Empirical effective interactions in the lower fp shell and upper sd shell

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Average values of matrix elements of the effective nucleon-nucleon interaction can be extracted from measured binding energies and spectra using a variety of simple techniques. Such techniques are applied to data in the mass range $A=28-64$ to yield information about the interaction energy of nucleons occupying the $2s_{1/2}$, $1d_{3/2}$, $1f_{7/2}$, and $2p_{3/2}$ shell-model orbits. The resulting interaction centroids are compared with values provided by other methods, such as shell-model fits to experimental energy levels, calculation of the G matrix from free nucleon-nucleon potentials, and extraction of interaction matrix elements directly from the spectra of nuclei with two nucleons relative to a closed shell. The empirical centroids are shown to be surprisingly accurate.

I. INTRODUCTION

The aim of the present work is to provide information about effective interactions, suitable for shell-model calculations around mass 40. The shell-model Hamiltonian is completely specified by a number of single-particle energies and a set of two-body matrix elements of the effective interaction (provided three-body and higher contributions are negligible). Several techniques have been developed to construct effective interactions, each with its advantages and disadvantages, but none has shown itself to be satisfactory in all respects. Since it has so far proven impossible to produce an effective Hamiltonian which is both sufficiently accurate for use in spectroscopic calculations and also firmly based theoretically, we follow here a purely empirical approach. For the most part we are concerned with extracting from experimental data values for centroids of the effective interaction; these act as constraints on matrix elements of the interaction found by other methods, as discussed below.

The interaction can, in principle, be calculated from the free nucleon-nucleon potential; this is the approach pioneered by Kuo [1] and Kuo and Brown [2], who produced realistic interactions based on the Hamada-Johnson potential. The basis of this method is the calculation of the Brueckner G matrix, with the addition of a limited number of model-space-dependent correction terms. However, serious doubts have been raised concerning the convergence of the perturbation expansion in the G matrix (see, for example, Ref. [3]). In any case, subsequent use of these interactions has shown that they suffer from serious defects, thus reducing their value for nuclear spectroscopy [4–9]. These defects are caused by incorrect centroids for the interaction between nucleons in different single-particle subshells; agreement between calculated spectra and data is improved significantly by *ad hoc* adjustments to these centroids [7–9]. A major aim of the present work is to allow such adjustments to be made reliably, by providing empirical estimates of centroids.

In an alternative approach to effective interactions, matrix elements of the effective interaction are treated as parameters to be adjusted in a fit to experimental energy levels. This approach is particularly successful in the sd shell [10,11], where it is possible to perform shell-model calculations in which all sd-shell basis states are retained. Wildenthal [11] has produced a single interaction which gives impressive agreement with experimental data for a variety of phenomena through the entire shell, provided a simple mass dependence of the interaction is assumed. One difficulty with this approach is that not all of the matrix elements of the interaction are equally well determined by the fitting procedure. In addition, there is no guarantee that an interaction which gives a reasonable fit to spectra will produce an equally reliable description of other phenomena, such as electromagnetic decay rates, which are more sensitive to details of the nuclear wave functions (although attempts have been made [12] to incorporate other observables into the fitting procedure). Moreover, it is difficult to extend this approach to heavier nuclei. For example, with present computational techniques it is impossible to perform untruncated fp shell calculations, except near the ends of the shell—fits must be done in a restricted mass range or in a highly truncated model space. In any case, there are 195 interaction matrix elements in the fp shell (compared with 63 for the sd shell); this alone would make the fitting procedure extremely difficult and time consuming.

A variation of this approach utilizes schematic interactions, such as the modified surface-delta interaction (MSDI) [13]. A particular form is assumed for the effective nucleon-nucleon force; this, together with further assumptions, allows two-body matrix elements of the interaction to be expressed analytically in terms of a small number of parameters. These parameters are determined by a fit to spectra, often for nuclei in a limited mass range, but in a large model space. Such an approach has been surprisingly successful in many instances.

A third approach is illustrated in a recent review by Daehnick [14]. Two-body matrix elements of the effective interaction are extracted directly from the spectra of nuclei with two nucleons outside a closed shell.

Measured nucleon spectroscopic factors are used to compensate, in part, for the lack of configurational purity of the nuclear wave functions. However, since the necessary experimental information is often not available, this procedure is not usually carried out in a consistent manner. The resulting uncertainty in the interaction matrix elements is estimated by Daehnick to be 0.3 MeV or more in most cases; moreover, significantly different values are obtained when the same matrix elements are extracted from two different nuclei.

In the present work, we exploit rather general relationships [15–17] which connect various centroids of the two-body interaction with the energy of an n-body system averaged over a fixed configuration of the appropriate nucleus or nuclei—values are obtained for the centroids if these average energies are computed from experimental binding energies and spectra. Thus reliable estimates may be extracted for interaction centroids even if the individual matrix elements cannot be deduced with the required accuracy. We have recently demonstrated [18–22] the usefulness of several variations of this technique in a variety of different situations. We present here an extension of this previous work to a much larger model space and a wider variety of situations; we also attempt to unify the results from the different versions of the technique; and finally we compare the interaction centroids produced by this technique with those from the other approaches discussed above.

II. EXTRACTION OF INTERACTION CENTROIDS

Our ultimate objective is a description of the properties of nuclei near the top of the *sd* shell and bottom of the *fp* shell. The full valence space necessarily spans two major shells, and the correct positioning of many states of nuclei in this mass region is particularly sensitive to the accuracy of cross-shell matrix elements. In order to limit the number of centroids which must be determined and to reduce the consequent computational difficulties, we have truncated the valence space by omitting the $1d_{5/2}$, $2p_{1/2}$, and $1f_{5/2}$ subshells. Such a truncation is in any case necessary for shell-model calculations.

The various techniques employed to extract interaction centroids require the utilization of appropriate subspaces of the full model space, often spanned by the individual subshells $2s_{1/2}$, $1d_{3/2}$, $1f_{7/2}$, and $2p_{3/2}$. In addition to ^{28}Si and ^{64}Ge, therefore, the ground states of ^{30}Si/S, ^{32}S, ^{36}S/Ca, ^{40}Ca, ^{48}Ca, ^{56}Ni, and ^{60}Ni are regarded as inert cores corresponding to closure of these subshells. Furthermore, ^{34}Si is also taken to be a good closed-shell nucleus, with excess neutrons filling both the $s_{1/2}$ and $d_{3/2}$ subshells. The possibility of ^{34}Si being a doubly magic nucleus was the subject of a recent experimental investigation [23]; its use as a core permits a significant extension to the range of values of (N,Z) that can be probed. The assumption of shell closure, as indicated by sum rules for single-nucleon spectroscopic factors [24], is apparently reasonable for ^{48}Ca and ^{36}S, questionable for ^{40}Ca, ^{32}S, and ^{60}Ni (proton core), and invalid for ^{30}Si, ^{28}Si, and, especially, ^{60}Ni (neutron core). The magnitude of the single-particle energy gap at the Fermi surface, which can be ascertained from tabulations of nuclear binding energies [25–27], suggests that the assumption of shell closure may also be valid for ^{56}Ni. The excitation energy of the first excited state of ^{64}Ge is small, indicating that this is not a good closed-shell nucleus.

