Pion-proton bremsstrahlung calculation and the "experimental" magnetic moment of $\Delta^{++}(1232)$

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(Received 21 May 1991)

A bremsstrahlung amplitude in the special two-energy-two-angle (TETAS) approximation, which is relativistic, gauge invariant, and consistent with the soft-photon theorem, is derived for the pion-proton bremsstrahlung ($\pi^+p\gamma$) process near the $\Delta^{++}(1232)$ resonance. In order to take into account bremsstrahlung emission from an internal Δ^{++} line with both charge and the anomalous magnetic moment λ_Δ, we have applied a radiation decomposition identity to modify Low's standard prescription for constructing a soft-photon amplitude. This modified procedure is very general; it can be used to derive the TETAS amplitude for any bremsstrahlung process with resonance. The derived TETAS amplitude is applied to calculate all $\pi^+p\gamma$ cross sections which can be compared with the experimental data. Treating λ_Δ as a free parameter in these calculations, we extract the "experimental" magnetic moment of the Δ^{++}, μ_Δ, from recent data. The extracted values of μ_Δ are $(3.7-4.2)e/(2m_p)$ from the University of California, Los Angeles data and $(4.6-4.9)e/(2m_p)$ from the Paul Scherrer Institute data. Here, m_p is the proton mass. These values are smaller than the value $5.58e/(2m_p)$, the "bare" magnetic moment predicted by the SU(6) model or the quark model, but they are close to the value $4.25e/(2m_p)$ predicted by the modified SU(6) model of Beg and Pais and to the value $(4.41-4.89)e/(2m_p)$ predicted by the corrected bag-model of Brown, Rho, and Vento. Using the extracted μ_Δ as an input for calculating $\pi^+p\gamma$ cross sections, we show that the overall agreement between the theoretical predictions calculated with the extracted μ_Δ and the experimental measurements is excellent. This agreement demonstrates that the TETAS amplitude can be used to describe almost all the available $\pi^+p\gamma$ data. Finally, we also treat λ_Δ as a complex quantity, $\lambda_\Delta=\lambda_R+i\lambda_I$, in order to estimate the contribution from the imaginary part λ_I. The best fit to the data gives $\lambda_I\approx 0$, independent of the choice of λ_R. This fact implies that further dynamical corrections to the TETAS amplitude from the open pion-proton channel are small. Therefore, there is a good reason to believe that the "experimental" magnetic moment, which is very close to the "bare" magnetic moment predicted by the modified SU(6) or the quark model with corrections, should be nearly equal to the "effective" magnetic moment.

I. INTRODUCTION

The pion-proton bremsstrahlung ($\pi^\pm p\gamma$) processes near the $\Delta(1232)$ resonance have been thoroughly studied both experimentally [1–5] and theoretically [6–29]. In addition to the investigation of the off-shell pion-proton interaction, interest in these processes was mainly motivated by the hope that these processes could be used to probe the electromagnetic properties of the Δ resonance. Moreover, owing to an unexpected large discrepancy between the experimental measurements and most of the theoretical calculations [2,25,28], much attention has also been focused on a search for a fundamental theory which can be used to describe the experimental observation.

We will confine our studies to the $\pi^+p\gamma$ process in this paper. Two experimental groups, the UCLA group [2] and the SIN group [5], have systematically measured the $\pi^+p\gamma$ cross sections which can be used to determine the magnetic moment of the Δ^{++} resonance, μ_Δ. To extract μ_Δ from the $\pi^+p\gamma$ data, one needs a valid bremsstrahlung amplitude which takes into account photon emission from the internal Δ^{++} line. Such an amplitude can be derived, in principle, from a dynamical model or from a fundamental theorem, known as the soft-photon theorem or the low energy theorem for photons. The theorem was first derived by Low [30] and was extended later by Adler and Dothan [31]. Various soft-photon amplitudes, which are consistent with the soft-photon theorem, have been constructed by using Low's prescription [30]. Low's prescription involves the following steps: (a) Obtain the external amplitude, $M_\mu^{(E)}$, from four external emission diagrams and expand $M_\mu^{(E)}$ in powers of the photon energy K. (b) Impose the gauge invariant condition, $M_\mu^{(I)}K^\mu=-M_\mu^{(E)}K^\mu$, to obtain the leading term (order K^0) of the internal amplitude, $M_\mu^{(I)}$. (c) Combine $M_\mu^{(E)}$ and $M_\mu^{(I)}$ to obtain the total bremsstrahlung amplitude, M_μ. The first two terms of the expansion of M_μ, which are independent of the off-shell effects, define a soft-photon amplitude, $M_\mu^{\rm SPA}$.

The most important feature of a soft-photon amplitude is that the amplitude can be calculated exactly in terms of the corresponding elastic amplitude and electromagnetic constants of the participating particles. We emphasize here that the soft-photon amplitude is an approximate amplitude which depends only on the complete elastic amplitude since the exact bremsstrahlung amplitude without any approximation, which is almost impossible to obtain, involves other terms which cannot be expressed in terms of the elastic amplitude. However, this does not mean that $M_\mu^{\rm SPA}$ can be uniquely defined. Since the soft-phonon theorem states nothing about the energy and the scattering angle at which the elastic amplitude should be evaluated, an infinite number of soft-photon amplitudes can be constructed [25]. Depending upon how many energies and scattering angles are involved, these amplitudes can be divided into the following classes [25]: the one-energy-one-angle (OEOA) amplitudes, the one-energy-two-angle (OETA) amplitudes, the two-energy-one-angle (TEOA) amplitudes, the two-energy-two-angle (TETA) amplitudes, and other amplitudes.

Recent studies [25,28,29,32] show that bremsstrahlung cross sections near a scattering resonance, such as $\pi^\pm p\gamma$ cross sections near the Δ resonance or $p\,^{12}C\gamma$ cross sections near either the 1.7-MeV resonance or the 0.5-MeV resonance, can be used to differentiate various soft-photon amplitudes. Thus, the combined $\pi^\pm p\gamma$ and $p\,^{12}C\gamma$ data can provide a very sensitive test of the validity of various soft-photon amplitudes (and other theoretical approximations and models). In fact, it has been found that the OEOA and OETA approximations have failed to adequately describe the combined data in the resonance region. The data can only be described by special two-energy amplitudes (i.e., those amplitudes which depend upon two special energies, the initial energy $\sqrt{s_i}$ and the final energy $\sqrt{s_f}$). Moreover, some special two-energy-two-angle (TETAS) amplitudes are found to give the best fit to the combined data. Here, the TETAS amplitudes are those soft-photon amplitudes which depend upon two special energies, $\sqrt{s_i}$ and $\sqrt{s_f}$, and two special scattering angles determined by t_p and t_q. (Here, t is the four-momentum transfer squared, and s_i, s_f, t_p, and t_q will be defined in the next section.) In other words, TETAS amplitudes depend only upon the elastic T matrix, evaluated at two special energies and two special scattering angles, but they are free of any derivative of the T matrix with respect to energy or scattering angle (or with respect to s or t).

The TETAS amplitudes have been investigated by Fischer and Minkowski [13], by Heller [23], and most recently by our group [28,29]. However, none of the amplitudes obtained by these authors can be used to determine μ_Δ from the $\pi^+ p\gamma$ data. As explained in Ref. [29], bremsstrahlung emissions from the internal Δ^{++} line involve two sources: One contribution comes from the charge of the Δ^{++} and another contribution is due to the magnetic moment of the Δ^{++}. Low's prescription can be applied to find an expression for the charge contribution, i.e., the charge contribution can be obtained from the external amplitude by imposing the gauge invariant condition. (The expressions for the charge contribution obtained by Fischer and Minkowski, Heller, and our group are all identical even though the expressions are written in different forms.) But it is very difficult to obtain an expression for the magnetic contribution by using Low's prescription. This is because, as pointed out in Ref. [29], the magnetic contribution involves an important term which depends upon the anomalous magnetic moment of the Δ, λ_Δ, and this λ_Δ-dependent term is separately gauge invariant. (If M_λ^μ is the λ_Δ-dependent term which is separately gauge invariant, then we have $M_\lambda^\mu K_\mu = 0$. In that case, M_λ^μ cannot be derived from the external amplitude by imposing the gauge invariant condition. Imposing the gauge invariant condition to determine the leading term of the internal amplitude is the most important step of Low's prescription.) This explains why a soft-photon amplitude which takes into account photon emission from the Δ^{++} (including both the charge contribution and the magnetic contribution) cannot be constructed by using Low's standard prescription. Since the amplitudes obtained in Refs. [13,23,28] do not have the λ_Δ-dependent term, these amplitudes cannot be used to extract λ_Δ or μ_Δ from the $\pi^+ p\gamma$ data.

The main purposes of this paper are the following.

(i) To present the details of the derivation of the TETAS amplitude for the $\pi^+ p\gamma$ process near the Δ^{++} resonance: This amplitude, which takes into account the bremsstrahlung emission from the internal Δ^{++} line with charge and anomalous magnetic moment λ_Δ, has been reported in Ref. [29] without derivation. Since Low's original prescription cannot be used to obtain an internal contribution which is separately gauge invariant, we have applied a radiation decomposition identity (a generalized Brodsky-Brown identity) [33] to modify Low's prescription. The first step in this modified procedure is exactly the same as Low's original prescription, i.e., to obtain the external amplitude M_μ^E and to expand it in powers of photon energy K. The second step is to obtain an internal contribution M_μ^Δ, which represents photon emission from the internal Δ^{++} line, and to split M_μ^Δ into four quasiexternal amplitudes by using the generalized Brodsky-Brown identity. The third step is to obtain an additional gauge invariant term M_μ^G by imposing the gauge invariant condition, $M_\mu^G K^\mu = -M_\mu^{E\Delta} K^\mu$. Here, $M_\mu^{E\Delta} = M_\mu^E + M_\mu^\Delta$. The last step is to obtain the total amplitude M_μ by combining $M_\mu^{E\Delta}$ with M_μ^G: $M_\mu = M_\mu^{E\Delta} + M_\mu^G$. The first two terms of the expansion of M_μ, which can be written in terms of the complete elastic T matrix, define the TETAS amplitude. It is this modified procedure which has been used to derive the TETAS amplitude for the $\pi^+ p\gamma$ process.

(ii) To extract the "experimental" magnetic moment of the Δ^{++}, μ_Δ, by fitting to 85% of the available $\pi^+ p\gamma$ data: We present the values of μ_Δ which have been extracted from 45 sets of the UCLA data [2] and 3 sets of the SIN data [5] by using the TETAS amplitude derived in (i). We show that the extracted values of μ_Δ, which are smaller than the "bare" magnetic moment predicted by the SU(6) model [34] or by the quark model, are in good agreement with the "bare" magnetic moment predicted by a modified SU(6) model with mass corrections [35] or by a corrected bag-model [36]. We have used the term "experimental" magnetic moment to describe the result obtained in this work for the following reason: The magnetic moment of the Δ^{++} obtained in this work is based upon the TETAS amplitude. In deriving this amplitude, we have ignored the emission from the internal pion-proton loop (or the open pion-proton channel). In a recent study using a nonrelativistic dynamical model, Heller *et al.* [26] have reported that an "effective" (or dressed) magnetic moment of the Δ^{++} can be defined if the contribution from the loop diagrams is involved. This effective moment, which is different from the "bare" moment predicted by the SU(6) or the quark model, is a complex and energy-dependent quantity. Although these authors have found that the imaginary part of the effective moment is not negligible, the problem of defining and calculating the effective moment for an off-shell unstable Δ^{++} particle remains unsolved mainly because they were unable to demonstrate that their model could be used to describe most of the $\pi^+ p\gamma$ data. (Generally speaking, the model-dependent approach is more complicated than the soft-photon approach. The MIT model, for example, involves a set of internal emission diagrams and some of these diagrams are very difficult to calculate. Since the calculated $\pi^+ p\gamma$ cross sections are found to be very sensitive to the precise form of the internal amplitude [28], a successful model-dependent calculation requires not only a good model but also an accurate method (or approximation) of calculating the whole set of internal diagrams.) The problem needs further study. Now, it is obvious that the effective moment cannot be calculated to arbitrary precision in any model-independent calculations since it is difficult to take into account the loop contribution in the soft-photon approximation. This is why the magnetic moment of the Δ^{++} extracted from the $\pi^+ p\gamma$ data by using the TETAS amplitude is an approximation with theoretical errors to the effective moment. Since it is also difficult to identify our magnetic moment with the "bare" moment, we have therefore used the "experimental" magnetic moment to describe the result obtained by us. However, we have also performed another experimental test by treating λ_Δ as a complex quantity, $\lambda_\Delta=\lambda_R+i\lambda_I$, in order to estimate the contribution from the imaginary part λ_I. Our best fit to the data gives $\lambda_I=0$, independent of λ_R. Since a small value of λ_I implies a small contribution from the loop diagrams, this perhaps surprising finding does provide a good reason for us to believe that the "experimental" magnetic moment is very close to the "effective" moment.

(iii) To demonstrate that the TETAS amplitude derived in (i) can be used to describe almost all the available data and hence it is valid in the energy region near the $\Delta^{++}(1232)$ resonance: Using the TETAS amplitude derived in (i) and the values of μ_Δ extracted in (ii) from the experimental data, we show that the overall agreement between theory and experiment is excellent. To the best of our knowledge, such an agreement has not been obtained previously. We also present some theoretical justification and physical meaning for the TETAS amplitude obtained in Ref. [28] (i.e., the amplitude M_μ(TETAS) given by Eq. (16) in Ref. [28]). Based on the theoretical justification and the fact that the amplitude obtained in Ref. [28] works so well for the $\pi^+ p\gamma$ process near the Δ^{++} resonance, we can argue that the magnetic moment of the Δ^{++} should be about $4.25e/(2m_p)$, the value predicted by the modified SU(6) model of Beg and Pais [35]. Using $\mu_\Delta=4.25e/(2m_p)$ as an input for the TETAS amplitude derived in this work, we show that the calculated $\pi^+ p\gamma$ cross sections are very close to those cross sections calculated using the amplitude obtained in Ref. [28].

The plan of this paper is as follows: In Sec. II, the most general TETAS amplitude, M_μ^{TETAS}, is derived for the $\pi^+ p\gamma$ process near the $\Delta^{++}(1232)$ resonance. In Sec. III, the "experimental" magnetic moment of the Δ^{++} is extracted from the UCLA data and the SIN data. A comparison between the experimental $\pi^+ p\gamma$ spectra and the calculated spectra is presented in Sec. IV. Section V is devoted to further studies and discussions. Two important issues are studied and discussed in this section: (a) We explain why the TETAS amplitude obtained in Ref. [28] works so well. (b) Treating λ_Δ as a complex quantity, $\lambda_\Delta=\lambda_R+i\lambda_I$, we show that the best fit to the UCLA data (at 298 MeV for counters G1–G10) gives $\lambda_I=0$, independent of the choice of λ_R. Our conclusion is given in the last section. There is an Appendix where the detailed expressions for some off-shell terms are given.

II. BREMSSTRAHLUNG AMPLITUDE

This section is divided into three subsections. In Sec. II A, we discuss the $\pi^+ p$ elastic scattering process [Fig. 1(a)]. We define the general form for the $\pi^+ p$ elastic T matrix, T, which is an important input for bremsstrahlung calculations. In the energy region of the Δ^{++} resonance, a tree diagram given by Fig. 1(b), $\pi^+ p\rightarrow\Delta^{++}\rightarrow\pi^+ p$, becomes the dominant elastic diagram. We derive the explicit expression for the T matrix corresponding to Fig. 1(b), $\tilde{T}$, which will be used to define a TETAS amplitude $\tilde{M}_\mu^{\text{TETAS}}$ for the $\pi^+ p\gamma$ process at the tree level. In Sec. II B, we treat Fig. 1(b) as a source graph to generate $\pi^+ p$ bremsstrahlung diagrams at the tree level [Figs. 2(a)–2(e)]. By using a generalized Brodsky-Brown identity [33] for photon emission from the internal Δ^{++} line [Fig. 2(e)], we derive the expression for $\tilde{M}_\mu^{\text{TETAS}}$ in terms of $\tilde{T}$ and the electromagnetic constants of π^+, p, and Δ^{++}. The amplitude $\tilde{M}_\mu^{\text{TETAS}}$ plays a vital role in our derivation of a more general TETAS amplitude, M_μ^{TETAS}, for the $\pi^+ p\gamma$ process. In Sec. II C, we use the modified Low procedure to derive the amplitude M_μ^{TETAS} which can be written in terms of the general form of the elastic T matrix T and the electromagnetic constants of π^+, p, and Δ^{++}. In deriving M_μ^{TETAS}, we have imposed a condition that M_μ^{TETAS} reduces to $\tilde{M}_\mu^{\text{TETAS}}$ in the energy region of the Δ^{++} resonance.

A. $\pi^+ p$ elastic scattering T matrix

We consider the $\pi^+ p\gamma$ process,

$$\pi^+(q_i^\mu)+P(p_i^\mu)\rightarrow\pi^+(q_f^\mu)+P(p_f^\mu)+\gamma(k^\mu)\ , \qquad (1)$$

where $q_i^\mu(q_f^\mu)$ and $p_i^\mu(p_f^\mu)$ are the initial (final) four-momenta of the pion and proton, respectively, and k^μ is the four-momentum of the emitted photon. These four-momenta satisfy energy-momentum conservation:

$$q_i^\mu + p_i^\mu = q_f^\mu + p_f^\mu + k^\mu \ . \tag{2}$$

In the limit when k approaches zero, the $\pi^+ p\gamma$ process reduces to the corresponding $\pi^+ p$ elastic scattering process,

$$\pi^+(q_i^\mu) + P(p_i^\mu) \rightarrow \pi^+(\bar{q}_f^\mu) + P(\bar{p}_f^\mu) \ , \tag{3}$$

where

$$\bar{p}_f^\mu = \lim_{k\to 0} p_f^\mu \ ,$$

$$\bar{q}_f^\mu = \lim_{k\to 0} q_f^\mu \ .$$

The energy-momentum conservation becomes

$$q_i^\mu + p_i^\mu = \bar{q}_f^\mu + \bar{p}_f^\mu \ . \tag{4}$$

A diagram which represents the $\pi^+ p$ elastic scattering process is shown in Fig. 1(a). In this diagram, T represents the $\pi^+ p$ elastic scattering T matrix. Although we are interested in the TETAS amplitude which depends only on the elastic (on-shell) T matrix, the exact bremsstrahlung amplitude without the soft-photon approximation involves off-shell T matrices. Thus, we have to show how a TETAS amplitude which is independent of the off-shell effects can be derived. All T matrices, on-shell or off-shell, can be written in terms of six Lorentz invariants as

$$T(s,t,p_i^2,q_i^2,p_f^2,q_f^2) \ . \tag{5}$$

Here, s is the total energy squared and t is the momentum transfer squared. For the $\pi^+ p$ elastic scattering process, the elastic T matrix depends only on two independent variables, s and t, since all four external lines (legs) are on their mass shells; i.e., the on-mass-shell conditions,

$$p_i^2 = p_f^2 = m_p^2 \tag{6}$$

and

$$q_i^2 = q_f^2 = m_\pi^2 \ ,$$

are satisfied. Here m_p and m_π are the masses of proton and pion, respectively. A half-off-shell T matrix is defined if one of the external lines is off its mass shell. For example, if $q_i^2 \neq m_\pi^2$, then we have a half-off-shell T matrix which can be written as

$$T(s,t,m_p^2,q_i^2,m_p^2,m_\pi^2) \ . \tag{7}$$

We can write the $\pi^+ p$ elastic T matrix in the standard form

$$T(s,t) \equiv T(s,t,m_p^2,m_\pi^2,m_p^2,m_\pi^2)$$
$$= A(s,t) + \tfrac{1}{2}(\not{q}_i + \bar{\not{q}}_f)B(s,t) \ , \tag{8}$$

where

$$s = (p_i + q_i)^2 = (\bar{p}_f + \bar{q}_f)^2$$

and

$$t = (\bar{p}_f - p_i)^2 = (\bar{q}_f - q_i)^2 \ .$$

If s and t are given (or if the incident energy and the scattering angle are known), the amplitudes $A(s,t)$ and $B(s,t)$ can be calculated in terms of $\pi^+ p$ phase shifts and inelasticities, determined by the $\pi^+ p$ elastic scattering experiments. The experimentally determined T matrix has been used as an input for all bremsstrahlung calculations using soft-photon amplitudes.

