Semiclassical description of alpha clustering in heavy nuclei

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A semiclassical model for α clustering in heavy nuclei is developed where the nucleus is viewed as a composite system of pairwise correlated nucleons interacting with α particles. The nucleons are treated microscopically by the use of a spherical shell-model basis while the α particles are considered as elementary bosons carrying angular momentum zero and one. The description of such a system is based on a time-dependent variational formalism, which yields BCS-like equations for the ground and some low-lying excited states. It is shown that the system may undergo transitions from a nucleon superfluid phase to intermediate phases characterized by the coexistence of few α clusters with a nucleon pair condensate. The model allows also to study in a random-phase-approximation context the fluctuations of the system around the BCS-like configurations. Two peculiar aspects emerge, the occurrence of collective states describing a wobbling motion of the α particles and the unusually large number of collective states describing a coherent motion of nucleons oscillating in phase with α particles. The numerical illustrative applications refer to ^{220}Ra.

I. INTRODUCTION

Alpha clustering in nuclei is a very old though still central subject in nuclear physics, which goes back to Gamow's pioneering paper on the quantum treatment of alpha-particle penetration through the Coulomb and the centrifugal barrier [1]. The underlying idea of Gamow as well as of later microscopic descriptions [2–4] of α-particle decay was that such a particle is already formed inside the nucleus before tunneling.

Alpha clustering plays an important role in the study of light nuclei [5–9] and in connection with the analysis of the asymmetric fission process [10–13].

Evidence in favor of alpha clustering in heavy nuclei has more recently come from systematics for ground-state widths [14] indicating large reduced alpha widths in the vicinity of the $Z=50$ and 82 shell closures and from α-transfer [15] reactions showing a large population of excited states. This evidence is particularly strong in Ra isotopes [16,17] and in ^{222}Th [18,19], where the occurrence of low-lying negative-parity states, the sequence of alternating parity high-spin levels, the enhancement of the $E1$ transitions connecting these states seem to represent the signature of molecular states in which the α clusters perform collective dipole oscillations [20].

Several microscopic formalisms based on alpha clustering have been developed to study the properties of some low-lying nuclear states. It is worth mentioning the quartet [21–24] and the quadrupole [25] models. Because of the intrinsic complexity of the problem, their actual application has been forcefully confined mostly to light nuclei. These microscopic approaches deal also with the internal structure of the cluster. It has been suggested in this respect [26,27] that not only $T=1$ but also $T=0$ interacting correlated pairs, which favor deformation, would be needed to describe four-body correlations, especially in heavy nuclei.

Nucleon pairing represents on its own a major example of clustering in nuclei. The superfluid nuclear properties are indeed explained [28] by assuming the nuclear ground state to be a condensate of nucleon pairs coupled to $J=0$, $T=1$. Also pairs with higher spin have been considered. In the interacting boson model (IBM) [29], for instance, the low-lying properties of nuclei in different regions have been explained in terms of strongly correlated $J=0$ and $J=2$ pairs of nucleons, assumed to form s and d bosons.

Pairing correlations and α clustering may coexist in a nucleus and may be considered in competition. The question of whether nuclei are a superfluid condensate of α particles rather than a nucleon pair condensate has been raised [26]. Analysis carried out within schematic models [30,31] inspired by the work of Nozières and Saint James [32] on the condensation of excitons versus biexcitons in semiconductors have suggested that a phase of α condensation may set in some heavy nuclei.

In this paper we intend to explore whether and how heavy nuclei undergo a transition from a nucleon pair condensate to an alpha condensate going through intermediate steps with a few alpha clusters coexisting with a pair condensate. Since a completely fermionic treatment of the problem is practically impossible in heavy nuclei, we developed a semiclassical model where the nucleus is described as a system of nucleons pairwise correlated interacting with α particles. The correlated nucleons are treated in BCS approximation, while the α particles are considered as elementary bosons. Although explicitly ignored, the nucleonic structure of the α clusters has been effectively accounted for by a proper choice of the different pieces forming the model Hamiltonian. Also the nuclear deformation, important in connection with α-cluster phenomena, has been simulated by blocking the BCS ground state with two-neutron single-particle creation operators.

A time-dependent variational formalism has been developed to study the static properties of the ($J^{\pi}=0^{+}$) ground state and of the $J^{\pi}=1^{-}$ and 2^{+} excited states. The same formalism allows naturally a description in random phase approximation (RPA) of the small oscillations of the system around the variational states, which lead to new types of collective excitations.

This project is achieved according to the following plan. The model Hamiltonian is described in Sec. II. Equations of motion for the classical coordinates describing the system are derived in Sec. III. The BCS-like equations are presented in Sec. IV while the RPA eigenvalue problem is formulated in Sec. V. The numerical applications referring to ^{220}Ra are commented upon in Sec. VI and a summary of the main results is presented in the concluding Sec. VII, where the final conclusions are drawn.

II. THE MODEL HAMILTONIAN

As stated in the Introduction we intend to study a coupled system of interacting nucleons and α particles. The nucleon subsystem consists of protons and neutrons moving in a spherical shell-model potential, with like nucleons interacting through a pairing force. We denote their single-particle energies by ϵ_j and the corresponding creation and annihilation fermion operators by $c_{jm}^{\dagger}$ and c_{jm}. The α particles are treated as elementary bosons moving in a mean field generated by their interaction with the surrounding medium and described by a spherical harmonic-oscillator well. For our purposes it is necessary to consider a truncated well such that only two bound states, the $0s$ and the $1p$ states, be allowed, the third lying at the edge of the continuum spectrum. The boson creation and annihilation operators for the α states are denoted by $A_{lm}^{\dagger}$ and A_{lm} and the α particles in the $0s$ and $1p$ are referred to as α_0 and α_1.

The full model Hamiltonian describing the interacting system of α particles and nucleons has the form

$$H=\sum(\varepsilon_{j_\tau}-\lambda_\tau)c_{j_\tau m_\tau}^{\dagger}c_{j_\tau m_\tau}+\sum_{J(=0,1)M}(\omega_J-4\lambda_\alpha)A_{JM}^{\dagger}A_{JM}-\tfrac{1}{4}\sum_{\tau=p,n}G_\tau P_\tau^{\dagger}P_\tau+X_0(P_p^{\dagger}P_n^{\dagger}A_0+\text{H.c.})$$
$$+X_1(\hat N_p+\hat N_n)(\hat N_{\alpha_0}+\hat N_{\alpha_1})+X_2\sum_{\mu}q_{2\mu}Q_{2-\mu}(-)^{\mu}-\lambda_2\hat J^{2}\ , \tag{2.1}$$

where $\tau=p$, n, $\hat N_p$, $\hat N_n$, and $\hat N_\alpha$ are the number operators for protons, neutrons, and α particles, respectively, $P_\tau^{\dagger}$ and their Hermitian conjugates P_τ are the pairing operators, $q_{2\mu}$ and $Q_{2\mu}$ the nucleon and alpha quadrupole operators, respectively, $\hat J$ the total angular momentum of the whole system, λ_τ, λ_α, and λ_2 are Lagrange multipliers. More explicitly the pairing and quadrupole operators have the form

$$\begin{aligned}P_\tau^{\dagger}&=\sum_{j_\tau}P_{j_\tau}^{\dagger}\\&=\sum_{j_\tau}\sum_{m_\tau}c_{j_\tau m_\tau}^{\dagger}c_{j_\tau -m_\tau}^{\dagger}s_{j_\tau m_\tau},\quad s_{jm}=(-1)^{j-m}\ ,\\q_{2\mu}&=\sum\langle jm|r^2Y_{2\mu}|j'm'\rangle c_{jm}^{\dagger}c_{j'm'}\ ,\\Q_{2\mu}&=\left[\frac{15}{32\pi}\right]^{1/2}\frac{\hbar}{M\omega_1}[(A_1^{\dagger}A_1^{\dagger})_{2\mu}+(A_1A_1)_{2\mu}\\&\qquad-2(A_1^{\dagger}A_1)_{2\mu}]\ ,\end{aligned} \tag{2.2}$$

where M stands for the mass of the α particle and ω_1 is the frequency of the α_1 particle.

The total angular momentum operator is

$$J_\mu=j_\mu^{(p)}+j_\mu^{(n)}+j_\mu^{(\alpha_1)}, \tag{2.3}$$

where the proton and neutron spins are given by

$$j_\mu^{(\tau)}=-\sum_{j_\tau}[j_\tau(j_\tau+1)]^{1/2}\frac{\hat{j}_\tau}{\hat{1}}(c_{j_\tau}^{\dagger}c_{j_\tau})_{1\mu}\ , \tag{2.4}$$

while the angular momentum carried by the α_1 system is

$$j_\mu^{(\alpha_1)}=\sqrt{2}(A_1^{\dagger}A_1)_{1\mu}\ . \tag{2.5}$$

[We use the tensorial coupling with the phase convention $(c_j^{\dagger}c_j)_{1\mu}=\sum C_{m_1m_2\mu}^{jj1}c_{jm_1}^{\dagger}c_{j-m_2}s_{jm_2}$. The other conventions are those of Rose. The Wigner-Eckart theorem is used in the form $\langle jm|T_{k\mu}|j'm'\rangle=C_{m'\mu m}^{j'kj}\langle j\|T_k\|j'\rangle$.]

The terms containing the Lagrange multipliers λ,s are constraint operators on the total spin J and the number of protons, neutrons, and α particles. Being $A_{lm}^{\dagger}$ and A_{lm} elementary boson operators, the Hamiltonian clearly breaks the gauge symmetry. In fact,

$$[H,\hat N_p+\hat N_n+4\hat N_{\alpha_0}+4\hat N_{\alpha_1}]\neq 0\ . \tag{2.6}$$

This breaking is enforced, as we will see, by the trial wave function, which violates also the rotational symmetry. Hence the need for introducing constraining terms which restore at least on average the two symmetries.

The Hamiltonian displays three interaction terms. These were chosen on the ground of the following physical motivations. (a) The first one having the coupling constant X_0 accounts for the fact that an α particle can be formed out of or can be split into two $T=1$ pairs of nucleons. (b) The second in X_1 simulates an interaction between α particles. Because of the constraint on the total number of constituents, $\hat{N}_p+\hat{N}_n$ can be expressed in terms of $\hat{N}_{\alpha_0}+\hat{N}_{\alpha_1}$ thereby generating terms in $(\hat{N}_{\alpha_0}+\hat{N}_{\alpha_1})^2$ specific of α-pairing interaction. (c) The quadrupole-quadrupole term is dictated by the fact that the α particles cluster in deformed nuclei. This is in fact a cooperative process. The formation of α particles reduces the number of pairs thereby favoring the deformation of the mean field and consequently of the nuclear shape. This in turn induces distortions on the motion of the α particles, whose trajectories are expected to lie close to the nuclear surfaces. It is indeed reasonable to assume that the α clusters are formed out of valence nucleons whose single-particle wave functions overlap each other mostly near the nuclear surface.

