary part of the forward (virtual) Compton amplitude shown in this figure. Here P is the momentum of the (on-shell) target of mass M. The wavy lines denote virtual photons which interact with nucleons of momentum k^μ. The lower part of the figure serves to define a density matrix, $\rho_{\alpha\beta}(P^\mu,k^\mu)$.

The analysis then shows that, in the asymptotic limit, one is able to extract this momentum distribution from the measured cross sections. The momentum distribution contains valuable information on short-range correlations and is very difficult to determine otherwise. In the analysis of the experimental data one is especially interested in high momenta (up to 0.7 and 0.8 GeV) and, therefore, we believe that the description of the target as a relativistic system is appropriate.

Rather than proceeding in the standard manner, we here undertake a calculation of the hadronic tensor that determines the response of the target to an electromagnetic probe. This tensor may be calculated in terms of the target density matrix. The form taken by the hadronic tensor allows us to define structure functions of the *nucleus* that are analogous to the structure functions of the *nucleon* defined in the study of deep inelastic electron-nucleon scattering.

By evaluating the nuclear structure functions in the scaling limit, we find an expression for the cross section which is very similar to that obtained when the target is treated as a nonrelativistic system. We then study some special forms of the relativistic density matrix and show under which assumptions our analysis reproduces the results of other authors. It is not our intention to review the large body of literature which deals with y scaling. We refer the reader to an excellent recent review [5], where extensive references to the literature may be found.

The organization of our work is as follows. In Sec. II we specify the form of the density matrix and calculate the nuclear structure functions. In Sec. III we evaluate these structure functions for some special forms of the density matrix and discuss their asymptotic limit. In Sec. IV we introduce the concept of y scaling. In Sec. V we describe a Lorentz-invariant double differential cross section that describes scattering from moving, off-mass-shell nucleons and we continue our discussion of y scaling. Section VI contains further discussion and conclusions.

II. THE DENSITY MATRIX AND THE NUCLEAR STRUCTURE FUNCTIONS

Assuming that quasielastic electron-nucleus scattering is dominated by the one-nucleon knockout process and considering the plane-wave impulse approximation, we find that the hadronic tensor $W^{\mu\nu}$ may be obtained from the evaluation of the Feynman diagram of Fig. 1. The lower part of the diagram does not depend on the interaction with the external photon, and it can be parameterized by a density matrix, $\rho_{\alpha\beta}(P^\mu,k^\mu)$, which determines the probability of finding a nucleon with momentum k^μ in a nucleus of momentum P^μ. In this approximation, the density matrix contains all the information about mean-field dynamics and nucleon correlations. When we consider a target of zero spin and isospin, the most general form for ρ which exhibits Lorentz covariance, parity, and time reversal invariance is

$$\rho(P^\mu,k^\mu)=\rho_1(k^2,P\cdot k)+\rho_2(k^2,P\cdot k)P_\mu\gamma^\mu +\rho_3(k^2,P\cdot k)k_\mu\gamma^\mu \ . \tag{2.1}$$

Here the $\rho_i(k^2,P\cdot k)$ are scalar functions. They do not depend on P^2, since $P^2=M^2$, where M is the mass of the target nucleus.

For our purpose we need not discuss the structure of ρ in other than the target rest frame. Here the total momentum is $P^\mu=(M,\mathbf{0})$ and Eq. (2.1) can be rewritten as

$$\rho(k^0,\mathbf{k})\equiv\frac{a(k^0,|\mathbf{k}|)m+b(k^0,|\mathbf{k}|)k^0\gamma^0-c(k^0,|\mathbf{k}|)\mathbf{k}\cdot\boldsymbol{\gamma}}{2m} \tag{2.2a}$$

$$\equiv\frac{A(k^0,|\mathbf{k}|)+B_\mu(k^0,|\mathbf{k}|)\gamma^\mu}{2m}\ , \tag{2.2b}$$

where m is the nucleon mass, A and B_μ may be written in terms of a, b, and c by comparing Eqs. (2.2a) and (2.2b). (Note that B_μ is not a four-vector.) The density matrix is normalized by requiring that the total baryon number be given correctly. Thus,

$$N+Z=\frac{1}{2}\int d^4k\,\mathrm{Tr}(\gamma^0\rho(k))\ , \tag{2.3a}$$

$$=\frac{1}{m}\int d^4k B^0(k^0,|\mathbf{k}|)\ , \tag{2.3b}$$

$$=\int d^4k\frac{k^0}{m}b(k^0,|\mathbf{k}|)\ , \tag{2.3c}$$

for a nucleus with N neutrons and Z protons. (In general, there is a different density matrix for neutrons and protons; however, we neglect that complication here.)

By evaluating the diagram of Fig. 1, we find the hadronic tensor

$$W^{\mu\nu}=\frac{1}{2}\int d^4k\frac{m}{E(\mathbf{p})}\delta(k^0+\omega-E(\mathbf{p})) \times\mathrm{Tr}\left[(\Gamma^\mu)^*\frac{\not p+m}{2m}\Gamma^\nu\rho(k)\right]\ , \tag{2.4}$$

where $q^\mu=(\omega,\mathbf{q})$ is the four-momentum of the virtual photon and $E(\mathbf{p})=\sqrt{|\mathbf{p}|^2+m^2}$ is the energy of the scattered nucleon, which has four-momentum $p=k+q$. Γ^μ is the usual photon-nucleon vertex,

$$\Gamma^{\mu}=\gamma^{\mu}F_1(Q^2)+i\sigma^{\mu\nu}q_\nu\frac{\kappa}{2m}F_2(Q^2)\ , \qquad (2.5)$$

which depends on the anomalous magnetic moments, $\kappa^{p,n}$, and the free-nucleon form factors, $F_{1,2}^{p,n}$, which are functions of the square of the four-momentum transfer, $Q^2=|\mathbf{q}|^2-\omega^2$. In Eqs. (2.4) and (2.5) we assume isospin symmetry, i.e., we use the same density matrix for neutrons and protons, and correspondingly the quantities κ and $F_{1,2}$ in Eq. (2.5) have to be averaged over the number of neutrons and protons. The vertex Γ^μ specifies the coupling of a free nucleon to a photon, since the corresponding form factors $F_{1,2}$ are determined by scattering experiments on free protons or quasi-free neutrons. This represents another approximation. (It is quite possible that the form factors $F_{1,2}$ and the structure of the vertex are modified when the nucleon is in a nucleus. However, at this time there is no consensus as to how medium modifications and off-shell effects should be introduced.)