In the energy region of the $\Delta^{++}(1232)$ resonance, the Feyman diagram given by Fig. 1(b) is the dominant contribution to the $\pi^+ p$ elastic process and the photon emission from the intermediate Δ^{++} line becomes significant in that region. This diagram, which will be treated as a source graph to generate photon emission diagrams at the tree level, is important in our derivation of the TETAS amplitude. The elastic T matrix corresponding to Fig. 1(b) has the form

$$\tilde{T} = [g\bar{q}_f^\rho] G_{\rho\alpha}(p) [g q_i^\alpha] \tag{9}$$

where g is the $\pi^+ p \Delta^{++}$ vertex, $p^\mu = p_i^\mu + q_i^\mu$,

$$G_{\rho\alpha}(p) = \frac{i d_{\rho\alpha}(p)}{p^2 - M_\Delta^2 + i\varepsilon} \ , \tag{10a}$$

$$d_{\rho\alpha}(p) = (\not{p} + M_\Delta)\left\{ g_{\rho\alpha} - \frac{1}{3}\gamma_\rho\gamma_\alpha - \frac{1}{3M_\Delta}(\gamma_\rho p_\alpha - \gamma_\alpha p_\rho) - \frac{2}{3M_\Delta^2} p_\rho p_\alpha \right\} + \frac{2}{3M_\Delta^2}(p^2 - M_\Delta^2)[\gamma_\rho p_\alpha - \gamma_\alpha p_\rho + (\not{p} + M_\Delta)\gamma_\rho\gamma_\alpha], \tag{10b}$$

and M_Δ is the mass of the Δ^{++}. In terms of s, t, p_i^2, q_i^2, $\bar{p}_f^2$, and $\bar{q}_f^2$, $\tilde{T}$ can be written as

$$\tilde{T} \equiv \tilde{T}(s,t,p_i^2,q_i^2,\bar{p}_f^2,\bar{q}_f^2)$$
$$= \tilde{A}(s,t,p_i^2,q_i^2,\bar{p}_f^2,\bar{q}_f^2) + \tfrac{1}{2}(\not{q}_i + \bar{\not{q}}_f)\tilde{B}(s,t,p_i^2,q_i^2,\bar{p}_f^2,\bar{q}_f^2) \ , \tag{11}$$

where

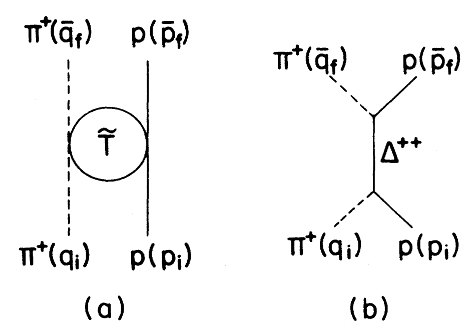


FIG. 1. (a) The graphic representation of the elastic $\pi^+ p$ process. (b) The one-particle s-channel exchange diagram (the dominant elastic diagram in the resonance region) for the $\pi^+ p$ process.

$$\begin{aligned}
&\tilde{A}(s,t,p_i^2,q_i^2,\bar{p}_f^2,\bar{q}_f^2)\\
&=\frac{ig^2}{s-M_\Delta^2+i\varepsilon}\left\{\frac{1}{2}(M_\Delta+m_p)\left[-t+q_i^2+\bar{q}_f^2-\frac{1}{3M_\Delta^2}(s-\bar{p}_f^2+\bar{q}_f^2)(s-p_i^2+q_i^2)\right]\right.\\
&\qquad\left.-\frac{1}{6M_\Delta}\bar{q}_f^2(s-p_i^2+q_i^2)+(s-p_i^2)\left[\frac{1}{6M_\Delta}(s-\bar{p}_f^2+\bar{q}_f^2)+\frac{2s-3M_\Delta^2}{3M_\Delta^2}(M_\Delta+m_p)\right]\right\} \qquad (12a)\\
&=\frac{ig^2}{s-M_\Delta^2+i\varepsilon}\left\{\frac{1}{2}(M_\Delta+m_p)\left[-t+q_i^2+\bar{q}_f^2-\frac{1}{3M_\Delta^2}(s-\bar{p}_f^2+\bar{q}_f^2)(s-p_i^2+q_i^2)\right]\right.\\
&\qquad\left.-\frac{1}{6M_\Delta}q_i^2(s-\bar{p}_f^2+\bar{q}_f^2)+(s-p_i^2)\left[\frac{1}{6M_\Delta}(s-p_i^2+q_i^2)+\frac{2s-3M_\Delta^2}{3M_\Delta^2}(M_\Delta+m_p)\right]\right\}, \qquad (12b)
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{B}(s,t,p_i^2,q_i^2,\bar{p}_f^2,\bar{q}_f^2)\\
&=\frac{ig^2}{s-M_\Delta^2+i\varepsilon}\left\{\frac{1}{2}\left[-t+q_i^2+\bar{q}_f^2-\frac{1}{3M_\Delta^2}(s-\bar{p}_f^2+\bar{q}_f^2)(s-p_i^2+q_i^2)\right]+\frac{1}{2}\left[\frac{2s}{3M_\Delta^2}-1-\frac{m_p}{3M_\Delta}\right](\bar{p}_f^2-p_i^2+q_i^2-\bar{q}_f^2)\right.\\
&\qquad\left.+\left[\frac{2s-3M_\Delta^2}{3M_\Delta^2}\right]\bar{q}_f^2-2m_p\left[\frac{1}{6M_\Delta}(s-\bar{p}_f^2+\bar{q}_f^2)+\frac{2s-3M_\Delta^2}{3M_\Delta^2}(M_\Delta+m_p)\right]\right\} \qquad (12c)\\
&=\frac{ig^2}{s-M_\Delta^2+i\varepsilon}\left\{\frac{1}{2}\left[-t+q_i^2+\bar{q}_f^2-\frac{1}{3M_\Delta^2}(s-\bar{p}_f^2+\bar{q}_f^2)(s-p_i^2+q_i^2)\right]+\frac{1}{2}\left[\frac{2s}{3M_\Delta^2}-1-\frac{m_p}{3M_\Delta}\right](p_i^2-\bar{p}_f^2+\bar{q}_f^2-q_i^2)\right.\\
&\qquad\left.+\left[\frac{2s-3M_\Delta^2}{3M_\Delta^2}\right]q_i^2-2m_p\left[\frac{1}{6M_\Delta}(s-p_i^2+q_i^2)+\frac{2s-3M_\Delta^2}{3M_\Delta^2}(M_\Delta+m_p)\right]\right\}. \qquad (12d)
\end{aligned}$$

In Eqs. (12a)–(12d), p_i^2, q_i^2, $\bar{p}_f^2$, and $\bar{q}_f^2$ satisfy the on-mass-shell conditions given by Eq. (6). Without imposing these on-mass-shell conditions explicitly, the expressions for $\tilde{A}$ and $\tilde{B}$ can be extended to define the half-off-shell T matrix later.

B. TETAS amplitude for the $\pi^+p\gamma$ process at the tree level

As we have already mentioned, Fig. 1(b) will be used as a source graph to generate photon emission diagrams at the tree level. Five diagrams generated by Fig. 1(b) are shown in Fig. 2. (We shall impose the gauge invariant condition later to take care of the remaining contributions.) The first four Feynman diagrams [2(a)–2(d)] represent the photon emissions from external pion lines and external proton lines and the last Feynman diagram [2(e)] represents photon emission from the internal Δ^{++} line.

From four external emission diagrams [2(a)–2(d)], we can define the following half-off-shell T matrices for π^+p interactions ($\pi^+p\rightarrow\Delta^{++}\rightarrow\pi^+p$):

$$\tilde{T}_a=[g(q_f+k)^\rho]G_{\rho\alpha}(p)(gq_i{}^\alpha)\,, \qquad (13a)$$

$$\tilde{T}_b=(gq_f{}^\rho)G_{\rho\alpha}(p')[g(q_i-k)^\alpha]\,, \qquad (13b)$$

$$\tilde{T}_c=(gq_f{}^\rho)G_{\rho\alpha}(p)(gq_i{}^\alpha)\,, \qquad (13c)$$

and

$$\tilde{T}_d=(gq_f{}^\rho)G_{\rho\alpha}(p')(gq_i{}^\alpha)\,. \qquad (13d)$$

Here, $p^\mu=(q_i+p_i)^\mu$, $p'^\mu=(q_i+p_i-k)^\mu=(q_f+p_f)^\mu$, and $G_{\rho\alpha}(p)$ are defined by Eq. (10a). It is easy to show that $\bar{u}(p_f,\nu_f)\tilde{T}_a u(p_i,\nu_i)$ and $\bar{u}(p_f,\nu_f)\tilde{T}_b u(p_i,\nu_i)$ can be written in terms of $\tilde{A}$ and $\tilde{B}$ [given by Eqs. (12a) and (12c), re-

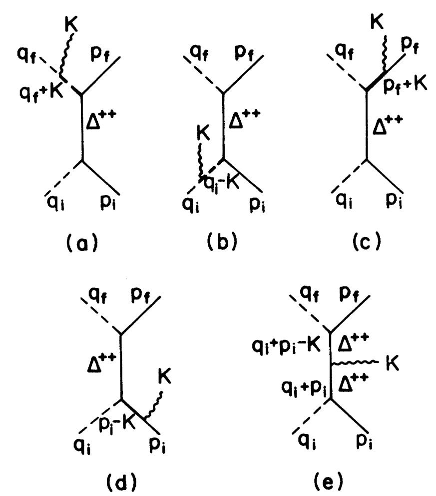


FIG. 2. Feynman diagrams for the $\pi^+p\gamma$ process at the tree level. These diagrams are generated from the source graph, Fig. 1(b).

spectively] as

$$\bar{u}\widetilde{T}_a u = \bar{u}\{\widetilde{A}[s_i,t_p,p_i^2,q_i^2,p_f^2,\Delta_a] + \tfrac{1}{2}(\not{q}_i+\not{q}_f+\not{k})\widetilde{B}[s_i,t_p,p_i^2,q_i^2,p_f^2,\Delta_a]\}u \quad (14a)$$

and

$$\bar{u}\widetilde{T}_b u = \bar{u}\{\widetilde{A}[s_f,t_p,p_i^2,\Delta_b,p_f^2,q_f^2] + \tfrac{1}{2}(\not{q}_i+\not{q}_f-\not{k})\widetilde{B}[s_f,t_p,p_i^2,\Delta_b,p_f^2,q_f^2]\}u \ , \quad (14b)$$

where

$$s_i=(p_i+q_i)^2=(p_f+q_f+k)^2 \ ,$$
$$s_f=(p_f+q_f)^2=(p_i+q_i-k)^2 \ ,$$
$$t_p=(p_f-p_i)^2 \ ,$$
$$\Delta_a=(q_f+k)^2=m_\pi^2+2q_f\cdot k \ ,$$

and

$$\Delta_b=(q_i-k)^2=m_\pi^2-2q_i\cdot k \ .$$

However, the expressions for $\widetilde{T}_c u(p_i,\nu_i)$ and $\bar{u}(p_f,\nu_f)\widetilde{T}_d$ involve extra off-shell amplitudes:

$$\widetilde{T}_c u = \{\widetilde{A}[s_i,t_q,p_i^2,q_i^2,\Delta_c,q_f^2] + \tfrac{1}{2}(\not{q}_i+\not{q}_f)\widetilde{B}[s_i,t_q,p_i^2,q_i^2,\Delta_c,q_f^2] + \tfrac{1}{2}(\not{p}_f+\not{k}-m_p)\widetilde{C}[s_i,t_q,p_i^2,q_i^2,\Delta_c,q_f^2]\}u \quad (15a)$$

and

$$\bar{u}\widetilde{T}_d = \bar{u}\{\widetilde{A}[s_f,t_q,\Delta_d,q_i^2,p_f^2,q_f^2] + \tfrac{1}{2}(\not{q}_i+\not{q}_f)\widetilde{B}[s_f,t_q,\Delta_d,q_i^2,p_f^2,q_f^2] + \tfrac{1}{2}\widetilde{C}[s_f,t_q,\Delta_d,q_i^2,p_f^2,q_f^2](\not{p}_i-\not{k}-m_p)\} \ , \quad (15b)$$

where

$$t_q=(q_f-q_i)^2 \ ,$$
$$\Delta_c=(p_f+k)^2=m_p^2+2p_f\cdot k \ ,$$
$$\Delta_d=(p_i-k)^2=m_p^2-2p_i\cdot k \ ,$$

and the extra off-shell amplitude $\widetilde{C}$ has the form

$$\widetilde{C}[s,t,p_i^2,q_i^2,p_f^2,q_f^2]=\frac{ig^2}{s-M_\Delta^2+i\varepsilon}\left\{\frac{1}{2}\left[-t+q_i^2+q_f^2-\frac{1}{3M_\Delta^2}(s-p_f^2+q_f^2)(s-p_i^2+q_i^2)\right] + \frac{1}{2}\left[\frac{2s}{3M_\Delta^2}-1-\frac{m_p}{3M_\Delta}\right](2s-p_i^2-p_f^2+q_i^2+q_f^2)-\left[\frac{2s-3M_\Delta^2}{3M_\Delta^2}\right]m_\pi^2 - 2m_p\left[\frac{2s-3M_\Delta^2}{3M_\Delta^2}(m_p+M_\Delta)+\frac{1}{6M_\Delta}(s-m_p^2+m_\pi^2)\right] - 2\not{q}_i\left[\frac{2s-3M_\Delta^2}{3M_\Delta^2}(m_p+M_\Delta)+\frac{1}{6M_\Delta}(s-m_p^2+m_\pi^2)\right]\right\} . \quad (16)$$

The expressions for $\widetilde{A}$ and $\widetilde{B}$ in Eq. (15b) are given by Eqs. (12a) and (12c), respectively, but the expressions for $\widetilde{A}$ and $\widetilde{B}$ in Eq. (15a) are given by Eqs. (12b) and (12d), respectively. Again, p_i^2, q_i^2, p_f^2, and q_f^2 in Eqs. (14), (15), and (16) satisfy the on-mass-shell conditions given by Eq. (6). Since we are interested in the soft-photon approximation, the extra off-shell amplitude involving $\widetilde{C}$ will be completely ignored later in our derivation of the TETAS amplitude. (Justification for neglecting the extra off-shell amplitude will be discussed again in the next section.)

From the expressions for T_x $(x=a,b,c,d)$ given by Eqs. (14) and (15), we can see that T_x depends on the square of the invariant mass Δ_x $(x=a,b,c,d)$ of the off-mass-shell leg on which the photon emission occurs. As k approaches zero, Δ_x reduces to (mass)^2 and T_x reduces to on-shell (elastic) T matrix. Since the TETAS amplitude which we wish to derive depends only on the on-shell T matrix [evaluated at four different sets: (s_i,t_p), (s_f,t_p), (s_i,t_q), and (s_f,t_q)], we must expand T_x in powers of k. Keeping only terms to order k, we obtain

$$\bar{u}\widetilde{T}_a u = \bar{u}\left[\widetilde{T}(s_i,t_p)+2q_f\cdot k\left[\frac{\partial\widetilde{T}_a}{\partial\Delta_a}\right]+\cdots\right]u \ , \quad (17a)$$

$$\bar{u}\widetilde{T}_b u = \bar{u}\left[\widetilde{T}(s_f,t_p)-2q_i\cdot k\left[\frac{\partial\widetilde{T}_b}{\partial\Delta_b}\right]+\cdots\right]u \ , \quad (17b)$$

$$\widetilde{T}_c u = \left[\widetilde{T}(s_i,t_q)+2p_f\cdot k\left[\frac{\partial\widetilde{T}_c'}{\partial\Delta_c}\right]+\widetilde{T}_{cc}(s_i,t_q)+\cdots\right]u \ , \quad (17c)$$

and

$$\bar{u}\widetilde{T}_d = \bar{u}\left[\widetilde{T}(s_f,t_q)-2p_i\cdot k\left[\frac{\partial\widetilde{T}_d'}{\partial\Delta_d}\right]+\widetilde{T}_{dc}(s_f,t_q)+\cdots\right] , \quad (17d)$$

where

$$\tilde{T}(s_i,t_p)=\tilde{A}(s_i,t_p,m_p^2,m_\pi^2,m_p^2,m_\pi^2)$$
$$+\tfrac{1}{2}(\not{q}_i+\not{q}_f+\not{K})\tilde{B}(s_i,t_p,m_p^2,m_\pi^2,m_p^2,m_\pi^2)\ , \tag{18a}$$

$$\tilde{T}(s_f,t_p)=\tilde{A}(s_f,t_p,m_p^2,m_\pi^2,m_p^2,m_\pi^2)$$
$$+\tfrac{1}{2}(\not{q}_i+\not{q}_f-\not{K})\tilde{B}(s_f,t_p,m_p^2,m_\pi^2,m_p^2,m_\pi^2)\ , \tag{18b}$$

$$\tilde{T}(s_i,t_q)=\tilde{A}(s_i,t_q,m_p^2,m_\pi^2,m_p^2,m_\pi^2)$$
$$+\tfrac{1}{2}(\not{q}_i+\not{q}_f)\tilde{B}(s_i,t_q,m_p^2,m_\pi^2,m_p^2,m_\pi^2)\ , \tag{18c}$$

$$\tilde{T}(s_f,t_q)=\tilde{A}(s_f,t_q,m_p^2,m_\pi^2,m_p^2,m_\pi^2)$$
$$+\tfrac{1}{2}(\not{q}_i+\not{q}_f)\tilde{B}(s_f,t_q,m_p^2,m_\pi^2,m_p^2,m_\pi^2)\ , \tag{18d}$$

$$\tilde{T}_{cc}(s_i,t_q)=\tfrac{1}{2}(\not{p}_f+\not{K}-m_p)\tilde{C}(s_i,t_q,m_p^2,m_\pi^2,m_p^2,m_\pi^2)$$
$$+(p_f\cdot k)(\not{p}_f+\not{K}-m_p)\left[\frac{\partial\tilde{C}}{\partial\Delta_c}\right]\ , \tag{18e}$$

$$\tilde{T}_{dc}(s_f,t_q)=\tfrac{1}{2}\tilde{C}(s_f,t_q,m_p^2,m_\pi^2,m_p^2,m_\pi^2)(\not{p}_i-\not{K}-m_p)$$
$$-(p_i\cdot k)\left[\frac{\partial\tilde{C}}{\partial\Delta_d}\right](\not{p}_i-\not{K}-m_p)\ , \tag{18f}$$

and $\tilde{T}_c'$ and $\tilde{T}_d'$ are defined by Eqs. (15a) and (15b), respectively, but without those terms involving $\tilde{C}$.