III. SEMICLASSICAL DESCRIPTION

Instead of solving the stationary eigenvalue problem we seek the solutions $\psi(t)$ of the corresponding time-dependent Schrödinger equation. One thus obtains information not only about the static properties of the stationary solutions but also about the dynamic of the excitation process. The time evolution of $\psi(t)$ is entirely determined by the variational principle

$$\delta\int_{t_0}^{t}\left\langle\psi_{f,\alpha}\left|i\hbar\frac{\partial}{\partial t'}-H\right|\psi_{f,\alpha}\right\rangle dt'=0\ , \tag{3.1}$$

provided the trial wave functions $\psi_{f,\alpha}$ span the whole Hilbert product space $S_f\otimes S_\alpha$ of fermion and α-particle states.

We choose variational states of the following type:

$$\begin{aligned}\psi_{f,\alpha}=\mathcal{N}\exp(Z_0A_0^\dagger-Z_0^*A_0)&\exp[Z_1(A_{11}^\dagger+A_{1-1}^\dagger)\\&-Z_1^*(A_{11}+A_{1-1})]\\&\times c_{j_1m_1}^\dagger c_{j_1-m_1}^\dagger\exp\left[\sum_{i=j_p,j_n}(z_iP_i^\dagger-z_i^*P_i)\right]|0\rangle_f|0\rangle_\alpha,\end{aligned} \tag{3.2}$$

where $\mathcal{N}$ denotes the normalization constant and $|0\rangle_f,|0\rangle_\alpha$ the vacua for nucleons and α particles.

The α components of the trial wave functions are coherent states with respect to the α_0 and α_1 "oscillations" and as such are suitable for a semiclassical description of the α system. These states clearly break gauge and rotational symmetries. The nucleon component is a BCS wave function which does not preserve the number of protons and neutrons. The BCS state is not deformed. The deformation however is simulated by the action of two neutron pair operators $c_{j_1m_1}^\dagger c_{j_1-m_1}^\dagger$, with j_1 and m_1 given on the BCS vacuum. The resulting state is a superposition of states with total spin $J=0,2,\ldots,2j_1-1$. The choice of neutron rather than proton blocking operators is dictated by the fact that heavy nuclei have an excess of valence neutrons.

The variational principle (3.1) provides a set of equations of motion for the classical coordinates Z_0,Z_1,z_i. Since the trial wave function breaks gauge and rotational symmetry, we shall select those solutions which obey the restrictions

$$\langle\psi_{f,\alpha}|\hat{N}_p+\hat{N}_n+4\hat{N}_{\alpha_0}+4\hat{N}_{\alpha_1}|\psi_{f,\alpha}\rangle=N_t\ , \tag{3.3}$$

$$\langle\psi_{f,\alpha}|\hat{J}^2|\psi_{f,\alpha}\rangle=J(J+1)\ , \tag{3.4}$$

where the values of N_t and J are given.

The parameters Z_0, Z_1, z_i, and their corresponding conjugates Z_0^*,Z_1^*,z_i^* are smooth complex functions of time. We will use the parametrization

$$Z_0=R_0e^{i\phi_0},\quad Z_1=R_1e^{i\phi_1},\quad z_k=\rho_ke^{i\varphi_k}\ ,\quad k=2,\ldots,n_s+1\ , \tag{3.5}$$

where n_s denotes the number of nucleon states. These complex variables define a manifold which plays the role of a classical phase space. The set of solutions of Eq. (3.1) and of their complex conjugates $M(t)=(Z_0(t),Z_1(t),z_2(t),z_3(t),\ldots;\ Z_0^*(t),Z_1^*(t),\ z_2^*(t),\ z_3^*(t),\ldots)$ defines a point in the classical phase space. Once we know $M(t_0)$ for a given time t_0, the set $\{M(t)\}_t$ is fully determined for any t and defines a classical trajectory of our system. The classical trajectories are contained in the energy surface defined by

$$\begin{aligned}\langle\psi_{f,\alpha}|H|\psi_{f,\alpha}\rangle&\equiv\mathcal{H}(Z_0,Z_1,z_2,z_3,\ldots;Z_0^*,Z_1^*,z_2^*,z_3^*,\ldots)\\&=E\ \text{(given)}\ .\end{aligned} \tag{3.6}$$

This equation expresses the fact that $\mathcal{H}$ is a constant of motion in accordance with any time-dependent formalism derived from a variational principle.

The matrix elements involved in (3.1) can be easily evaluated if one uses the quasiparticle representation

$$e^{T_f}c_{km}^\dagger e^{-T_f}=u_kc_{km}^\dagger-s_{km}v_kc_{k-m}\equiv a_{km}^\dagger\ , \tag{3.7}$$

where

$$\begin{aligned}\exp[T_f]|0\rangle_f&=\exp\left[\sum_{i=j_p,j_n}(z_iP_i^\dagger-z_i^*P_i)\right]|0\rangle_f\\&\equiv|\mathrm{BCS}\rangle\ .\end{aligned} \tag{3.8}$$

The u and v coefficients are related to the particle phase-space coordinates by

$$u_k=\cos2\rho_k,\quad v_k=\sin2\rho_ke^{-i\varphi_k}\ . \tag{3.9}$$

Using the fact that $e^{T_f}|0\rangle_f$ is a vacuum state for the quasiparticle operators $a_{km}^\dagger$ and that the trial wave functions $\psi_{f,\alpha}$ are eigenstates of the operators A_0 and $A_{1\pm1}$

$$A_0\psi_{f,\alpha}=Z_0\psi_{f,\alpha},\quad A_{11}\psi_{f,\alpha}=A_{1-1}\psi_{f,\alpha}=Z_1\psi_{f,\alpha}\ , \tag{3.10}$$

one easily obtains for $\mathcal{H}$ the expression given in Appendix A. In terms of the classical coordinates the constraints (3.3) and (3.4) on the number of particles and on the total spin have the expressions

$$2\sum|v_{j_r}|^2\Omega_r+2u_{j_1}^2+4|Z_0|^2+8|Z_1|^2=N_t\ , \quad (3.11)$$

$$2u_{j_1}^2C_{j_1m_1}+4|Z_1|^2=J(J+1), \quad (3.12)$$

where $\Omega_k=(2j_k+1)/2$ and $C_{j_1m_1}$ is given by Eq. (A3).

The classical equations of motion can be put in canonical form if they are written in terms of the conjugate coordinates (r_k,φ_k) defined by (3.5) and by

$$r_0=R_0^2,\quad r_1=2R_1^2,\quad r_k=(\Omega_k-\delta_{kj_1})\sin^2 2\rho_k\ ,$$
$$k=2,3,\ldots,n_s+1\ . \quad (3.13)$$

We also put

$$\varphi_0=\phi_0,\quad \varphi_1=\phi_1\ , \quad (3.14)$$

so as to have a unitary notation for α and nucleon coordinates. The equations of motion for these new coordinates are

$$\frac{\partial\mathcal{H}}{\partial r_k}=-\dot{\varphi}_k,\quad \frac{\partial\mathcal{H}}{\partial\varphi_k}=\dot{r}_k\ , \quad (3.15)$$

where the overdot denotes time derivative. Their explicit form is given in Appendix B.

Expressed in these new coordinates, the constraints (3.11) and (3.12) become

$$2\sum_{k=2}^{n_s+1} r_k+2+4(r_0+r_1)=N_t\ , \quad (3.16)$$

$$\left[1-\frac{r_{j_1}}{\Omega_{j_1}-1}\right]C_{j_1m_1}+r_1=\frac{J(J+1)}{2}\ . \quad (3.17)$$

According to Appendix B, r_1 is a constant of motion. This is a classical counterpart of the fact that quantum mechanically the Hamiltonian (2.1) commutes with $\hat{N}_{\alpha_1}$:

$$[H,\hat{N}_{\alpha_1}]=0\ . \quad (3.18)$$

(Here we consider only the $\Delta N=0$ component of the quadrupole operator associated to the α system.) Using the equations of motion given in Appendix B we get

$$\sum_{k=2}^{n_s+1}\dot{r}_k+2\dot{r}_0=0\ , \quad (3.19)$$

which states that at the classical level the total number of particles is preserved. Constraint (3.16) is therefore automatically satisfied by the classical solutions. The same does not hold, however, for the spin constraint, since $\dot{r}_{j_1}\neq 0$, as indicated by the equations listed in Appendix B. Constraint (3.17) can therefore be satisfied only in the stationary points of $\mathcal{H}$.

Before closing this section we would like to comment on the transformation relating the old coordinates (z,z^*) to the new ones (r,φ). This transformation is not defined for $r=0$, where the functions at the right-hand side (rhs) of the equations in Appendix B become singular and therefore cannot be used to describe the behavior of the system around the origin of the phase-space coordinates. Being highly nonlinear, the classical equations of motion can be solved only by adopting some approximation. We shall first search for the stationary points of the energy surface and then select the minimum energy points which satisfy the constraints (3.16) and (3.17). Let us denote one of the minima of $\mathcal{H}$ by $(\mathring{r}_k,\mathring{\varphi}_k)$, $k=0,1,\ldots,n_s+1$. After expanding the rhs of the equations in Appendix B around $(\mathring{r}_k,\mathring{\varphi}_k)$ and keeping only the linear terms we can easily integrate the resulting system of equations. The corresponding solutions are closed trajectories surrounding $(\mathring{r}_k,\mathring{\varphi}_k)$. The coordinates (r_k,φ_k) are, therefore, not suitable when the minimum point has at least one vanishing component. This happens when the angular momentum of the composite system is zero. For $J=0$ indeed, Eq. (3.17) is satisfied by $\mathring{r}_1=\mathring{u}_{j_1}=0$. In this case we have either to use the (z,z^*) representation and keep the constraint (3.17) or to keep the (r,φ) representation but change the constraint such that the minimum has nonvanishing components. We opt for the second alternative and require that the ground-state mean value of the static quadrupole operator

$$\bar{Q}_{20}=\left[\frac{16\pi}{5}\right]^{1/2}\left[\sum_{jm}\langle jm|r^2Y_{20}|jm\rangle c_{jm}^\dagger c_{jm}+\sum_M\langle 1M|r^2Y_{20}|1M\rangle A_{1M}^\dagger A_{1M}\right] \quad (3.20)$$

be the experimental value Q_0^{expt}.