Writing

$$T^{\mu\nu}=\frac{1}{2}\mathrm{Tr}\left[(\Gamma^\mu)^*\frac{p_\alpha\gamma^\alpha+m}{2m}\Gamma^\nu\frac{A+B_\beta\gamma^\beta}{2m}\right] \qquad (2.6)$$

for the trace in Eq. (2.4), we find, after some calculations, that

$$\begin{aligned}T^{\mu\nu}=&\frac{F_1^2}{2m^2}\{p^\mu B^\nu+p^\nu B^\mu+[mA-(p\cdot B)]g^{\mu\nu}\}\\&+\frac{\kappa F_1F_2}{4m^3}\{m(q^\mu B^\nu+q^\nu B^\mu)-A(p^\mu q^\nu+p^\nu q^\mu)+2[A(p\cdot q)-m(B\cdot q)]g^{\mu\nu}\}\\&+\frac{\kappa^2F_2^2}{8m^4}(Q^2(p^\mu B^\nu+p^\nu B^\mu)+(p\cdot q)(q^\mu B^\nu+q^\nu B^\mu)+(B\cdot q)(p^\mu q^\nu+p^\nu q^\mu)\\&\qquad-[mA+(p\cdot B)]q^\mu q^\nu-\{Q^2[mA+(p\cdot B)]+2(p\cdot q)(B\cdot q)\}g^{\mu\nu})\ .\end{aligned} \qquad (2.7)$$

At this point we see that our calculation is not fully gauge invariant, i.e., $q_\mu T^{\mu\nu}\neq 0$. We remedy this defect by replacing $T^{\mu\nu}$ with $\hat{T}^{\mu\nu}$, where [6]

$$\hat{T}^{\mu\nu}=\left[g^{\mu\alpha}+\frac{q^\mu q^\alpha}{Q^2}\right]T_{\alpha\beta}\left[g^{\beta\nu}+\frac{q^\beta q^\nu}{Q^2}\right]\ . \qquad (2.8)$$

This yields

$$\hat{T}^{\mu\nu}=T_1\left[g^{\mu\nu}+\frac{q^\mu q^\mu}{Q^2}\right]+T_2(\hat{p}^\mu\hat{B}^\nu+\hat{p}^\nu\hat{B}^\mu)\ , \qquad (2.9)$$

where $\hat{p}^\mu$ and $\hat{B}^\mu$ are defined using the relation

$$\hat{X}^\mu=X^\mu+\frac{(X\cdot q)}{Q^2}q^\mu\ , \qquad (2.10)$$

with $Q^2=-q^2$. Further, T_1 and T_2 of Eq. (2.9) are given by

$$\begin{aligned}T_1=&\frac{F_1^2}{2m^2}[mA-(p\cdot B)]+\frac{\kappa F_1F_2}{2m^3}[A(p\cdot q)-m(B\cdot q)]\\&-\frac{\kappa^2F_2^2}{8m^4}\{Q^2[mA+(p\cdot B)]+2(p\cdot q)(B\cdot q)\}\end{aligned} \qquad (2.11a)$$

and

$$T_2=\frac{1}{2m^2}\left[F_1^2+\frac{Q^2}{4m^2}\kappa^2F_2^2\right]\ . \qquad (2.11b)$$

The cross section for inclusive, inelastic, unpolarized electron-nucleus scattering can be expressed in terms of the hadronic structure functions $W_{1,2}$ through

$$\frac{d^2\sigma}{dE'\,d\Omega}=\sigma_{\mathrm{Mott}}[W_2+2W_1\tan^2[(\tfrac{1}{2}\theta)]\ , \qquad (2.12)$$

where E' is the energy of the electron after the scattering process, θ is the scattering angle, and σ_{Mott} is the well-known Mott cross section. The *nuclear* structure functions W_1 and W_2 characterize the symmetric part of the hadronic tensor,

$$W^{\mu\nu}=-W_1\left[g^{\mu\nu}+\frac{q^\mu q^\nu}{Q^2}\right]+W_2\left[\frac{\hat{P}^\mu\hat{P}^\nu}{M^2}\right]\ , \qquad (2.13)$$

where M is again the target mass, P^μ is the four-momentum of the target, and $\hat{P}^\mu$ is defined in Eq. (2.10). From Eq. (2.13) we find

$$W_1=\frac{1}{2}\left[\frac{\hat{P}_\mu\hat{P}_\nu}{\gamma M^2}-\left[g_{\mu\nu}+\frac{q_\mu q_\nu}{Q^2}\right]\right]W^{\mu\nu}\ , \qquad (2.14a)$$

$$W_2=\frac{1}{2\gamma}\left[\frac{3\hat{P}_\mu\hat{P}_\nu}{\gamma M^2}-\left[g_{\mu\nu}+\frac{q_\mu q_\nu}{Q^2}\right]\right]W^{\mu\nu}\ , \qquad (2.14b)$$

where $\gamma=\hat{P}_\mu\hat{P}^\mu/M^2$. Thus, we can write

$$W_{1,2}=\int d^4k\frac{m}{E(\mathbf{p})}\delta[k^0+\omega-E(\mathbf{p})]h_{1,2}\ , \qquad (2.15)$$

with the kernels

$$h_1=\frac{1}{2}\left[\frac{\hat{P}_\mu\hat{P}_\nu}{\gamma M^2}-\left[g_{\mu\nu}+\frac{q_\mu q_\nu}{Q^2}\right]\right]\hat{T}^{\mu\nu}\ , \qquad (2.16a)$$

$$h_2=\frac{1}{2\gamma}\left[\frac{3\hat{P}_\mu\hat{P}_\nu}{\gamma M^2}-\left[g_{\mu\nu}+\frac{q_\mu q_\nu}{Q^2}\right]\right]\hat{T}^{\mu\nu}\ . \qquad (2.16b)$$

Making use of Eqs. (2.9) and (2.11), we finally obtain

$$h_1=\frac{F_1^2+(Q^2/4m^2)\kappa^2F_2^2}{2m^2}\left[\frac{(\hat{P}\cdot\hat{p})(\hat{P}\cdot\hat{B})}{\gamma M^2}-(\hat{p}\cdot\hat{B})\right]$$
$$+\frac{F_1^2}{2m^2}[(p\cdot B)-mA]$$
$$+\frac{\kappa F_1F_2}{2m^3}[m(B\cdot q)-A(p\cdot q)]$$
$$+\frac{\kappa^2F_2^2}{8m^4}\{Q^2[mA+(p\cdot B)]+2(p\cdot q)(B\cdot q)\}\ , \qquad (2.17a)$$

$$h_2=\frac{F_2^2+(Q^2/4m^2)\kappa^2F_2^2}{2\gamma m^2}\left[\frac{3(\hat{P}\cdot\hat{p})(\hat{P}\cdot\hat{B})}{\gamma M^2}-(\hat{p}\cdot\hat{B})\right]. \qquad (2.17b)$$