The external scattering amplitude corresponding to the four external diagrams [2(a)–2(d)] at the tree level can be written in terms of the half-off-shell T matrices $\tilde{T}_a$, $\tilde{T}_b$, $\tilde{T}_c$, and $\tilde{T}_d$ as

$$\tilde{M}_\mu^{(E)}=\bar{u}(p_f,\nu_f)\left[\frac{Q_a q_{f\mu}}{q_f\cdot k}\tilde{T}_a-\tilde{T}_b\frac{Q_b q_{i\mu}}{q_i\cdot k}+\frac{Q_c(p_f+R_f)_\mu}{p_f\cdot k}\tilde{T}_c-\tilde{T}_d\frac{Q_d(p_i+R_i)_\mu}{p_i\cdot k}\right]u(p_i,\nu_i)\ , \tag{19}$$

where $Q_a=Q_b$ represents the charge of pion, $Q_c=Q_d$ represents the charge of proton, and $R_{i\mu}$ and $R_{f\mu}$ have the form

$$\varepsilon^\mu R_{i\mu}=\tfrac{1}{4}[\not{\varepsilon},\not{K}]+\frac{\lambda_p}{8m_p}\{[\not{\varepsilon},\not{K}],\not{p}_i\}\ , \tag{20a}$$

and

$$\varepsilon^\mu R_{f\mu}=\tfrac{1}{4}[\not{\varepsilon},\not{K}]+\frac{\lambda_p}{8m_p}\{[\not{\varepsilon},\not{K}],\not{p}_f\}\ . \tag{20b}$$

In Eqs. (20a) and (20b), ε^μ is the photon polarization, λ_p is the anomalous magnetic moment of proton, and we have used $[X,Y]\equiv XY-YX$ and $\{X,Y\}\equiv XY+YX$. (Note that $R_{i\mu}$, $R_{f\mu}$, and R_μ [Eq. (43)] defined in this paper are slightly different from those defined in Ref. [29]. There is a sign difference between the two definitions since $[\not{\varepsilon},\not{K}]=-[\not{K},\not{\varepsilon}]$.) The factors $[Q_c(p_f+R_f)_\mu/p_f\cdot k]$ and $[-Q_d(p_i+R_i)_\mu/p_i\cdot k]$ in Eq. (19) are obtained from the following relations:

$$\bar{u}(p_f,\nu_f)[-iQ_c\Gamma_\mu][i/(\not{p}_f+\not{K}-m_p)]=\bar{u}(p_f,\nu_f)[Q_c(p_f+R_f)_\mu/p_f\cdot k]\ , \tag{21}$$

which describes photon emission by an outgoing proton line with charge Q_c, anomalous magnetic moment λ_p, and momentum p_f^μ [Fig. 3(a)], and

$$[i/(\not{p}_i-\not{K}-m_p)](-iQ_d\Gamma_\mu)u(p_i,\nu_i)=[-Q_d(p_i+R_i)_\mu/p_i\cdot k]u(p_i,\nu_i)\ , \tag{22}$$

which describes photon emission by an incoming proton line with charge Q_d, anomalous magnetic moment λ_p, and momentum p_i^μ [Fig. 3(b)]. Here, Γ_μ is the (on-shell) electromagnetic vertex,

$$\Gamma_\mu=\gamma_\mu-i\lambda_p\sigma_{\mu\nu}k^\nu/(2m_p)\ , \tag{23}$$

with

$$\sigma_{\mu\nu}=i[\gamma_\mu,\gamma_\nu]/2\ .$$

It is easy to show that $R_{i\mu}$ and $R_{f\mu}$ are separately gauge invariant, i.e., they satisfy

$$R_i\cdot k=R_f\cdot k=0\ . \tag{24}$$

The internal amplitude (at the tree level) $\tilde{M}_\mu^{(\Delta)}$ corresponding to Fig. 2(e) has the form

$$\varepsilon^\mu\tilde{M}_\mu^{(\Delta)}=\bar{u}(p_f,\nu_f)[gq_f^\rho]G_{\rho\sigma}(p')[-i(Q_b+Q_d)\Gamma_\mu^{\sigma\beta}\varepsilon^\mu]\times G_{\beta\alpha}(p)[gq_i^\alpha]u(p_i,\nu_i)\ , \tag{25}$$

where $p^\mu=q_i^\mu+p_i^\mu$, $p'^\mu=p^\mu-k^\mu=q_f^\mu+p_f^\mu$, $G_{\rho\sigma}(p')$ is the propagator for the Δ^{++} given by Eq. (10a), and $\Gamma_\mu^{\sigma\beta}\varepsilon^\mu$ is the electromagnetic vertex for the Δ^{++} (in the Rarita-Schwinger formalism but neglecting the contribution from the electric quadrupole and magnetic octupole moment of the Δ^{++}),

$$\Gamma_\mu^{\sigma\beta}\varepsilon^\mu=\left[\not{\varepsilon}+\frac{\lambda_\Delta}{2M_\Delta}\not{\varepsilon}\not{K}\right]g^{\sigma\beta}-\tfrac{1}{3}\not{\varepsilon}\gamma^\sigma\gamma^\beta-\tfrac{1}{3}(\gamma^\sigma\varepsilon^\beta-\gamma^\beta\varepsilon^\sigma)\ . \tag{26}$$

In Eq. (26), λ_Δ is the anomalous magnetic moment of the Δ^{++}. The amplitude $\tilde{M}_\mu^{(\Delta)}$ can be decomposed into four quasiexternal amplitudes by using a generalized Brodsky-Brown decomposition identity [33]. To do this, we introduce an operator $\Lambda^{\sigma\beta}(p)$,

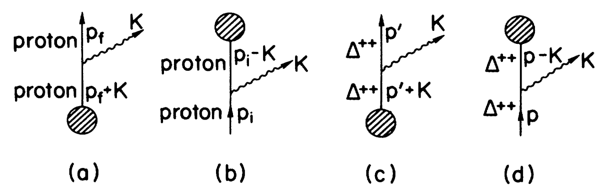


FIG. 3. Photon emission by an incoming proton (Δ^{++}) or outgoing proton (Δ^{++}).

$$\Lambda^{\sigma\beta}(p)=(\not{p}-m_\Delta)g^{\sigma\beta}-\tfrac{1}{3}(\gamma^\sigma p^\beta+\gamma^\beta p^\sigma)+\tfrac{1}{3}\gamma^\sigma(\not{p}+M_\Delta)\gamma^\beta , \quad (27)$$

which satisfies the condition

$$d_{\rho\sigma}(p)\Lambda^{\sigma\beta}(p)=(p^2-M_\Delta^2)g_\rho^\beta , \quad (28)$$

where $d_{\rho\sigma}(p)$ is defined by Eq. (10b). It is easy to prove the following useful relations:

$$\frac{d_{\rho\sigma}(p')}{p'^2-M_\Delta^2}\Gamma_\mu^{\sigma\beta}\varepsilon^\mu d_{\beta\alpha}(p)=\frac{d_{\rho\sigma}(p')}{p'^2-M_\Delta^2}[\Gamma_\mu^{\sigma\beta}\varepsilon^\mu d_{\beta\alpha}(p)+\Lambda^{\sigma\beta}(p')\Gamma_{\mu,\beta\alpha}\varepsilon^\mu]-g_\rho^\beta\Gamma_{\mu,\beta\alpha}\varepsilon^\mu , \quad (29)$$

$$d_{\rho\sigma}(p')\Gamma_\mu^{\sigma\beta}\varepsilon^\mu\frac{d_{\beta\alpha}(p)}{p^2-M_\Delta^2}=[d_{\rho\sigma}(p')\Gamma_\mu^{\sigma\beta}\varepsilon^\mu+\Gamma_{\mu,\rho\sigma}\varepsilon^\mu\Lambda^{\sigma\beta}(p)]\frac{d_{\beta\alpha}(p)}{p^2-M_\Delta^2}-\Gamma_{\mu,\rho\sigma}\varepsilon^\mu g_\alpha^\sigma , \quad (30)$$

and

$$\frac{d_{\rho\sigma}(p')}{p'^2-M_\Delta^2}\Gamma_\mu^{\sigma\beta}\varepsilon^\mu d_{\beta\alpha}(p)-d_{\rho\sigma}(p')\Gamma_\mu^{\sigma\beta}\varepsilon^\mu\frac{d_{\beta\alpha}(p)}{p^2-M_\Delta^2}=2(p'\cdot k)\frac{d_{\rho\sigma}(p')}{p'^2-M_\Delta^2}\Gamma_\mu^{\sigma\beta}\varepsilon^\mu\frac{d_{\beta\alpha}(p)}{p^2-M_\Delta^2} . \quad (31)$$

Combining Eqs. (29), (30), and (31), we find

$$\frac{d_{\rho\sigma}(p')}{p'^2-M_\Delta^2}\Gamma_\mu^{\sigma\beta}\varepsilon^\mu\frac{d_{\beta\alpha}(p)}{p^2-M_\Delta^2}=\frac{1}{2p'\cdot k}\frac{d_{\rho\sigma}(p')}{p'^2-M_\Delta^2}[\Gamma_\mu^{\sigma\beta}\varepsilon^\mu d_{\beta\alpha}(p)+\Lambda^{\sigma\beta}(p')\Gamma_{\mu,\beta\alpha}\varepsilon^\mu]$$
$$-\frac{1}{2p\cdot k}[d_{\rho\sigma}(p')\Gamma_\mu^{\sigma\beta}\varepsilon^\mu+\Gamma_{\mu,\rho\sigma}\varepsilon^\mu\Lambda^{\sigma\beta}(p)]\frac{d_{\beta\alpha}(p)}{p^2-M_\Delta^2} , \quad (32)$$

which gives the following decomposition identity:

$$G_{\rho\sigma}(p')(Q_\Delta\Gamma_\mu^{\sigma\beta}\varepsilon^\mu)G_{\beta\alpha}(p)=iG_{\rho\sigma}(p')\left[\frac{Q_\Delta O^\sigma{}_\alpha}{2p'\cdot k}\right]+\left[-\frac{Q_\Delta O'^\beta_\rho}{2p\cdot k}\right]iG_{\beta\alpha}(p) , \quad (33)$$

where

$$O^\sigma{}_\alpha=\Gamma_\mu^{\sigma\beta}\varepsilon^\mu d_{\beta\alpha}(p)+\Lambda^{\sigma\beta}(p')\Gamma_{\mu,\beta\alpha}\varepsilon^\mu , \quad (34)$$

$$O'^\beta_\rho=d_{\rho\sigma}(p')\Gamma_\mu^{\sigma\beta}\varepsilon^\mu+\Gamma_{\mu,\rho\sigma}\varepsilon^\mu\Lambda^{\sigma\beta}(p) , \quad (35)$$

and

$$Q_\Delta=Q_b+Q_d=Q_a+Q_c . \quad (36)$$

It should be pointed out that the factor $[Q_\Delta O^\sigma{}_\alpha/(2p'\cdot k)]$ in Eq. (33) can also be obtained from the following expression,

$$\bar{u}_\sigma^{(\Delta)}(p',\lambda)[-iQ_\Delta(\Gamma_\mu^{\sigma\beta}\varepsilon^\mu)]\frac{id_{\beta\alpha}(p)}{p^2-M_\Delta^2}=\bar{u}_\sigma^{(\Delta)}(p',\lambda)\left[\frac{Q_\Delta O^\sigma{}_\alpha}{2p'\cdot k}\right] , \quad (37)$$

which describes photon emission by an outgoing Δ line with charge Q_Δ, anomalous magnetic moment λ_Δ, and momentum $p^\mu=(p'+k)^\mu$. [See Fig. 3(c).] Here the vector-spinors $\bar{u}_\sigma^{(\Delta)}$ satisfy the following conditions:

$$\begin{aligned}&\bar{u}_\sigma^{(\Delta)}(p',\lambda)(\not{p}'-M_\Delta)=0 ,\\&\bar{u}_\sigma^{(\Delta)}(p',\lambda)(p'^2-M_\Delta^2)=0 ,\\&\bar{u}_\sigma^{(\Delta)}(p',\lambda)\gamma^\sigma=0 ,\\&\bar{u}_\sigma^{(\Delta)}(p',\lambda)p'^\sigma=0 ,\end{aligned} \quad (38)$$

and

$$\bar{u}_\sigma^{(\Delta)}(p',\lambda)\Lambda^{\sigma\beta}(p')=0 .$$

The proof is very simple. The left-hand side of Eq. (37) can be rewritten in the form

$$\bar{u}_{\sigma}^{(\Delta)}(p',\lambda)Q_{\Delta}[\Gamma_{\mu}^{\sigma\beta}\varepsilon^{\mu}d_{\beta\alpha}(p)+\Lambda^{\sigma\beta}(p')\Gamma_{\mu,\beta\alpha}\varepsilon^{\mu}-\Lambda^{\sigma\beta}(p')\Gamma_{\mu,\beta\alpha}\varepsilon^{\mu}]\frac{1}{p^2-M_{\Delta}^2}$$
$$=\bar{u}_{\sigma}^{(\Delta)}(p',\lambda)Q_{\Delta}[O^{\sigma}{}_{\alpha}-\Lambda^{\sigma\beta}(p')\Gamma_{\mu,\beta\alpha}\varepsilon^{\mu}]\frac{1}{p^2-M_{\Delta}^2} . \quad (39)$$

Since $p^2=(p'+k)^2=M_{\Delta}^2+2p'\cdot k$ and $\bar{u}_{\sigma}^{(\Delta)}\Lambda^{\sigma\beta}=0$, we obtain Eq. (37). Similarly, we can also show that

$$\frac{id_{\rho\sigma}(p')}{p'^2-M_{\Delta}^2}[-iQ_{\Delta}\Gamma_{\mu}^{\sigma\beta}\varepsilon^{\mu}]u_{\beta}^{(\Delta)}(p_i,\lambda)=\left[-\frac{Q_{\Delta}O_{\rho}'^{\beta}}{2p\cdot k}\right]u_{\beta}^{(\Delta)}(p_i,\lambda) , \quad (40)$$

which describes photon emission by an incoming Δ line with charge Q_{Δ}, anomalous magnetic moment λ_{Δ}, and momentum p^{μ}. [See Fig. 3(d).]

If we substitute the expression for $\Gamma_{\mu}^{\sigma\beta}\varepsilon^{\mu}$ given by Eq. (26) into Eqs. (34) and (35), we find

$$O^{\sigma}{}_{\alpha}=(2p'\cdot\varepsilon+2R\cdot\varepsilon)g^{\sigma}{}_{\alpha}+E^{\sigma}{}_{\alpha} \quad (41)$$

and

$$O_{\rho}'^{\beta}=(2p\cdot\varepsilon+2R\cdot\varepsilon)g_{\rho}{}^{\beta}+E_{\rho}'^{\beta} , \quad (42)$$

where the expressions for $E^{\sigma}{}_{\alpha}$ and $E_{\rho}'^{\beta}$ are given in Appendix A and

$$R\cdot\varepsilon=\tfrac{1}{4}[\not\varepsilon,\not k]+\frac{\lambda_{\Delta}}{8M_{\Delta}}\{[\not\varepsilon,\not k],\not p\} . \quad (43)$$

In Eq. (43), $p^{\mu}=(p_i+q_i)^{\mu}$ and we have used $[X,Y]\equiv XY-YX$ and $\{X,Y\}\equiv XY+YX$. Again, it is easy to verify that R^{μ} is separately gauge invariant, $R\cdot k=0$. If we compare Eq. (43) with Eqs. (20a) and (20b), we find that $R_i\cdot\varepsilon$, $R_f\cdot\varepsilon$, and $R\cdot\varepsilon$ can be written in the same form. Inserting Eq. (33) [with the expressions for $O^{\sigma}{}_{\alpha}$ and $O_{\rho}'^{\beta}$ given by Eqs. (41) and (42), respectively] into Eq. (25) and remembering that charge is conserved [Eq. (36)], we obtain

$$\varepsilon^{\mu}\widetilde{M}_{\mu}^{(\Delta)}=-\bar{u}(p_f,\nu_f)\left[Q_a\left[\frac{p'\cdot\varepsilon+R\cdot\varepsilon}{p'\cdot k}\right]\widetilde{T}_a-Q_B\widetilde{T}_b\left[\frac{p\cdot\varepsilon+R\cdot\varepsilon}{p\cdot k}\right]\right.$$
$$\left.+Q_c\left[\frac{p'\cdot\varepsilon+R\cdot\varepsilon}{p'\cdot k}\right]\widetilde{T}_c-Q_d\widetilde{T}_d\left[\frac{p\cdot\varepsilon+R\cdot\varepsilon}{p\cdot k}\right]\right]u(p_i,\nu_i)+\bar{u}(p_f,\nu_f)[\varepsilon\cdot\widetilde{D}]u(p_i,\nu_i) , \quad (44)$$

where

$$\varepsilon\cdot\widetilde{D}=Q_a\left[\frac{p'\cdot\varepsilon+R\cdot\varepsilon}{p'\cdot k}\right](gk^{\rho})G_{\rho\alpha}(p)(gq_i{}^{\alpha})+Q_b(gq_f{}^{\rho})G_{\rho\alpha}(p')(gk^{\alpha})\left[\frac{p\cdot\varepsilon+R\cdot\varepsilon}{p\cdot k}\right]$$
$$+(Q_b+Q_d)(gq_f{}^{\rho})\left[\frac{G_{\rho\sigma}(p')E^{\sigma}{}_{\alpha}}{2p'\cdot k}-\frac{E_{\rho}'^{\beta}G_{\beta\alpha}(p)}{2p\cdot k}\right](gq_i{}^{\alpha}) . \quad (45)$$

Since $Q_a=Q_b$ and $Q_c=Q_d$, it is easy to show that the leading term in Eq. (44) (i.e., the amplitude $\widetilde{M}_{\mu}^{(\Delta)}$ without those terms involving $\varepsilon\cdot R$ and $\varepsilon\cdot\widetilde{D}$) is of order k^0 and is independent of k^{μ} when $k\to 0$. Thus $\widetilde{M}_{\mu}^{(\Delta)}$ has no kinematic singularity at $k=0$.