This yields the relation

$$u_{j_1}^2\tilde{C}_{j_1m_1}+r_1=-\frac{M\omega_1}{\hbar}Q_0^{\text{expt}}\ , \quad (3.21)$$

where

$$\tilde{C}_{j_1m_1}=-4\frac{M\omega_1}{m\tilde{\omega}_0}(N+\tfrac{3}{2})C^{j_1 2 j_1}_{1/2\,0\,1/2}C^{j_1 2 j_1}_{m_1 0 m_1}\ . \quad (3.22)$$

Here m and M stand for the nucleon and α-particle masses, respectively, $\tilde{\omega}_0$ denotes the spherical shell-model frequency, and N is the principal quantum number for the neutron orbital (j_1m_1). With these new constraints the $J=0$ case is formally identical to that of $J\neq 0$ if we replace $C_{j_1m_1}$ with $\tilde{C}_{j_1m_1}$ and $J(J+1)/2$ with $-(M\omega_1/\hbar)Q_0^{\text{expt}}$.

Since r_1 is a constant of motion, the problem in the (r,φ) representation is simplified by fixing its value. One solves, indeed, only $2n_s+2$ equations describing the motion of the coordinates (r_k,φ_k) with $k\neq 1$. Taking $r_1=0$ for the $J=0$ case we get from Eq. (3.21) a nonvanishing solution for u_{j_1} provided $\tilde{C}_{j_1m_1}$ and Q_0^{expt} have opposite signs. So for $m_1=\frac{1}{2}$, Q_0^{expt} should be positive since

$\tilde{C}_{j_1 1/2} < 0$, while for $m_1 = j_1$ one obtains nonvanishing solutions for u_{j_1} only for oblate deformations.

IV. BCS TREATMENT OF THE COMPOSITE SYSTEM OF NUCLEONS AND α PARTICLES

As already mentioned, the classical equations of motion, before being solved, are linearized around the minimum points of the energy surface ($\mathcal{H} = E$). The stationary points satisfy the equations

$$\frac{\partial \mathcal{H}}{\partial r_k} = 0, \quad \frac{\partial \mathcal{H}}{\partial \varphi_k} = 0 . \tag{4.1}$$

These will be solved separately for $J = 0$ and $J \neq 0$. Let us consider the $J = 0$ case. Since r_1 is a constant of motion and accounts for the number of α_1 particles in the positive-parity $J = 0$ state, we take $\mathring{r}_1 = 0$. Equation (3.21) then determines $\mathring{u}_{j_1}$. For spherical nuclei it gives $\mathring{u}_{j_1} = 0$ and $\mathring{v}_{j_1} = 1$.

It is well known that the BCS equations produce real solutions for the gap parameter. We suppose that the gap remains real even when nucleons interact with α particles. A sufficient condition for this to occur is $\mathring{\varphi}_k = 0$, $k \neq j_1$. For $k = j_1$ one can take $\mathring{\varphi}_{j_1} = 0$ when $\mathring{v}_{j_1} = 1$ and $\mathring{\varphi}_{j_1} = \pi/2$ if $\mathring{v}_{j_1} \neq 1$. The v and u coefficients satisfy the equations

$$|\mathring{v}_k|^2 = \frac{1}{2}\left[1 - \frac{|\varepsilon_k - \tilde{\lambda}_\tau|}{\sqrt{(\varepsilon_k - \tilde{\lambda}_\tau)^2 + \tilde{\Delta}_\tau^2}}\right], \quad \mathring{u}_k^2 = 1 - |\mathring{v}_k|^2 , \tag{4.2}$$

where

$$\tilde{\lambda}_\tau = \lambda_\tau - X_1 \mathring{r}_0, \quad \tilde{\Delta}_\tau = \mathring{\Delta}_\tau - 4X_0\sqrt{\mathring{r}_0}\frac{\mathring{\Delta}_{\tau'}}{G_{\tau'}} , \tag{4.3}$$

with τ' taking the complementary values of τ.

For a given $\mathring{r}_0$ the gap parameters $\mathring{\Delta}_\tau$ and the Lagrange multipliers λ's are determined by the following equations:

$$G_\tau \sum_{j_\tau} \Omega_\tau \mathring{u}_{j_\tau} |\mathring{v}_{j_\tau}| = \mathring{\Delta}_\tau ,$$

$$2\sum_{j_n} \Omega_n |\mathring{v}_{j_n}|^2 = N - 2\mathring{r}_0 ,$$

$$2\sum_{j_p} \Omega_p |\mathring{v}_{j_p}|^2 = Z - 2\mathring{r}_0 , \tag{4.4}$$

$$\omega_0 - 4\lambda_\alpha + \frac{4X_0}{G_n G_p}\frac{\mathring{\Delta}_n \mathring{\Delta}_p}{\sqrt{\mathring{r}_0}} + X_1(N_t - 4\mathring{r}_0) = 0 ,$$

$$\omega_1 - 4\lambda_\alpha + X_1(N_t - 4\mathring{r}_0) + \tilde{X}_2\left[1 - \left[\frac{2}{2j_1 - 1}\right]\mathring{r}_{j_1}\right] - 2\lambda_2 = 0 ,$$

where $\tilde{X}_2$ is defined by Eq. (A4).

For $J \neq 0$ we keep the condition that Δ_n and Δ_p are real numbers. Again this is true if $\varphi_k = 0$ for any value of $k > 1$. Also the angular variables φ_0 and φ_1 can be put equal to zero if we require that the momentum of the α particle, which is proportional to $(A_k^+ - A_k)$ $(k = 0, 1)$, vanishes in the ground state.

Under these conditions the occupation probabilities are given by

$$|\mathring{v}_k|^2 = \frac{1}{2}\left[1 - \frac{|\tilde{\varepsilon}_k - \tilde{\lambda}_\tau|}{\sqrt{(\tilde{\varepsilon}_k - \tilde{\lambda}_\tau)^2 + \tilde{\Delta}_\tau^2}}\right], \quad k \neq j_1 , \tag{4.5}$$

$$|\mathring{v}_{j_1}|^2 = \frac{1}{2}\left[1 - \frac{|\tilde{\tilde{\varepsilon}}_{j_1} - \tilde{\lambda}_n|}{\sqrt{(\tilde{\tilde{\varepsilon}}_{j_1} - \tilde{\lambda}_n)^2 + \tilde{\tilde{\Delta}}_n^2}}\right] , \tag{4.6}$$

where $\tilde{\lambda}_\tau, \tilde{\Delta}_\tau$ have the expressions (4.3), while $\tilde{\varepsilon}_k$, $\tilde{\tilde{\varepsilon}}_{j_1}$, $\tilde{\tilde{\Delta}}_n$ are given by

$$\tilde{\varepsilon}_k = \varepsilon_k + X_1 \mathring{r}_1 - G_\tau \mathring{v}_k^2 ,$$

$$\tilde{\tilde{\varepsilon}}_{j_1} = \tilde{\varepsilon}_{j_1} + \lambda_2 \frac{C_{j_1 m_1}}{\Omega_{j_1} - 1} + \frac{G_n}{2}(u_{j_1}^2 - v_{j_1}^2) + \tilde{X}_2 \frac{\mathring{r}_1}{2j_1 - 1} , \tag{4.7}$$

$$\tilde{\tilde{\Delta}}_n = \tilde{\Delta}_n - G_n \mathring{u}_{j_1} \mathring{v}_{j_1} .$$

For given r_0 and r_1 the equations for Δ's and λ's are

$$G_\tau \sum_{j_\tau} \Omega_\tau \mathring{u}_{j_\tau} |\mathring{v}_{j_\tau}| = \mathring{\Delta}_\tau ,$$

$$2\mathring{u}_{j_1}^2 + 2\sum_{j_n} \Omega_n |\mathring{v}_{j_n}|^2 = N - 2(\mathring{r}_0 + \mathring{r}_1) = N_n' ,$$

$$2\sum_{j_p} \Omega_p |\mathring{v}_{j_p}|^2 = Z - 2(\mathring{r}_0 + \mathring{r}_1) = N_p' ,$$

$$4(r_0 + r_1) = N_t - N_p' - N_n' = 4N_\alpha , \tag{4.8}$$

$$2\mathring{u}_{j_1}^2 C_{j_1 m_1} + 2\mathring{r}_1 = J(J+1) ,$$

$$\omega_0 - 4\lambda_\alpha + \frac{X_0}{2\sqrt{\mathring{r}_0}}\left[-4\mathring{u}_{j_1}|\mathring{v}_{j_1}|\frac{2\mathring{\Delta}_p}{G_p} + \frac{8\mathring{\Delta}_n \mathring{\Delta}_p}{G_n G_p}\right] + X_1[N_t - 4\mathring{r}_0 - 4\mathring{r}_1] = 0 ,$$

$$\omega_1 - 4\lambda_\alpha + X_1[N_t - 4\mathring{r}_0 - 4\mathring{r}_1] + \tilde{X}_2\left[1 - \frac{2}{2j_1 - 1}\mathring{r}_{j_1}\right] - 2\lambda_2 = 0 .$$

It is understood that Eqs. (4.5), (4.6), (4.7), and (4.8) are to be solved iteratively.

Once the stationary equations (4.1) are solved for $J = 0$ and $J \neq 0$ and a stable solution is found for each value of J we have to linearize the equations of motion around the minima and solve the resulting equations. This is fully equivalent to the standard RPA procedure for any many-body system. The specific feature of our case is that we deal with a composite system of nucleons and α particles.

V. RPA DESCRIPTION

The linearization of the equations of motion (3.15) leads to a system of equations for the deviations of the phase-space coordinates from their static values:

$$q_k = r_k - \mathring{r}_k, \quad p_k = \varphi_k - \mathring{\varphi}_k \ . \tag{5.1}$$

The result is

$$\dot{p}_k = \sum_{\substack{i=0\\ i\neq 0}}^{n_s+1} \mathcal{A}_{ki} q_i, \quad \dot{q}_k = \sum_{\substack{i=0\\ i\neq 1}}^{n_s+1} \mathcal{B}_{ki} p_i \ , \tag{5.2}$$

where the coefficients $\mathcal{A}_{ki}$ and $\mathcal{B}_{ki}$ are listed in Appendix C.