These kernels can also be expressed in terms of the functions $a(k^0,|\mathbf{k}|)$, $b(k^0,|\mathbf{k}|)$, and $c(k^0,|\mathbf{k}|)$ that we used to characterize the density matrix in Eq. (2.2a). When we take into account that we are performing our analysis in the laboratory system, so that $P^\mu=(M,\mathbf{0})$, and also note that the outgoing nucleon is on mass shell, i.e., $p^2=(k+q)^2=m^2$, we find

$$h_1=\frac{F_1^2+(Q^2/4m^2)\kappa^2F_2^2}{2m^2}\left[b(k^0)^2-c\frac{(\mathbf{k}\cdot\mathbf{q})^2}{|\mathbf{q}|^2}\right]$$
$$+\frac{F_1^2+\kappa F_1F_2+\frac{2[k^0\omega-(\mathbf{k}\cdot\mathbf{q})]-Q^2}{4m^2}\kappa^2F_2^2}{2m^2}$$
$$\times[bk^0\omega-c(\mathbf{k}\cdot\mathbf{q})]$$
$$+\left[\frac{Q^2}{8m^2}\kappa^2F_2^2-\frac{1}{2}F_1^2+\frac{Q^2-k^0\omega+(\mathbf{k}\cdot\mathbf{q})}{2m^2}\kappa F_1F_2\right]a\ , \qquad (2.18a)$$

$$h_2=\frac{F_1^2+(Q^2/4m^2)\kappa^2F_2^2}{m^2}$$
$$\times\left[\left[bk^0-c\frac{\omega(\mathbf{k}\cdot\mathbf{q})}{|\mathbf{q}|^2}\right]\left[k^0-\frac{\omega(\mathbf{k}\cdot\mathbf{q})}{|\mathbf{q}|^2}\right]\right.$$
$$\left.+c\frac{Q^2}{2|\mathbf{q}|^2}\left[|\mathbf{k}|^2-\frac{(\mathbf{k}\cdot\mathbf{q})^2}{|\mathbf{q}|^2}\right]\right]. \qquad (2.18b)$$

This is the final result of this section. In the following we will calculate h_1 and h_2 for some special, simple forms of the density matrix and we will also consider the expressions for h_1 and h_2 obtained in the scaling limit.

III. THE STRUCTURE FUNCTIONS IN THE SCALING LIMIT

The simplest approximation we can make for the quasielastic scattering process, as depicted in Fig. 1, is that the virtual photon of momentum q interacts with a quasifree *on-shell* nucleon of momentum k. In this approximation the struck nucleon is on its mass shell before and after the scattering process, i.e.,

$$k^2=(k+q)^2=m^2\ , \qquad (3.1)$$

and, in both cases, is represented by a Dirac plane wave. Equation (3.1) serves to fix the zeroth component of the four-momentum of the struck nucleon to $k^0=\sqrt{|\mathbf{k}|^2+m^2}$. In the target rest frame, the density matrix in Eq. (2.2) is then

$$\rho(k^0,\mathbf{k})=n(|\mathbf{k}|)\delta(k^0-\sqrt{|\mathbf{k}|^2+m^2})\frac{\not{k}+m}{2m}\ , \qquad (3.2)$$

where $n(|\mathbf{k}|)$ is a momentum distribution. The functions which parameterize the density matrix, $a(k^0,|\mathbf{k}|)$, $b(k^0,|\mathbf{k}|)$, and $c(k^0,|\mathbf{k}|)$, are

$$a(k^0,|\mathbf{k}|)=b(k^0,|\mathbf{k}|)=c(k^0,|\mathbf{k}|)$$
$$=n(|\mathbf{k}|)\delta(k^0-\sqrt{|\mathbf{k}|^2+m^2}) \qquad (3.3)$$

in this case. We find for h_1 and h_2 of Eq. (2.18)

$$h_1=n(|\mathbf{k}|)\delta(k^0-E)\left[w_1^N+w_2^N\left[\frac{|\mathbf{k}|^2|\mathbf{q}|^2-(\mathbf{k}\cdot\mathbf{q})^2}{2m^2|\mathbf{q}|^2}\right]\right] \qquad (3.4a)$$

and

$$h_2=n(|\mathbf{k}|)\delta(k^0-E)w_2^N$$
$$\times\left[\left[\frac{E|\mathbf{q}|^2-\omega(\mathbf{k}\cdot\mathbf{q})}{m|\mathbf{q}|^2}\right]^2+\frac{Q^2}{2m^2|\mathbf{q}|^2}\left[|\mathbf{k}|^2-\frac{(\mathbf{k}\cdot\mathbf{q})^2}{|\mathbf{q}|^2}\right]\right], \qquad (3.4b)$$

where we used the relation $E\omega-(\mathbf{k}\cdot\mathbf{q})=Q_2/2$, which follows from Eq. (3.1). Here, $E=\sqrt{|\mathbf{k}|^2+m^2}$ is the energy of the struck nucleon before the interaction, and the functions

$$w_1^N=\frac{Q^2}{4m^2}(F_1+\kappa F_2)^2\ , \qquad (3.5a)$$

$$w_2^N=F_1^2+\frac{Q^2}{4m^2}\kappa^2F_2^2 \qquad (3.5b)$$

are related to the structure functions describing *elastic* scattering on free nucleons. The result presented in Eq. (3.4) was given in an earlier publication of our group [7]. There we calculated the structure functions of the deuteron in terms of the structure functions of the nucleon. We also employed the relativistic impulse approximation and the same simple ansatz for the density matrix that we use here. We see from Eq. (3.4) that, in general, the structure functions mix, i.e., h_1 is a function of both w_1^N and w_2^N. However, this mixing vanishes in the deep inelastic or scaling limit, where Q^2 and ω tend towards infinity, while $x=Q^2/(2M\omega)$ stays finite. In that limit, Eq. (3.4) yields

$$h_1 \to n(|\mathbf{k}|)\delta(k^0-E)w_1^N , \tag{3.6a}$$

$$h_2 \to n(|\mathbf{k}|)\delta(k^0-E)w_2^N \left[\frac{Mx}{m}\right]^2 . \tag{3.6b}$$

We now express w_1^N and w_2^N in terms of the electric and magnetic form factors of the nucleon, G_E and G_M, and obtain

$$w_1^N = \frac{Q^2}{4m^2} G_M^2 , \tag{3.7a}$$

$$w_2^N = \frac{G_E^2 + (Q^2/4m^2)G_M^2}{1+Q^2/4m^2} . \tag{3.7b}$$

Then, in the scaling limit,

$$w_1^N \to \frac{Q^2}{4m^2} G_M^2 , \tag{3.8a}$$

$$w_2^N \to G_M^2 , \tag{3.8b}$$

and we finally find

$$h_1 \to \frac{Q^2}{4m^2} G_M^2 n(|\mathbf{k}|)\delta(k^0-E) , \tag{3.9a}$$

$$h_2 \to \left[\frac{Mx}{m}\right]^2 G_M^2 n(|\mathbf{k}|)\delta(k^0-E) . \tag{3.9b}$$

The kinematic factor $(Mx/m)^2$, where x is the Bjorken scaling variable that characterizes deep inelastic electron-nucleon scattering, will appear again later in this work. This kinematic factor was discussed in detail in an earlier publication on this topic [4].