Since our aim is to construct centroids of the *nuclear* interaction, the effects of the Coulomb force must first be eliminated. We have therefore employed the procedures of Refs. [28] and [29] to estimate the Coulomb contribution to the energies of nuclear states.

A. Definition of centroids

Diagonal matrix elements of the effective interaction for spin J and isospin T are denoted $V_{JT}(pq) = \langle pq|V|pq;JT\rangle$, where p and q label the single-particle orbits. From the set of matrix elements for fixed (p,q) we can construct the following average values. The simple average of all matrix elements $V_{JT}(pq)$ for fixed (p,q) is

$$\bar{V}(pq)=\frac{\sum_{JT}(2J+1)(2T+1)V_{JT}(pq)}{\sum_{JT}(2J+1)(2T+1)}\ , \tag{1}$$

with the sums extending over all values of J and T for which the matrix elements $V_{JT}(pq)$ exist. We can also form averages for fixed isospin:

$$\bar{V}^{T}(pq)=\sum_{J}(2J+1)V_{JT}(pq)\Big/\sum_{J}(2J+1)\ ; \tag{2}$$

if $p=q$ the sums include even (odd) values of J for $T=1(0)$. The average matrix element in the neutron-proton formalism is found by including in the sums over J matrix elements for both $T=0$ and $T=1$:

$$\bar{V}^{np}(pq)=\sum_{J}(2J+1)V_{J}(pq)\Big/\sum_{J}(2J+1)\ . \tag{3}$$

Any two of the centroids so far defined can be regarded as independent; the remainder are given by relationships which can be derived by explicitly evaluating the sums in these equations. We find

$$(4j+\delta_{pq})\bar{V}(pq)=(j+\delta_{pq})\bar{V}^{T=0}(pq)+3j\bar{V}^{T=1}(pq)\ , \tag{4a}$$

$$(2j+\delta_{pq})\bar{V}^{np}(pq)=(j+\delta_{pq})\bar{V}^{T=0}(pq)+j\bar{V}^{T=1}(pq)\ , \tag{4b}$$

where j is the angular momentum of a nucleon in subshell p. Note that these relationships are independent of j if $p\neq q$.

For a system of two nucleons occupying the same single-particle orbit, further average matrix elements may be defined. The seniority quantum number v possesses two possible values, namely, 0 and 2. We can therefore classify the two-body matrix elements with respect to both T and v, denoting the averages for $T=1$, $v=2$, and $T=0$, $v=2$ by

$$\bar{V}_{2}(pp)=\sum_{J}(2J+1)V_{J1}(pp)\Big/\sum_{J}(2J+1)\ , \tag{5a}$$

$$\bar{V}_{1}(pp)=\sum_{J}(2J+1)V_{J0}(pp)\Big/\sum_{J}(2J+1)\ , \tag{5b}$$

respectively. The sums in Eq. (5a) are restricted to even values of $J>0$ and those in Eq. (5b) to odd J. There is in addition a single two-particle state $|p^2;01\rangle$ with $T=1$, $v=0$ and the corresponding diagonal matrix element is denoted $V_0(pp)$. The additional interaction averages (5) are, of course, related to those already defined. For example, directly from Eqs. (2) and (5b) we find

$$\bar{V}^{T=0}(pp)=\bar{V}_1(pp)\ , \tag{6a}$$

so that the introduction of $\bar{V}_1$ provides no extra information. However, the separate specification of V_0 and $\bar{V}_2$ more completely describes the interaction for identical nucleons occupying the same subshell. The $T=1$ centroid is, in terms of these quantities, given by

$$j(2j+1)\bar{V}^{T=1}(pp)=V_0(pp)+(j+1)(2j-1)\bar{V}_2(pp), \tag{6b}$$

where j is the angular momentum of a nucleon in subshell p.

B. Centroids from single-particle spectra

Useful information about effective interactions can be found from single-particle spectra by comparing single-particle energies relative to different inert cores. Consider a single nucleon in the $f_{7/2}$ subshell; its energy relative to ^{32}S differs from its energy with respect to ^{40}Ca since the latter contains an additional contribution from the interaction of the $f_{7/2}$ nucleon with the fully occupied $d_{3/2}$ subshell. Comparison of the energies of the relevant $\frac{7}{2}^-$ states of ^{33}S and ^{41}Ca therefore yields information about the $f_{7/2}$-$d_{3/2}$ interaction. Details concerning the extraction of interaction centroids from single-particle spectra are given elsewhere [21,22]; here we simply summarize the necessary equations.

We represent two closed-shell nuclei by A_0 and A_0' and consider three different possibilities. In the first, the core A_0' comprises A_0 plus the subshell p completely filled with protons and neutrons; both cores have $Z=N$. Relative to A_0', the energy of a single neutron in any subshell q is

$$\epsilon_q^{\nu}(A_0')=\epsilon_q^{\nu}(A_0)+(2N_p-\delta_{pq})\bar{V}(pq)\ , \tag{7}$$

where $\epsilon_q^{\nu}(A_0)$ is the energy of the same neutron with respect to A_0, and $N_p=2j_p+1$. If q refers to a subshell below the Fermi energy of the core, then ϵ_q^{ν} represents a suitably defined single-hole energy. Equation (7) is equally valid for single-proton energies ϵ_q^{π} provided the Coulomb energy is treated correctly [22]. In the second possibility, the cores A_0 and A_0' differ through N_p *neutrons* occupying the subshell p, with $N\neq Z$ for one of the cores and $N=Z$ for the other. Relative to these two cores, a neutron in the subshell q has energies related by

$$\epsilon_q^{\nu}(A_0')=\epsilon_q^{\nu}(A_0)+(N_p-\delta_{pq})\bar{V}^{T=1}(pq)\ , \tag{8}$$

whereas a proton in subshell q has energies given by

$$\epsilon_q^{\pi}(A_0')=\epsilon_q^{\pi}(A_0)+N_p\bar{V}^{np}(pq)\ . \tag{9}$$

If, on the other hand, *protons* rather than neutrons fill subshell p, then the roles of protons and neutrons are reversed so that π and ν in Eqs. (8) and (9) must be interchanged.