These equations can be easily solved. Let us define the transformation

$$\begin{bmatrix} \mathcal{P} \\ \mathcal{Q} \end{bmatrix} = \begin{bmatrix} \mathcal{G} & \mathcal{F} \\ \mathcal{U} & \mathcal{V} \end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} , \tag{5.3}$$

which relates the coordinates p and q to a new set of coordinates P and Q fulfilling the following equations:

$$\dot{P} = -\omega Q, \quad \dot{Q} = \omega P \ . \tag{5.4}$$

These equations of motion yield a simple relation between the row vectors involved in (5.3):

$$\mathcal{U} = -\mathcal{G}, \quad \mathcal{V} = \mathcal{F} \ . \tag{5.5}$$

Using (5.3), (5.4), and (5.5) one obtains the following eigenvalue equations:

$$\begin{bmatrix} \mathcal{O} & \mathcal{A} \\ \mathcal{B} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \end{bmatrix} = \omega \begin{bmatrix} \mathcal{F} \\ \mathcal{G} \end{bmatrix} \tag{5.6}$$

which determines the vectors $\mathcal{F}$ and $\mathcal{G}$ as well as the eigenvalue ω.

Equation (5.6) admits a spurious solution with $\omega = 0$. This corresponds to the breaking of the symmetry property expressed by

$$[H, \hat{N}_p + \hat{N}_n + 4\hat{N}_{\alpha_0}] = 0 \tag{5.7}$$

and reflects a well-known property of the RPA equations of motion.

Let us label the $n_s + 1$ physical solutions of the RPA equations according to the ordering sequence

$$\omega_1 \le \omega_2 \le \cdots \le \omega_{n_s+1} \ . \tag{5.8}$$

The corresponding eigenstates are denoted by $\mathcal{F}_k$ and $\mathcal{G}_k$, with components $\mathcal{F}_{k,i}, \mathcal{G}_{k,i}$ $(i = 0, 2, 3, \ldots, n_s + 1)$. Once the energies ω_k are determined, Eqs. (5.4) can be easily integrated. The solutions are

$$Q_k = \mathring{Q}_k \sin(\omega_k t + \delta_k), \quad P_k = \mathring{Q}_k \cos(\omega_k t + \delta_k) \ , \tag{5.9}$$

where $\mathring{Q}_k$ and δ_k are integration constants.

The classical energy $\mathcal{G}$ can be easily expressed in terms of the conjugate coordinates (Q_k, P_k) after expanding $\mathcal{H}(r, \varphi)$ in terms of (q_k, p_k) up to second order.

In fact, the linearized equations (5.2) also have a canonical form:

$$\frac{\partial \mathcal{H}}{\partial p}(q,p) = \dot{q}, \quad \frac{\partial \mathcal{H}}{\partial q}(q,p) = -\dot{p} \ . \tag{5.10}$$

If we interpret Q_k and P_k as components of the column vectors (Q) and (P), respectively, and $\mathcal{G}$ and $\mathcal{F}$ as $(n_s + 1)$-dimensional matrices, Eq. (5.3) defines a unitary transformation and can therefore be reversed giving (q,p) in terms of the RPA conjugate variables (Q,P). The resulting classical Hamiltonian $\mathcal{H}$ is a quadratic form in Q_k and P_k.

This classical function can be quantized by one of the following procedures.

(a) Using for Q_k and P_k their expressions (5.9), one can check that $\mathcal{H}$ is independent of time and of the phases δ_k and depends only on the squared amplitudes $\mathring{Q}_k$. These can be fixed by using the Bohr-Sommerfeld quantization procedure, which generates a discrete spectrum. The classical function $\mathcal{H}$ assumes indeed the following expression:

$$\mathcal{H}_{n_0, n_2, \ldots, n_s+1} = \sum_{k \neq 1} n_k \hbar\omega_k \ , \tag{5.11}$$

where $n_k \in N$.

(b) Alternatively one can regard the canonical variables (Q_k, P_k) as quantal operators satisfying the commutation relations

$$[Q_k, P_k] = i\hbar \ . \tag{5.12}$$

The classical function $\mathcal{H}(Q,P)$ becomes then a quantal Hamiltonian. This can be brought into diagonal form through the canonical transformation

$$Q_k = a_k \sqrt{2}(B_k^\dagger + B_k), \quad P_k = \frac{i\hbar}{a_k\sqrt{2}}(B_k^\dagger - B_k) \ , \tag{5.13}$$

where the new operators have a boson character:

$$[B_k, B_k^\dagger] = 1 \ . \tag{5.14}$$

The constants a_k are chosen so that the coefficients of the cross terms $B_k^\dagger B_k^\dagger + \text{H.c.}$ vanish. The resulting Hamiltonian is

$$H_B = \sum_k \hbar\omega_k^{(J)}(B_{J,k}^\dagger B_{J,k} + \tfrac{1}{2}) \ , \tag{5.15}$$

where the label J indicates that the variational procedure is carried out for a given value of the total spin.

This is nothing but the zero-order boson representation of the initial Hamiltonian describing the system of mutually interacting fermions and α particles. It gives the same excitation energies of the semiclassical Hamiltonian (5.11), but has in addition the zero-point energy term.

The $J = 0^+, 2^+, 1^-$ states, though obtained by solving separately three different variational equations, are approximately mutually orthogonal. We then assume that the boson operators $B^\dagger, B$ corresponding to different values of J commute with each other.

The present QRPA approach, formally identical to the usual many-body QRPA description, deals with a composite system of nucleons and α_0 particles. Both nucleons and α's are treated on an equal footing. Even the classical coordinates for nucleons and α_0 particles have a similar meaning, being in both cases mean values of their number operators

$$r_0 = \langle \psi_{f,\alpha} | \hat{N}_{\alpha_0} | \psi_{f,\alpha} \rangle ,$$
$$r_j = \left\langle \psi_{f,\alpha} \left| \sum_m c^\dagger_{jm} c_{jm} \right| \psi_{f,\alpha} \right\rangle . \tag{5.16}$$

VI. NUMERICAL APPLICATIONS AND DISCUSSION

The formalism described in the previous sections was applied to ^{220}Ra.

The single-particle space chosen consisted of two major spherical shells for both protons $(4\hbar\tilde{\omega}_0, 5\hbar\tilde{\omega}_0)$ and neutrons $(5\hbar\tilde{\omega}_0, 6\hbar\tilde{\omega}_0)$. The strengths of the proton- and neutron-pairing interactions were fitted to the mass differences of ^{220}Ra and the neighboring odd nuclei. Since the BCS quantities depend only on the difference $\omega_1 - \omega_0$, this was determined as follows [33]. The α-particle formation amplitude was computed microscopically and then interpreted as an eigenfunction of a Hamiltonian with an effective potential. The potential so obtained could be assimilated to an oscillator well. The computed energy distance between the first two bound states of this potential, about 5.63 MeV, was assigned to the energy difference $\omega_1 - \omega_0$. The coupling constant X_1 of the term simulating the pairing interaction between α particles was fitted [30] to the mass difference of ^{220}Ra and the average mass of ^{222}Ra and ^{222}Th with the result $X_1 = 0.32$ MeV. The X_0 and X_2 strengths were determined by imposing that the variational states $J=1$ and $J=2$ lie above the ground $J=0$ state by an amount which is equal to the observed low-lying excitation energies E_{1^-} and E_{2^+}, respectively [34]. The resulting values were $X_0 = 2\times 10^{-4}$ MeV and $X_2 = -0.01$ MeV.

The BCS equations were solved successively for $J=0,1,2$. Being a constant of motion, the number of α_1 particles was chosen to be $r_1 = 0$ for $J=0,2$ and $r_1 = 1$ for $J=1$. Correspondingly the $J=0$ and 2 states resulted to have positive parity while the $J=1$ state assumed a negative parity.

The occupation probability for the single-particle states is plotted in Fig. 1 for protons and in Fig. 2 for neutrons for $J=0$ and $J=2$. While the protons behave similarly in the $J=0$ and $J=2$ states, the neutrons have a sharp Fermi surface in the $J=2$ state but a diffuse one in the ground state. This reflects the fact that only neutron alignment is responsible for exciting states with $J\neq 0$. For the $J=0$ case we plotted in Fig. 3(a) the gap parameter as a function of the number of α_0 particles. A transition from a nucleon superfluid to a normal phase is registered for $r_0 = 3$, which corresponds to the double magic closure reached by the nucleons. The dependence of the proton gap parameter Δ_p on the pairing strength G_p and on X_0 is illustrated in Figs. 3(b) and 3(c), respectively. The dependence of Δ_n on G_n and X_0 is shown in Figs. 3(d) and 3(e). One may notice that, as G_p decreases or X_0 increases, Δ_p reaches a secondary minimum at $r_0 = 6$ corresponding to a shell subclosure for protons. Such a behavior can be easily understood since the nucleon pairing correlations tend to stiffen the system against α clustering while the term in X_0 favors the clustering of a neutron pair with a proton pair. The neutron gap Δ_n is rather insensitive to changes in X_0 but varies with G_n even more than in the proton case. By a modest increase of G_n the first minimum disappears, while a

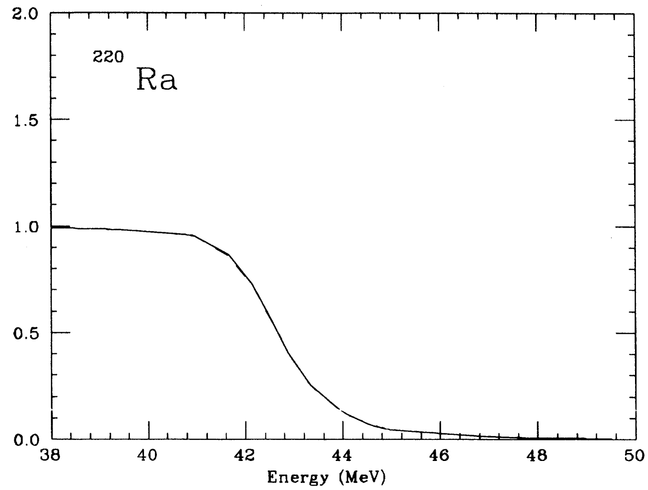

FIG. 1. The proton occupation probability $(|v_j|^2)$ as a function of the corresponding energies for ^{220}Ra. The dashed line refers to $J=2$, the solid line to $J=0$. The parameters have the following values (in MeV): $\omega_1 - \omega_0 = 5.63$, $G_n = 0.1$, $G_p = 0.14$, $X_0 = 2\times 10^{-4}$, $X_1 = 0.32$, and $X_2 = -0.01$.

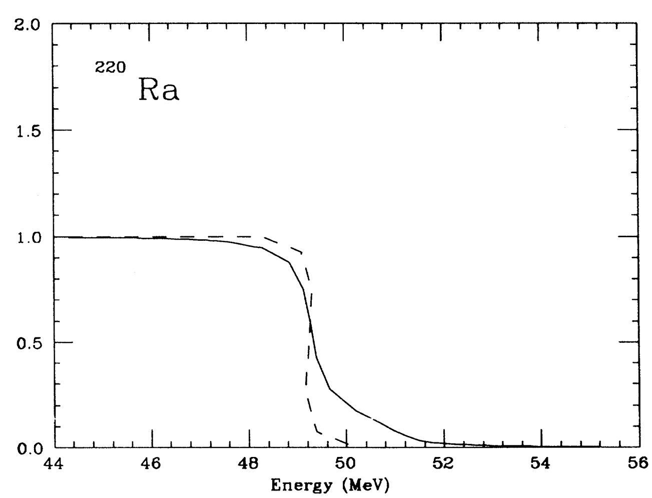

FIG. 2. The same as in Fig. 1 but for neutrons.