We now go on and consider the case where the nucleon is off its mass shell before the interaction. In the spirit of describing quasielastic electron-nucleus scattering as a sum over scattering processes on individual free nucleons, we still use Dirac plane waves for the wave function of the struck nucleon. However, we consider an off-shell characterization of the struck nucleon by introducing an effective mass

$$m^* = \sqrt{(k^0)^2 - |\mathbf{k}|^2} \neq m . \tag{3.10}$$

This quantity can only be defined if the nucleon is not too far off its mass shell, and thus has $|\mathbf{k}| < |k^0|$. In this case we find the density matrix

$$\rho(k^0,\mathbf{k}) = n(k^0,|\mathbf{k}|)\frac{\not{k}+m^*}{2m} . \tag{3.11}$$

This differs from Eq. (3.2) by the replacement of m with m^* in the numerator. Further, $n(k^0,|\mathbf{k}|)$ is a generalized momentum distribution, which now depends both on k^0 and $|\mathbf{k}|$. This momentum distribution is the probability of finding a nucleon with energy $E=k^0$ and three-momentum $\mathbf{k}$ in the nucleus. We find the functions $a(k^0,|\mathbf{k}|)$, $b(k^0,|\mathbf{k}|)$, and $c(k^0,|\mathbf{k}|)$, which parametrize the density matrix [see Eq. (2.2)], to be

$$\begin{aligned} a(k^0,|\mathbf{k}|) &= \frac{m^*}{m} n(k^0,|\mathbf{k}|) , \\ b(k^0,|\mathbf{k}|) &= c(k^0,|\mathbf{k}|) = n(k^0,|\mathbf{k}|) , \end{aligned} \tag{3.12}$$

and the kernels of the structure functions are then

$$h_1 = n(k^0,|\mathbf{k}|)\left[\tilde{w}_1^N + w_2^N \left[\frac{|\mathbf{k}|^2|\mathbf{q}|^2 - (\mathbf{k}\cdot\mathbf{q})^2}{2m^2|\mathbf{q}|^2}\right]\right] , \tag{3.13a}$$

$$h_2 = n(k^0,|\mathbf{k}|)w_2^N \left[\left[\frac{k^0|\mathbf{q}|^2 - \omega(\mathbf{k}\cdot\mathbf{q})}{m|\mathbf{q}|^2}\right]^2 + \frac{Q^2}{2m^2|\mathbf{q}|^2}\left[|\mathbf{k}|^2 - \frac{(\mathbf{k}\cdot\mathbf{q})^2}{|\mathbf{q}|^2}\right]\right] . \tag{3.13b}$$

Here we have used $k^0\omega - (\mathbf{k}\cdot\mathbf{q}) = (Q^2+m^2-m^{*2})/2$, which follows from the fact that the struck nucleon is on its mass shell after the absorption of the virtual photon; w_2^N is again the structure function for elastic scattering off a free nucleon, as defined in Eq. (3.5b). However, the other structure function, $\tilde{w}_1^N$, is a modified form which appears when the nucleon is off shell:

$$\begin{aligned} w_1^N &= \frac{Q^2}{4m^2}(F_1+\kappa F_2)^2 \to \tilde{w}_1^N , \\ \tilde{w}_1^N &= \frac{Q^2+(m^*-m)^2}{4m^2}\left[F_1 + \frac{m^*+m}{2m}\kappa F_2\right]^2 . \end{aligned} \tag{3.14}$$

This expression reduces to the w_1^N given in Eq. (3.5a) when we set $m^*=m$.

From Eqs. (3.13) and (3.14) we see that, in the scaling limit, we find exactly the same expression as in the on-shell case, since the extra terms, which are functions of m^*-m, are small when compared to Q^2. Therefore, we obtain, for off-shell nucleons,

$$h_1 \to \frac{Q^2}{4m^2} G_M^2 n(k^0,|\mathbf{k}|) , \tag{3.15a}$$

$$h_2 \to \left[\frac{Mx}{m}\right]^2 G_M^2 n(k^0,|\mathbf{k}|) . \tag{3.15b}$$

We conclude that, in the scaling limit and for the simple parametrization used here, it does not matter whether we parametrize the density matrix with on-shell or with off-shell spinors.

We complete this section by deriving the asymptotic limit for h_1 and h_2, without making any simplifying assumptions concerning the form of the density matrix. [We evaluate Eq. (2.18) in the deep inelastic limit, where Q^2 and ω tend towards infinity, while $x=Q^2/(2M\omega)$ stays finite, and restrict ourselves to the terms of leading order.] First, we reexpress F_1 and F_2 in terms of the electric and magnetic form factors G_E and G_M,

$$F_1 = \frac{G_E + (Q^2/4m^2)G_M}{1+Q^2/4m^2} , \tag{3.16a}$$

$$\kappa F_2 = \frac{G_M - G_E}{1+Q^2/4m^2} . \tag{3.16b}$$

This shows that F_2 is of the order of $1/Q^2$ compared to F_1 and, therefore, we can neglect all terms containing F_2 in Eq. (2.18). Furthermore, we see that, in the scaling limit, F_1 reduces to the magnetic form factor G_M.

Only the second term in h_1 [see Eq. (2.18a)] is of the order of Q^2, whereas the two other terms are of the order of 1; thus, it is only this second term which contributes in the scaling limit. In the case of h_2, the first term is of the order of 1, whereas the second term is of the order of $1/Q^2$, and thus only the first term survives. In the scaling limit, we have $\omega/|\mathbf{q}|\approx 1$ and we find

$$h_1 \to \frac{Q^2}{4m^2} G_M^2 \frac{2[b(k^0,|\mathbf{k}|)k^0\omega - c(k^0,|\mathbf{k}|)(\mathbf{k}\cdot\mathbf{q})]}{Q^2} , \tag{3.17a}$$

$$h_2 \to \left[\frac{Mx}{m}\right]^2 G_M^2 \frac{2[b(k^0,|\mathbf{k}|)k^0\omega - c(k^0,|\mathbf{k}|)(\mathbf{k}\cdot\mathbf{q})]}{Q^2} , \tag{3.17b}$$

where we used $x=Q^2/(2M\omega)$ and

$$k^0\omega - (\mathbf{k}\cdot\mathbf{q}) \to \frac{Q^2}{2} . \tag{3.18}$$

Recall that the struck nucleon is on its mass shell after the interaction, i.e., $(k+q)^2=m^2$. The above result can also be expressed in terms of B^μ. From Eq. (2.2) we see that $B^\mu=(bk^0,c\mathbf{k})$, and Eq. (3.17) then reads

$$h_1 \to \frac{Q^2}{4m^2} G_M^2 \frac{2[q\cdot B(k^0,|\mathbf{k}|)]}{Q^2} , \tag{3.19a}$$

$$h_2 \to \left[\frac{Mx}{m}\right]^2 G_M^2 \frac{2[q\cdot B(k^0,|\mathbf{k}|)]}{Q^2} . \tag{3.19b}$$

This is the final result of this section and it demonstrates how the calculation of the structure functions is related to the structure of the density matrix of the target when the target is described as a relativistic system.

