$p(\gamma,\pi^0)$ cross section and the low energy theorem

J. C. Bergstrom
Saskatchewan Accelerator Laboratory, University of Saskatchewan, Saskatoon, Saskatchewan, Canada S7N OWO
(Received 10 April 1991)

The reported breakdown of the low energy theorem (LET) for $p(\gamma,\pi^0)$ as deduced from recent measurements at Mainz and earlier at Saclay is reexamined and found to be premature. Both qualitative and quantitative arguments are presented that suggest the conventional LET disagrees with experiment at the 8–12 % level. There is, however, a strong quenching of $E_{0^+}(\pi^0)$ near the π^+ threshold which is described by a semiphenomenological rescattering calculation based on the integral equation for the transition operator. The results are in good agreement with the values of $E_{0^+}(\pi^0)$ deduced from the Mainz experiment, as well as the differential cross sections. Comparisons are made with other data up to $E_\gamma=300$ MeV.

I. INTRODUCTION

Measurements of the reaction $p(\gamma,\pi_0)p$ near threshold, first at Saclay [1], and more recently at Mainz [2], produced cross sections much smaller than had been anticipated on the basis of the so-called low energy theorem (LET). In fact, it is claimed that the data demonstrate a clear and spectacular "violation" of the LET itself, and this conclusion has in turn spawned an extensive theoretical industry [3–9] which attempts to blame the discrepancies on all manner of phenomenon from simple final-state interactions to a severe breakdown of the partial conservation of axial-vector current (PCAC).

Near threshold, the photopion cross section from the nucleon may be written as

$$\left.\frac{d\sigma}{d\Omega}\right|_{q\to 0}=\frac{q}{k}|E_{0^+}|^2 ,$$

where q and k are the pion and photon momenta in the c.m. frame, and E_{0^+} is the S-wave electric dipole amplitude. The low energy theorems, which derive quite generally from the fundamental premises of gauge invariance and PCAC, usually predict E_{0^+} in terms of a power series in the parameter $\mu=m_\pi/M$, where M is the nucleon mass [10]. In the case of neutral pion production from the proton, the LET is model independent up to, and including, terms of order μ^2, and is given by

$$E_{0^+}=\frac{ef}{m_\pi}\left[-\mu+\frac{\mu^2}{2}(3+\kappa_p)\right] , \qquad (1.1)$$

where $e^2=1/137$, $f^2\approx 0.080$, and $\kappa_p=1.79$. Model-dependent terms, such as the PCAC-breaking vector-meson exchanges, enter at order μ^3. Terms of this and higher order, although generally estimated to be small, are excluded from the LET to conform with the strictly model-independent spirit of the theorem. Thus, there was some concern when the Saclay group [1] reported the value

$$E_{0^+}(\pi^0)=(-0.5\pm 0.3)\times 10^{-3}/m_\pi , \qquad (1.2)$$

while the LET prediction is

$$E_{0^+}(\pi^0)=-2.27\times 10^{-3}/m_\pi . \qquad (1.3)$$

(Throughout this paper, m_π refers to the charged pion mass.) From the subsequent experiment at Mainz, Beck *et al.* [2] report results consistent with the Saclay data, although no threshold amplitude is specifically mentioned.

Our purpose in the present paper is to reexamine the question of whether there is a large discrepancy between experiment and the LET prediction, as has been claimed. In our discussion, we will concentrate solely on the recent very high quality data from Mainz. Using quite reasonable assumptions, we will demonstrate in the first part of our paper that the total experimental cross section displays only a small disagreement with the LET, at about the 8% level. It must be understood that we are referring here to the LET *at* π^0 *threshold* where it is defined [10,11]; it generally has little predictive power above threshold, a point often overlooked. It is our contention that the interesting physics in $p(\gamma,\pi^0)p$ is not at π^0 threshold, but rather in the region of the π^+ threshold where the S-wave part of the cross section is highly suppressed. In the second part of our paper, we will examine the role rescattering can play in accounting for the suppression of E_{0^+} near π^+ threshold, and consider the implications for the LET at π^0 threshold. As a consequence, we will show that the E_{0^+} amplitudes reported by Beck *et al.* are not incompatible with the constraints of the low energy theorem.