Now let us add the internal amplitude $\widetilde{M}_{\mu}^{(\Delta)}$ given by Eq. (44) to the external amplitude $\widetilde{M}_{\mu}^{(E)}$ given by Eq. (19). We find

$$\varepsilon^{\mu}\widetilde{M}_{\mu}^{(E\Delta)}=\varepsilon^{\mu}\widetilde{M}_{\mu}^{(E)}+\varepsilon^{\mu}\widetilde{M}_{\mu}^{(\Delta)}$$
$$=\varepsilon^{\mu}\widetilde{M}_{\mu}^{\mathrm{TETA}}+\bar{u}(p_f,\nu_f)[\varepsilon^{\mu}\cdot\widetilde{D}_{\mu}]u(p_i,\nu_i) , \quad (46)$$

where

$$\widetilde{M}_{\mu}^{\mathrm{TETA}}=\bar{u}(p_f,\nu_f)\left[Q_a\left[\frac{q_{f\mu}}{q_f\cdot k}-\frac{(p_f+q_f+R)_{\mu}}{(p_f+q_f)\cdot k}\right]\widetilde{T}_a-Q_b\widetilde{T}_b\left[\frac{q_{i\mu}}{q_i\cdot k}-\frac{(p_i+q_i+R)_{\mu}}{(p_i+q_i)\cdot k}\right]\right.$$
$$\left.+Q_c\left[\frac{(p_f+R_f)_{\mu}}{p_f\cdot k}-\frac{(p_f+q_f+R)_{\mu}}{(p_f+q_f)\cdot k}\right]\widetilde{T}_c-Q_d\widetilde{T}_d\left[\frac{(p_i+R_i)_{\mu}}{p_i\cdot k}-\frac{(p_i+q_i+R)_{\mu}}{(p_i+q_i)\cdot k}\right]\right]u(p_i,\nu_i) , \quad (47)$$

which is defined in terms of four half-off-shell T matrices, $\widetilde{T}_a$, $\widetilde{T}_b$, $\widetilde{T}_c$, and $\widetilde{T}_d$. Although $\widetilde{M}^{\mathrm{TETA}}$ is gauge invariant,

$$\tilde{M}_{\mu}^{\rm TETA}k^{\mu}=0\ , \tag{48}$$

the amplitude $\tilde{M}_{\mu}^{(E\Delta)}$ does not satisfy the gauge invariant condition since

$$\tilde{D}_{\mu}k^{\mu}\neq 0\ .$$

That $\tilde{M}_{\mu}^{(E\Delta)}$ is not gauge invariant is not surprising because there are other Feynman diagrams for $\pi^{+}p\gamma$ process which are not shown in Fig. 2. To obtain the total bremsstrahlung amplitude $\tilde{M}_{\mu}$ which is completely gauge invariant, we have to impose the gauge invariant condition:

$$\tilde{M}_{\mu}=\tilde{M}_{\mu}^{(E\Delta)}+\tilde{M}_{\mu}^{(G)}\ , \tag{49}$$

$$\tilde{M}_{\mu}k^{\mu}=(\tilde{M}_{\mu}^{(E\Delta)}+\tilde{M}_{\mu}^{(G)})k^{\mu}=0\ . \tag{50}$$

The additional gauge term $\tilde{M}_{\mu}^{(G)}$, which is required to make the total amplitude $\tilde{M}_{\mu}$ gauge invariant, can be determined from the gauge invariant condition, Eq. (50). Such calculations are very lengthy and the final expression for $\tilde{M}_{\mu}$ can be written as

$$\varepsilon^{\mu}\tilde{M}_{\mu}=\varepsilon^{\mu}\tilde{M}_{\mu}^{\rm TETA}+\varepsilon^{\mu}\tilde{M}_{\mu}{}^{x}\ , \tag{51}$$

where

$$\begin{aligned}\varepsilon^{\mu}\tilde{M}_{\mu}{}^{x}=\bar{u}(p_f,\nu_f)\Bigg[&Q_a g^2\left[\frac{(p_f+q_f)^{\rho}(\varepsilon_{\rho}k^{\beta}-\varepsilon^{\beta}k_{\rho})+R\cdot\varepsilon k^{\beta}}{(p_f+q_f)\cdot k}\right]G_{\beta\alpha}(p_i+q_i)q_i{}^{\alpha}\\ &+Q_b g^2 q_f{}^{\rho}G_{\rho\sigma}(p_f+q_f)\left[\frac{(k^{\sigma}\varepsilon_{\alpha}-k_{\alpha}\varepsilon^{\sigma})(p_i+q_i)^{\alpha}+k^{\sigma}R\cdot\varepsilon}{(p_i+q_i)\cdot k}\right]\\ &+(Q_b+Q_d)g^2 q_f{}^{\rho}\left[\frac{G_{\rho\sigma}(p_f+q_f)\tilde{C}^{\sigma}_{\alpha\mu}\varepsilon^{\mu}}{2(p_f+q_f)\cdot k}-\frac{\varepsilon^{\mu}\tilde{C}'^{\beta}_{\rho\mu}G_{\beta\alpha}(p_i+q_i)}{2(p_i+q_i)\cdot k}\right]q_i{}^{\alpha}\Bigg]u(p_i,\nu_i)\ ,\end{aligned} \tag{52}$$

and the expressions for $\tilde{C}^{\sigma}_{\alpha\mu}$ and $\tilde{C}'^{\beta}_{\rho\mu}$ are given in Appendix A.

It is clear that the amplitude $\tilde{M}_{\mu}{}^{x}$ is gauge invariant but it cannot be written in terms of the $\pi^{+}p$ elastic T matrix. This amplitude will be ignored in the soft-photon approximation mainly because it cannot be calculated if the $\pi^{+}p$ elastic T matrix and the electromagnetic constants of p, π^{+} and Δ^{++} are the only input for the $\pi^{+}p\gamma$ calculation. In order to estimate the contribution from $\tilde{M}_{\mu}{}^{x}$, we have used two amplitudes, $\tilde{M}_{\mu}$ and $\tilde{M}_{\mu}^{\rm TETA}$, to calculate the $\pi^{+}p\gamma$ cross sections. The average cross sections over G1–G19 at 298 MeV have been calculated. When two results are compared, the difference between the two calculations is within 11%. If $\tilde{M}_{\mu}{}^{x}$ is ignored, then the total amplitude $\tilde{M}_{\mu}$ reduces to $\tilde{M}_{\mu}^{\rm TETA}$. As we have already mentioned, the amplitude $\tilde{M}_{\mu}^{\rm TETA}$ involves the half-off-shell T matrices. It can be calculated if we have a dynamical model from which the half-off-shell T matrix elements can be determined. However, since we are interested in the on-shell soft-photon approximation in this work, the possibility of developing a new approximation based on the off-shell amplitude $\tilde{M}_{\mu}^{\rm TETA}$ will not be discussed here.

With the help of the expansions given by Eqs. (17a)–(17d), $\tilde{M}_{\mu}^{\rm TETA}$ can be expanded as follows:

$$\tilde{M}_{\mu}^{\rm TETA}=\tilde{M}_{\mu}^{\rm TETAS}+\tilde{M}_{\mu}^{\rm off}\ , \tag{53}$$

where

$$\begin{aligned}\tilde{M}_{\mu}^{\rm TETAS}=\bar{u}(p_f,\nu_f)\Bigg[&Q_a\left[\frac{q_{f\mu}}{q_f\cdot k}-\frac{(p_f+q_f+R)_{\mu}}{(p_f+q_f)\cdot k}\right]\tilde{T}(s_i,t_p)-Q_b\tilde{T}(s_f,t_p)\left[\frac{q_{i\mu}}{q_i\cdot k}-\frac{(p_i+q_i+R)_{\mu}}{(p_i+q_i)\cdot k}\right]\\ &+Q_c\left[\frac{(p_f+R_f)_{\mu}}{p_f\cdot k}-\frac{(p_f+q_f+R)_{\mu}}{(p_f+q_f)\cdot k}\right]\tilde{T}(s_i,t_q)\\ &-Q_d\tilde{T}(s_f,t_q)\left[\frac{(p_i+R_i)_{\mu}}{p_i\cdot k}-\frac{(p_i+q_i+R)_{\mu}}{(p_i+q_i)\cdot k}\right]\Bigg]u(p_i,\nu_i)\end{aligned} \tag{54}$$

and

$$\tilde{M}_\mu^{\text{off}}=\bar{u}(p_f,\nu_f)\left\{2Q_a\left[q_{f\mu}-(q_f\cdot k)\frac{(p_f+q_f)_\mu}{(p_f+q_f)\cdot k}\right]\left[\frac{\partial \tilde{T}_a}{\partial\Delta_a}\right]+2Q_b\left[\frac{\partial\tilde{T}_b}{\partial\Delta_b}\right]\left[q_{i\mu}-(q_i\cdot k)\frac{(p_i+q_i)_\mu}{(p_i+q_i)\cdot k}\right]\right.$$
$$+2Q_c\left[p_{f\mu}-(p_f\cdot k)\frac{(p_f+q_f)_\mu}{(p_f+q_f)\cdot k}\right]\left[\frac{\partial \tilde{T}'_c}{\partial\Delta_c}\right]+2Q_d\left[\frac{\partial\tilde{T}'_d}{\partial\Delta_d}\right]\left[p_{i\mu}-(p_i\cdot k)\frac{(p_i+q_i)_\mu}{(p_i+q_i)\cdot k}\right]$$
$$+Q_c\left[\frac{(p_f+R_f)_\mu}{p_f\cdot k}-\frac{(p_f+q_f+R)_\mu}{(p_f+q_f)\cdot k}\right]\tilde{T}_{cc}(s_i,t_q)$$
$$\left.-Q_d\tilde{T}_{dc}(s_f,t_q)\left[\frac{(p_i+R_i)_\mu}{p_i\cdot k}-\frac{(p_i+q_i+R)_\mu}{(p_i+q_i)\cdot k}\right]+\cdots\right\}u(p_i,\nu_i)\ . \tag{55}$$

$\tilde{M}_\mu^{\text{TETAS}}$ defined by Eq. (54) is the special (on-shell) TETA amplitude for the $\pi^+p\gamma$ at the tree level. It is gauge invariant,

$$\tilde{M}_\mu^{\text{TETAS}}k^\mu=0\ , \tag{56}$$

and it depends only upon the elastic T matrix, evaluated at (s_i,t_p), (s_f,t_p), (s_i,t_q), and (s_f,t_q). Moreover, it is easy to show that $\tilde{M}_\mu^{\text{TETAS}}$ has no kinematic singularity at $k=0$. The amplitude $\tilde{M}_\mu^{\text{off}}$ given by Eq. (55), on the other hand, is an off-shell amplitude. It depends upon the off-shell derivatives and also upon the extra off-shell amplitudes involving $\tilde{C}$. Thus, if we ignore those terms which cannot be expressed in terms of the elastic T matrix (i.e., the amplitude $\tilde{M}_\mu{}^x$) and those terms which involve off-shell effects (i.e., the amplitude $\tilde{M}_\mu^{\text{off}}$), then we obtain the TETAS amplitude $\tilde{M}_\mu^{\text{TETAS}}$,

$$\tilde{M}_\mu\approx\tilde{M}_\mu^{\text{TETAS}}\ , \tag{57}$$

which can be evaluated exactly in terms of the π^+p elastic T matrix and the electromagnetic constants of p, π^+ and Δ^{++}.

For the purpose of comparison between the modified procedure and Low's standard (original) procedure for deriving a soft-photon amplitude, let us derive a different version of the TETAS amplitude by using the standard procedure. We first expand $\tilde{M}_\mu^{(E)}$ in powers of k. Substituting Eqs. (17a)–(17d) into Eq. (19), we obtain

$$\tilde{M}_\mu^{(E)}=\bar{u}(p_f,\nu_f)\left[Q_a\frac{q_{f\mu}}{q_f\cdot k}\tilde{T}(s_i,t_p)-\tilde{T}(s_f,t_p)Q_b\frac{q_{i\mu}}{q_i\cdot k}+Q_c\frac{(p_f+R_f)_\mu}{p_f\cdot k}\tilde{T}(s_i,t_q)-\tilde{T}(s_f,t_q)Q_d\frac{(p_i+R_i)_\mu}{p_i\cdot k}\right.$$
$$+2Q_aq_{f\mu}\left[\frac{\partial\tilde{T}_a}{\partial\Delta_a}\right]+2Q_bq_{i\mu}\left[\frac{\partial\tilde{T}_b}{\partial\Delta_b}\right]+2Q_cp_{f\mu}\left[\frac{\partial\tilde{T}'_c}{\partial\Delta_c}\right]+2Q_dp_{i\mu}\left[\frac{\partial\tilde{T}'_d}{\partial\Delta_d}\right]$$
$$\left.+(\text{terms involving }\tilde{C})+\cdots\right]u(p_i,\nu_i)\ . \tag{58}$$

Since those terms involving $\tilde{C}$ will be completely ignored later, we shall neglect them in the rest of our derivation. The second step is to obtain the leading term of the internal amplitude $\tilde{M}_\mu^{(I)}$ by imposing the gauge invariant condition:

$$k^\mu\tilde{M}_\mu^{(I)}=-k^\mu\tilde{M}_\mu^{(E)}$$
$$=-\bar{u}(p_f,\nu_f)\left[\tilde{T}^{(EK)}+2Q_aq_f\cdot k\left[\frac{\partial\tilde{T}_a}{\partial\Delta_a}\right]+2Q_bq_i\cdot k\left[\frac{\partial\tilde{T}_b}{\partial\Delta_b}\right]\right.$$
$$\left.+2Q_cp_f\cdot k\left[\frac{\partial\tilde{T}'_c}{\partial\Delta_c}\right]+2Q_dp_i\cdot k\left[\frac{\partial\tilde{T}'_d}{\partial\Delta_d}\right]+\cdots\right]u(p_i,\nu_i)\ , \tag{59}$$

where

$$\tilde{T}^{(EK)}=Q_a\tilde{T}(s_i,t_p)-Q_b\tilde{T}(s_f,t_p)+Q_c\tilde{T}(s_i,t_q)-Q_d\tilde{T}(s_f,t_q) \tag{60}$$

and we have $Q_a=Q_b$ and $Q_c=Q_d$ for the $\pi^+p\gamma$ process. The amplitude $\tilde{T}^{(EK)}$ has been studied by Fischer and Minkowski [13] and also by Heller [23]. For example, Heller has used the mean value theorem for derivatives,

$$T(s_i,t_p)-T(s_f,t_p)=(s_i-s_f)\frac{\partial T(s_0,t_p)}{\partial s}=2(q_i+p_i)\cdot k\frac{\partial T(s_0,t_p)}{\partial s},\quad s_f\le s_0\le s_i\ , \tag{61a}$$

and

$$T(s_i,t_q)-T(s_f,t_q)=(s_i-s_f)\frac{\partial T(s_0',t_q)}{\partial s}=2(q_i+p_i)\cdot k\frac{\partial T(s_0',t_q)}{\partial s}, \quad s_f\le s_0'\le s_i \ , \tag{61b}$$

to obtain

$$\tilde{T}^{(EK)}=2Q_a(q_i+p_i)\cdot k\frac{\partial T(s_0,t_p)}{\partial s}+2Q_c(q_i+p_i)\cdot k\frac{\partial T(s_0',t_q)}{\partial s} \ . \tag{62}$$

Inserting Eq. (62) into Eq. (59), we find

$$\tilde{M}_\mu^{(I)}=\bar{u}(p_f,\nu_f)\left[-2Q_a(q_i+p_i)_\mu\frac{\partial T(s_0,t_p)}{\partial s}-2Q_c(q_i+p_i)_\mu\frac{\partial T(s_0',t_q)}{\partial s}\right.$$
$$\left.-2Q_aq_{f\mu}\left[\frac{\partial \tilde{T}_a}{\partial \Delta_a}\right]-2Q_bq_{i\mu}\left[\frac{\partial \tilde{T}_b}{\partial \Delta_b}\right]-2Q_cp_{f\mu}\left[\frac{\partial \tilde{T}_c'}{\partial \Delta_c}\right]-2Q_dp_{i\mu}\left[\frac{\partial \tilde{T}_d'}{\partial \Delta_d}\right]+\cdots\right]u(p_i,\nu_i) \ . \tag{63}$$

If we apply Eqs. (61a) and (61b) again, we can rewrite $\tilde{M}^{(I)}$ in the form

$$\tilde{M}_\mu^{(I)}=\bar{u}(p_f,\nu_f)\left[-Q_a\frac{(p_f+q_f)_\mu}{(p_f+q_f)\cdot k}\tilde{T}(s_i,t_p)+Q_b\tilde{T}(s_f,t_p)\frac{(p_i+q_i)_\mu}{(p_i+q_i)\cdot k}-Q_c\frac{(p_f+q_f)_\mu}{(p_f+q_f)\cdot k}\tilde{T}(s_i,t_q)\right.$$
$$+Q_d\tilde{T}(s_f,t_q)\frac{(p_i+q_i)_\mu}{(p_i+q_i)\cdot k}-2Q_aq_{f\mu}\left[\frac{\partial \tilde{T}_a}{\partial \Delta_a}\right]-2Q_bq_{i\mu}\left[\frac{\partial \tilde{T}_b}{\partial \Delta_b}\right]$$
$$\left.-2Q_cp_{f\mu}\left[\frac{\partial \tilde{T}_c'}{\partial \Delta_c}\right]-2Q_dp_{i\mu}\left[\frac{\partial \tilde{T}_d'}{\partial \Delta_d}\right]+\cdots\right]u(p_i,\nu_i) \ . \tag{64}$$

Here, we have used the fact that $(q_i+p_i)\cdot k=(q_f+p_f)\cdot k$, $(q_i+p_i)\cdot \varepsilon=(q_f+p_f)\cdot \varepsilon$, $Q_a=Q_b$, and $Q_c=Q_d$. Finally, we add $\tilde{M}_\mu^{(E)}$ [Eq. (58)] and $\tilde{M}_\mu^{(I)}$ [Eq. (64)] to obtain the total amplitude $\tilde{M}_\mu'$:

$$\tilde{M}_\mu'=\bar{u}(p_f,\nu_f)\left[Q_a\left[\frac{q_{f\mu}}{q_f\cdot k}-\frac{(p_f+q_f)_\mu}{(p_f+q_f)\cdot k}\right]\tilde{T}(s_i,t_p)-Q_b\tilde{T}(s_f,t_p)\left[\frac{q_{i\mu}}{q_i\cdot k}-\frac{(p_i+q_i)_\mu}{(p_i+q_i)\cdot k}\right]\right.$$
$$\left.+Q_c\left[\frac{(p_f+R_f)_\mu}{p_f\cdot k}-\frac{(p_f+q_f)_\mu}{(p_f+q_f)\cdot k}\right]\tilde{T}(s_i,t_q)-Q_d\tilde{T}(s_f,t_q)\left[\frac{(p_i+R_i)_\mu}{p_i\cdot k}-\frac{(p_i+q_i)_\mu}{(p_i+q_i)\cdot k}\right]+\cdots\right]u(p_i,\nu_i) \ , \tag{65}$$

which is to be compared with the amplitude $\tilde{M}_\mu$ obtained by using the modified procedure,

$$\tilde{M}_\mu\approx\tilde{M}_\mu^{\text{TETAS}}+\tilde{M}_\mu^{\text{off}} \ , \tag{66}$$

where $\tilde{M}_\mu^{\text{TETAS}}$ and $\tilde{M}_\mu^{\text{off}}$ are given by Eqs. (54) and (55), respectively. [But we neglect those terms involving $\tilde{T}_{cc}$ and $\tilde{T}_{dc}$ in Eq. (55) in this comparison.] Two substantial differences can be observed: (i) Equation (65) shows that the first two terms in the series expansion of the amplitude $\tilde{M}_\mu'$ in powers of k are independent of off-shell derivatives. Equation (66), on the other hand, shows that the amplitude $\tilde{M}_\mu$ does depend upon off-shell derivatives of order k^0. (ii) The amplitude $\tilde{M}_\mu$ has extra terms involving R_μ. These terms represent photon emission from the internal Δ^{++} line with spin-$\frac{3}{2}$ and the anomalous magnetic moment λ_Δ. The amplitude $\tilde{M}_\mu'$ does not include any R_μ term which is separately gauge invariant, $R_\mu k^\mu=0$. This is because the standard procedure [or the gauge invariant condition, Eq. (59)] cannot be used to determine an internal term which is separately gauge invariant. [Note that both $\tilde{M}_\mu$ and $\tilde{M}_\mu'$ include internal terms which are proportional to either $(q_i+p_i)_\mu$ or $(q_f+p_f)_\mu$. These internal terms represent photon emission from the charge of the interval Δ^{++} line.]