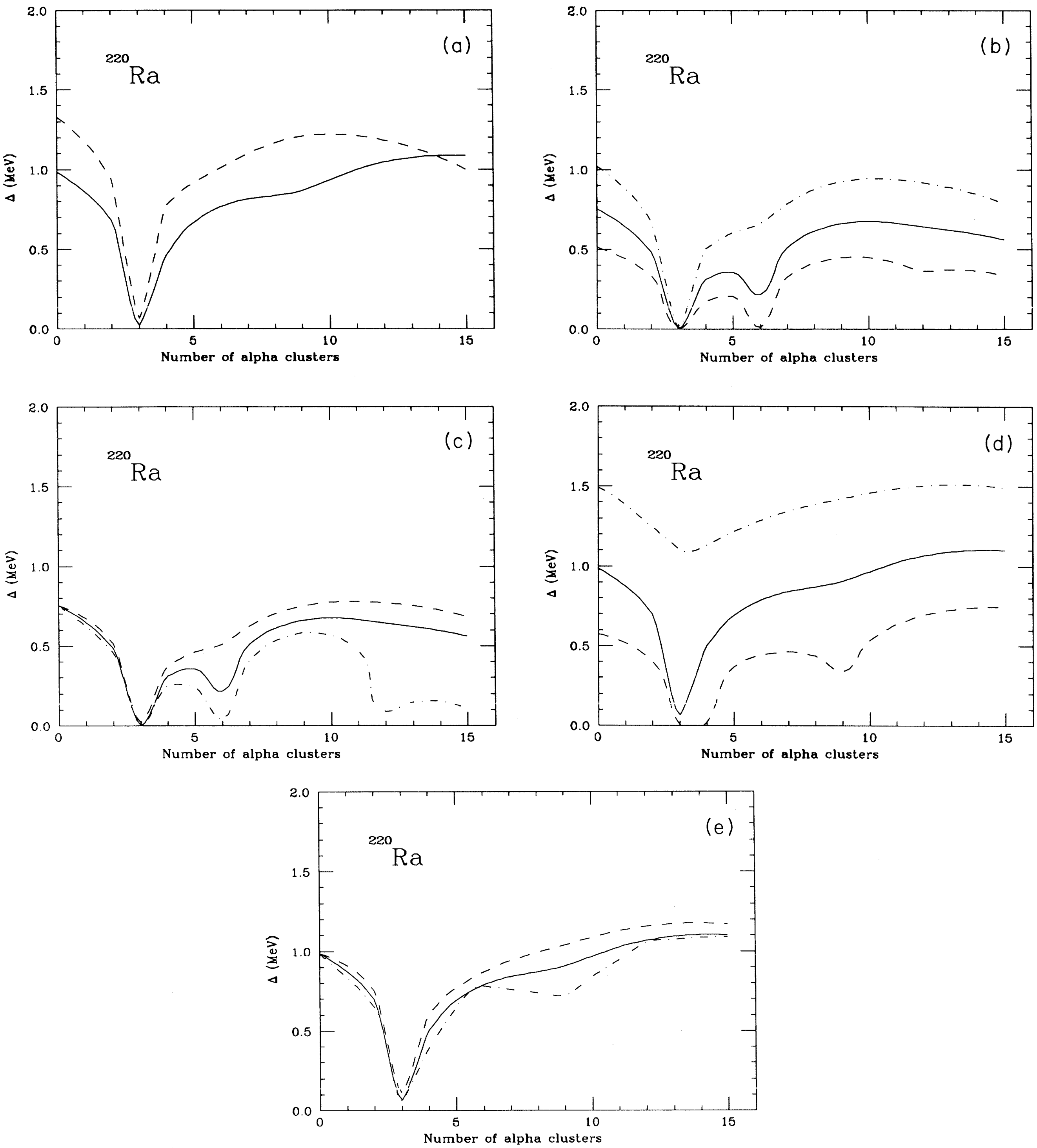

FIG. 3. (a) The gap parameter for protons (dashed line) and neutrons (solid line) as a function of the average number of α_0 particles, r_0. The parameters are those of Fig. 1 for $J^{\pi}=0^{+}$. (b) The gap parameter for protons Δ_p as a function of the average number of α_0 particles for three distinct values (in MeV) for G_p: 0.1 (solid line), 0.08 (dashed line), 0.12 (dash-dotted line). The remaining parameters are $\omega_1-\omega_0=5$, $G_n=0.1$, $X_0=X_2=0$, and $X_1=0.3$. (c) The gap parameter for protons as a function of the average number of α_0 particles for three distinct values (in MeV) of X_0: 0 (solid line), -0.001 (dashed line), $+0.001$ (dash-dotted line). Here $G_p=0.1$ and the other parameters are the same as in (b). (d) The gap parameter for neutrons as a function of the average number of α_0 particles for three different values (in MeV) of G_n: 0.1 (solid line), 0.08 (dashed line), 0.12 (dash-dotted line). Here $G_p=0.1$ and the parameters are the same as in (b). (e) The gap parameters for neutrons as a function of the average number of α_0 particles for three different values (in MeV) of X_0: 0.1 (solid line), 0.08 (dashed line), 0.12 (dash-dotted line). The remaining parameters are those from Fig. 1.

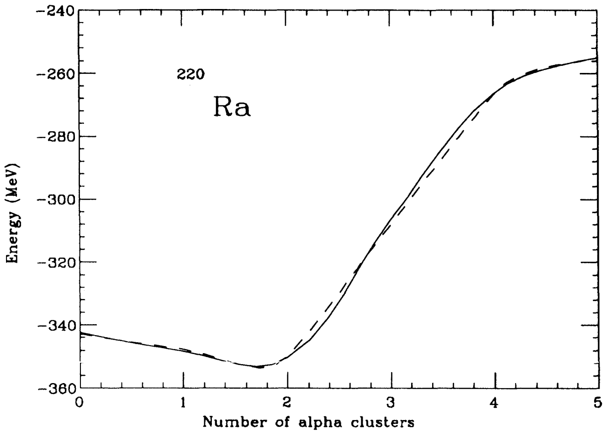

FIG. 4. The energies of states 0^+ and 1^-, obtained by the variational equation, as functions of r_0, the number of α_0 particles. The strength parameters are the same as for Fig. 1.

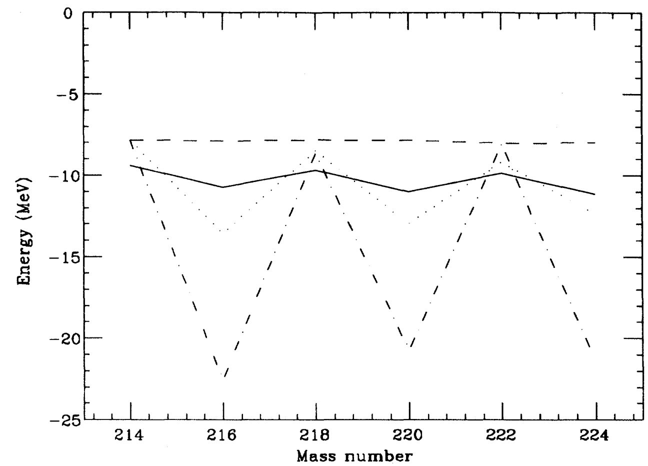

FIG. 6. Even-odd staggering computed with three different values (in MeV) of the strength parameters X_1: 0.2 (dashed), 0.3 (point-like), 0.4 (dash-dotted). The other parameters are given the following values (in MeV): $\omega_1-\omega_0=5$, $G_n=G_p=0.1$, $X_0=X_2=0$.

small reduction generates a secondary minimum at $Z=70$ and $N=114$ corresponding to a shell subclosure.

The above results clearly show the competition between nucleon pairing and α clustering. This is more remarkable since in our formalism the appearance of an α particle does not induce explicitly a blocking of the single-particle orbits, but only affects the occupation probability amplitudes via the restrictions coming from the particle number conservation.

The energy of the $J=0$ ground and $J=1$ states are plotted in Fig. 4 as a function of the number of α_0 particles. Both energies have a minimum at $r_0=2$ corresponding to a formation of an α cluster coexistence with the nucleon superfluid phase. The difference between the two energies corresponding to the minimum point is just the experimental value $E_{1^-}=0.41$ MeV.

Since the strength parameters supposedly do not vary significantly in going to neighboring nuclei, we have calculated the BCS ground-state energy of neighboring even-even nuclei by keeping the strengths constant. We obtained (Fig. 5) a sawtooth behavior which follows rather closely the experimental behavior. The agreement may be improved if we allow the strength parameters to vary with A. Such a staggering effect is indeed very sensitive to the X_1 parameter. We plotted for the sake of illustra-

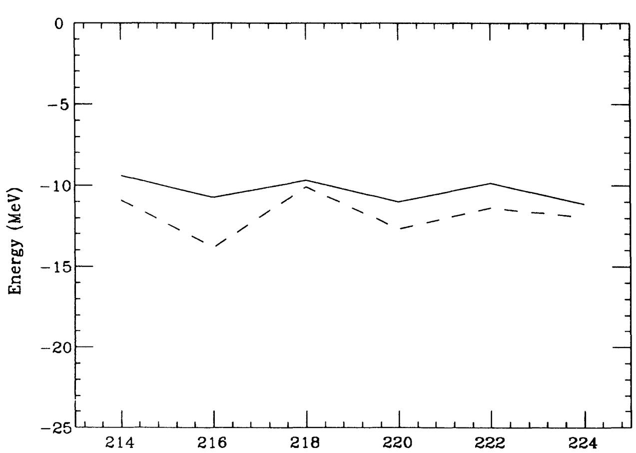

FIG. 5. Even-odd staggering predicted by the present formalism (dashed line) compared with the experimental data [35] (solid line). The strength parameters are those determined for ^{220}Ra.