There have been several attempts in the past to understand the so-called violation of the LET by invoking final-state interactions (FSI) through rescattering, wherein a π^+ is produced which subsequently undergoes a charge exchange to yield a neutral pion, for example. Most of these calculations have either relied on a K-matrix approach [6], or have started with the integral equation for the T matrix [5]. The typical K-matrix method implicitly assumes that the off-shell πN charge-exchange amplitude below π^+ threshold is equivalent to the on-shell amplitude above threshold, and as a result the predicted FSI *vanish* at π^+ threshold and are *largest* at the π^0 threshold (i.e., they yield a Wigner cusp). This has led some to believe that one should compare the LET with the $p(\gamma,\pi^0)p$ cross section at the π^+ threshold, not at the π^0 threshold.

The other approach, based on the integral equation for the transition operator, depends on a model for the off-energy-shell behavior of the πN charge-exchange amplitude. However, unlike the K-matrix calculations, the resulting FSI are largest near the π^+ threshold and tend to be much smaller at the π^0 threshold. Of course, a criticism of the T-matrix approach is that while formally resembling a dispersion relation, the principle part integral generally is not dominated by the low energy behavior of the amplitudes, which is unsettling from the physics point of view. Nevertheless, as we will demonstrate, the Mainz data clearly show the S-wave multipole E_{0^+} is most strongly modified near the π^+ threshold, and assuming rescattering is responsible, calls into question the K-matrix approach.

We will show that the $p(\gamma,\pi^0)$ cross section can be described by a rescattering model based on the integral equation for the transition operator. The resulting expression is therefore similar to that given by Nozawa *et al.* [5]. We assume that the charge-exchange mechanism is the dominant process, and that π^0 rescattering is negligible. The on-shell values of all relevant amplitudes in the rescattering integral were constrained by the available experimental data. The major unknown physics is the off-energy-shell behavior (strictly speaking, the half-off-shell behavior) of the charge-exchange amplitude. Rather than constructing a specific form factor from fundamental interactions, we have turned to the photoproduction data and considered what sort of off-shell behavior might be required in order to describe it. Accordingly, we have tried various purely phenomenological models for the off-energy-shell form factor, only one of which comes close to giving a reasonable accounting of the data. The ultimate purpose of the exercise is to estimate the *explicit* rescattering effects at π^0 threshold by using most of the (total) cross-section data, and thus make a statement about the departure from the LET prediction.

By *explicit* rescattering effects at threshold, we mean those FSI not already *implicitly* included in the *conventional* LET's. We have adopted the point of view of Naus *et al.* [11] and others [10], that in the absence of pion and nucleon mass splittings, rescattering effects are implicitly included in the conventional LET's and must not be added as extra corrections. (All PCAC-derived LET's neglect the pion mass splittings). However, when the mass degeneracies are removed it is not clear if this is still correct; perhaps the FSI at threshold are modified slightly. It is this possible change that will be termed the *explicit* rescattering effects and whose magnitude we have attempted to determine from the data. One may argue that LET's based on chiral symmetry and extrapolated to finite pion mass automatically contain *all* rescattering effects even in the presence of isospin splitting, so what is being proposed here would amount to double counting. That may turn out to be true, but until LET's are established that include isospin splitting from the beginning, the matter is still debatable in our opinion. Note that we are referring here to changes of order μ and μ^2. Although the explicit FSI at these orders must vanish at threshold in the limit of equal pion and equal nucleon masses, they need not vanish above threshold since the LET's are strictly valid only at threshold. Effects of order higher than μ^2 may or may not vanish at threshold; however, in the particular model we adopt later, all orders vanish.

The preceding paragraph justifies our ignoring π^0 rescattering effects (aside from the fact that the $\pi^0 p$ scattering length is extremely small); such effects are already largely contained in the LET.

The values of E_{0^+} presented by Beck *et al.* [2] were deduced from the pion angular distributions $d\sigma/d\Omega$, which inherently contain interference terms between large and poorly known small multipoles. We have chosen to analyze the total cross sections specifically to avoid such interference terms, and then predict the angular distributions. While some differences exist, on the whole the experimental angular distributions are described fairly well.