C. General TETAS amplitude and modified Low procedure

The result obtained in the preceding section will be used as an important guide to develop a modified procedure for constructing a more general TETA amplitude for the $\pi^+p\gamma$ process. The first step in the modified procedure is exactly the same as the one used in Low's standard procedure. We obtain the external amplitude $M_\mu^{(E)}$ from four external emission diagrams, Figs. 4(a)–4(d), which are generated from the source diagram shown in Fig. 1(a). We find

$$M_\mu^{(E)}=\bar{u}(p_f,\nu_f)\left[\frac{Q_aq_{f\mu}}{q_f\cdot k}T_a-T_b\frac{Q_bq_{i\mu}}{q_i\cdot k}+\frac{Q_c(p_f+R_f)_\mu}{p_f\cdot k}T_c-T_d\frac{Q_d(p_i+R_i)_\mu}{p_i\cdot k}\right]u(p_i,\nu_i) \ , \tag{67}$$

where $R_{i\mu}$ and $R_{f\mu}$ are defined by Eqs. (20a) and (20b), respectively, and T_x $(x=a,b,c,d)$, which are the half-off-shell T

matrices, can be written in the form

$$\bar{u}T_a u \equiv \bar{u}T[s_i,t_p,p_i^2=m_p^2,q_i^2=m_\pi^2,p_f^2=m_p^2,\Delta_a=(q_f+k)^2]u$$

$$=\bar{u}\{A(s_i,t_p,m_p^2,m_\pi^2,m_p^2,\Delta_a)+\tfrac{1}{2}(\not{q}_i+\not{q}_f+\not{K})B(s_i,t_p,m_p^2,m_\pi^2,m_p^2,\Delta_a)\}u \ , \tag{68a}$$

$$\bar{u}T_b u \equiv \bar{u}T[s_f,t_p,p_i^2=m_p^2,\Delta_b=(q_i-k)^2,p_f^2=m_p^2,q_f^2=m_\pi^2]u$$

$$=\bar{u}\{A(s_f,t_p,m_p^2,\Delta_b,m_p^2,m_\pi^2)+\tfrac{1}{2}(\not{q}_i+\not{q}_f-\not{K})B(s_f,t_p,m_p^2,\Delta_b,m_p^2,m_\pi^2)\}u \ , \tag{68b}$$

$$T_c u \equiv T[s_i,t_q,p_i^2=m_p^2,q_i^2=m_\pi^2,\Delta_c=(p_f+k)^2,q_f^2=m_\pi^2]u$$

$$=\{A(s_i,t_q,m_p^2,m_\pi^2,\Delta_c,m_\pi^2)+\tfrac{1}{2}(\not{q}_i+\not{q}_f)B(s_i,t_q,m_p^2,m_\pi^2,\Delta_c,m_\pi^2)+\tfrac{1}{2}(\not{p}_f+\not{K}-m_p)C(s_i,t_q,m_p^2,m_\pi^2,\Delta_c,m_\pi^2)\}u \ , \tag{68c}$$

and

$$\bar{u}T_d \equiv \bar{u}T[s_f,t_q,\Delta_d=(p_i-k)^2,q_i^2=m_\pi^2,p_f^2=m_p^2,q_f^2=m_\pi^2]$$

$$=\bar{u}\{A(s_f,t_q,\Delta_d,m_\pi^2,m_p^2,m_\pi^2)+\tfrac{1}{2}(\not{q}_i+\not{q}_f)B(s_f,t_q,\Delta_d,m_\pi^2,m_p^2,m_\pi^2)+\tfrac{1}{2}C'(s_f,t_q,\Delta_d,m_\pi^2,m_p^2,m_\pi^2)(\not{p}_i-\not{K}-m_p)\} \ . \tag{68d}$$

In Eqs. (69c) and (69d), the expressions for T_c and T_d have extra off-shell terms involving amplitude C or C'. These extra off-shell terms vanish on the mass-shell. The expressions for off-shell amplitudes A, B, C, and C' are much more complicated than those for off-shell amplitudes $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ at the tree level. [See Eqs. (12) and (16).] However, since the Feynman diagrams given by Figs. 2(a)–2(e) are the dominant contribution to the $\pi^+ p\gamma$ process in the energy region of the $\Delta^{++}(1232)$ resonance, we expect that A reduces to $\tilde{A}$, B reduces to $\tilde{B}$, and C and C' reduce to $\tilde{C}$ when Figs. 4(a)–4(e) reduced to Figs. 2(a)–2(e), respectively. The on-shell values of the amplitudes A, B, C, and C' are defined by

$$A(s_\alpha,t_\beta)\equiv A(s_\alpha,t_\beta,m_p^2,m_\pi^2,m_p^2,m_\pi^2) \ , \tag{69a}$$

$$B(s_\alpha,t_\beta)\equiv B(s_\alpha,t_\beta,m_p^2,m_\pi^2,m_p^2,m_\pi^2) \ , \tag{69b}$$

$$C(s_i,t_q)\equiv C(s_i,t_q,m_p^2,m_\pi^2,m_p^2,m_\pi^2) \ , \tag{69c}$$

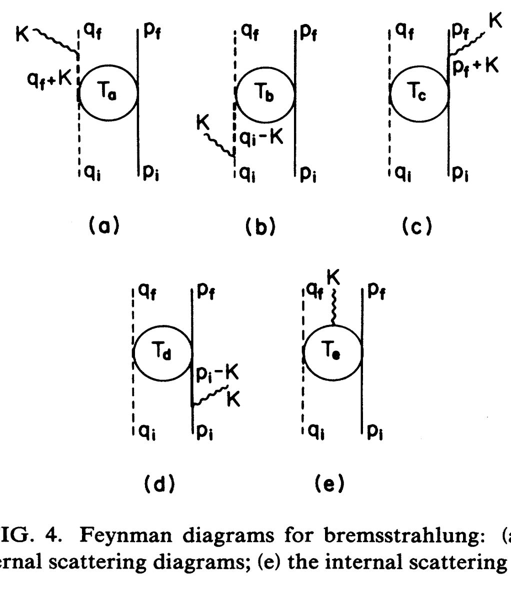


FIG. 4. Feynman diagrams for bremsstrahlung: (a)–(d) the external scattering diagrams; (e) the internal scattering diagram.

and

$$C'(s_f,t_q)\equiv C'(s_f,t_q,m_p^2,m_\pi^2,m_p^2,m_\pi^2) \ , \tag{69d}$$

with $\alpha=i$ or f and $\beta=p$ or q. The on-shell amplitudes $A(s_\alpha,t_\beta)$ and $B(s_\alpha,t_\beta)$, which determine the on-shell (elastic) π^+p T matrix $T(s_\alpha,t_\beta)$, can be calculated in terms of π^+p phase shifts and inelasticities, determined by the π^+p elastic scattering experiments. Since the expressions for $C(s_i,t_q)$ and $C'(s_f,t_q)$ are not known, all extra terms involving C and C' have been completely ignored in the on-shell soft-photon approximation.

The second step is to find an internal amplitude $M_\mu^{(\Delta)}$ which represents photon emission from the intermediate Δ^{++} line. The idea is to write $M_\mu^{(\Delta)}$ as a linear combination of T_x $(x=a,b,c,d)$ and D:

$$\varepsilon^\mu M_\mu^{(\Delta)}=\bar{u}(p_f,\nu_f)(Y_aT_a+T_bY_b+Y_cT_c+T_dY_d+\varepsilon\cdot D)u(p_i,\nu_i) \ , \tag{70}$$

where Y_x $(x=a,b,c,d)$ are the coefficients to be determined and $\varepsilon\cdot D$ represents the remainder of other terms which cannot be written in terms of T_x. To determine Y_x $(x=a,b,c,d)$, we demand that $M_\mu^{(\Delta)}$ reduce to $\tilde{M}_\mu^{(\Delta)}$ given by Eq. (44) when Fig. 4(e) reduces to Fig. 2(e). Since T_x $(x=a,b,c,d)$ reduce to $\tilde{T}_x$ and D reduces to $\tilde{D}$, we find

$$Y_a=-Q_a\frac{\varepsilon\cdot(p_f+q_f)+\varepsilon\cdot R}{(p_f+q_f)\cdot k} \ ,$$

$$Y_b=Q_b\frac{\varepsilon\cdot(p_i+q_i)+\varepsilon\cdot R}{(p_i+q_i)\cdot k} \ , \tag{71}$$

$$Y_c=-Q_c\frac{\varepsilon\cdot(p_f+q_f)+\varepsilon\cdot R}{(p_f+q_f)\cdot k} \ ,$$

and

$$Y_d = Q_d \frac{\varepsilon\cdot(p_i+q_i)+\varepsilon\cdot R}{(p_i+q_i)\cdot k} .$$

Combining $M_\mu^{(E)}$ with $M_\mu^{(\Delta)}$, we obtain

$$\varepsilon^\mu M_\mu^{(E\Delta)} = \varepsilon^\mu M_\mu^{\text{TETA}} + \bar{u}(p_f,\nu_f)(\varepsilon^\mu D_\mu)u(p_i,\nu_i) , \tag{71a}$$

where

$$M_\mu^{\text{TETA}} = \bar{u}(p_f,\nu_f)\left[Q_a \left[\frac{q_{f\mu}}{q_f\cdot k} - \frac{(p_f+q_f+R)_\mu}{(p_f+q_f)\cdot k} \right] T_a - Q_b T_b \left[\frac{q_{i\mu}}{q_i\cdot k} - \frac{(p_i+q_i+R)_\mu}{(p_i+q_i)\cdot k} \right] + Q_c \left[\frac{(p_f+R_f)_\mu}{p_f\cdot k} - \frac{(p_f+q_f+R)_\mu}{(p_f+q_f)\cdot k} \right] T_c - Q_d T_d \left[\frac{(p_i+R_i)_\mu}{p_i\cdot k} - \frac{(p_i+q_i+R)_\mu}{(p_i+q_i)\cdot k} \right] \right] u(p_i,\nu_i) . \tag{71b}$$

Since D_μ cannot be expressed in terms of T_x, it has been ignored in the soft-photon approximation.

The third step is to impose the gauge invariant condition,

$$k^\mu(M_\mu^{(E\Delta)} + M_\mu^{(G)}) = 0 , \tag{72a}$$

in order to obtain an additional gauge term $M_\mu^{(G)}$ so that the total amplitude, $M_\mu = M_\mu^{(E\Delta)} + M_\mu^{(G)}$, will be completely gauge invariant. Since

$$k^\mu M_\mu^{\text{TETA}} = 0 ,$$

the condition (72a) gives

$$k^\mu M_\mu^{(G)} = -k^\mu D_\mu , \tag{72b}$$

which shows that $M_\mu^{(G)}$ can be determined if the detailed expression for D_μ is known. However, since D_μ has been ignored in the soft-photon approximation, $M_\mu^{(G)}$ will also be ignored in our derivation. Thus, the total amplitude has the form

$$M_\mu = M_\mu^{\text{TETA}} \tag{73}$$

if D_μ and $M_\mu^{(G)}$ are neglected. Equation (73) shows that the off-shell TETA amplitude M_μ^{TETA} is an approximate amplitude which can be rigorously derived for the $\pi^+ p\gamma$ process.

Finally, to obtain a special on-shell TETAS amplitude M_μ^{TETAS}, we have to expand T_x ($x=a,b,c,d$). We obtain

$$M_\mu^{\text{TETA}} = M_\mu^{\text{TETAS}} + M_\mu^{\text{off}} , \tag{74}$$

where

$$M_\mu^{\text{TETAS}} = \bar{u}(p_f,\nu_f)\left[Q_a \left[\frac{q_{f\mu}}{q_f\cdot k} - \frac{(p_f+q_f+R)_\mu}{(p_f+q_f)\cdot k} \right] T(s_i,t_p) - Q_b T(s_f,t_p) \left[\frac{q_{i\mu}}{q_i\cdot k} - \frac{(p_i+q_i+R)_\mu}{(p_i+q_i)\cdot k} \right] + Q_c \left[\frac{(p_f+R_f)_\mu}{p_f\cdot k} - \frac{(p_f+q_f+R)_\mu}{(p_f+q_f)\cdot k} \right] T(s_i,t_q) - Q_d T(s_f,t_q) \left[\frac{(p_i+R_i)_\mu}{p_i\cdot k} - \frac{(p_i+q_i+R)_\mu}{(p_i+q_i)\cdot k} \right] \right] u(p_i,\nu_i) , \tag{75}$$

and M_μ^{off} represents the rest of the other terms which include off-shell derivatives of the amplitudes A and B and extra off-shell terms involving amplitudes C and C'. The amplitude M_μ^{TETAS}, Eq. (75), is identical to the one given by Eq. (1) in Ref. [29]. Because of different definitions for $R_{i\mu}$, $R_{f\mu}$, and R_μ used in this work, there is a sign difference for those terms involving $R_{i\mu}$, $R_{f\mu}$, and R_μ in the expression for M_μ^{TETAS}. In Eq. (75), $T(s_i,t_p)$, $T(s_f,t_p)$, $T(s_i,t_q)$, and $T(s_f,t_q)$ are the elastic $\pi^+ p$ T matrices, evaluated at (s_i,t_p), (s_f,t_p), (s_i,t_q), and (s_f,t_q), respectively. (See Ref. [28] for a discussion on the calculation of these T matrices.) We have shown that if we neglect those terms which cannot be expressed in terms of the T matrix (i.e., $\varepsilon\cdot D$) and ignore all off-shell terms (i.e., $\varepsilon\cdot M^{\text{off}}$), then we obtain the amplitude M_μ^{TETAS} which can be calculated exactly in terms of the $\pi^+ p$ elastic T matrix and the electromagnetic constants of p, π^+, and Δ^{++}.

III. THE "EXPERIMENTAL" MAGNETIC MOMENT OF $\Delta^{++}(1232)$ EXTRACTED FROM EXPERIMENTAL DATA

We have used the special two-energy-two-angle amplitude, M_μ^{TETAS} given by Eq. (75), to calculate $\pi^+ p\gamma$ cross sections as a function of photon energy k,

$$\frac{d^3\sigma}{d\Omega_\pi d\Omega_\gamma dk}=\frac{J}{(2\pi)^5}\int \delta^4(p_i+q_i-p_f-q_f-k)\left\{\frac{1}{2}\sum_{\text{pol,spin}}(M_\mu^{\text{TETAS}}\varepsilon^\mu)^\dagger(M_\nu^{\text{TETAS}}\varepsilon^\nu)\right\}d^4F\ , \tag{76}$$

where

$$J=e^2m_p^2/[(p_i\cdot q_i)^2-m_\pi^2m_p^2]^{1/2}\ ,$$

$$d^4F=[q_f^2dq_f/(2E_\pi)][d^3\mathbf{p}_f/(2E_p)][k^2/(2k)]\ ,$$

$$E_\pi=(m_\pi^2+\mathbf{q}_f^2)^{1/2}\ ,$$

and

$$E_p=(m_p^2+\mathbf{p}_f^2)^{1/2}\ .$$

In these calculations, the anomalous magnetic moment of the Δ^{++}, λ_Δ, has been treated as a free parameter and it is to be determined from the UCLA data [2] at three bombarding energies, 269, 298, and 324 MeV, and the SIN data [5] at 299 MeV.

The UCLA group has used 19 photon counters, G_i $(i=1-19)$, to measure $\pi^+p\gamma$ differential cross sections. As a result, 18 sets of cross sections have been obtained for each bombarding energy. (Cross sections for the photon counter G16 have not been determined.) In each set, the UCLA data are given at the following photon energies: $k_1=22.5$ MeV, $k_2=40$ MeV, $k_3=60$ MeV, $k_4=80$ MeV, $k_5=100$ MeV, $k_6=120$ MeV, and $k_7=140$ MeV. We shall use the "spectrum G_i" to label the set of cross sections obtained from the photon counter G_i. Thus, if we define $E_1=269$ MeV, $E_2=298$ MeV, and $E_3=324$ MeV, then the UCLA data can be denoted by $\sigma^{\text{UCLA}}(E_i,G_j,k_l)$, which represents cross section at the bombarding energy E_i $(i=1,2,3)$ and the photon energy k_l $(l=1,\ldots,7)$ for the spectrum G_j $(j=1,\ldots,19)$. The corresponding theoretical cross section, calculated using Eq. (76), will be denoted by $\sigma^{\text{th}}(E_i,G_j,k_l)$.

Using the experimental cross sections $\sigma^{\text{UCLA}}(E_i,G_j,k_l)$ and the theoretical cross sections $\sigma^{\text{th}}(E_i,G_j,k_l)$, we first calculated the following average cross sections:

$$\sigma_{1-10}^{\text{UCLA}}(E_i,k_l)=\sum_{j=1}^{10}\sigma^{\text{UCLA}}(E_i,G_j,k_l)/10\ , \tag{77a}$$

$$\sigma_{1-10}^{\text{th}}(E_i,k_l)=\sum_{j=1}^{10}\sigma^{\text{th}}(E_i,G_j,k_l)/10\ , \tag{77b}$$

$$\sigma_{11-15}^{\text{UCLA}}(E_i,k_l)=\sum_{j=11}^{15}\sigma^{\text{UCLA}}(E_i,G_j,k_l)/5\ , \tag{77c}$$

$$\sigma_{11-15}^{\text{th}}(E_i,k_l)=\sum_{j=11}^{15}\sigma^{\text{th}}(E_i,G_j,k_l)/5\ , \tag{77d}$$

$$\sigma_{1-15}^{\text{UCLA}}(E_i,k_l)=\sum_{j=1}^{15}\sigma^{\text{UCLA}}(E_i,G_j,k_l)/15\ , \tag{77e}$$

and

$$\sigma_{1-15}^{\text{th}}(E_i,k_l)=\sum_{j=1}^{15}\sigma^{\text{th}}(E_i,G_j,k_l)/15\ , \tag{77f}$$

for all photon energies k_l $(l=1,\ldots,7)$ at three bombarding energies, E_i $(i=1,2,3)$. The values of $\sigma_{1-10}^{\text{UCLA}}(E_i,k_l)$, $\sigma_{11-15}^{\text{UCLA}}(E_i,k_l)$, and $\sigma_{1-15}^{\text{UCLA}}(E_i,k_l)$, without including the experimental errors, are shown in Table I. We then use these cross sections to define the following average deviations:

$$D_{1-10}(E_i,\lambda_\Delta)=\sum_l\frac{|\sigma_{1-10}^{\text{UCLA}}(E_i,k_l)-\sigma_{1-10}^{\text{th}}(E_i,k_l)|}{\sigma_{1-10}^{\text{UCLA}}(E_i,k_l)}\ , \tag{78a}$$

$$D_{11-15}(E_i,\lambda_\Delta)=\sum_l\frac{|\sigma_{11-15}^{\text{UCLA}}(E_i,k_l)-\sigma_{11-15}^{\text{th}}(E_i,k_l)|}{\sigma_{11-15}^{\text{UCLA}}(E_i,k_l)}\ , \tag{78b}$$

and

$$D_{1-15}(E_i,\lambda_\Delta)=\sum_l\frac{|\sigma_{1-15}^{\text{UCLA}}(E_i,k_l)-\sigma_{1-15}^{\text{th}}(E_i,k_l)|}{\sigma_{1-15}^{\text{UCLA}}(E_i,k_l)}\ . \tag{78c}$$

Since there are three bombarding energies, we obtain nine deviation functions, which are all functions of λ_Δ. Varying the value of λ_Δ, we find nine deviation curves. As shown in Figs. 5(a)–5(c), each of these nine deviation curves clearly exhibits a minimum point. The values of λ_Δ at these minimum points are as follows: $\lambda_\Delta^{1-10}(E_1)=1.53$, $\lambda_\Delta^{1-10}(E_2)=1.47$, $\lambda_\Delta^{1-10}(E_3)=1.20$, $\lambda_\Delta^{11-15}(E_1)=1.90$, $\lambda_\Delta^{11-15}(E_2)=2.00$, $\lambda_\Delta^{11-15}(E_3)=1.44$, $\lambda_\Delta^{1-15}(E_1)=1.86$, $\lambda_\Delta^{1-15}(E_2)=1.58$, and $\lambda_\Delta^{1-15}(E_3)=1.36$. Taking an average, we have

$$\lambda_\Delta=\sum_{i=1}^{3}\lambda_\Delta^{1-10}(E_i)/3=1.4 \tag{79a}$$

for spectra G1–G10,

$$\lambda_\Delta=\sum_{i=1}^{3}\lambda_\Delta^{11-15}(E_i)/3=1.8 \tag{79b}$$

for spectra G11–G15, and

$$\lambda_\Delta=\sum_{i=1}^{3}\lambda_\Delta^{1-15}(E_i)/3=1.6 \tag{79c}$$

for spectra G1–G15. Using these results for λ_Δ, the value of the "experimental" magnetic moment of the $\Delta^{++}(1232)$, μ_Δ, can be calculated. We find

$$\mu_\Delta = 2(1+\lambda_\Delta)\frac{e}{2M_\Delta}$$
$$= 2(1+\lambda_\Delta)\frac{m_p}{M_\Delta}\left[\frac{e}{2m_p}\right]$$
$$= \begin{cases} 3.7\dfrac{e}{2m_p} & \text{for spectra G1–G10 ,} \\ 4.0\dfrac{e}{2m_p} & \text{for spectra G1–G15 ,} \\ 4.2\dfrac{e}{2m_p} & \text{for spectra G11–G15 .} \end{cases} \tag{80}$$

This gives us a range of the value of μ_Δ which can be extracted from the UCLA data.