The basis of the technique is that the energies ϵ_q^{ν} and ϵ_q^{π} should be deduced from experimentally determined binding energies [25–27, 30–40] so that the interaction centroids may be extracted using Eqs. (7)–(9). There are, however, a number of complications in the utilization of the technique; these are discussed fully in Refs. [21] and [22]. A complete application of the technique to nuclei with masses $A=28$–64 is also presented in Ref. [22], where further details can be found; the results are reproduced in Table I together with additional information. Extracted centroids are shown for each pair of cores (A_0, A_0') separately in order to reveal any dependence on model space or mass and also to facilitate comparison with interaction centroids from other calculations. Cer-

TABLE I. Centroids (in MeV) of the interaction $V(pq)$ from single-particle spectra.

pq	A_0, A_0'	$\bar{V}^{np}(pq)$[a]	$\bar{V}^{T=1}(pq)$[a]	$\bar{V}^{T=0}(pq)$[b]
ss	28,30	−2.28	−2.14	−2.33
	30,32	−2.23	−2.02	−2.30
	34,36		−1.36	
dd	30,34	−1.37	−0.32	−2.00
	32,36	−1.53	−0.42	−2.19
	36,40	−1.44	−0.30	−2.12
ff	40,48	−1.07	−0.23	−1.72
	48,56	−0.84	0.03	−1.51
pp	56,60	−1.02	−0.37	−1.41
	60,64	−0.91	−0.24(13)	−1.31(8)
sd	28,30	−1.47	0.31	−3.25
	30,32	−1.01	0.74	−2.76
	34,36	−1.18	0.44	−2.80
	30,34	−1.22		
	32,36	−1.01	0.51	−2.53
	36,40	−1.28	0.29	−2.85
df	30,34		0.26	
	32,36	−1.07	0.35	−2.49
	36,40	−1.02	0.40	−2.44
	40,48	−0.98	0.24	−2.20
	48,56	−0.84	0.39	−2.07
fp	40,48	−0.78	0.16	−1.72
	48,56	−0.64	0.30	−1.58
	56,60	−0.59	0.42	−1.60
sf	28,30	−0.95	0.70	−2.60
	30,32	−1.13	0.52	−2.78
	34,36	−0.94		
	40,48	−0.62	0.56	−1.80
	48,56	−0.77	0.40	−1.94
dp	32,36		0.44	
	36,40	−0.69		
	56,60	−0.49		
sp	28,30	−1.15	0.24	−2.54
	30,32	−1.18	0.21	−2.57
	56,60	−0.54		

[a]Extracted from data using Eqs. (8) and (9) with inert cores A_0 and A_0'.
[b]Calculated from $\bar{V}^{np}$ and $\bar{V}^{T=1}$ using Eqs. (4).

tain centroids are evidently sensitive to the choice of model space; see, for example, $\bar{V}^{T=1}(ff)$ and $\bar{V}^{T=1}(sd)$. In addition, some mass dependence of the interaction is also apparent; see particularly the lower part of Table I, where there is a wide spread in the values of (A_0, A_0').

C. Centroids using seniority formalism

Additional information about interaction centroids for $p=q$ may be extracted provided the nuclear wave functions can be classified according to the seniority quantum number. In the case of n identical nucleons in a subshell of angular momentum $j_p \leq \frac{7}{2}$, coupled to spin J and seniority v, the exact ground-state energy [15] is given by

$$E(n,v)=n\epsilon_p+\eta_p n(n-1)/2+[n/2]\gamma_p \ , \tag{10}$$

where $[N]$ represents the largest integer not exceeding N. The model parameters η_p and γ_p are related to the fixed-seniority centroids of Eqs. (5) through

$$V_0(pp)=\eta_p+\gamma_p \ , \tag{11a}$$

$$\bar{V}_2(pp)=\eta_p+\gamma_p/2(j_p+1) \ . \tag{11b}$$

Seniority is a valid quantum number for identical nucleons occupying any of the active subshells included in the present work; hence, by fitting measured ground-state binding energies, Eq. (10) may be used to extract the interaction centroids $\bar{V}_2(pp)$ and $V_0(pp)$ with p equal to $s_{1/2}$, $d_{3/2}$, $f_{7/2}$, or $p_{3/2}$. The $T=1$ centroids may then be calculated from Eq. (6b).

Owing to the wealth of data available and the large number of cores in the total space considered, the same interaction may be extracted in several different subspaces. These subspaces are labeled (A_0, A_0') where A_0 again represents the core in which the subshell p is unoccupied and A_0' the corresponding core with p filled with identical nucleons. Equation (10) has previously [15,20,41] been used to extract interaction centroids for $p=f_{7/2}$ with (A_0, A_0') equal to either (40,48) or (48,56), and also [20] for $p=d_{3/2}$ with (A_0, A_0') equal to either (32,36) or (36,40). A full discussion of the difficulties that may be encountered in the analysis is given in Ref. [20]. Interaction centroids extracted with Eq. (10) in the present work are displayed in Table II. Also given is the number of data used to produce each centroid; the results may be less accurate if N is small. Some centroids will also be less reliable due to the crudeness of the process used to estimate Coulomb energies. The model-space dependence of $\bar{V}^{T=1}(ff)$ is clearly illustrated in Table II; it is seen to be due to variations in all matrix elements for the $T=1$ interaction. Further data for extremely neutron-rich nuclei would increase the reliability of estimates of some centroids, permitting this model-space dependence to be more firmly established.

In the case of n nonidentical nucleons in a subshell of angular momentum $j_p \leq \frac{3}{2}$, coupled to spin J, seniority v, and in addition to isospin T, the exact ground-state [15] is given by

$$E(n,v,T)=n\epsilon_p+\alpha_p n(n-1)/2+[n/2]\gamma_p + \{T(T+1)-3n/4\}\beta_p \ , \tag{12}$$

where

$$\eta_p=\alpha_p+\beta_p/2 \tag{13}$$

and

$$\bar{V}_1(pp)=\alpha_p-3\beta_p/2-\gamma_p/2(j_p+1) \ , \tag{14}$$

with $\bar{V}_1(pp)$ defined by Eq. (5b). Seniority is a good quantum number for states constructed by placing both neutrons and protons simultaneously in the $d_{3/2}$ or $p_{3/2}$ subshells, but not the $f_{7/2}$ subshell.

Equation (12) has previously [15,20] has been used to

TABLE II. Centroids (in MeV) of the interaction $V(pq)$ from the seniority formalism, using Eq. (10) for identical nucleons in the subshell p; A_0 and A_0' are the cores with p empty and filled, respectively.

pq	A_0, A_0'	N[a]	$\bar{V}^{T=1}(pq)$	$V_0(pq)$	$\bar{V}_2(pq)$
ss	28,30	2	−2.14		
	30,34	2	−2.02		
	34,36	2	−1.36(5)		
dd	30,34	4	−0.30	−2.81	0.20
	32,36	4	−0.41	−2.83	0.07
	36,40	4	−0.28	−2.26	0.12
	44,48	2		−3.67	
ff	36,44	4	−0.39	−3.42	−0.28
	40,48	8	−0.24	−3.01	−0.13
	48,56	8	0.04	−2.69	0.14
	52,60	5	0.34	−2.73	0.48
pp	48,52	3	0.15	−1.20	0.41
	56,60	4	−0.36	−2.16	−0.01
	60,64	3		−2.00	

[a]Number of data used to extract centroids, sometimes less than the maximum possible due to missing data.