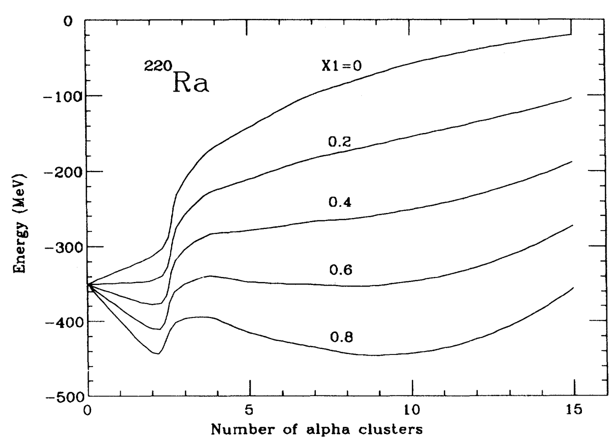

FIG. 7. The BCS ground-state energy of ^{220}Ra as a function of r_0 for five values of X_1. The remaining parameters are the same as in Fig. 6.

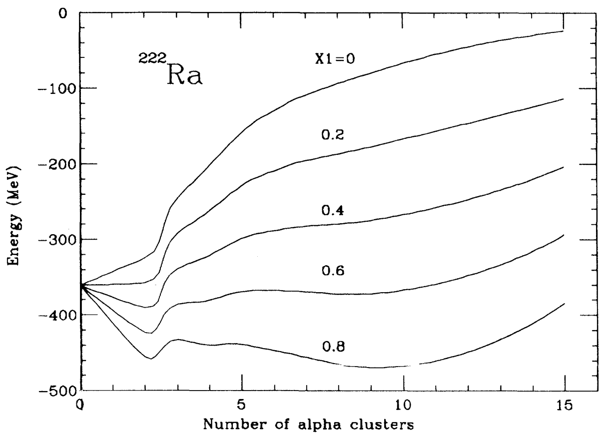

FIG. 8. The same as in Fig. 7 but for ^{222}Ra.

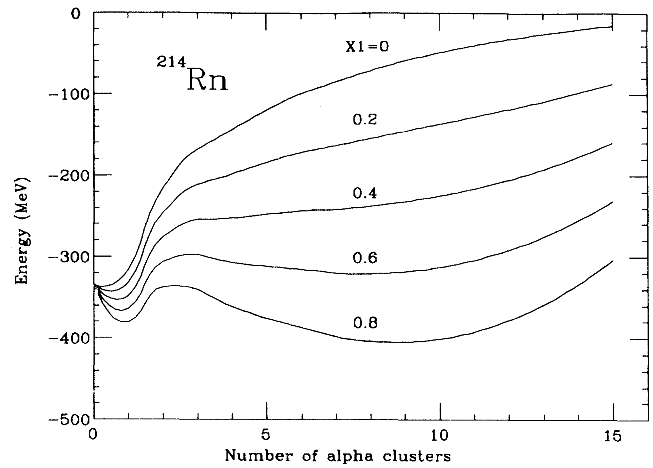

FIG. 10. The same as in Fig. 7 but for ^{214}Rn.

tion the even-odd staggering for three sets of parameters which differ from each other by the value of X_1 (Fig. 6). One sees that for small values of X_1 the teeth disappear implying that the nucleon pairs are unpaired. In order to analyze further the role of the X_1 parameter in setting the α condensate we studied the dependence of the BCS ground-state energy ($\mathring{r}_1=0$) on the number of α_0 particles for $X_0=X_2=0$ MeV and for different values of X_1 (Fig. 7). For $X_1\geq 0.6$ MeV a broad secondary minimum appears, which corresponds to the ground state if $X_1\geq 0.8$ MeV. The flat curve around the minimum reflects the fact that, given the large degeneracy in the single-particle spectrum around $Z=70$ and $N\simeq 114$, the shell effects have a reduced importance with respect to pairing and α-α interactions.

The position of the first minimum does not depend on the number of neutrons, as it can be seen by comparing Fig. 7 with Figs. 8 and 9 referring to ^{218}Ra and ^{222}Ra. It depends instead on the number of active protons as illustrated in Figs. 10–12 for Rn isotopes having two protons less than for Ra isotopes and in Figs. 13 and 14 for ^{222}Th and ^{224}Th with two protons more. The minimum for the two cases is in fact shifted back and forth, respectively, by one unit. It is indeed reached at $r_0=1$ for Rn and at $r_0=3$ for Th. For ^{212}Po which has one α_0 outside the double magic shell there is no first minimum (Fig. 15), consistently with the fact that this nucleus is unstable against α emission. These features persist also in ^{214}Po (Fig. 16).

The RPA-like results for ^{220}Ra reveal the following features. For $J=0$ and $J=1$ there are low-lying RPA states with dominant components describing the harmonic oscillations of α_0 around the stationary $J=0$ and $J=1$ states, respectively. For $J=0$ the oscillation energy of

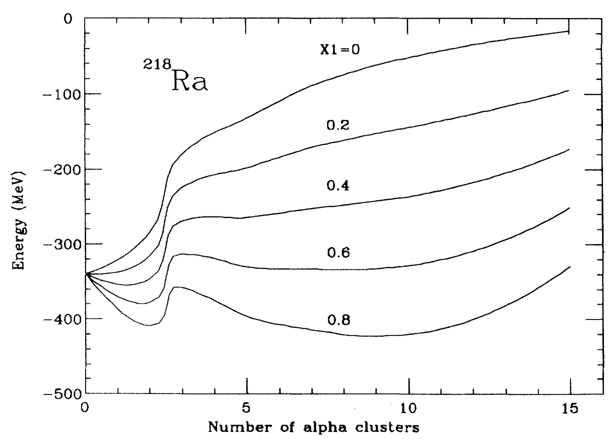

FIG. 9. The same as in Fig. 7 but for ^{218}Ra.

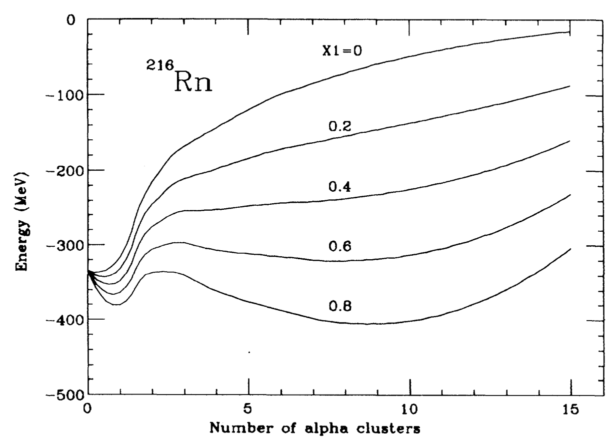

FIG. 11. The same as in Fig. 7 but for ^{216}Rn.

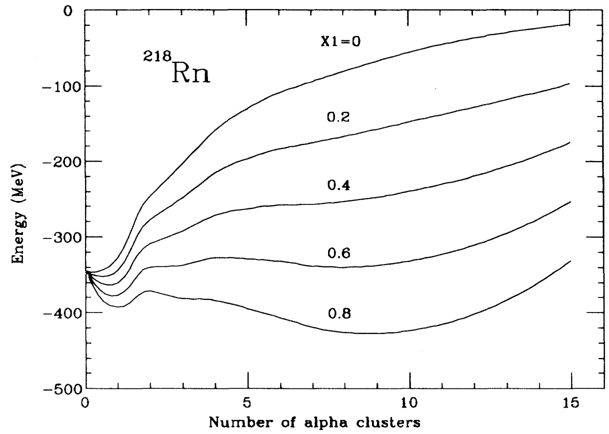

FIG. 12. The same as in Fig. 7 but for ^{218}Rn.

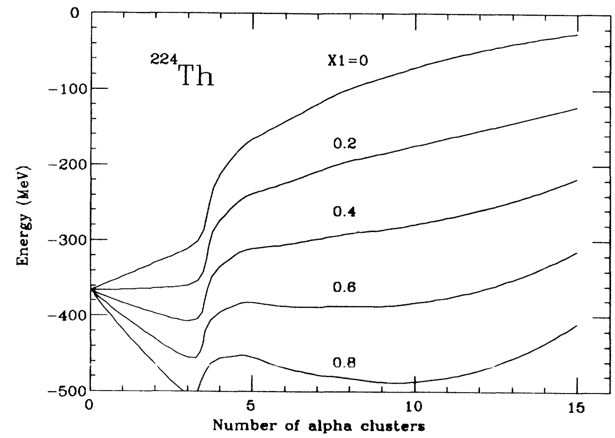

FIG. 14. The same as in Fig. 7 but for ^{224}Th.

the α_0 particle is 0.364, while for $J=1$ the α_0 particle oscillates with an energy of about 0.309 MeV. These states describe clearly a wobbling motion. We shall therefore call them wobbling α vibrational states.

The $J=2$ case is completely different. Among the 24 RPA states there are 9 collective states describing a coherent motion of several quasiparticle pairs oscillating in phase with the α_0 particles. The occurrence of a large number of collective states was quite unexpected in view of the fact that the usual many-body RPA predicts very few collective states, one for a pure system of protons or neutrons and two, at most, for a proton-neutron system [35]. The present result may simulate a transition to deformation induced by α clustering.

For the sake of illustration we give in Table I the energies of the RPA collective states describing the fluctuations around the $J=2$ variational state and having large α_0 components.

It is interesting to analyze the zero-point energy of the "almost" pure α_0 vibrating states

$$E_v=\tfrac{1}{2}\omega_1^0+\tfrac{1}{2}\omega_1^1+E_1=0.736 \text{ MeV} .$$

This is very close to the value of E_v determined by an empirical procedure adopted in Ref. [13], where the zero-point energy of the α particle was introduced as a fitting parameter to correct the Q value entering the WKB integral of the α-penetration probability. The empirical value necessary to get a good agreement with the experimental lifetimes $^{220}\text{Ra}\rightarrow{}^{216}\text{Ru}$ was $E_v=0.72$ MeV. Our formalism may then provide a theoretical justification to such a fitting procedure.

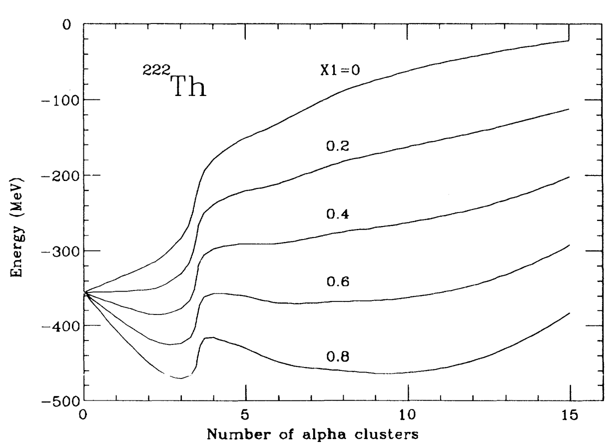

FIG. 13. The same as in Fig. 7 but for ^{222}Th.