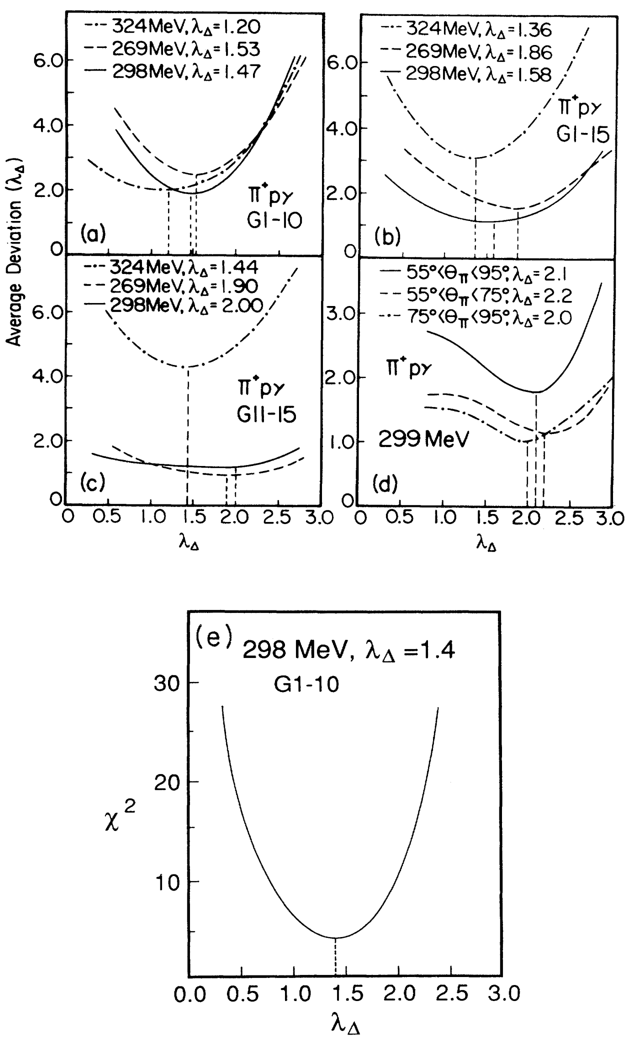


FIG. 5. Average deviation as a function of λ_Δ. The deviation curves shown in (a), (b), and (c) are obtained from the UCLA data while the curves shown in (d) are from the SIN data. The minimum point on each curve determines an extracted value of λ_Δ. (e) The χ^2 as a function of λ_Δ. The UCLA data at 298 MeV are used to calculate the χ^2 curve, which shows a clear minimum point at $\lambda_\Delta = 1.4$.

Of course, the UCLA data sets can be analyzed together to yield a single value of μ_Δ. This has been done and the value is $3.8e/(2m_p)$. It should be pointed out that if the χ^2 as a function of λ_Δ were used to determine the value of λ_Δ, then we would also obtain about the same result as that obtained by using the deviation function defined by Eq. (78). For spectra G1–G10 at 298 MeV, for example, the χ^2 fit gives $\lambda_\Delta = 1.4$ while the method based on Eq. (78) gives $\lambda_\Delta \equiv \lambda_\Delta^{1-10}(E_2) = 1.47$. In Fig. 5(e), we show the χ^2 curve with a clear minimum point at $\lambda_\Delta = 1.4$. This curve is to be compared with the solid curve exhibited in Fig. 5(a).

We have also extracted the value of μ_Δ from the SIN data. The SIN group has measured the $\pi^+ p\gamma$ cross sections at 299 MeV. Depending upon the angular regions for the outgoing pions, the group has obtained three sets of cross sections. We shall call the set for $55° < \theta_\pi < 95°$ as the first set, the set for $55° < \theta_\pi < 75°$ as the second set, and the set for $75° < \theta_\pi < 95°$ as the third set. In each set, the SIN data are given at the following photon energies: $k_1' = 27.5$ MeV, $k_2' = 42.5$ MeV, $k_3' = 57.5$ MeV, $k_4' = 72.5$ MeV, $k_5' = 87.5$ MeV, $k_6' = 102.5$ MeV, and $k_7' = 117.5$ MeV. Thus, the SIN data will be denoted by $\sigma_i^{\rm SIN}(k_j')$ and the corresponding theoretical cross section by $\sigma_i^{\rm th}(k_j')$. Here, $k_j'(j = 1, \ldots, 7)$ are photon energies and the subscript i indicates the set number $(i = 1,2,3)$. The values of $\sigma_i^{\rm SIN}(k_j')$, without including the experimental errors, are shown in Table I. Using $\sigma_i^{\rm SIN}(k_j')$ and $\sigma_i^{\rm th}(k_j')$, we define three deviation functions:

$$D_1(\lambda_\Delta) = \sum_j \frac{|\sigma_1^{\rm SIN}(k_j') - \sigma_1^{\rm th}(k_j')|}{\sigma_1^{\rm SIN}(k_j')} , \tag{81a}$$

$$D_2(\lambda_\Delta) = \sum_j \frac{|\sigma_2^{\rm SIN}(k_j') - \sigma_2^{\rm th}(k_j')|}{\sigma_2^{\rm SIN}(k_j')} , \tag{81b}$$

and

$$D_3(\lambda_\Delta) = \sum_j \frac{|\sigma_3^{\rm SIN}(k_j') - \sigma_3^{\rm th}(k_j')|}{\sigma_3^{\rm SIN}(k_j')} , \tag{81c}$$

which are all functions of λ_Δ. Varying the value of λ_Δ, we obtain three deviation curves. As shown in Fig. 5(d), each curve has a minimum point. The values of λ_Δ at these minimum points are

$$\lambda_\Delta = 2.1$$

for the first set, $55° < \theta_\pi < 95°$,

$$\lambda_\Delta = 2.2$$

for the second set, $55° < \theta_\pi < 75°$, and

$$\lambda_\Delta = 2.0$$

for the third set, $75° < \theta_\pi < 95°$. The values of μ_Δ calculated from λ_Δ are as follows

TABLE I. The UCLA data and the SIN data: the values of $\sigma_{1-10}^{\mathrm{UCLA}}(E_i,k_l)$, $\sigma_{11-15}^{\mathrm{UCLA}}(E_i,k_l)$, $\sigma_{1-15}^{\mathrm{UCLA}}(E_i,k_l)$, and $\sigma_i^{\mathrm{SIN}}(k_j')$.

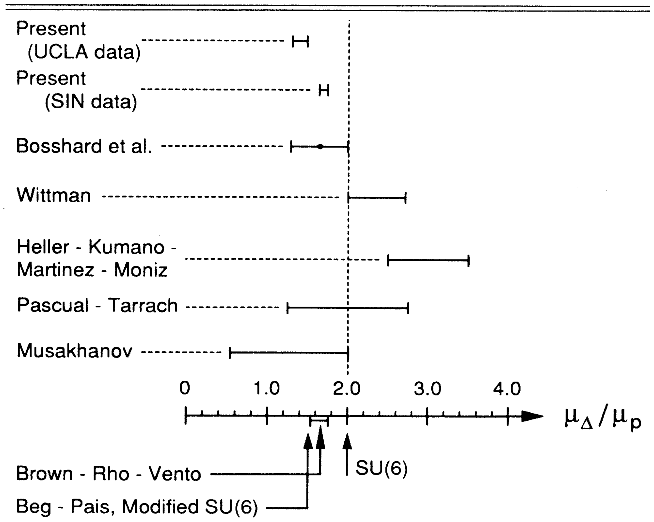
Photon energy (MeV)		22.5	40.0	60.0	80.0	100.0	120.0	140.0
269	average	2.53	1.65	1.25	0.79	0.28		
(UCLA)	G1–10	±0.41	±0.27	±0.21	±0.15	±0.08		
	average G11–15	31.8	15.1	8.8	5.8	2.5	1.2	
	average G1–15	12.3	6.11	3.77	2.44	1.03	0.55	
298	average	1.78	1.03	1.03	0.74	0.44	0.44	
(UCLA)	G1–10	±0.28	±0.17	±0.15	±0.13	±0.09	±0.12	
	average G11–15	25.0	14.1	7.62	5.68	3.38	1.14	
	average G1–15	9.50	5.41	3.21	2.39	1.44	0.61	
324	average	1.05	1.04	0.66	0.77	0.30	0.10	
(UCLA)	G1–10	±0.31	±0.22	±0.17	±0.18	±0.11	±0.06	
	average G11–15	22.6	8.30	4.80	2.22	1.64	0.58	0.38
	average G1–15	8.24	3.45	2.05	1.25	0.74	0.43	
Photon energy (MeV)		27.5	42.5	57.5	72.5	87.5	102.5	117.5
299	average	1.57	1.24	1.40	1.21	1.12	0.85	0.71
(SIN)	55°–95°	±0.23	±0.18	±0.16	±0.14	±0.13	±0.10	±0.09
	average	1.23	1.31	1.40	1.20	1.27	0.83	0.82
	55°–75°	±0.28	±0.21	±0.18	±0.16	±0.17	±0.13	±0.11
	average	1.91	1.17	1.40	1.22	0.90	0.90	
	75°–95°	±0.30	±0.24	±0.20	±0.18	±0.16	±0.13	

$$\mu_\Delta = 2(1+\lambda_\Delta)\frac{m_p}{M_\Delta}\left[\frac{e}{2m_p}\right] = \begin{cases} 4.7\dfrac{e}{2m_p} & \text{for } 55^\circ < \theta_\pi < 95^\circ\ , \\ 4.9\dfrac{e}{2m_p} & \text{for } 55^\circ < \theta_\pi < 75^\circ\ , \\ 4.6\dfrac{e}{2m_p} & \text{for } 75^\circ < \theta_\pi < 95^\circ\ . \end{cases} \tag{82}$$

The range of μ_Δ determined by the SIN data is therefore $4.6e/(2m_p) \le \mu_\Delta \le 4.9e/(2m_p)$. If, on the other hand, all the SIN data sets are analyzed together to yield a single value of μ_Δ, we find $\mu_\Delta = 4.6e/(2m_p)$.

It is clear that the values of μ_Δ extracted from either the UCLA data or the SIN data are smaller than the "bare" magnetic moment, $\mu_\Delta = 5.58e/(2m_p)$, predicted by the SU(6) model [34] and the quark model. However, as pointed out by the UCLA group, a modified SU(6) model (with mass corrections) suggested by Beg and Pais [35] predicts $\mu_\Delta = (m_p/M_\Delta) \times 5.58e/(2m_p) = 4.25e/(2m_p)$. Moreover, Meyer *et al.* have also pointed out that bag-model corrections to the quark model [36] give $\mu_\Delta = (4.41-4.89)e/(2m_p)$. Thus the values of μ_Δ extracted from the data [the average value of μ_Δ determined from both the UCLA and the SIN data is $4.35e/(2m_p)$] are in much better agreement with the value predicted by the modified SU(6) model or the quark model with corrections. The values of μ_Δ previously obtained by other authors were 3.6±2.0 by Musakhanov [15], 5.6±2.1

TABLE II. Compilation of μ_Δ/μ_p results obtained by different groups using various approximations and methods. $\mu_p = 2.79e/(2m_p)$.



by Pascual and Tarrach [24], 7.0–9.8 by Heller *et al.* [26], and 5.58–7.53 by Wittman [27] in units of $e/(2m_p)$. All of these results were extracted from the UCLA data using quite different approximations and methods. Most recently, by fitting the asymmetry data to predictions calculated in the MIT model, Bosshard *et al.* have found $\mu_\Delta = 4.58 \pm 0.33 e/(2m_p)$ [37]. A comparison of these results is shown in Table II. For other theoretical predictions, we refer to an article published recently by Krivoruchenko *et al.* [38].

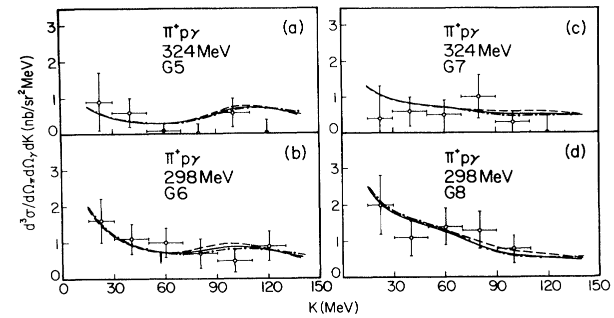


FIG. 7. Same as Fig. 6, but for G5, G6, G7, and G8.

IV. THEORETICAL $\pi^+p\gamma$ CROSS SECTIONS AND COMPARISON TO EXPERIMENTAL DATA

Using the values of μ_Δ extracted from the experimental data [Eqs. (80) and (82)] as input, we have applied the amplitude M_μ^{TETAS} to calculate all $\pi^+p\gamma$ cross sections which can be compared with the experimental data at the five bombarding energies, 165, 269, 298, 299, and 324 MeV. Some of these calculations are shown in Figs. 6–12. In these figures, the calculated cross sections at 269, 298, and 324 MeV are compared with the UCLA data of Nefkens *et al.* and the calculated cross sections at 165 and 299 MeV are compared with the UCLA data of Smith *et al.* [3] and the SIN data, respectively.

For G11–G17, the calculated cross sections are insensitive to the variation of μ_Δ between $3.7e/(2m_p)$ and $4.2e/(2m_p)$. As shown in Figs. 8, 9, and 10, the calculations using $\mu_\Delta = 3.7$, 4.0, and $4.2e/(2m_p)$ give almost identical spectra. Although different values of μ_Δ $[3.7e/(2m_p) \le \mu_\Delta \le 4.2e/(2m_p)]$ predict spectra which are slightly different at $k > 70$ MeV for G1–G10, Figs. 6, 7, and 8 show that the difference is smaller than the experimental errors. The overall agreement between theory and the UCLA data is excellent. This fact can also be seen from the following χ^2 values. We have calculated the χ^2 values for those UCLA cross sections shown in Figs. 6–9 using $\mu_\Delta = 4.0e/(2m_p)$ as an input for all theoretical predictions. The χ^2 values are 1.6(0.9), 0.4, 4.1, 5.3(2.1), 0.8(0.3), 0.5, 0.6(0.5), 0.5, 1.0, 0.6, 2.8, 8.1(1.8), 3.6(1.6), 0.7, and 1.0 for G1 at 298 MeV [Fig. 6(a)], G2 at 298 MeV [Fig. 6(b)], G3 at 269 MeV [Fig. 6(c)], G4 at 269 MeV [Fig. 6(d)], G5 at 324 MeV [Fig. 7(a)], G6 at 298 MeV [Fig. 7(b)], G7 at 324 MeV [Fig. 7(c)], G8 at 298 MeV [Fig. 7(d)], G9 at 298 MeV [Fig. 8(a)], G10 at 269 MeV [Fig. 8(b)], G11 at 298 MeV [Fig. 8(c)], G12 at 269 MeV [Fig. 8(d)], G13 at 269 MeV [Fig. 9(a)], G14 at 269 MeV [Fig. 9(b)], and G15 at 269 MeV [Fig. 9(c)], respectively. The χ^2 values in parentheses are obtained from the calculation which does not include the last datum with zero cross section. Thus, the TETAS amplitude with values of μ_Δ in the range from $3.7e/(2m_p)$ to $4.2e/(2m_p)$ can be used to describe all $\pi^+p\gamma$ data obtained by the UCLA group except for the measurements obtained for G18. Here, we must point out that our predicted cross sections for G18 at 269, 298, and 324 MeV are quite different from the UCLA data. [These three spectra for G18 are the only exceptions. The rest of other spectra at 165, 269, 298, and 324 MeV for G_i ($i = 1, \ldots, 19$ but $i \neq 18$) are in excellent agreement with the UCLA data.] However, the agreement is much better for G18 at 165 MeV. This comparison is shown in Fig. 10(b).

In Fig. 11, we present the results of our calculation using $\mu_\Delta = 4.6e/(2m_p)$, $4.7e/(2m_p)$, and $4.9e/(2m_p)$ at 299 MeV. These results are compared with the SIN data. Using $\mu_\Delta = (4.6, 4.9)e/(2m_p)$ as input for theoretical predictions, we have calculated the χ^2 values for the three sets of cross sections shown in Fig. 11. The χ^2 values [corresponding to $\mu_\Delta = (4.6, 4.9)e/(2m_p)$] are (4.3,3.4),

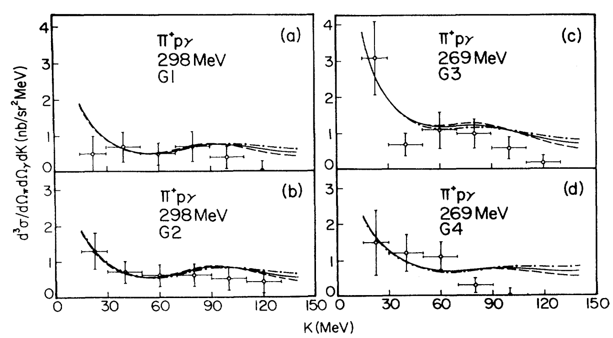


FIG. 6. The $\pi^+p\gamma$ cross sections as a function of photon energy k for G1, G2, G3, and G4. The dashed, solid, and dot-dashed curves are calculated with $\mu_\Delta = (4.2, 4.0, 3.7)e/(2m_p)$, respectively. The UCLA data are from Ref. [2].

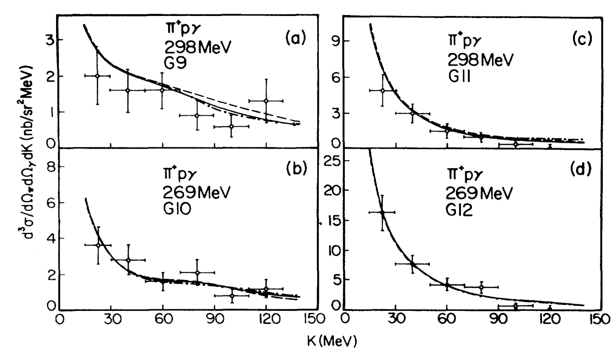


FIG. 8. Same as Fig. 6, but for G9, G10, G11, and G12.

(2.6,2.8), and (2.2,2.1) for the second set [Fig. 11(a)], the third set [Fig. 11(b)], and the first set [Fig. 11(c)], respectively. We therefore conclude that the SIN data can be described by the TETAS amplitude with the value of μ_Δ between $4.6e/(2m_p)$ and $4.9e/(2m_p)$.