TABLE III. Centroids (in MeV) of the interaction $V(pq)$ from the seniority formalism using Eq. (12) for nonidentical nucleons in the subshell p; A_0 and A_0' are the cores with p empty and filled, respectively.

pq	A_0, A_0'	N[a]	$\bar{V}^{T=0}(pq)$	$V_0(pq)$	$\bar{V}_2(pq)$	$\bar{V}^{T=1}(pq)$
dd	32,40	11	−2.07	−2.67	0.12	−0.35
ff	40,56	34	−1.65	−3.69	0.04	−0.10
pp	56,64	6	−1.44	−2.16	0.00	−0.36

[a]Number of data used to extract centroids.

deduce interaction centroids for $p=d_{3/2}$. In our earlier work [20] we also attempted to apply Eq. (12) to the $f_{7/2}$ subshell, but ground states for which seniority is not automatically a good quantum number were not well described by the equation. Results obtained using Eq. (12) in the present work are shown in Table III. In the case $p=d_{3/2}$ the ground states of all odd-A and even-even nuclei have been included, whereas for the $p_{3/2}$ subshell nuclei with more than two valence protons were omitted due to a lack of information about Coulomb energies. The data set for the $f_{7/2}$ analysis includes ground states for which seniority may not be an exact quantum number—in addition to even N, even Z nuclei we also included all odd-A nuclei, even in cases for which the ground state did not have the expected spin-parity $\frac{7}{2}^-$ (the energy of the lowest $\frac{7}{2}^-$ state was used). Comparison of centroids from Table III with averages over appropriate subspaces in Table II shows that reasonably consistent results are obtained for $T=1$, even for the $f_{7/2}$ data.

D. Centroids from the weak-coupling model

Interaction centroids for $p\neq q$ may be determined through use of the weak-coupling model of Bansal and French [42,43]. Consider a particle-hole state of the nucleus (A,T), with n particles and m holes relative to the closed-shell nucleus A_0; the particles are coupled to spin J_n and isospin T_n and the holes to spin J_m and isospin T_m, and these are in turn coupled to total spin J and isospin T. Bansal and French assumed that the residual particle-hole interaction is sufficiently weak that the energy of the n particles is equal to the energy of the nuclear state (A_0+n,J_n,T_n) relative to the core A_0; this may be taken from experiment. Similarly, the energy of the m holes is given by the energy of the nuclear state (A_0-m,J_m,T_m). The particle-hole interaction itself is expanded in isoscalar and isovector parts:

$$V_{\rm ph}=-a+b\mathbf{T}_n\cdot\mathbf{T}_m\ , \tag{15}$$

where, in terms of the particle-particle centroids defined in Sec. II A, the model parameters a and b are most usefully given by

$$\bar{V}^{np}(pq)=a_{pq}-b_{pq}/4\ , \tag{16a}$$

$$\bar{V}(pq)=a_{pq}\ , \tag{16b}$$

where p and q now label the subshells in which the particles and holes reside.

The weak-coupling model has been employed in numerous nuclear structure studies (for example, see Refs. [18] and [44–48]), either to extract information about the effective interaction or to predict the approximate position of selected particle-hole states using a given interaction. The model can be used in a variety of ways; in the present work we follow the approach of Refs. [18] and [44]. Consider a nucleus with n nucleons occupying subshell p outside core (A_0,Z_0). The energy required to remove m *protons* from the filled subshell q to form the n-particle m-hole configuration of the nucleus (A,Z) is [18]

$$R_{m\pi}(A,Z)=R_{m\pi}(X)-m(A+m-A_0)\bar{V}^{np}(pq) -m(Z+m-Z_0)b_{pq}/2\ , \tag{17}$$

where X is the state obtained by removing m protons from subshell q of the core (A_0,Z_0). Similarly, the binding energy of the n-particle m-hole configuration of the nucleus (A,N) formed by binding n *neutrons* in subshell p to a nucleus with m holes in subshell q is [18]

$$B_{n\nu}(A,N)=B_{n\nu}(Y)+n(A_0+n-A)\bar{V}^{np}(pq) +n(N_0+n-N)b_{pq}/2\ , \tag{18}$$

where Y is the state formed by adding n neutrons in subshell p to the core (A_0,N_0). Many other equations can be derived for different types of particle-hole state—see Refs. [18] and [44] for several examples—but the two presented here are the most useful. In this work, we have determined from experimental data [25–27, 30–40] the energies $R_{m\pi}$ and $B_{n\nu}$ for a large number of n-particle, m-hole states in the mass range 29 to 63. Equations (17) and (18) have then been used to deduce the interaction centroids through least-squares fits for fixed m and fixed n, respectively. A similar approach has previously [18,44] been employed to extract the centroids of the $d_{3/2}$-$f_{7/2}$ interaction.

Equation (17) is most accurate when the n nucleons in subshell p are all neutrons; similarly, Eq. (18) is most accurate when all m holes in subshell q are proton holes [18]. In each case only $\bar{V}^{np}(pq)$ is determined, since $Z=Z_0-m$ and $N=N_0+n$, respectively. We have therefore extracted estimates of $\bar{V}^{np}(pq)$ using only data for which $Z=Z_0-m$ or $N=N_0+n$, averaging over all possible values of m and n. The results of this analysis are summarized in Table IV. The uncertainties in $\bar{V}^{np}(pq)$ should be small; values for different m or n vary by less

TABLE IV. Centroids (in MeV) of the interaction $V(pq)$ from the weak-coupling model.

pq	Data[a]	N[b]	$\bar{V}^{np}(pq)$	$\bar{V}^{T=1}(pq)$	$\bar{V}^{T=0}(pq)$
sd	part(n)	11	−1.08		
	part(1)	4	−1.03	0.26	−2.31
	hole(m)	10	−1.11		
	hole(1)	8	−1.08	0.21(6)	−2.36(6)
df	part(n)	32	−0.95		
	part(1)	10	−1.03	0.35	−2.40
	hole(m)	27	−0.93		
	hole(1)	18	−0.97	0.36(5)	−2.31
fp	part(n)	24	−0.68		
	part(1)	12	−0.69	0.16	−1.54
	hole(m)	28	−0.63		
	hole(1)	3	−0.59		
sf	part(1)	4	−1.13	0.70	−2.96
	hole(1)	14[c]	−0.80	0.45	−2.05
dp	part(1)	5	−0.69	0.42	−1.78
	hole(1)	3	−0.49		
sp	part(1)	5	−1.17	0.22	−2.57
	hole(1)	3	−0.54		