In the two special cases we have studied in this section, B_μ was proportional to k_μ and, as $2(k\cdot q)/Q^2 \to 1$ in the scaling limit [see Eq. (3.18)], we found the same result for the structure functions, whether we used on-shell or off-shell spinors. Clearly, Eqs. (3.9) and (3.15) can easily be obtained from the more general result given in Eq. (3.19).

IV. *y* SCALING

In the last section we calculated asymptotic expressions for the structure functions in the scaling limit. We found [see Eq. (3.19)] that the kernels of those structure functions can be written as the product of the projection of the vector part of the density matrix on the photon momentum times some kinematic factors. In this section we want to relate the latter result to the concept of y scaling, and we will show that the cross section for quasielastic electron-nucleus scattering can be written as an integral over a spectral function (which specifies the energy and momentum distribution in the nucleus) times an electron-nucleon cross section [8]. In Sec. V we present a semiclassical interpretation for this electron-nucleon cross section. This cross section is seen to depend on the motion and the off-shell nature of the nucleon in the nucleus.

When we consider quasielastic electron-nucleus scattering in the plane-wave impulse approximation, the final state includes only the free, knocked-out nucleon and a recoiling $A-1$-body spectator system, which may be excited with some energy ϵ. Often the $A-1$-body system is taken to be a single spectator nucleus and both the struck nucleon and the spectator nucleus are considered to be on their mass shells [8]. The spectator nucleus is then ascribed some effective mass, M^*_{A-1}, such that

$$\sqrt{M^{*2}_{A-1}+|\mathbf{k}|^2}=\sqrt{M^2_{A-1}+|\mathbf{k}|^2}+\epsilon . \tag{4.1}$$

The corresponding process is depicted in Fig. 2. Energy conservation fixes the struck nucleon's energy k^0 to

$$k^0=M-\sqrt{M^2_{A-1}+|\mathbf{k}|^2}-\epsilon , \tag{4.2}$$

where M is the mass of the target nucleus.

In this approximation we neglect all interactions between the struck nucleon and the spectator nucleus in the final state. We also neglect excitations of the struck particle and any interaction of the virtual photon with constituents other than a single nucleon.

In all discussions of y scaling based upon nonrelativistic dynamics [5], the struck nucleon is treated as a quasifree particle. One considers the nucleus to be a gas of off-shell nucleons having some energy and momentum distribution, which can be parametrized by means of a spectral function $S(|\mathbf{k}|,\epsilon)$. Here $\mathbf{k}$ is the three-momentum of the struck nucleon before the interaction with the photon, and ϵ is the excitation energy of the $A-1$-body spectator in the final state.

In our formalism, this means that we can represent the struck nucleon by a Dirac plane wave with some effective mass m^*,

$$m^*=\sqrt{(k^0)^2-|\mathbf{k}|^2} . \tag{4.3}$$

Here m^* differs from the free-nucleon mass m due to the fact that energy conservation has already fixed k^0, as can be seen from Eq. (4.2). Thus, we write the density matrix as

$$\rho(k^0,\mathbf{k})=\int_0^\infty d\epsilon\, S(|\mathbf{k}|,\epsilon)\delta(k^0-M+\sqrt{M^2_{A-1}+|\mathbf{k}|^2}+\epsilon) \times\frac{\not{k}+m^*}{2m} , \tag{4.4}$$

which is analogous to Eq. (3.11). However, the generalized momentum distribution $n(k^0,|\mathbf{k}|)$ is now expressed

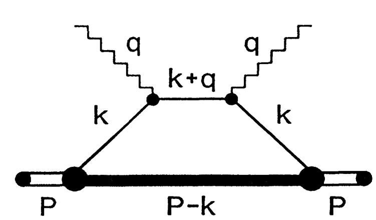

FIG. 2. Forward (virtual) Compton amplitude in the case the residual system is an $A-1$-body nucleus of mass M_{A-1} and excitation energy ϵ.

through a spectral function $S(|\mathbf{k}|,\epsilon)$ and an integral over all possible excitation energies ϵ of the spectator nucleus.

As was outlined in detail in Sec. III, the structure of the density matrix is such that the following expressions for the kernels of the structure functions are obtained:

$$h_1 \to \frac{Q^2}{4m^2} G_M^2 \int_0^\infty d\epsilon\, S(|\mathbf{k}|,\epsilon) \times \delta(k^0 - M + \sqrt{M_{A-1}^2 + |\mathbf{k}|^2} + \epsilon) , \quad (4.5a)$$

$$h_2 \to \left[\frac{Mx}{m}\right]^2 G_M^2 \int_0^\infty d\epsilon\, S(|\mathbf{k}|,\epsilon) \times \delta(k^0 - M + \sqrt{M_{A-1}^2 + |\mathbf{k}|^2} + \epsilon) , \quad (4.5b)$$

in the scaling limit. From this we can calculate the structure functions themselves using

$$W_{1,2} = \int d^4k \frac{m}{E(\mathbf{k}+\mathbf{q})} \delta\left[k^0 + \omega - E(\mathbf{k}+\mathbf{q})\right] h_{1,2} , \quad (4.6)$$

with $E(\mathbf{k}+\mathbf{q}) = \sqrt{|\mathbf{k}+\mathbf{q}|^2 + m^2}$.

The integral of Eq. (4.6) contains two delta functions, one from the on-shell characterization of the struck nucleon and one from the fact that the spectator nucleus is on shell [see Eq. (4.5)]. If we use one delta function to eliminate the k^0 integration in Eq. (4.6), the other delta function will fix the angle θ between $\mathbf{k}$ and $\mathbf{q}$. We have

$$\frac{m}{E(\mathbf{k}+\mathbf{q})} \delta\left[k^0 + \omega - E(\mathbf{k}+\mathbf{q})\right] = \frac{m}{|\mathbf{k}||\mathbf{q}|} \delta\left[\cos\theta - \Omega(|\mathbf{k}|,\epsilon,|\mathbf{q}|,\omega)\right] , \quad (4.7)$$

where ω is the energy of the virtual photon and

$$\Omega(|\mathbf{k}|,\epsilon,|\mathbf{q}|,\omega) = \frac{\left[M + \omega - \sqrt{M_{A-1}^2 + |\mathbf{k}|^2} - \epsilon\right]^2 - m^2 - |\mathbf{k}|^2 - |\mathbf{q}|^2}{2|\mathbf{k}||\mathbf{q}|} . \quad (4.8)$$

Details can be found in Ref. [5], where the same kinematics is used. (See also Ref. [8].) The delta function in Eq. (4.8) can only contribute to the integral if $|\Omega(|\mathbf{k}|,\epsilon,|\mathbf{q}|,\omega)| \le 1$. This restricts both the integration over $|\mathbf{k}|$ and the integration over ϵ.