Finally, we have also used $\mu_\Delta = 5.58e/(2m_p)$, the value predicted by SU(6), SU(3), and the naive quark model, to calculate $\pi^+ p\gamma$ spectra at 298 MeV for G7, G14, G15, and G1–G10 (the average cross section over the ten photon counters G1 to G10). As shown in Fig. 12, these results are compared with the calculations using $\mu_\Delta = (3.7, 4.0, 4.2)e/(2m_p)$ and also with the calculations using an approximate TETAS amplitude [M_μ(TETAS)] obtained in Ref. [28]. [The expression for M(TETAS) is given by Eq. (83) in the next section.] This comparison reveals three important facts: (i) For G14 and G15, all calculations give similar results which are in excellent agreement with the UCLA data. This confirms our statement that the calculated cross sections for G11–G19 are insensitive to the variation of μ_Δ. (ii) The data for G7 and G1–G10 can be used to differentiate between the calculation using $\mu_\Delta = 5.58e/(2m_p)$ and the calculation using $\mu_\Delta = (3.7-4.2)e/(2m_p)$. The latter is in better agreement with the data. For $k > 70$ MeV, all calculations with $\mu_\Delta = 5.58e/(2m_p)$ are in complete disagreement

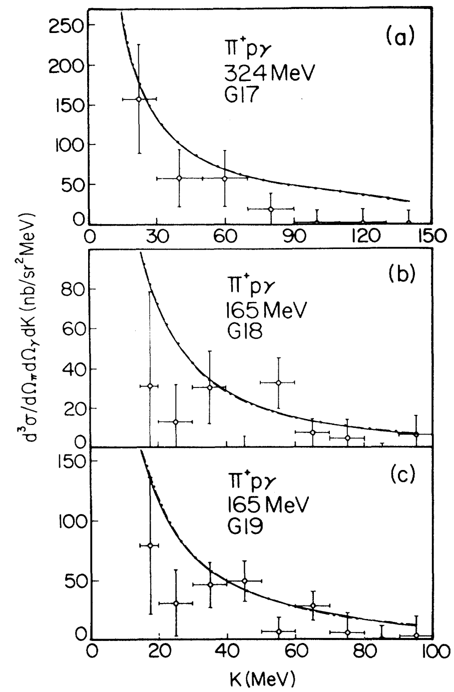


FIG. 10. Same as Fig. 6, but for G17, G18, and G19.

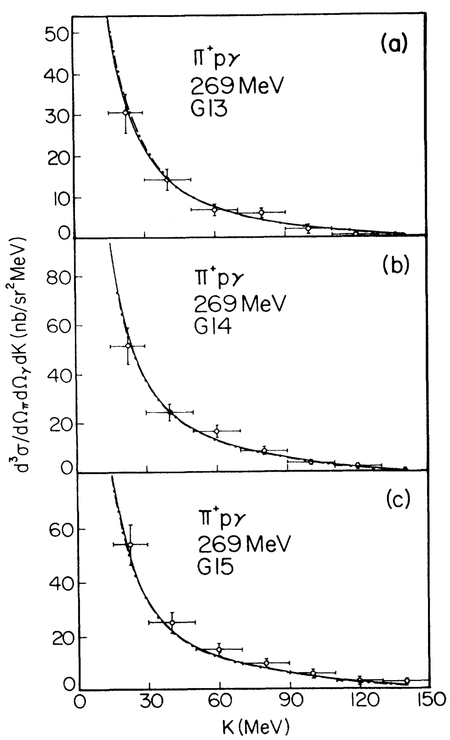


FIG. 9. Same as Fig. 6, but for G13, G14, and G15.

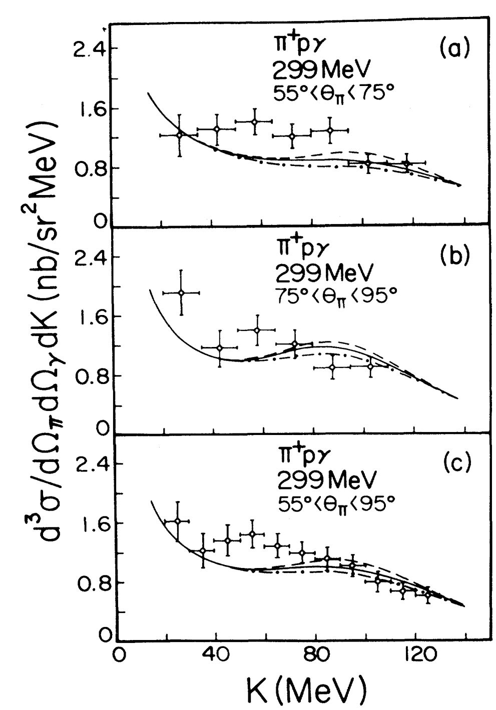


FIG. 11. The $\pi^+ p\gamma$ cross sections as a function of k. The dashed, solid, and dot-dashed curves are calculated with $\mu_\Delta = (4.9, 4.7, 4.6)e/(2m_p)$, respectively. The SIN data are from Ref. [5].

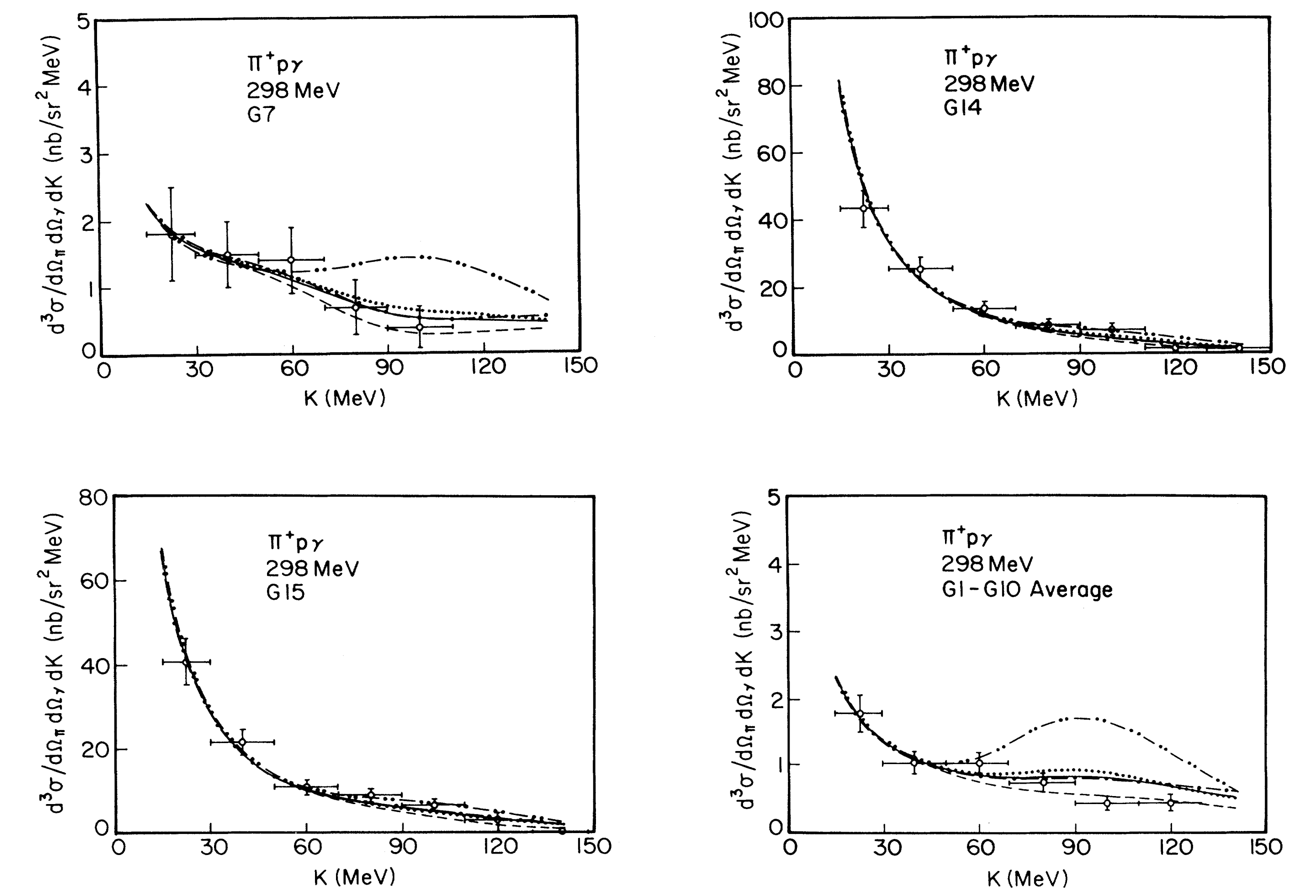


FIG. 12. The $\pi^+p\gamma$ cross sections as a function of k for G7 (a), G14 (b), G15 (c), and G1–G10 average (d). The dot-dashed, solid, dotted, and dash-double-dotted curves are calculated with $\mu_\Delta=(3.7,4.0,4.2,5.58)e/(2m_p)$, respectively. The dashed curve is calculated using an approximate TETAS amplitude given by Eq. (83) (which is identical to the amplitude M_μ(TETAS) given by Eq. (16) in Ref. [28]).

with the data. (iii) The spectra calculated using an approximate TETAS amplitude obtained in Ref. [28] are very close to those calculated using the amplitude M_μ^{TETAS} given by Eq. (75) with the value of μ_Δ between $3.7e/(2m_p)$ and $4.2e/(2m_p)$.

V. DISCUSSION

It has been reported in Ref. [28] that almost all of the $\pi^\pm p\gamma$ cross sections obtained by the UCLA group can be described by a TETAS amplitude of the form

$$M_\mu(\mathrm{TETAS})=\bar{u}(p_f,\nu_f)\left[Q_a\left[\frac{q_{f\mu}}{q_f\cdot k}-\frac{(q_f+p_f+R_f)_\mu}{(q_f+p_f)\cdot k}\right]T(s_i,t_p)-Q_bT(s_f,t_p)\left[\frac{q_{i\mu}}{q_i\cdot k}-\frac{(q_i+p_i+R_i)_\mu}{(q_i+p_i)\cdot k}\right]\right.$$
$$+Q_c\left[\frac{p_{f\mu}+R_{f\mu}}{p_f\cdot k}-\frac{(q_f+p_f+R_f)_\mu}{(q_f+p_f)\cdot k}\right]T(s_i,t_q)$$
$$\left.-Q_dT(s_f,t_q)\left[\frac{p_{i\mu}+R_{i\mu}}{p_i\cdot k}-\frac{(q_i+p_i+R_i)_\mu}{(q_i+p_i)\cdot k}\right]\right]u(p_i,\nu_i)\;, \qquad (83)$$

where $R_{i\mu}$ and $R_{f\mu}$ are defined by Eqs. (20a) and (20b), respectively. This amplitude, which cannot be rigorously derived, is slightly different from the amplitude M_μ^{TETAS} given by Eq. (75). In Sec. IV, as shown in Fig. 12, we have found that the $\pi^+p\gamma$ cross sections calculated with the amplitude M_μ(TETAS) are very close to the cross sections predicted by the amplitude M_μ^{TETAS} if the value of μ_Δ used in M_μ^{TETAS} is about $4e/(2m_p)$. To understand why the amplitude M_μ(TETAS) works so well and why the two amplitudes, M_μ(TETAS) and M_μ^{TETAS}, can give similar results, let us compare these two amplitudes carefully. From Eqs. (75) and (83), we can see that both amplitudes have the same form for the external contribution but they differ from one another in the expression for the internal contribution. They can predict about the same cross sections only under the following condition:

$$R_{i\mu} \approx R_{\mu} \approx R_{f\mu} \ . \tag{84}$$

To study this condition without any approximation is very difficult. Fortunately, a good approximation can be found. If we replace $\not{p}_i$, $\not{p}_f$, and $\not{p}$ in the expressions for $R_{i\mu}$, $R_{f\mu}$, and R_{μ} [Eqs. (20a), (20b), and (43)] by m_p, m_p, and M_{Δ}, respectively, then we find

$$R_{i\mu} = R_{f\mu} = R_{\mu} \tag{85}$$

provided that

$$\lambda_{\Delta} = \lambda_p \ . \tag{86}$$

This implies that the two amplitudes can produce about the same result if the magnetic moment of the Δ^{++} (treated as a parameter in the amplitude M_{μ}^{TETAS}) is

$$\begin{aligned} \mu_{\Delta} &= 2(1+\lambda_{\Delta})e/(2M_{\Delta}) \\ &= 2(1+\lambda_p)\frac{m_p}{M_{\Delta}}\frac{e}{2m_p} \\ &= 2(1+1.79)\frac{938}{1232}\frac{e}{2m_p} \\ &= 4.25e/(2m_p) \ , \end{aligned}$$

which is exactly the value predicted by the modified SU(6) model of Beg and Pais [35]. This value also agrees very well with the average value of μ_{Δ}, $4.35e/(2m_p)$, extracted from both the UCLA data and the SIN data. Thus, the fact that the UCLA data can be described by the amplitude M_{μ}(TETAS) suggests that the value of μ_{Δ} is about $4e/(2m_p)$.

It is obvious that the modified Low procedure can be applied to obtain TETAS amplitudes for other bremsstrahlung processes near a scattering resonance. For example, the TETAS amplitude for the $p\,^{12}C\gamma$ process has the same expression as the amplitude M_{μ}^{TETAS} given by Eq. (75) but without those terms involving $R_{i\mu}$, $R_{f\mu}$, and R_{μ} [28]. This is because the contribution from $R_{i\mu}$, $R_{f\mu}$, and R_{μ} terms is negligible for the low energy $p\,^{12}C\gamma$ process near either the 1.7-MeV resonance [39–41] or the 0.5-MeV resonance [42]. As shown in Ref. [28], those gauge terms involving $(p_i+q_i)_{\mu}$ or $(p_f+q_f)_{\mu}$ represent photon emissions from the charge of the intermediate $^{13}N^*$ resonance.

As we have already mentioned, the effective moment which is a complex quantity has been studied by Heller *et al.* [26]. We cannot calculate this moment to arbitrary precision in this work since it is difficult to take into account the loop contribution in the soft-photon approximation. Nevertheless, we have done a numerical study by treating λ_{Δ} in Eq. (75) as a complex quantity, $\lambda_{\Delta} = \lambda_R + i\lambda_I$, in order to estimate the contribution from the imaginary part λ_I. We have chosen λ_{Δ} to be $1.47+i\lambda_I$, $1.6+i\lambda_I$, and $2.4+i\lambda_I$. By varying λ_I from -1.0 to 1.0 in each case, we have used the UCLA data (at 298 MeV for counters G1–G10) to calculate average deviations as a function of λ_I. As shown in Fig. 13, we have obtained three deviation curves which have the same interesting feature. The value of the average deviation decreases rapidly as λ_I increases from -1.0 to zero and then it increases rapidly as λ_I increases from zero to 1.0. Thus, the minimum points for all three average deviation curves are around $\lambda_I = 0$, independent of the choice of λ_R, indicating that the best fit to the UCLA data (at 298 MeV for counters G1–G10) can be obtained by choosing λ_{Δ} to be a real quantity, as we have done in this work. This result implies that the dynamical contribution (photon emission from the π^+p loop) to the imaginary part λ_I is very small. If there is no dynamical contribution to λ_I, we also expect very little dynamical contribution to the real part λ_R. We may therefore conclude that the whole dynamical contribution would be small and hence the "experimental" magnetic moment should be very close to the effective moment.

To understand why the best fit to the UCLA data can be obtained only if λ_{Δ} is chosen to be a real quantity, we have performed another study. Our numerical investigation of the amplitude M_{μ}^{TETAS} reveals that the best agreement between theory and experiment is obtained when the contribution from the R_{μ}-dependent terms cancels the total contribution from those terms involving $R_{i\mu}$ and $R_{f\mu}$ in Eq. (75). This cancellation occurs when μ_{Δ} is around $4e/(2m_p)$. However, no cancellation is possible if λ_{Δ} is chosen to be a complex quantity with a large imaginary part since the anomalous magnetic moment of proton λ_p is a real quantity ($\lambda_p = 1.79$). This explains why the minimum point is always found around $\lambda_I = 0$, in-

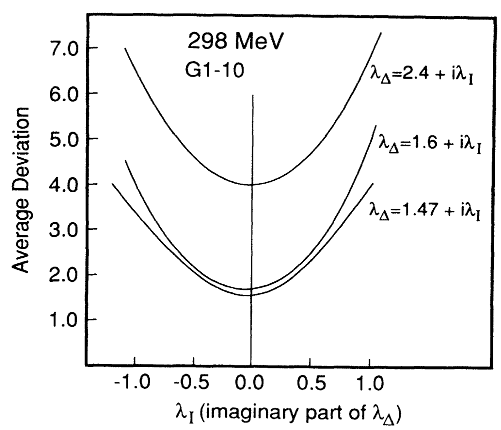


FIG. 13. Average deviation as a function of λ_I (the imaginary part of λ_{Δ}). Using the amplitude M_{μ}^{TETAS} but treating λ_{Δ} as a complex quantity, $\lambda_{\Delta} = \lambda_R + i\lambda_I$, the average deviations are calculated as a function of λ_I for $\lambda_R = 1.47$, 1.6, and 2.4. The experimental cross sections used in these calculations are the UCLA data at 298 MeV for the photon counters G1–G10. Each deviation curve shows a clear minimum point at $\lambda_I = 0$.

dependent of the choice of λ_R, if the average deviation is plotted as a function of λ_I. As we have already pointed out, our numerical study also indicates that the spectra calculated by using Eq. (75) agree very well with those spectra predicted by Eq. (83) (which is identical to Eq. (16) of Ref. [28]) if μ_Δ used in Eq. (75) is about $4e/(2m_p)$. Both results are in excellent agreement with the experimental data.

Now let us discuss what would happen if those terms involving $R_{i\mu}$, $R_{f\mu}$, and R_μ are cancelled out precisely. We would obtain an amplitude M_μ^{TETAS} with $R_{i\mu}=R_{f\mu}=R_\mu=0$. Such an amplitude was first proposed by Heller [23] and it was discussed in great details in Ref. [28] (Heller's amplitude is identical to Eq. (3) of Ref. [28]). It is a well known fact that Heller's amplitude can be successfully applied to describe both the $\pi^\pm p\gamma$ data and the $p\,{}^{12}C\gamma$ data. This fact may have two possible implications that are consistent with our findings. (i) The cancellation between the contribution from the magnetic moment of the Δ^{++} (including all possible loop corrections) and the contribution from the magnetic moment of proton exists. (ii) The imaginary part of the effective magnetic moment of the Δ^{++} is small and the real part is $(3.7\text{–}4.9)e/(2m_p)$. In short, the data seem to suggest that dynamical corrections from the loop diagrams are small. In other words, our best fit implies that the effective magnetic moment of the Δ^{++} should be nearly equal to not only the "experimental" moment obtained in this work but also the bare moment given by the modified SU(6) model or the quark model with corrections. The problem requires further careful studies.

VI. CONCLUSION

We conclude the following.

(i) We have derived a radiation decomposition identity for bremsstrahlung emission from an internal Δ^{++} line with an anomalous magnetic moment λ_Δ. We show how this identity can be applied to modify Low's standard prescription for constructing soft-photon amplitudes for bremsstrahlung processes.

(ii) Using the modified Low procedure, we have derived a TETAS amplitude, M_μ^{TETAS} given by Eq. (75), for the $\pi^+p\gamma$ process near the $\Delta^{++}(1232)$ resonance. This TETAS amplitude has many interesting features: (1) It is relativistic, gauge invariant, and consistent with the soft-photon theorem. (2) It depends only on the elastic T matrix, evaluated at four different sets of (s,t): (s_i,t_p), (s_i,t_q), (s_f,t_p), and (s_f,t_q), but it is free of any derivative of T with respect to s or t. (3) It takes into account bremsstrahlung emissions from (a) the incoming pion and the outgoing pion (with charge $+e$), (b) the incoming proton and the outgoing proton (with charge $+e$ and the anomalous magnetic moment λ_p), (c) the internal Δ^{++} line (with charge $+2e$ and the anomalous magnetic moment λ_Δ), and (d) other sources by imposing the gauge invariant condition.