[a]Centroids extracted using Eq. (18) for particle binding energies for $n=1$ or for all n [part(1) and part(n), respectively], or using Eq. (17) for hole creation energies for $m=1$ or all m [hole(1) and hole(m), respectively].
[b]Number of data used to extract the centroids.
[c]Data for ^{39}K omitted (incompatible with remaining data).

than about 0.1 MeV in general, although there is some evidence of systematic behavior. The parameter b_{pq} is determined, less accurately, by including data for all values of Z or N (although reliable data are available only for $m=1$ or $n=1$). The remaining centroids listed in Table IV were computed by combining the values of $\bar{V}^{np}(pq)$ and b_{pq} extracted from these data. These centroids are significantly less reliable, partly because b_{pq} is less well determined by the fits and partly because some values of b_{pq} are sensitive to the procedure used to calculate Coulomb energies. Recent data on extremely neutron-rich nuclei, such as ^{34,35}Si, ^{37,38}S, ^{35,37}P, and ^{46,47}Ar, can be used with equations presented here to produce further estimates for some interaction centroids. The results are consistent with those presented in Table IV.

E. Summary

Although interaction centroids for $p=q$ may be found using either of the methods described in Secs. II B and II C, the approaches differ in several respects. First, the seniority method allows separate determination of $V_0(pp)$ and $\bar{V}_1(pp)$, although if N is small only $V_0(pp)$ may be accurately specified. Secondly, the seniority method has the advantage that more data are used, so that extracted centroids are not as sensitive to inaccuracies in individual data values. If corresponding $T=1$ centroids in Tables I and II are compared, it is seen that very similar results are obtained from the two methods. The data used to produce the $T=1$ centroids listed in Table I form a subset of the data used to construct Table II. If $N=2$ the two data sets are identical, so this similarity is not unexpected; however, the agreement for larger N is encouraging. $T=0$ centroids from the seniority method, listed in Table III, may also be compared with values from single-particle energies by averaging the centroids of Table I over appropriate subspaces; the agreement is excellent.

Centroids for $p\neq q$ may be determined using the methods of Secs. II B and II D; values listed in Table IV can be compared to those in Table I. The approach based on the weak-coupling model, in which larger model spaces are implicitly assumed, washes out the sensitivity to model space previously noted, so that all estimates of the centroids for $(pq)=(sd)$, (df), and (fp) are essentially equal. The mass dependence of the centroids is, however, still apparent. Values given in Tables I and IV for the interactions (sp) and (dp) are similar because of the large overlap in the data sets used. Provided the correct comparison is made, good agreement is also found for the remaining centroids. Differences are usually less than 100 keV, although the weak-coupling estimate of $\bar{V}^{T=0}(sf)$ is about 230 keV more attractive, whereas the values of $\bar{V}^{T=0}(sd)$ and $\bar{V}^{T=1}(sd)$ are approximately 0.5 MeV less attractive and 0.2 MeV less repulsive, respectively.

The information extracted for interaction centroids using the empirical techniques of this section is summarized in Table V; uncertainties are quoted where these are thought to exceed about 0.2 MeV. In computing best values for a given centroid, we have assigned increased weighting to estimates considered more reliable. In addition, centroids extracted from data outside the quoted

TABLE V. Summary for empirical interaction centroids (in MeV).

pq	A, A' [a]	$\bar{V}^{np}(pq)$	$\bar{V}^{T=0}(pq)$	$\bar{V}^{T=1}(pq)$	$V_0(pq)$	$\bar{V}_2(pq)$
ss	28,34		−2.30	−2.05		
dd	30,40		−2.10	−0.35	−2.6(3)[b]	0.1
ff[c]	40,56		−1.65	−0.10	−3.0(3)	0.0
pp[d]	56,64		−1.40	−0.35	−2.1(3)	0.0
sd[e]	30,40	−1.15	−2.55(30)	0.30		
df	32,56	−1.00	−2.35	0.35		
fp	40,60	−0.65	−1.60	0.30		
sf	28,36	−1.05	−2.75	0.65		
dp	36,40	−0.70[f]	−1.80	0.40		
sp	28,32	−1.15[f]	−2.55	0.25		

[a]Centroids extracted using data in the mass range A to A'.
[b]See Table II for conflicting value for $(A, A')=(44,48)$.
[c]See Table II for conflicting values for $(A, A')=(36,44)$ and (52,60).
[d]See Table II for conflicting values for $(A, A')=(48,52)$.
[e]See Table I for conflicting values for $(A, A')=(28,30)$.
[f]See Table I for conflicting values for $(A, A')=(56,60)$.

mass regions have been excluded from the averages; some conflicting values are listed in footnotes to the table.

III. CENTROIDS OF OTHER INTERACTIONS

In this section we compute centroids of interactions produced using the various other techniques mentioned in the Introduction.

A. Effective interactions from two-particle spectra

Two-body matrix elements of the effective interaction can be extracted directly from the spectra of nuclei with two particles, two holes, or a particle-hole pair relative to an inert core, using

$$V^{JT}(pq)=[E(pq;JT)-E_0]-[E(p;j_p)-E_0] -[E(q;j_q)-E_0]\ . \quad (19)$$

E_0 is the energy of the core, and $E(p;j_p)$ and $E(q;j_q)$ give the single-particle contributions to the energy $E(pq;JT)$ of the two-particle nucleus. A two-particle Coulomb energy must be subtracted from $V^{JT}(pq)$ if both particles are protons. In the case of particle-hole states, Eq. (19) yields particle-hole matrix elements of the effective interaction which must be converted to particle-particle form through use of the appropriate Pandya transform. The configuration $(pq;JT)$ will of course be fragmented over several energy levels of the two-particle nucleus, through mixing with other configurations within the model space. The center of gravity $E(pq;JT)$ of the configuration must be calculated by using spectroscopic factors for one-nucleon transfer reactions with the appropriate single-particle nucleus as target. This restricts the use of the technique to situations in which the relevant single-particle nuclei are stable.

The use of Eq. (19) is less straightforward if the assumption of shell closure is not valid. It is usually assumed that the procedure for calculating the center of gravity $E(pq;JT)$ automatically corrects for fragmentation of the two-particle configuration caused by mixing with core-excited configurations of the two-particle nucleus. In addition, the energies $E(p,j_p)$ and $E(q,j_q)$ are usually replaced by the centers of gravity of the corresponding single-particle strength, as deduced from measured spectroscopic factors with the core as target. But, in fact, it is impossible to correct for core breaking in a consistent way. For example, in calculating $E(p;j_p)$ one should also include strength reached by coupling a nucleon to excited states of the core nucleus; similarly, $E(pq;JT)$ should also include strength due to coupling a nucleon to excited states of the single-particle nucleus. Moreover, the energy E_0 of the closed-shell configuration is no longer adequately represented by the ground-state energy of the core.