It is usual practice to call the lower limit of the $|\mathbf{k}|$ integration $-y$. The y-scaling regime is then defined as the kinematic region where $|\mathbf{q}|$ tends towards infinity, which y stays finite. It is a straightforward exercise to show that the y-scaling limit is equivalent to the Bjorken limit, where both Q^2 and ω tend towards infinity, while $x = Q^2/(2M\omega)$ remains finite. In this limit, upon combining Eqs. (4.5) to (4.7), we find W_1 and W_2 to be

$$W_1 = \frac{Q^2}{4m^2} G_M^2 2\pi \frac{m}{|\mathbf{q}|} \int_{-y}^\infty |\mathbf{k}| d|\mathbf{k}| \int_0^{\epsilon_m} d\epsilon\, S(|\mathbf{k}|,\epsilon) , \quad (4.9a)$$

$$W_2 = \left[\frac{Mx}{m}\right]^2 G_M^2 2\pi \frac{m}{|\mathbf{q}|} \int_{-y}^\infty |\mathbf{k}| d|\mathbf{k}| \int_0^{\epsilon_m} d\epsilon\, S(|\mathbf{k}|,\epsilon) . \quad (4.9b)$$

Here

$$y = \frac{M(1-x)}{2} - \frac{M_{A-1}^2}{2M(1-x)} \quad (4.10a)$$

and

$$\epsilon_m = y + |\mathbf{k}| + \sqrt{M_{A-1}^2 + y^2} - \sqrt{M_{A-1}^2 + |\mathbf{k}|^2} . \quad (4.10b)$$

The nuclear structure functions W_1 and W_2 determine the cross section, since

$$\frac{d^2\sigma}{dE'\,d\Omega} = \sigma_{\mathrm{Mott}}[W_2 + 2W_1 \tan^2(\tfrac{1}{2}\theta)] . \quad (4.11)$$

Upon inverting Eq. (4.10a), and using Eqs. (4.9) and (4.11), we find, in the scaling limit,

$$\frac{d^2\sigma}{dE'\,d\Omega} = \sigma_{\mathrm{red}} \frac{m}{|\mathbf{q}|} F(y) , \quad (4.12)$$

where we have defined the reduced cross section

$$\sigma_{\mathrm{red}} = \sigma_{\mathrm{Mott}} \left[\left[\frac{M - \sqrt{M_{A-1}^2 + y^2} - y}{m} \right]^2 G_M^2 + \frac{Q^2}{2m^2} G_M^2 \tan^2\left[\frac{1}{2}\theta\right] \right] , \quad (4.13)$$

and the scaling function

$$F(y) = 2\pi \int_{-y}^\infty |\mathbf{k}| d|\mathbf{k}| \int_0^{\epsilon_m} d\epsilon\, S(|\mathbf{k}|,\epsilon) . \quad (4.14)$$

This is exactly the same result as appears in Refs. [5] and [8]. However, the difference between our derivation and that presented in Ref. [5] is that, in Ref. [5], Eq. (4.12) was obtained by "smearing" an off-shell electron-nucleon cross section (the cross section cc1 of de Forest [9]) with a spectral function $S(|\mathbf{k}|,\epsilon)$. In contrast, we began our analysis by introducing a density matrix for the nucleons and it is only in the scaling limit that we obtain simple folding expressions for W_1, W_2 and the cross section [see Eqs. (4.9) and (4.14)]. In the nonasymptotic case, which is more the rule than the exception, given the experimental situation, there is not only correction terms arising from the limits of the integrations, $-y$ and ϵ_m, but there are, in addition, correction terms which arise in the kernels of the integrals that yield W_1 and W_2. It is only in the deep inelastic limit that these corrections vanish so that our derivation agrees with the conventional, nonrelativistic analysis [8].

Another difference is that our derivation provides an expression for the reduced cross section, σ_{red} of Eq. (4.13), while that quantity appears as an additional input in the calculation of Ref. [5]. As will be discussed in sec. V, the reduced cross section of Eq. (4.13) is simply the scaling limit of the electron-nucleon cross section, evaluated for a moving nucleon at a very special kinematic point. In this we agree with the analysis of Ref. [5].

When we evaluate the cc1 cross section of de Forest at this special kinematic point, and consider the leading order terms only, we end up with an expression for $\sigma_{\rm red}$ which differs from Eq. (4.13) by the appearence of a single kinematic factor. This feature of our analysis will be discussed in detail in Sec. V.

V. THE REDUCED ELECTRON-NUCLEON CROSS SECTION

In this section we want to show how the reduced cross section, introduced in Eq. (4.13), can be derived as an off-shell extrapolation of the well-known Rosenbluth cross section [10], which is to be evaluated for a moving target nucleon. The procedure is analogous to that used in our previous work [3,4], where we studied two pointlike, spin-zero "nucleons" bound to form a scalar "deuteron". There we could also show [4] that the electron-nucleon cross section that appears when evaluating the hadronic structure functions in the scaling limit, can as well be derived from the Mott cross section by rewriting the latter in the appropriate frame of reference and evaluating it for moving, off-shell "nucleons". In order to be able to transform the cross section to another Lorentz frame, we first have to write it in a covariant manner. (For this purpose we introduce the Mandelstam variables. Further details may be found in our previous work on this topic [4].)

As we are now considering nucleons of spin one-half, we start with the Rosenbluth formula [10]

$$\left\{\frac{d^2\sigma}{dE'\,d\Omega}\right\}_{\rm lab}=\frac{\alpha^2\cos^2(\theta/2)}{4E^2\sin^4(\theta/2)}\times\left[\left[F_1^2+\frac{Q^2}{4m^2}\kappa^2F_2^2\right]+\frac{Q^2}{2m^2}(F_1+\kappa F_2)^2\tan^2\left[\frac{1}{2}\theta\right]\right] \tag{5.1}$$

that describes elastic scattering of relativistic electrons, with incident energy E, from free on-shell nucleons at rest in the laboratory frame. θ is the scattering angle and the first factor on the right-hand side of Eq. (5.1) is the Mott cross section. The energy-conserving delta function, which usually appears in the elastic cross section, is here suppressed, since we have taken that delta function into the integral that is used to calculate the hadronic structure functions [see Eq. (4.6)].