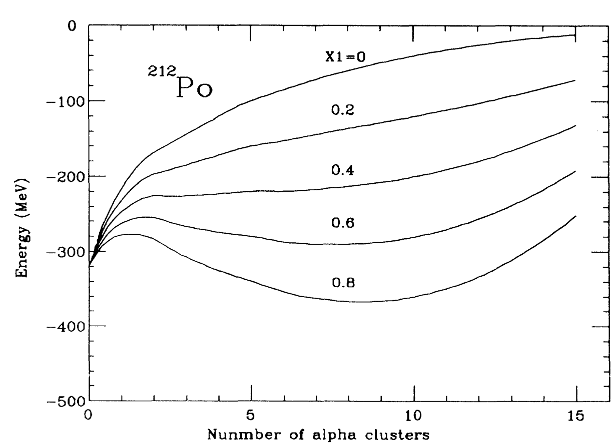

FIG. 15. The same as in Fig. 7 but for ^{212}Po.

TABLE I. Energies (in MeV) $\omega_n^{(2)}$ of the RPA states describing harmonic vibrations around the $J=2$ variational state. The amplitudes a_{α_0} of the α_0 components contained in these states are also shown.

$\omega_n^{(2)}$	2.07	5.63	7.83	7.95	9.19	10.15	10.67	11.13	12.27
a_{α_0}	0.55	0.61	0.59	0.75	0.68	0.74	0.83	0.79	0.87

VII. CONCLUSIONS

Using a time-dependent variational method we derived BCS- and RPA-like equations for a composite system of nucleons and α particles, carrying angular momentum zero (α_0) and one (α_1). The two subsystems interact with each other by monopole-monopole and quadrupole-quadrupole terms.

The variational method was applied to study some low-lying states of ^{220}Ra. Two spherical major shells were used for protons ($4\hbar\tilde{\omega}_0$ and $5\hbar\tilde{\omega}_0$) and for neutrons ($5\hbar\tilde{\omega}_0$ and $6\hbar\tilde{\omega}_0$). The strengths of the interaction between nucleons and α particles were fixed by fitting the energies of the first 1^- and 2^+ states and the mass difference between ^{220}Ra and the average mass of ^{222}Ra and ^{222}Th. The energy difference $\omega_1-\omega_0$ was taken from a microscopic calculation of the amplitude for α-particle preformation. The proton and neutron pairing strengths were fixed in the standard way by fitting the mass differences to the neighboring odd nuclei.

The analysis of the gap parameters as a function of the average number of α_0 particles r_0 indicates that a transition to a normal phase occurs at $r_0=3$, corresponding to a double magic shell closure. The static solution however gives an intermediate phase with a number of α_0 particles $r_0=2$. This value does not change for a fixed Z by varying N by two units. A nice agreement with the experimental data is obtained for the even-odd mass staggering.

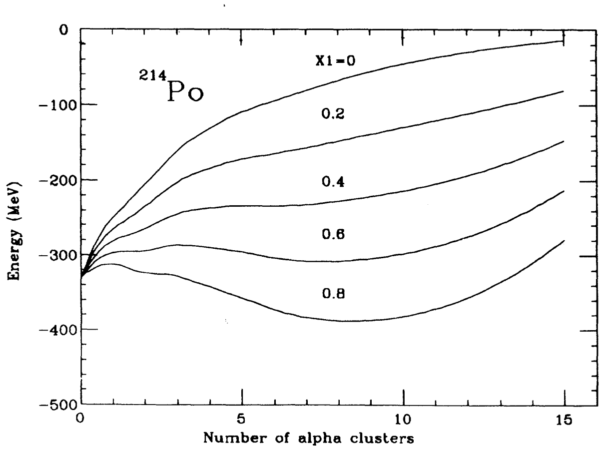

FIG. 16. The same as in Fig. 7 but for ^{214}Po.

The α condensate may set in mainly because of the term in X_1 simulating a two-body interaction between α particles. Indeed, the ground-state energy has no minimum (as function of r_0) for small values of X_1, one minimum for $X_1\in[0.3,0.5]$ MeV and two minima for $X_1\geq 0.5$ MeV.

The RPA equations yield two types of collective states. Some of them describe a wobbling motion of α particles around the stationary $J=0$ and $J=1$ states. Many others having $J=2$ are generated by a coherent motion of nucleons oscillating coherently with the α particles. In ^{220}Ra, among 24 RPA physical states, we identified 9 collective states with large α-particle components (>0.55). This is at variance with standard RPA for a nucleon system which predicts very few collective states.

A quite remarkable RPA result consists in that the zero-point energy of the α_0 vibrations is in surprisingly good agreement with the fitted value obtained in a phenomenological WKB description of the penetration probability for the α-decay process.

The present model can be improved by using a deformed basis for the nucleon subsystem. This would certainly render more explicit the interplay between nuclear deformation and α-clustering phenomena. We expect the position of the first energy minimum to be little affected by adopting such a deformed basis. On the other hand other secondary minima may occur especially far from the shell closure. A deformed basis may also induce fragmentation and damping of the RPA modes. It would be therefore helpful for testing the nature of the states obtained within the present context.

In order to make a more reliable test of the asserted reduced importance of the shell effects in the region around the broad minimum, it would be desirable to account for the fermionic structure of the α particles. We expect, for instance, that the exchange terms coming from antisymmetrizing alike nucleons belonging to different clusters or to a cluster and the rest of the nucleus have an increasing inhibiting effect versus α clustering as the number of α clusters increases. Treating the α particles microscopically in heavy nuclei is, however, a prohibitive task. One may hope to account for these exchange terms only in some approximate effective fashion.

In view of the approximations made, the present model appears to be a valid tool for carrying out a systematic study of the gross features of the low-lying states of the heavy nuclei which are candidates for exhibiting α-like correlations. The next step is to see if the model can account satisfactorily for α-decay processes and α-transfer reactions. This analysis is in progress.

APPENDIX A

We give here the explicit expression of the mean value of the Hamiltonian (2.1) using the trial wave functions $\psi_{f,\alpha}$ [Eq. (3.2)]:

$$\begin{aligned}\langle \psi_{f,\alpha}|H|\psi_{f,\alpha}\rangle \equiv \mathcal{H} = {} & 2(\varepsilon_{j_1}-\lambda_n)u_{j_1}^2+2\sum_{j_\tau}\left[\varepsilon_{j_\tau}-\lambda_\tau-\frac{G_\tau}{2}|v_{j_\tau}|^2\right]\Omega_\tau|v_{j_\tau}|^2\\ & -\frac{|\Delta_p|^2}{G_p}-\frac{|\Delta_n|^2}{G_n}+u_{j_1}(v_{j_1}^*\Delta_n+v_{j_1}\Delta_n^*)-G_n u_{j_1}^2(1+2|v_{j_1}|^2)+(\omega_0-4\lambda_\alpha)|Z_0|^2\\ & +(\omega_1-4\lambda_\alpha)|Z_1|^2+4\frac{X_0}{G_p}\left[-u_{j_1}(Z_0 v_{j_1}\Delta_p+Z_0^* v_{j_1}^*\Delta_p^*)+\frac{1}{G_n}(Z_0\Delta_n\Delta_p+Z_0^*\Delta_n^*\Delta_p^*)\right]\\ & +2X_1[|Z_0|^2+2|Z_1|^2]\left[u_{j_1}^2+\sum_{j_\tau}\Omega_\tau|v_{j_\tau}|^2\right]+\tilde{X}_2 u_{j_1}^2|Z_1|^2-2\lambda_2(u_{j_1}^2 C_{j_1m_1}+2|Z_1|^2)\ ,\end{aligned} \tag{A1}$$

where Δ_τ is the gap parameter

$$\Delta_\tau=\tfrac{1}{2}G_\tau\sum_{k_\tau}\Omega_{k_\tau}\sin4\rho_{k_\tau}e^{-i\varphi_{k_\tau}}, \tag{A2}$$

$C_{j_1m_1}$ stands for the following sum:

$$C_{j_1m_1}=\sum_{J=\text{even}}J(J+1)(C^{j_1j_1J}_{m_1-m_10})^2\ , \tag{A3}$$

and $\tilde{X}_2$ is defined by

$$\tilde{X}_2=-X_2\left[\frac{5}{\pi}\right]^{1/3}\frac{\hbar}{m\omega_1}\langle j_1\|r^2Y_2\|j_1\rangle C^{j_12j_1}_{m_10m_1}\ . \tag{A4}$$

Here m stands for the nucleon mass and X_2 is the quadrupole coupling constant.

APPENDIX B

We give the explicit form of the equations of motion in the coordinates (r_k,φ_k):

$$\begin{aligned}-\dot{\varphi}_0={} & \omega_0-4\lambda_\alpha+\frac{X_0}{2\sqrt{r_0}}\left[-\frac{2}{G_p}\sin4\rho_{j_1}(\Delta_p e^{i(\varphi_0-\varphi_{j_1})}+\Delta_p^* e^{-i(\varphi_0-\varphi_{j_1})})+\frac{4}{G_nG_p}(\Delta_n\Delta_p e^{i\varphi_0}+\Delta_p^*\Delta_n^* e^{-i\varphi_0})\right]\\ & +X_1\left[2\cos^2 2\rho_{j_1}+\sum_{j_\tau}2\Omega_{j_\tau}\sin^2 2\rho_{j_\tau}\right]\ ,\end{aligned}$$

$$-\dot{\varphi}_1=\omega_1-4\lambda_\alpha+X_1\left[2\cos^2 2\rho_{j_1}+\sum_{j_\tau}2\Omega_{j_\tau}\sin^2 2\rho_{j_\tau}\right]+\tilde{X}_2\left[1-\frac{2}{2j_1-1}r_{j_1}\right]-2\lambda_2\ ,$$

$$-\dot{\varphi}_k=2(\varepsilon_k-\lambda_\tau)+2X_1(r_0+r_1)-2G_{\tau_k}(1-\delta_{kj_1})\sin^2 2\rho_k+\delta_{kj_1}G_n\left[2\frac{\lambda_2}{G_n}\frac{C_{j_1m_1}}{\Omega_{j_1}-1}+\cos^2 2\rho_{j_1}-3\sin^2 2\rho_{j_1}\right]$$

$$-\frac{\cos 4\rho_k}{\sin 4\rho_k}\left[\Delta_{\tau_k}e^{i\varphi_k}+\Delta^*_{\tau_k}e^{-i\varphi_k}-G_n\sin 4\rho_{j_1}\cos(\varphi_{j_1}-\varphi_k)\delta_{\tau_k n}(1-\delta_{kj_1})\right.$$