Daehnick [14], in a systematic study of effective interactions throughout the entire Periodic Table, used Eq. (19) to compute matrix elements of the effective interaction for many different model spaces. Only a few of his results are relevant to the present work, namely, those for $(d_{3/2})^2$ and $(d_{3/2}f_{7/2})$ multiplets relative to the core ^{32}S, and $(d_{3/2})^{-2}$, $(d_{3/2})^{-1}f_{7/2}$, and $(f_{7/2})^2$ multiplets relative to the core ^{40}Ca. Daehnick sometimes included corrections to the single-particle energies to account for core breaking, but no further corrections were applied. In order to provide a fuller comparison with the present work, we have applied the same technique to a much wider selection of two-particle spectra in the mass region of interest here. We have included all cases for which the relevant single-particle nucleus is stable and where sufficient spectroscopic information concerning the two-particle multiplets exists to make the results meaningful. The details of this study are presented elsewhere [24]; the resulting interaction centroids are listed in Table VI. Unlike Daehnick [14], we have not attempted to correct for the effects of core breaking on the single-particle energies; results from Refs. [14] and [24] therefore differ, although the difference is usually insignificant.

TABLE VI. Centroids (in MeV) of the interaction $V(pq)$ from two-particle spectra, calculated from Eq. (19); A denotes the nucleus with two particles (or two holes or a particle-hole pair) relative to the core A_0.

pq	A_0, A	$V_0(pq)$	$\bar{V}_2(pq)$	$\bar{V}^{T=1}(pq)$	$\bar{V}^{T=0}(pq)$	$\bar{V}^{np}(pq)$
ss	28,30			−0.78	−2.03	
	30,28			−1.17		
	30,30				−2.54	
	30,32			−1.28		
	32,30			−0.69	−1.87	
dd	32,34[a]	−2.67	0.08	−0.38	−2.22	
	32,34	−2.78	0.11	−0.37	−2.24	
	36,36	−2.13	−0.53	−0.80	−2.37	
	36,38	−2.11	0.14	−0.23		
	40,38	−2.36	0.18	−0.25	−2.14	
	40,38[a]	−2.36	0.22	−0.21	−2.00	
ff	40,42[a]	−2.32	−0.19	−0.26	−1.78	
	40,42	−2.61	−0.22	−0.30	−1.68	
	60,58	−2.51	0.60	−0.48		
sd	28,30			0.29	−2.26	−0.98
	30,30			0.33	−3.99	−1.83
	30,32			0.70	−2.44	−0.87
	32,32			0.77	−3.14	−1.18
	36,36			0.46		
	40,38			0.07	−1.76	−0.84
df	32,34[a]			−0.11	−2.45	−1.28
	32,34			0.39	−1.99	−0.80
	36,38					−1.00
	40,40			0.47	−2.46	−1.15
	40,40[a]			0.21	−2.62	−1.10
fp	40,42			0.15		
sf	28,30			0.61	−1.95	−0.67
	30,30			0.78	−3.38	−1.30
	30,32			0.58	−2.72	−1.07
	32,32			0.57	−2.93	−1.18
	40,40					−0.56
dp	40,40					−0.15

[a]From Ref. [14]; remainder from Ref. [24].

As can be seen from Table VI, the differences between interaction centroids for different cores are surprisingly small in general, indicating that the effects of core breaking can usually be ignored. In fact, correcting for fragmentation of the single-particle energies, using measured spectroscopic factors as described above, worsens the agreement considerably. In most cases, calculations [24] show that the shift induced in the centroids by this correction is actually canceled by a corresponding shift of the opposite sign in the energy of the closed-shell configuration E_0 (as computed using appropriate data for transfer reactions to 0^+ states of the core nucleus). Evidently, the contributions to these shifts due to strength based on excited states also cancel approximately; this cannot, of course, be checked using spectroscopic data.

B. Interactions from shell-model fits

The most successful effective interactions are generally those which result from shell-model fits to experimental energy levels. A recent example is the mass-dependent interaction of Wildenthal [11], who performed an essentially unconstrained fit to data from the entire sd shell, including all allowed sd-shell configurations in the calculations. The centroids of this interaction, scaled to $A=40$, and several other fitted interactions [7, 49-56] are listed in Table VII. The results are evidently sensitive to the model space used and to the data fitted.

As already noted in the Introduction, an unconstrained fit is not possible for the fp-shell interaction. The most extensive investigation of this interaction is that of Richter, van der Merwe, Julies, and Brown [55], who carried out a constrained fit to nuclei with $A=41$ to 49, including all possible fp-shell configurations. The centroids quoted in Table VII are computed from their interaction FPMI3; results for the interaction FPD6 are similar, except that the $T=1$ centroids for $p_{3/2}$-$p_{3/2}$ matrix elements are about 0.5 MeV more attractive. McGrory [53] studied the properties of nuclei with $A=42-44$ in the complete fp shell, using the realistic interaction of Kuo and Brown [2]. However, certain matrix elements of the interaction were allowed to vary in

TABLE VII. Centroids (in MeV) from shell-model fits to experimental spectra.

pq	Ref.	$\bar{V}^{np}(pq)$	$\bar{V}^{T=0}(pq)$	$\bar{V}^{T=1}(pq)$	$V_0(pq)$	$\bar{V}_2(pq)$
ss	11		−2.57	−1.67		
dd	11		−1.92	−0.33	−1.72	−0.05
	49			−0.25	−2.10	0.12
	50		−2.39	−0.07	−1.71	0.26
ff	49			−0.17	−2.55	−0.08
	7			−0.20	−2.11	−0.13
	51			−0.18	−2.64	−0.09
	52			−0.25	−2.95	−0.15
	53		−1.67	−0.27	−2.22	−0.20
	54		−1.29			−0.26
	55		−1.38	−0.19	−2.19	−0.12
pp	49			−0.71	−1.57	−0.54
	51			−0.23	−1.40	0.0
	55		−1.52	−0.52	−1.26	−0.38
sd	11	−1.09	−2.16	−0.02		
df	51	−1.01	−2.47	0.45		
	56	−1.17	−2.36	0.02		
	49	−0.99				
	52	−0.97				
fp	49			0.09		
	52			0.15		
	7			0.21		
	51			0.07		
	54	−0.71	−1.52	0.10		
	55	−0.46	−1.08	0.16		
sf	52	−0.74				
dp	56	−0.76	−1.60	0.08		
	49	−0.64				
	52	−0.98				

order to produce better agreement with data; the resulting centroids are listed in the table. The fitted interaction of van Hees and Glaudemans [54] is from a study of $(f_{7/2})^n$ states of $A=52-55$ nuclei with at most one particle allowed in the remaining fp-shell orbitals. There have been many shell-model studies of the calcium isotopes. Federman and Talmi [51] performed fits in a model space comprising the $f_{7/2}$ and $p_{3/2}$ subshells with all the interaction matrix elements allowed to vary. McGrory, Wildenthal, and Halbert [7] enlarged the model space to include all fp-shell orbitals plus the $g_{9/2}$ subshell; most matrix elements were held at the Kuo-Brown [2] values, but some where allowed to vary to improve the fit of selected energy levels; these are quoted in Table VII.