In close analogy to our previous work, we now rewrite Eq. (5.1) in terms of the Mandelstam variables

$$\begin{aligned} s&=(p+k)^2=m^2+2mE\ ,\\ t&=(p-p')^2=-2EE'(1-\cos\theta)\ ,\\ u&=(k-p')^2=m^2-2mE'\ , \end{aligned} \tag{5.2}$$

where $p=(E,\mathbf{p})$ and $p'=(E',\mathbf{p}')$ are the four-momenta of the electron before and after the scattering process; $k=(m,0)$ and $k'=(m+E-E',\mathbf{p}-\mathbf{p}')$ are the four-momenta of the struck nucleon before and after the interaction. Here we have neglected the electron mass (i.e., $|\mathbf{p}|-E$ and $|\mathbf{p}'|=E'$).

Using the relation

$$\frac{d^2\sigma}{dt\,du}=\left\{\frac{d^2\sigma}{dE'\,d\Omega}\right\}_{\rm lab}2\pi\frac{d(E',\cos\theta)}{d(t,u)}\ , \tag{5.3}$$

we find, in agreement with Eq. (4.56) of Ref. [11], the Lorentz invariant cross section

$$\frac{d^2\sigma}{dt\,du}=\frac{4\pi\alpha^2}{2mt^2}\left[\frac{(s-m^2)(m^2-t-u)+st}{(s-m^2)^2}\times\left[F_1^2-\frac{t}{4m^2}\kappa^2F_2^2\right]+\frac{t^2}{2(s-m^2)^2}(F_1+\kappa F_2)^2\right]\ . \tag{5.4}$$

As was outlined in detail in Sec. IV, quasielastic electron-nucleus scattering can be understood as an incoherent sum of individual scattering processes on single, quasi-free nucleons. The condition that the $A-1$-body spectator nucleus stays on its mass shell fixes the energy of the struck nucleon, and we found its four-momentum before the scattering process to be

$$k=(M-\sqrt{M_{A-1}^2+|\mathbf{k}|^2}-\epsilon,\mathbf{k})\ , \tag{5.5}$$

where ϵ is the excitation energy of the spectator nucleus and M is the target mass. (The corresponding diagram is show in Fig. 2.) We are here interested in calculating the cross section for elastic electron scattering from the moving, off-shell nucleons of the target.

In the following we will evaluate Eq. (5.4) at a very special kinematic point, where

$$|\mathbf{k}|=-y\quad\text{and}\quad\epsilon=0\ . \tag{5.6}$$

This corresponds to the lower limit of the integral for the response function, $F(y)$ [see Eq. (4.14)]. As the spectral function $S(|\mathbf{k}|,\epsilon)$, which appears in this integral, falls off quite rapidly with energy and momentum, it might be a good approximation to evaluate the single-nucleon cross section at the point where its contribution to the integral for $F(y)$ has a maximum. (In Ref. [5] it was demonstrated that this approximation is accurate to within 5%, if one uses the cc1 cross section of de Forest.) Furthermore, we work in the scaling limit, and we keep only the leading order terms in an expansion in $|\mathbf{q}|^{-1}$.

Using Eq. (5.2) and k from Eq. (5.5), where we now set $\epsilon=0$ and $|\mathbf{k}|=-y$, we find the Mandelstam variables to be

$$\begin{aligned} s&=M^2-2M\sqrt{M_{A-1}^2+y^2}+M_{A-1}^2+2E(M-\sqrt{M_{A-1}^2+y^2}-y)\ ,\\ t&=-2EE'(1-\cos\theta)\ ,\\ u&=M^2-2M\sqrt{M_{A-1}^2+y^2}+M_{A-1}^2-2E'(M-\sqrt{M_{A-1}^2+y^2}-y\cos\theta)\ . \end{aligned} \tag{5.7}$$

As can be seen from Eq. (5.5), the struck nucleon is off its mass shell, i.e., $k^2 \neq m^2$. Therefore we have to modify the invariant cross section of Eq. (5.4) by replacing m^2 by k^2 wherever it appears. This yields the following off-shell extrapolation of the invariant cross section:

$$\left[\frac{d^2\sigma}{dt\,du}\right]=\frac{4\pi\alpha^2}{2mt^2}\left[\frac{(s-k^2)(k^2-t-u)+st}{(s-k^2)^2}\times\left[F_1^2-\frac{t}{4k^2}\kappa^2F_2^2\right]+\frac{t^2}{2(s-k^2)^2}(F_1+\kappa F_2)^2\right]. \tag{5.8}$$

As we are only interested in the scaling limit, we can neglect the form factor F_2 and we can replace F_1 by the magnetic form factor G_M. We invert Eq. (5.3) to yield

$$\sigma_{\text{red}}=\left[\frac{d^2\sigma}{dt\,du}\right]\frac{1}{2\pi}\frac{d(t,u)}{d(E',\cos\theta)}, \tag{5.9}$$

and we finally find the asymptotic limit of the reduced electron-nucleon cross section:

$$\sigma_{\text{red}}=\sigma_{\text{Mott}}\left[\left[\frac{M-\sqrt{M_{A-1}^2+y^2}-y}{m}\right]^2G_M^2+\frac{Q^2}{2m^2}G_M^2\tan^2\left[\frac{1}{2}\theta\right]\right]. \tag{5.10}$$

In addition, the modification in the flux of incoming electrons, as seen by the moving nucleon, was taken into account by the appearance of a flux factor, $\phi_{\text{mov}}/\phi_{\text{rest}}$. Thus, we made the replacement

$$\sigma_{\text{red}}\rightarrow\sigma_{\text{red}}\frac{\phi_{\text{mov}}}{\phi_{\text{rest}}}. \tag{5.11}$$

Details can be found in Sec. V of our previous publication on this topic [4].

We note that the result of this derivation of the reduced electron-nucleon cross section fully agrees with the expression we found in Sec. IV [see Eq. (4.13)], where we calculated the hadronic structure functions in the scaling limit. We also see that evaluating the cross section at the special point, $|\mathbf{k}|=-y$ and $\epsilon=0$, is the correct procedure, at least up to leading order in $|\mathbf{q}|^{-1}$.

In almost all nonrelativistic calculations of y scaling, the cross section designated cc1 by de Forest [9] is used for the reduced electron-nucleon cross section, σ_{red}. Integrated over the azimuthal angle ϕ_k one has

$$\sigma_{\text{cc1}}=\sigma_{\text{Mott}}\left\{\left[\frac{Q^4}{|\mathbf{q}|^4}\left[\frac{E(\mathbf{k}')+\bar{E}(\mathbf{k})}{2m}\right]^2+\frac{Q^2}{2m^2|\mathbf{q}|^2}|\mathbf{k}'|^2\sin^2\gamma\right]\left[F_1^2+\frac{\bar{Q}^2}{4m^2}\kappa^2F_2^2\right]+\frac{Q^2}{4m^2|\mathbf{q}|^2}(\bar{Q}^2-Q^2)(F_1+\kappa F_2)^2+\left[\frac{|\mathbf{k}'|^2\sin^2\gamma}{m^2}\left[F_1^2+\frac{\bar{Q}^2}{4m^2}\kappa^2F_2^2\right]+\frac{\bar{Q}^2}{2m^2}(F_1+\kappa F_2)^2\right]\tan^2\left[\frac{1}{2}\theta\right]\right\}. \tag{5.12}$$