(iii) We have used the amplitude M_μ^{TETAS} to calculate $\pi^+p\gamma$ cross sections as a function of photon energy K, $d^3\sigma/d\Omega_\pi d\Omega_\gamma dK$, at five bombarding energies, 165, 269, 298, 299, and 324 MeV. Treating λ_Δ as a free parameter in these calculations, the "experimental" magnetic moment of the Δ^{++}, μ_Δ, has been extracted from 45 sets of the UCLA data and 3 sets of the SIN data. The extracted values of μ_Δ are

$$\mu_\Delta=\begin{cases}3.7e/(2m_p) & \text{for photon counters G1–G10}\ ,\\ 4.0e/(2m_p) & \text{for photon counters G1–G15}\ ,\\ 4.2e/(2m_p) & \text{for photon counters G11–G15}\ ,\end{cases}$$

from the UCLA data and

$$\mu_\Delta=\begin{cases}4.6e/(2m_p) & \text{for } 75^\circ<\vartheta_\pi<95^\circ\ ,\\ 4.7e/(2m_p) & \text{for } 55^\circ<\vartheta_\pi<95^\circ\ ,\\ 4.9e/(2m_p) & \text{for } 55^\circ<\vartheta_\pi<75^\circ\ ,\end{cases}$$

from the SIN data. These extracted values of μ_Δ [the average is $4.35e/(2m_p)$] are smaller than the value $5.58e/(2m_p)$, the "bare" magnetic moment predicted by the SU(6) model or the quark model, but they are close to the value $4.25e/(2m_p)$ predicted by the modified SU(6) model of Beg and Pais and also in accord with the value $(4.41-4.89)e/(2m_p)$ obtained by Brown, Rho, and Vento.

(iv) Using the amplitude M_μ^{TETAS} and the values of μ_Δ extracted from the experimental data, we have calculated all $\pi^+p\gamma$ cross sections which can be compared with the UCLA data and the SIN data. In general, the agreement between the theoretical predictions and the experimental measurements is excellent. This agreement demonstrates that the amplitude M_μ^{TETAS} is valid and it can be used to describe almost all the available $\pi^+p\gamma$ data near the $\Delta(1232)$ resonance.

(v) We have also treated λ_Δ as a complex quantity, $\lambda_\Delta=\lambda_R+i\lambda_I$, in order to estimate the contribution from the imaginary part λ_I. The best fit to the data gives $\lambda_I=0$, independent of the choice of λ_R. This finding suggests that further dynamical corrections to the amplitude M_μ^{TETAS} from the open pion-proton channel are small and hence the "effective" moment, the "experimental" moment, and the "bare" moment predicted by the modified SU(6) model of Beg and Pais should have about the same value.

(vi) We have shown that the approximate amplitude given by Eq. (83), an amplitude used in Ref. [28], is justified. This explains why the amplitude (used in Ref. [28]) works remarkably well for the $\pi^+p\gamma$ process. We have also explained why the amplitude M_μ^{TETAS} given by Eq. (75) can be used to describe $p\,{}^{12}C\gamma$ cross sections near either the 1.7-MeV resonance or the 0.5-MeV resonance.

ACKNOWLEDGMENTS

We wish to thank Professor Robert W. Brown for many stimulating discussions. This work was supported in part by the City University of New York Professional Staff Congress–Board of Higher Education Faculty Research Award Program.

APPENDIX A

The expression for $E^{\sigma}{}_{\alpha}$, E'^{β}_{ρ}, $\tilde{C}^{\sigma}_{\alpha\mu}$, and $\tilde{C}'^{\beta}_{\rho\mu}$ are as follows:

$$
\begin{aligned}
E^{\sigma}{}_{\alpha}=\;&\gamma^{\sigma}\gamma_{\alpha}[-2\not{\varepsilon}\not{p}'/9-4p'\cdot\varepsilon/9-2\not{\varepsilon}\not{k}/3+p'\cdot\varepsilon(\not{p}'+\not{k})/(9M_{\Delta})\\
&\quad-(p'^2+2p'\cdot k)\not{\varepsilon}/(9M_{\Delta})+2(p'^2+2p'\cdot k)(-p'\cdot\varepsilon+\not{\varepsilon}\not{p}'+\not{\varepsilon}\not{k})/(9M_{\Delta}^2)]\\
&+\varepsilon^{\sigma}\gamma_{\alpha}[-\not{p}'-4\not{k}/3-2M_{\Delta}/3+(p'^2+2p'\cdot k)/(3M_{\Delta})+2(p'^2+2p'\cdot k)(\not{p}'+\not{k})/(3M_{\Delta}^2)]\\
&+\gamma^{\sigma}\varepsilon_{\alpha}[7\not{p}'/9+\not{k}+2M_{\Delta}/9+2(p'^2+2p'\cdot k)/(9M_{\Delta})-4(p'^2+2p'\cdot k)(\not{p}'+\not{k})/(9M_{\Delta}^2)]\\
&+\varepsilon^{\sigma}(p'+k)_{\alpha}[\tfrac{2}{3}+2(\not{p}'+\not{k})/(3M_{\Delta})-2(p'^2+2p'\cdot k)/(3M_{\Delta}^2)]+2\varepsilon^{\sigma}k_{\alpha}/3\\
&+(p'+k)^{\sigma}\varepsilon_{\alpha}[-\tfrac{14}{9}-2(\not{p}'+\not{k})/(3M_{\Delta})+4(p'^2+2p'\cdot k)/(3M_{\Delta}^2)]-4k^{\sigma}\varepsilon_{\alpha}/9\\
&+\gamma^{\sigma}(p'+k)_{\alpha}[-\not{\varepsilon}/9-4\not{\varepsilon}(\not{p}'+\not{k})/(9M_{\Delta})+2p'\cdot\varepsilon/(9M_{\Delta})+2(p'\cdot\varepsilon\not{p}'+p'\cdot\varepsilon\not{k}+p'^2\not{\varepsilon}+2p'\cdot k\not{\varepsilon})/(9M_{\Delta}^2)]-5\gamma^{\sigma}k_{\alpha}\not{\varepsilon}/9\\
&+(p'+k)^{\sigma}\gamma_{\alpha}\not{\varepsilon}[\tfrac{1}{3}+(\not{p}'+\not{k})/(3M_{\Delta})-2(p'^2+2p'\cdot k)/(3M_{\Delta}^2)]+2k^{\sigma}\gamma_{\alpha}\not{\varepsilon}/3\\
&-(p'+k)^{\sigma}(p'+k)_{\alpha}2\not{\varepsilon}(\not{p}'+\not{k}+M_{\Delta})/(3M_{\Delta}^2)\\
&+\lambda_{\Delta}/(2M_{\Delta})\{\gamma^{\sigma}\gamma_{\alpha}[-(\not{\varepsilon}\not{k}\not{p}'+\not{p}'\not{\varepsilon}\not{k})/3+2\not{\varepsilon}\not{k}(\not{p}'+M_{\Delta})(p'^2-M_{\Delta}^2)/(3M_{\Delta}^2)]\\
&\qquad+p'^{\sigma}\gamma_{\alpha}\not{\varepsilon}\not{k}(2p'^2-M_{\Delta}\not{p}'-4M_{\Delta}^2)/(3M_{\Delta}^2)-\gamma^{\sigma}p'_{\alpha}\not{\varepsilon}\not{k}(2p'^2-M_{\Delta}\not{p}'-4M_{\Delta}^2)/(3M_{\Delta}^2)\\
&\qquad+(4k^{\sigma}\varepsilon_{\alpha}-4\varepsilon^{\sigma}k_{\alpha}+2\varepsilon^{\sigma}\gamma_{\alpha}\not{k}-2\gamma^{\sigma}\varepsilon_{\alpha}\not{k}+2\gamma^{\sigma}k_{\alpha}\not{\varepsilon}-2k^{\sigma}\gamma_{\alpha}\not{\varepsilon})(\not{p}'+M_{\Delta})(2p'^2-3M_{\Delta}^2)/(3M_{\Delta}^2)\\
&\qquad+[2\not{k}(\varepsilon^{\sigma}p'_{\alpha}-p'^{\sigma}\varepsilon_{\alpha})+2\not{\varepsilon}(p'^{\sigma}k_{\alpha}-k^{\sigma}p'_{\alpha})](2\not{p}'-3M_{\Delta})(\not{p}'+M_{\Delta})/(3M_{\Delta}^2)\\
&\qquad-2p'^{\sigma}p'_{\alpha}\not{\varepsilon}\not{k}(\not{p}'+M_{\Delta})/(3M_{\Delta}^2)\}\;,
\end{aligned}
$$

$$
\begin{aligned}
E'^{\beta}_{\rho}=\;&[-2\not{p}\not{\varepsilon}/9-4p\cdot\varepsilon/9-2\not{\varepsilon}\not{k}/3+p\cdot\varepsilon(\not{p}-\not{k})/(9M_{\Delta})-(p^2-2p\cdot k)\not{\varepsilon}/(9M_{\Delta})\\
&\quad+2(p^2-2p\cdot k)(-p\cdot\varepsilon+\not{p}\not{\varepsilon}-\not{k}\not{\varepsilon})/(9M_{\Delta}^2)]\gamma_{\rho}\gamma^{\beta}\\
&+[-\not{p}+4\not{k}/3-2M_{\Delta}/3+(p^2-2p\cdot k)/(3M_{\Delta})+2(p^2-2p\cdot k)(\not{p}-\not{k})/(3M_{\Delta}^2)]\gamma_{\rho}\varepsilon^{\beta}\\
&+[7\not{p}/9-\not{k}+2M_{\Delta}/9+2(p^2-2p\cdot k)/(9M_{\Delta})-4(p^2-2p\cdot k)(\not{p}-\not{k})/(9M_{\Delta}^2)]\varepsilon_{\rho}\gamma^{\beta}\\
&+[\tfrac{2}{3}+2(\not{p}-\not{k})/(3M_{\Delta})-2(p^2-2p\cdot k)/(3M_{\Delta}^2)](p-k)_{\rho}\varepsilon^{\beta}-2k_{\rho}\varepsilon^{\beta}/3\\
&+[-\tfrac{14}{9}-2(\not{p}-\not{k})/(3M_{\Delta})+4(p^2-2p\cdot k)/(3M_{\Delta}^2)]\varepsilon_{\rho}(p-k)^{\beta}+4\varepsilon_{\rho}k^{\beta}/9\\
&+[-\not{\varepsilon}/9-4(\not{p}-\not{k})\not{\varepsilon}/(9M_{\Delta})+2p\cdot\varepsilon/(9M_{\Delta})\\
&+2(p\cdot\varepsilon\not{p}-p\cdot\varepsilon\not{k}+p^2\not{\varepsilon}-2p\cdot k\not{\varepsilon})/(9M_{\Delta}^2)](p-k)_{\rho}\gamma^{\beta}+5\not{\varepsilon}k_{\rho}\gamma^{\beta}/9\\
&+[\tfrac{1}{3}+(\not{p}-\not{k})/(3M_{\Delta})-2(p^2-2p\cdot k)/(3M_{\Delta}^2)]\not{\varepsilon}\gamma_{\rho}(p-k)^{\beta}-2\not{\varepsilon}\gamma_{\rho}k^{\beta}/3\;-2(\not{p}-\not{k}+M_{\Delta})\not{\varepsilon}(p-k)_{\rho}(p-k)^{\beta}/(3M_{\Delta}^2)\\
&+\lambda_{\Delta}/(2M_{\Delta})\{[-(\not{\varepsilon}\not{k}\not{p}+\not{p}\not{\varepsilon}\not{k})/3+2(\not{p}+M_{\Delta})\not{\varepsilon}\not{k}(p^2-M_{\Delta}^2)/(3M_{\Delta}^2)]\gamma_{\rho}\gamma^{\beta}\\
&\qquad+(2p^2-M_{\Delta}\not{p}-4M_{\Delta}^2)\not{\varepsilon}\not{k}\gamma_{\rho}p^{\beta}/(3M_{\Delta}^2)-(2p^2-M_{\Delta}\not{p}-4M_{\Delta}^2)\not{\varepsilon}\not{k}p_{\rho}\gamma^{\beta}/(3M_{\Delta}^2)\\
&\qquad+(2p^2-3M_{\Delta}^2)(\not{p}+M_{\Delta})(4k_{\rho}\varepsilon^{\beta}-4\varepsilon_{\rho}k^{\beta}+2\not{k}\varepsilon_{\rho}\gamma^{\beta}-2\not{k}\gamma_{\rho}\varepsilon^{\beta}+2\not{\varepsilon}\gamma_{\rho}k^{\beta}-2\not{\varepsilon}k_{\rho}\gamma^{\beta})/(3M_{\Delta}^2)\\
&\qquad+(2\not{p}-3M_{\Delta})(\not{p}+M_{\Delta})[2\not{k}(\varepsilon_{\rho}p^{\beta}-p_{\rho}\varepsilon^{\beta})+2\not{\varepsilon}(p_{\rho}k^{\beta}-k_{\rho}p^{\beta})]/(3M_{\Delta}^2)\\
&\qquad-2(\not{p}+M_{\Delta})\not{\varepsilon}\not{k}p_{\rho}p^{\beta}/(3M_{\Delta}^2)\}\;,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{C}^{\sigma}_{\alpha\mu}(p',k)=&\gamma^{\sigma}\gamma_{\alpha}[2(p'^2-3M_\Delta^2)\gamma_\mu\not{k}/(9M_\Delta^2)+(p'_\mu\not{k}-\gamma_\mu p'\cdot k)/(9M_\Delta)]\\
&+\gamma_\alpha(g^\sigma_\mu\not{k}-k^\sigma\gamma_\mu)(-7M_\Delta^2-M_\Delta\not{p}'+4p'^2)/(6M_\Delta^2)\\
&+\gamma^\sigma(g_{\alpha\mu}\not{k}-k_\alpha\gamma_\mu)(15M_\Delta^2+4M_\Delta\not{p}'-6p'^2)/(18M_\Delta^2)\\
&+\gamma^\sigma(p'_\mu k_\alpha-p'\cdot k g_{\alpha\mu})2(M_\Delta+\not{p}')/(9M_\Delta^2)+p'_\alpha(g_\mu{}^\sigma\not{k}-k^\sigma\gamma_\mu)(2M_\Delta+\not{p}')/(3M_\Delta^2)\\
&+(g_\mu{}^\sigma k_\alpha-k^\sigma g_{\alpha\mu})(5M_\Delta^2+2M_\Delta\not{p}'-3p'^2)/(3M_\Delta^2)+p'^\sigma(g_{\alpha\mu}\not{k}-k_\alpha\gamma_\mu)\not{p}'/(3M_\Delta^2)+p'^\sigma\gamma_\alpha\gamma_\mu\not{k}/(3M_\Delta)\\
&+\gamma^\sigma p'_\alpha[-4M_\Delta\gamma_\mu\not{k}+2(p'_\mu\not{k}-\gamma_\mu p'\cdot k)]/(9M_\Delta^2)-2p'^\sigma p'_\alpha\gamma_\mu\not{k}/(3M_\Delta^2)\\
&+\lambda_\Delta/(2M_\Delta)\{\gamma^\sigma\gamma_\alpha[-(\not{p}'\gamma_\mu\not{k}+\gamma_\mu\not{k}\not{p}')/3+2(p'^2-M_\Delta^2)\gamma_\mu\not{k}(\not{p}'+M_\Delta)/(3M_\Delta^2)]\\
&\qquad+p'^\sigma\gamma_\alpha\gamma_\mu\not{k}(2p'^2-M_\Delta\not{p}'-4M_\Delta^2)/(3M_\Delta^2)-\gamma^\sigma p'_\alpha\gamma_\mu\not{k}(2p'^2-M_\Delta\not{p}'-4M_\Delta^2)/(3M_\Delta^2)\\
&\qquad+(2p'^2-3M_\Delta^2)[4(k^\sigma g_{\alpha\mu}-g^\sigma_\mu k_\alpha)+2\gamma_\alpha(g^\sigma_\mu\not{k}-k^\sigma\gamma_\mu)-2\gamma^\sigma(g_{\alpha\mu}\not{k}-k_\alpha\gamma_\mu)](\not{p}'+M_\Delta)/(3M_\Delta^2)\\
&\qquad+[2(g_\mu{}^\sigma\not{k}-k^\sigma\gamma_\mu)p'_\alpha-2p'^\sigma(g_{\alpha\mu}\not{k}-k_\alpha\gamma_\mu)](2\not{p}'-3M_\Delta)(\not{p}'+M_\Delta)/(3M_\Delta^2)\\
&\qquad-2p'^\sigma p'_\alpha\gamma_\mu\not{k}(\not{p}'+M_\Delta)/(3M_\Delta^2)\}\ ,
\end{aligned}
$$

$$
\begin{aligned}
\tilde{C}'^{\beta}_{\rho\mu}(p,k)=&[2(p^2-3M_\Delta^2)\gamma_\mu\not{k}/(9M_\Delta^2)-(p_\mu\not{k}-p\cdot k\gamma_\mu)/(9M_\Delta)]\gamma_\rho\gamma^\beta+(7M_\Delta^2+M_\Delta\not{p}-4p^2)(\not{k}g^\beta{}_\mu-\gamma_\mu k^\beta)\gamma_\rho/(6M_\Delta^2)\\
&+(-15M_\Delta^2-4M_\Delta\not{p}+6p^2)(\not{k}g_{\rho\mu}-\gamma_\mu k_\rho)\gamma^\beta/(18M_\Delta^2)+2(-M_\Delta-\not{p})(p_\mu k_\rho-p\cdot k g_{\rho\mu})\gamma^\beta/(9M_\Delta^2)\\
&+(-2M_\Delta-\not{p})(\not{k}g^\beta{}_\mu-\gamma_\mu k^\beta)p_\rho/(3M_\Delta^2)+(-5M_\Delta^2-2M_\Delta\not{p}+3p^2)(k_\rho g^\beta{}_\mu-g_{\rho\mu}k^\beta)/(3M_\Delta^2)\\
&-\not{p}(\not{k}g_{\rho\mu}-k_\rho\gamma_\mu)p^\beta/(3M_\Delta^2)+\gamma_\mu\not{k}\gamma_\rho p^\beta/(3M_\Delta)\\
&+[-4M_\Delta\gamma_\mu\not{k}-2(p_\mu\not{k}-\gamma_\mu p\cdot k)]p_\rho\gamma^\beta/(9M_\Delta^2)-2\gamma_\mu\not{k}p_\rho p^\beta/(3M_\Delta^2)\\
&+\lambda_\Delta/(2M_\Delta)\{[-(\not{p}\gamma_\mu\not{k}+\gamma_\mu\not{k}\not{p})/3+2(p^2-M_\Delta)(\not{p}+M_\Delta)\gamma_\mu\not{k}/(3M_\Delta^2)]\gamma_\rho\gamma^\beta\\
&\qquad+(2p^2-M_\Delta\not{p}-4M_\Delta^2)\gamma_\mu\not{k}\gamma_\rho p^\beta/(3M_\Delta^2)-(2p^2-M_\Delta\not{p}-4M_\Delta^2)\gamma_\mu\not{k}p_\rho\gamma^\beta/(3M_\Delta^2)\\
&\qquad+(2p^2-3M_\Delta^2)(\not{p}+M_\Delta)[4(k_\rho g^\beta{}_\mu-g_{\rho\mu}k^\beta)+2(\not{k}g_{\rho\mu}-\gamma_\mu k_\rho)\gamma^\beta+2(\gamma_\mu k^\beta-\not{k}g^\beta{}_\mu)\gamma_\rho]/(3M_\Delta^2)\\
&\qquad+(2\not{p}-3M_\Delta)(\not{p}+M_\Delta)[2(\not{k}g_{\rho\mu}-\gamma_\mu k_\rho)p^\beta+2(\gamma_\mu k^\beta-\not{k}g^\beta{}_\mu)p_\rho]/(3M_\Delta^2)\\
&\qquad-2(\not{p}+M_\Delta)\gamma_\mu\not{k}p_\rho p^\beta/(3M_\Delta^2)\}\ .
\end{aligned}
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