$$\left.-4X_0\sqrt{r_0}\left[\frac{1}{G_{\tau'_k}}(\Delta_{\tau'_k}e^{i(\varphi_0-\varphi_k)}+\Delta^*_{\tau'_k}e^{-i(\varphi_0-\varphi_k)})-\delta_{\tau_k p}\sin 4\rho_{j_1}\cos(\varphi_0-\varphi_{j_1}-\varphi_k)\right]\right]_{\tau_k^p}$$

$$-\delta_{kj_1}\tilde{X}_2\frac{2}{2j_1-1}r_1,\quad k>1\ ,$$

$$\dot{r}_0=2iX_0\sqrt{r_0}\left[-\frac{1}{G_p}\sin 4\rho_{j_1}(\Delta_p e^{i(\varphi_0-\varphi_{j_1})}-\Delta_p^* e^{-i(\varphi_0-\varphi_{j_1})})+\frac{2}{G_nG_p}(\Delta_n\Delta_p e^{i\varphi_0}-\Delta_n^*\Delta_p^* e^{-i\varphi_0})\right],\quad \dot{r}_1=0\ ,$$

$$\dot{r}_k=(\Omega_k-\delta_{kj_1})\sin 4\rho_k\left[-\frac{i}{2}(\Delta_{\tau_k}e^{+i\varphi_k}-\Delta^*_{\tau_k}e^{-i\varphi_k})+\delta_{\tau_k n}\frac{G_n}{2}\sin 4\rho_{j_1}\sin(\varphi_{j_1}-\varphi_k)\right.$$

$$\left.-2iX_0\sqrt{r_0}\left[\frac{1}{G_{\tau'_k}}(\Delta_{\tau'_k}e^{i(\varphi_0-\varphi_k)}-\Delta^*_{\tau'_k}e^{-i(\varphi_0-\varphi_k)})-i\delta_{\tau_k p}\sin 4\rho_{j_1}\sin(\varphi_0-\varphi_{j_1}-\varphi_k)\right]\right],\quad k>1\ .$$

In the above equations we have introduced the new notation τ'_k which takes the value $\tau'_k=p$ when $\tau_k=n$ and vice versa.

APPENDIX C

Here we give the explicit expressions of the matrices $\mathcal{A}_{ki}$ and $\mathcal{B}_{ki}$ entering the linearized equations of motion (5.2):

$$\mathcal{A}_{00}=\frac{2X_0}{\mathring{r}_0^{3/2}}\left[\mathring{u}_{j_1}|\mathring{v}_{j_1}|-\frac{\mathring{\Delta}_n}{G_n}\right]\frac{\mathring{\Delta}_p}{G_p}\ ,\quad \mathcal{A}_{0k}=\mathcal{A}_{k0}=2X_1+\frac{2X_0}{\sqrt{\mathring{r}_0}}\frac{\mathring{u}_k^2-|\mathring{v}_k|^2}{\mathring{u}_k|\mathring{v}_k|}\left[\frac{\mathring{\Delta}_{\tau'_k}}{G_{\tau'_k}}-\delta_{\tau_k,p}\mathring{u}_{j_1}|\mathring{v}_{j_1}|\right],\quad k>1\ ,$$

$$\mathcal{A}_{1k}=\mathcal{A}_{k1}=2X_1,\quad k>1,\quad k\neq j_1;\quad \mathcal{A}_{1,j_1}=-\frac{2}{2j_1-1}\tilde{X}_2\ ,$$

$$\mathcal{A}_{kk_1}=\delta_{kk_1}[2(\Omega_k-\delta_{kj_1})u_k^3|v_k|^3]^{-1}\left[-4G_{\tau_k}\mathring{u}_k^3|\mathring{v}_k|^3+\mathring{\Delta}_{\tau_k}-4X_0\sqrt{\mathring{r}_0}\frac{\Delta_{\tau'_k}}{G_{\tau'_k}}-\mathring{u}_{j_1}|\mathring{v}_{j_1}|(\delta_{\tau_k n}G_n-4\delta_{\tau_k p}X_0\sqrt{\mathring{r}_0})\right]$$

$$+G_{\tau_{k_1}}\left[-\delta_{\tau_{k_1},\tau_k}+\delta_{\tau_{k_1},\tau'_k}\frac{4X_0\sqrt{\mathring{r}_0}}{G_{\tau'_k}}\right]\frac{u_k^2-|v_k|^2}{2u_k|v_k|}\frac{u_{k_1}^2-|v_{k_1}|^2}{u_{k_1}|v_{k_1}|}\ ,$$

$$\mathcal{B}_{00}=-8X_0\sqrt{\mathring{r}_0}\frac{\mathring{\Delta}_p}{G_p}\left[\frac{\mathring{\Delta}_n}{G_n}-\mathring{u}_{j_1}|\mathring{v}_{j_1}|\right],$$

$$\mathcal{B}_{0k}=\mathcal{B}_{k0}=8X_0\sqrt{\mathring{r}_0}\mathring{u}_k|\mathring{v}_k|(\Omega_k-\delta_{kj_1})\left[\frac{\mathring{\Delta}_{\tau'}}{G_{\tau'}}-\mathring{u}_{j_1}|\mathring{v}_{j_1}|\delta_{\tau_k p}\right],$$

$$\mathcal{B}_{kk_1}=\delta_{kk_1}\left[\mathring{\Delta}_{\tau_k}-4X_0\sqrt{\mathring{r}_0}\frac{\mathring{\Delta}_{\tau'_k}}{G_{\tau'_k}}-\mathring{u}_{j_1}|\mathring{v}_{j_1}|(G_n\delta_{\tau_k n}-4X_0\sqrt{\mathring{r}_0}\delta_{\tau_k p})\right]-\mathring{u}_{k_1}|\mathring{v}_{k_1}|(\Omega_{k_1}-\delta_{kj_1})(G_{\tau_k}+4X_0\sqrt{\mathring{r}_0}\delta_{\tau_k\tau_{k_1}}),$$

$$k,k_1>1\ .$$

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[1] G. Gamow, Z. Phys. **51**, 204 (1928).
[2] H. J. Mang, Phys. Rev. **119**, 1069 (1960).
[3] A. Arima and S. Yoshida, Nucl. Phys. **A219**, 475 (1974).

[4] T. Fliessbach and H. J. Mang, Nucl. Phys. **A263**, 75 (1976).
[5] K. Wildermuth and Th. Kanellopoulos, Nucl. Phys. **7**, 150 (1958); **9**, 449 (1958).
[6] D. M. Brink, in *Many-Body Description of the Nuclear Structure and Reactions,* edited by C. Bloch (Academic, New York, 1966), p. 247.
[7] D. M. Brink, H. Friedrich, A. Weiguny, and C. W. Wong, Phys. Lett. **33B**, 143 (1970).
[8] S. B. Khadkikar, Phys. Lett. **36B**, 451 (1971).
[9] W. Banhoff, H. Schultheis, and R. Schultheis, Phys. Rev. C **22**, 861 (1980).
[10] H. Faissner and K. Wildermuth, Nucl. Phys. **58**, 177 (1964).
[11] H. Faissner and K. Wildermuth, Phys. Lett. **2**, 212 (1962).
[12] G. Gönnenwein, H. Schultheis, R. Schultheis, and K. Wildermuth, Z. Phys. A **278**, 15 (1976).
[13] A. Sandulescu, D. N. Poenaru, and W. Greiner, Yad. Fiz. **11**, 1334 (1980) [Sov. J. Part. Nucl. **11**, 528 (1980)]; D. N. Poenaru, M. Ivascu, A. Sandulescu, and W. Greiner, Phys. Rev. C **32**, 5 (1985); A. Sandulescu, J. Phys. G **15**, 529 (1989).
[14] E. Roecki, Nucl. Phys. **A400**, 131c (1983).
[15] J. Jänecke, F. D. Bechetti, D. Overway, J. D. Cossairt, and R. L. Spross, Phys. Rev. C **23**, 101 (1981).
[16] M. Gai, J. F. Ennis, M. Ruscev, E. C. Schloemer, B. Shivakumar, S. M. Sterbenz, N. Tsoupas, and D. A. Bromley, Phys. Rev. Lett. **51**, 646 (1983).
[17] J. F. Shriner, Jr., P. D. Cottle, J. F. Ennis, M. Gai, D. A. Bromley, J. W. Olness, E. K. Warburton, L. Hildingsson, M. A. Quader, and D. B. Fossan, Phys. Rev. C **32**, 1888 (1985).
[18] W. Bonin, M. Dahlinger, S. Glienke, E. Kankeleit, M. Krämer, D. Habs, B. Schwartz, and H. Backe, Z. Phys. A **310**, 249 (1983).
[19] D. Ward, G. D. Dracoulis, J. R. Leigh, R. J. Charity, D. J. Hinde, and J. O. Newton, Nucl. Phys. **A406**, 591 (1983).
[20] F. Iachello and A. D. Jackson, Phys. Lett. **108B**, 151 (1982).
[21] M. Danos and V. Gillet, Phys. Rev. **161**, 1034 (1967).
[22] A. Arima and V. Gillet, Ann. Phys. (N.Y.) **66**, 117 (1971).
[23] M. Danos and V. Gillet, Z. Phys. **249**, 294 (1972).
[24] M. Cauvin, V. Gillet, F. Soulmagnon, and M. Danos, Nucl. Phys. **A361**, 192 (1981).
[25] J. Eichler and M. Yamamura, Nucl. Phys. **A182**, 33 (1972).
[26] G. G. Dussel, R. J. Liotta, and R. Perazzo, Nucl. Phys. **A388**, 606 (1982).
[27] G. G. Dussel, A. J. Fendrik, and C. Pomar, Phys. Rev. C **34**, 196 (1986).
[28] See, for instance, P. Ring and P. Schuck, *The Nuclear Many-Body Problem* (Springer-Verlag, New York, 1980).
[29] See, for instance, A. Arima and F. Iachello, Annu. Rev. Nucl. Part. Sci. **31**, 75 (1981).
[30] Y. K. Gambhir, P. Ring, and P. Schuck, Phys. Rev. Lett. **51**, 1235 (1983).
[31] P. Curutchet, J. Dukelsky, G. G. Dussel, and F. J. Fendrik, Phys. Rev. C **40**, 2361 (1989).
[32] P. Nozières and D. Saint James, J. Phys. (Paris) **43**, 1133 (1982).
[33] D. S. Delion, A. Insolia, and R. J. Liotta (unpublished).
[34] See Ref. [17].
[35] A. A. Raduta, V. Ceausescu, A. Gheorghe, and M. S. Papa, Nucl. Phys. **A427**, 1 (1984).
[36] H. H. Wapstra and K. Bos, At. Data Nucl. Data Tables **19**, 177 (1977).