Warburton and co-workers are currently studying various properties of nuclei around mass 40 in shell-model calculations in a space that spans both the sd and fp shell, with restrictions on subshell occupancies. Warburton, Becker, Millener, and Brown have constructed a hybrid interaction [56] in which critical cross-shell matrix elements are fitted to appropriate energy levels in mass-40 nuclei. Centroids computed from these matrix elements are given in Table VII. The remaining centroids in the table are from shell-model calculations in which a few particles are excited from the sd shell to the fp shell, with all matrix elements of the interaction treated as free parameters. Gloeckner, Lawson, and Serduke [49] concentrated on the properties of $^{39-42}$Ar in shell-model calculations with neutrons in the $f_{7/2}$ and $p_{3/2}$ subshells and two proton holes restricted to the $d_{3/2}$ orbital. The model space of Johnstone [52] was spanned by the $f_{7/2}$ subshell with either a hole in the $d_{3/2}$ or $s_{1/2}$ orbital or one particle in the $p_{3/2}$ orbital and one hole in the $d_{3/2}$ orbital; the fp part of the interaction is the result of a fit to calcium isotopes whereas the cross-shell matrix elements are from a fit to $^{39-47}$K. Erné [50] carried out a shell-model study of nuclei near the top of the sd shell, in a space containing only the $d_{3/2}$ subshell with at most one particle excited to the $f_{7/2}$ orbital.

C. The modified surface-delta interaction

The original surface-delta interaction required the specification of two parameters, A_0 and A_1, essentially the strengths of the $T=0$ and $T=1$ interactions, respectively. Values for these parameters can be determined from fits to the spectra of nuclei. In order to improve the accuracy of predicted ground-state binding energies, two further parameters, B_0 and B_1, are needed; these are added to the expressions for diagonal $T=0$ and $T=1$ interaction matrix elements, respectively [57]. However, since the MSDI is frequently fitted to data for a rather

limited mass range, these additional parameters cannot be reliably determined (they are at least partially absorbed by the single-particle energies).

Using properties of angular momentum recoupling coefficients, formulas for interaction centroids can be derived from the usual expressions [57] for two-body matrix elements of the MSDI. For $p \neq q$ we find

$$\bar{V}^{T=0}(pq) = B_0 - \tfrac{3}{2}A_0 \ , \tag{20a}$$

$$\bar{V}^{T=1}(pq) = B_1 - \tfrac{1}{2}A_1 \ , \tag{20b}$$

independent of p and q. Similarly, for $p = q$

$$\bar{V}^{T=0}(pp) = B_0 - \tfrac{3}{2}A_0(2j+1)/2(j+1) \ , \tag{21a}$$

$$\bar{V}^{T=1}(pp) = B_1 - \tfrac{1}{2}A_1(2j+1)/2j \ , \tag{21b}$$

where j is the angular momentum of the nucleon in subshell p. The seniority averages are

$$V_0(pp) = B_1 - \tfrac{1}{2}A_1(2j+1) \ , \tag{22a}$$

$$\bar{V}_2(pp) = B_1 - \tfrac{1}{2}A_1(2j+1)/2(j+1) \ . \tag{22b}$$

The latter relationship is meaningful only for $j > \frac{1}{2}$.

There have been numerous shell-model studies of nuclei in the mass region of interest in which the modified surface-delta interaction has been fitted to energy levels. Centroids from some of these studies are listed in Table VIII. In the first group of calculations [13,58], active particles were restricted to *sd*-shell configurations. In the second set of calculations [59,60] the properties of nuclei near mass 40 were calculated, assuming a ^{28}Si core and allowing partial occupation of the $f_{7/2}$ and $p_{3/2}$ orbitals. Similarly, in the third study [61, 62], the properties of negative-parity states of nuclei with $A = 35$ to 39 were calculated assuming a ^{32}S core and allowing a single excitation to the $f_{7/2}$ and $p_{3/2}$ orbitals. The fit was performed separately for each mass, producing interactions which differed greatly; the centroids quoted in Table VIII are average values, with consequent uncertainties of about 0.8 MeV in all $T=0$ centroids and $V_0(dd)$ and of about 1.8 MeV in $V_0(ff)$. Finally, the fourth set of calculations [63], for isotopes of nickel and copper, assumed a ^{48}Ca core with the $p_{3/2}$, $f_{5/2}$, and $p_{1/2}$ orbitals active.

Because of the mass dependence of the interaction, centroids from the final set of calculations differ significantly from the others. In any case, there are large differences in the results from the remaining calculations, particularly for the matrix elements V_0. These are two further points of interest. First, MSDI centroids for $p \neq q$ are independent of p and q—see Eqs. (20); the approximate validity of this is evident from Tables V–VII. Second, MSDI estimates of the matrix elements V_0 are in general considerably smaller than estimates presented in Tables V–VII; this is also true of $\bar{V}^{T=1}(ss) = V_0(ss)$. Note that values of V_0 from realistic interactions are similarly small—see Secs. III D and IV.

Closed-form expressions have been derived for centroids of the modified surface-delta interaction; see Eqs. (20)–(22). It is therefore possible, at least in principle, to produce an MSDI fit to the empirical centroids summarized in Table V. In this way we could construct *all* matrix elements of an interaction with reliable centroids; in particular, we would obtain empirical estimates for the otherwise elusive off-diagonal matrix elements. But in practice there are many difficulties. With only four parameters, the MSDI simply does not have sufficient freedom to describe accurately the empirical centroids, especially those with $T=1$. The results of an attempted fit

TABLE VIII. Centroids (in MeV) of the modified surface-delta interaction.

pq	Ref.	$\bar{V}^{T=0}(pq)$	$\bar{V}^{T=1}(pq)$	$V_0(pq)$	$\bar{V}_2(pq)$
ss	13,58	−2.20	−0.50		
	59,60	−2.11	−0.13		
	61,62	−2.5	−0.3		
dd	13,58	−2.28	−0.14	−1.58	0.15
	59,60	−2.23	0.10	−0.8	0.28
	61,62	−2.6	0.09	−1.4	0.39
ff	59,60	−2.32	0.16	−2.2	0.25
	61,62	−2.6	0.19	−3.7	0.34
	63	−1.43	0.15	−1.67	0.21
pp	63	−1.40	0.10	−0.61	0.24
$p \neq q$	13,58	−2.40	0.04		
	59,60	−2.42	0.21		
	61,62	−2.7	0.28		
	63	−1.48	0.18		
ss	a	−2.18	−1.06		
dd	a	−2.30	−0.44	−2.92	0.1
ff	b	−1.59	−0.14	−3.28	0.0
pp	b	−1.47	−0.23	−1.45	0.0
$p \neq q$	a	−2.47	−0.13		
	b	−1.73	−0.08		

[a]MSDI fit to empirical centroids for $A \leq 40$ (see text).
[b]MSDI fit to empirical centroids for $A \geq 40$ (see text).

are shown in the lower part of Table VIII; mass dependence of the interaction has been included by fitting two sets of centroids separately. Comparison with values in Table V confirms that, although the fit to $T=0$ centroids is adequate, it is impossible to fit the matrix elements $V_0(ss)$ and $V_0(pp)$. In addition, it is impossible to fit simultaneously $\bar{V}_2$ for $p=q$ and $\bar{V}^{T=1}$ for $p\neq q$.