Here, γ is the angle between the photon momentum $\mathbf{q}$ and the momentum of the struck nucleon after the interaction, $\mathbf{k}'=\mathbf{k}+\mathbf{q}$, and $E(\mathbf{k}')=\sqrt{|\mathbf{k}'|^2+m^2}$. The off-shell features are contained in the difference between

$$E(\mathbf{k})=M-\sqrt{M_{A-1}^2+|\mathbf{k}|^2}-\epsilon \tag{5.13a}$$

[see Eq. (5.5)] and

$$\bar{E}(\mathbf{k})=\sqrt{|\mathbf{k}|^2+m^2}, \tag{5.13b}$$

and, thus, also in the difference between

$$Q^2=|\mathbf{q}|^2-(E(\mathbf{k}')-E(\mathbf{k}))^2 \tag{5.14a}$$

and

$$\bar{Q}^2=|\mathbf{q}|^2-(E(\mathbf{k}')-\bar{E}(\mathbf{k}))^2. \tag{5.14b}$$

When we evaluate σ_{cc1} for the special, minimal kinematics, as defined in Eq. (5.6), we find in the scaling limit

$$\sigma_{\text{cc1}}=\frac{\sqrt{y^2+m^2}-y}{M-\sqrt{M_{A-1}^2+y^2}-y}\sigma_{\text{red}}, \tag{5.15}$$

with σ_{red} from Eq. (5.10). Here we again considered only the leading order terms in an expansion in $|\mathbf{q}|^{-1}$, replaced F_1 with G_M, and neglected the form factor F_2. This reduced electron-nucleon cross section σ_{cc1} differs just by one factor of

$$\frac{\sqrt{y^2+m^2}-y}{M-\sqrt{M_{A-1}^2-y^2}-y}$$

from the expression σ_{red} we found by either evaluating the hadronic structure functions [see Sec. IV and Eq. (4.13)] or by transforming the Rosenbluth formula and evaluating it at the special kinematic point, $|\mathbf{k}|=-y$ and $\epsilon=0$ [see Eq. (5.10)]. This can be understood in the following manner: de Forest's off-shell extrapolation proceeds by replacing the actual energy of the struck nucleon $E(\mathbf{k})$ [see Eq. (5.13a)], which is fixed by the condition that the $A-1$-body spectator is on its mass shell, by the energy $\bar{E}(\mathbf{k})$ [see Eq. (5.13b)], which is just the energy the struck nucleon of three-momentum k would have, if it were on its mass shell. This then also leads to the difference between the actual Q^2 and the effective $\bar{Q}^2$, as can be seen from Eq. (5.14). de Forest thus approximates the electromagnetic properties of the off-shell nucleon by those of an on-shell nucleon. Following that approach, in our relativistic derivation we should not work with off-shell spinors, as we did for the evaluation of the hadronic structure functions in Sec. IV, and which finally led to Eq. (4.13), but we should use effective on-shell spinors. Thus, instead of writing

$$B^\mu \propto k^\mu = (E(\mathbf{k}), \mathbf{k}) \tag{5.16}$$

for the "vector part" of the density matrix, defined in Eq. (2.2b), we should use an *effective* $\bar{B}^\mu$, where now

$$\bar{B}^\mu \propto \bar{k}^\mu = (\bar{E}(\mathbf{k}), \mathbf{k}) \ , \tag{5.17}$$

with $\bar{E}(\mathbf{k})$ from Eq. (5.13b).

It was shown in Sec. III that, in the scaling limit, the relevant quantity which governs the behavior of the hadronic structure functions and the cross section is the term $2(B\cdot q)/Q^2$. But the condition that the struck nucleon is on its mass shell after the interaction fixes the scaling limit of $2(k\cdot q)/Q^2$,

$$\frac{2(k\cdot q)}{Q^2} \to 1 \ . \tag{5.18}$$

This is not the limit of $2(\bar{k}\cdot q)/Q^2$; when we evaluate the latter term for the special kinematics of Eq. (5.6), and consider the scaling limit only, we find

$$\frac{2(\bar{k}\cdot q)}{Q^2} \to \frac{\sqrt{y^2+m^2}-y}{M-\sqrt{M_{A-1}^2+y^2}-y} \ . \tag{5.19}$$

This is just the factor by which our reduced electron-nucleon cross section σ_{red} [Eqs. (4.13) and (5.10)] differs from the cross section cc1 of de Forest. Therefore we have demonstrated how both procedures can be related. For example, if we evaluate the hadronic structure functions with effective on-shell spinors, as defined in Eq. (5.17), we would obtain an expression for the reduced electron-nucleon cross section which differs from the one we derived in Sec. IV by a factor of

$$\frac{\sqrt{y^2+m^2}-y}{M-\sqrt{M_{A-1}^2+y^2}-y} \ .$$

In that case our reduced electron-nucleon cross section would be the same as de Forest's cc1. However, we remark that the choice of the correct procedure, using off-shell spinors as we did in Sec. IV [see Eq. (4.4)] or using effective on-shell spinors, i.e., replacing k in Eq. (4.4) with $\bar{k}$, is not straightforward. As long as there is no consistent derivation of the off-shell photon-nucleon vertex, there will always be at least some arbitrariness in calculations of the type presented here.

VI. DISCUSSION

In this work we have suggested that if one is to use y scaling to study high-momentum components in nuclei, one should describe the target nucleus using a relativistic formalism. We have also based our analysis on calculating the hadronic tensor of the target. That is a more satisfactory procedure than beginning with an expression which folds some electron-nucleon cross section with the target's spectral function [5,8]. Indeed, if one proceeds by calculating the hadronic tensor, the origin of various kinematic factors is readily understood [4].

Our analysis identifies the assumptions that are necessary in order to make contact with the standard formulation of the theory. We have also seen how, in the scaling limit, the calculation of the cross section is related to the structure of the relativistic density matrix.

We have carried out our analysis for particularly simple forms of the density matrix [see Eq. (3.2) or (3.11)] in Sec. III. However, we also presented results for the general form given in Eq. (2.2). Information concerning the density matrix of finite systems [12] may be obtained from self-consistent *relativistic* models of the nuclear ground state (Hartree, Hartree-Fock, or Brueckner Hartree-Fock [13,14]). Further work is needed, however, if one is to understand how short-range correlations affect the high-momentum components of the density matrix [15] of a finite nucleus.

Finally, we note that the physical meaning of the mass M_{A-1} is unclear if the momentum of the struck nucleon is large. In that case one expects a complex final state with more than two on-shell particles (nucleon plus spectator). The formalism has not been extended to treat multiparticle final states. Whether such extensions are needed to interpret experimental data in terms of fundamental models of nuclear structure remains to be seen.

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