D. Realistic interactions

Since the early work of Kuo [1] and of Kuo and Brown [2], there have been many attempts to construct effective interactions based on free nucleon-nucleon potentials; see Refs. [64–69] for a selection of recent examples. The immediate aim of most of these attempts was not to produce an interaction suitable for shell-model calculations, but rather to investigate some particular aspect of the method, such as the use of potentials generally regarded as more accurate, inclusion of further correction terms, and improved computational techniques for the G matrix and core-polarization terms. Centroids from several realistic interactions are presented in Table IX, which also lists the model space and nucleon-nucleon potential used in each case.

IV. DISCUSSION AND CONCLUSION

In this section we compare the empirical centroids discussed in Sec. II with the centroids of interactions discussed in Sec. III. Of course, *all* interactions should ultimately be compared with the results of unconstrained shell-model fits to a wide range of data in a configuration space which is as large as possible, since such calculations produce the best description of the experimental data. In making comparisons between different interactions it is important to take proper account of differing model spaces. It should be noted that the summary Table V omits empirical centroids determined in more exotic model spaces; these can be found in the separate Tables I–IV.

The computational techniques employed in constructing realistic effective interactions have undoubtedly improved substantially since the original work of Kuo [1] and Kuo and Brown [2] and, in addition, a variety of reliable bare nucleon-nucleon potentials is now available. However, it is evident from a comparison of centroids in Table IX with empirical estimates in Tables V–VII that none of the realistic interactions is sufficiently accurate for use in large-scale shell-model calculations. One particular difficulty is that the magnitude of the pairing matrix element V_0 is consistently underestimated, by more than 1 MeV in some instances. Moreover, several other centroids differ from empirical estimates by as much as 500 keV; errors of this magnitude lead to shell-model spectra in considerable disagreement with observation and nuclear wave functions which are essentially useless for the calculation of other nuclear properties.

The modified surface-delta interaction, with only four adjustable parameters, lacks the flexibility needed to con-

TABLE IX. Centroids (in MeV) of realistic interactions.

pq	Ref.	$\bar{V}^{np}(pq)$	$\bar{V}^{T=0}(pq)$	$\bar{V}^{T=1}(pq)$	$V_0(pq)$	$\bar{V}_2(pq)$
ss	a		−2.20	−0.97		
	b		−2.53	−1.32		
	c		−3.18	−1.95		
	d		−2.96	−1.54		
dd	a		−1.36	0.23	−0.38	0.36
	b		−1.72	0.25	−0.70	0.44
	c		−1.95	−0.07	−0.81	0.08
	d		−2.06	−0.44	−1.19	−0.29
ff	e			−0.28	−2.26	−0.21
	f		−1.15	−0.13	−1.81	−0.07
pp	e			−0.82	−1.65	−0.65
	f		−1.47	−0.52	−1.21	−0.38
sd	a	−0.81	−1.85	0.24		
	b	−0.76	−1.83	0.30		
	c	−1.18	−2.24	−0.12		
	d	−1.28	−2.30	−0.26		
df	g	−0.79	−1.54	−0.04		
fp	e			−0.29		
	f	−0.54	−0.97	−0.10		
sf	g	−0.53	−1.20	0.14		

[a]Sommermann, Muther, Tam, Kuo, and Faessler [64], Reid potential, *sd* space.
[b]Sommermann *et al.* [64], Bonn potential, *sd* space.
[c]Kuo [1], Hamada-Johnston potential, *sd* space.
[d]Skouras and Varvitsiotis [65], Sussex potential, *sd* space.
[e]Waroquier, Heyde, Van Isacker, and Vincx [66], extended Skyrme potential, $fp+g_{9/2}$ space.
[f]Kuo and Brown [2], Hamada-Johnston potential, *fp* space.
[g]Kuo and Brown [2], Hamada-Johnston potential, *fp* space with one hole in the *sd* shell.

sistently provide a good description of experimental data (although it has certainly been more than adequate in calculations for very light nuclei). As with realistic interactions, the prediction of the pairing interaction is a major difficulty. Nonetheless, the MSDI is useful in providing estimates for off-diagonal matrix elements, which are often difficult to determine using other empirical methods.

The technique of Sec. III A in which interaction matrix elements are extracted directly from two-nucleon spectra, is the closest in spirit to the methods of Sec. II. The centroids listed in Table VII, averaged over different cores, can be compared with the results of Sec. II, summarized in Table V. The largest discrepancy is for the centroid $\bar{V}^{T=1}(ss)$; the agreement is otherwise surprisingly good. We note that the methods of Sec. II are significantly easier to implement, although the technique discussed in Sec. III A has the considerable advantage of yielding values for individual matrix elements.

Finally, the agreement between the centroids of the summary Table V and results for fitted interactions in Table VIII is equally impressive in most cases. There are two exceptions. First, estimates of V_0 from the present work are too attractive by about 0.5 MeV, and second, empirical estimates of $\bar{V}^{T=1}$ for $p \neq q$ are consistently too repulsive by a small amount.

In summary, interaction centroids from the various empirical techniques presented in Sec. II are surprisingly accurate. A major problem is encountered in the accurate extraction of the single matrix element $V_0(pp)$ for all p. In fact, the large spread in values for V_0 found in all the tables indicates that this problem is apparently common to almost all methods of constructing effective interactions. The preferred method for extracting centroids for $p = q$ is based on the seniority formalism described in Sec. II C, whereas centroids for $p \neq q$ are most accurately determined with the weak-coupling model of Sec. II D. Nonetheless, the method discussed in Sec. II B, using measured single-particle energies, is easier to implement and produces reasonable estimates of centroids for both $p = q$ and $p \neq q$. Empirical methods are particularly useful in providing input to large-scale fits to experimental spectra. This is especially important as the size of the model space and database both increase; the number of iterations required for convergence to the final interaction can be decreased considerably through careful choice of starting values, thereby rendering such calculations more attractive.

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