Finally, Beck *et al.* produced two sets of E_{0^+} multipoles, depending on the specifics of their analysis. The first set, which they appear to favor, suggests a very strong suppression of the multipole just above π^0 threshold. On the other hand, we find the second set to be more consistent with the present analysis.

II. GENERAL FEATURES OF THE DATA

As noted in the Introduction, we will be working exclusively with the total $p(\gamma,\pi^0)$ data from the recent experiment performed at Mainz. The data themselves were taken from Ref. [12]. In this section, we wish to present the data in a form such that we might readily extrapolate to the π^0 threshold in a model-independent way.

The total cross section near threshold is dominated by the $l=0$ and $l=1$ partial waves, and may be written in terms of the fundamental multipole amplitudes as

$$\sigma=4\pi\left[\frac{q}{k}\right](|E_{0^+}|^2+2|M_{1^+}|^2+|M_{1^-}|^2+6|E_{1^+}|^2)\ . \tag{2.1}$$

The $l=1$ amplitudes are *assumed* to vary smoothly and monotonically as a function of the product qk at low energy, and their imaginary parts are negligible in the energy region of the data. Now, since $|M_{1^-}|^2$ and $|E_{1^+}|^2$ are much smaller than the dominant magnetic dipole $|M_{1^+}|^2$, we will lump all three together:

$$2|M_{1^+}|^2+|M_{1^-}|^2+6|E_{1^+}|^2=2f_0^2(qk)^2, \tag{2.2}$$

where f_0 is a scale factor with units $10^{-3}/m_\pi^3$. Estimates of f_0 for a pure magnetic dipole transition vary widely, but an average value is roughly $f_0=8\times10^{-3}/m_\pi^3$.

We can now recast Eq. (2.1) in the form

$$(\mathrm{Re}E_{0^+})^2=\left[\frac{\sigma}{4\pi}\right]\left[\frac{k}{q}\right]-2f_0^2(qk)^2-(\mathrm{Im}E_{0^+})^2\ . \quad (2.3)$$

The first term, $(\sigma/4\pi)(k/q)$, is plotted as circles in Fig. 1, and the second term, $2f_0^2(qk)^2$, is represented by the solid and dashed lines for $f_0=7.9$ and 8.0, respectively (from now on, we will suppress the units of f_0 for brevity). The third term in Eq. (2.3) is the imaginary part of E_{0^+}, and presumably it is negligible below the π^+ threshold at $E_\gamma=151.44$ MeV. It is negligible below π^+ threshold in both the K-matrix and integral-equation methods of calculating final-state interactions, but there is as yet no experimental evidence to confirm this. Anyway, to keep things simple we will ignore $\mathrm{Im}E_{0^+}$ until the detailed analysis of the following sections.

The square root of Eq. (2.3) should yield $\mathrm{Re}E_{0^+}$ below π^+ threshold, and this is presented in Fig. 2 for $f_0=8.0$. Extrapolation by eye to the π^0 threshold ($E_\gamma=144.67$ MeV) gives the estimate

$$\mathrm{Re}E_{0^+}|_{\mathrm{thr}}\approx-(2.1\pm0.2)\times10^{-3}/m_\pi\ , \quad (2.4)$$

which is indicated by the dashed line in the figure. The sign of $\mathrm{Re}E_{0^+}$ is based on the LET value, as well as the conclusion of Beck *et al.* [2].

Also shown in Fig. 2 are the two sets of values for $\mathrm{Re}E_{0^+}$ deduced by Beck *et al.*, solution 1 (boxes) and solution 2 (triangles). Based on the points closest to threshold, solution 2 is clearly preferred within the context of our P-wave assumptions.

The slope of the data below the π^+ threshold in Fig. 2 depends on the choice of f_0, but for $f_0=7.7-8.2$ the extrapolation to π^0 threshold gives essentially the same estimates as Eq. (2.4). The upper limit $f_0=8.2$ is about as far as one can go without cutting deeply into the data points shown in Fig. 1. The lower limit $f_0=7.7$ corresponds to the lower bound defined by the errors on the P-wave multipoles given in Refs. [1] and [12]. Such a low value would produce a serious discrepancy between the E_{0^+} results of Beck *et al.* and those deduced from Eq. (2.3) in the 151–154 MeV region.

As we said, the P-wave amplitudes are assumed to vary smoothly and monotonically in the low energy range. Large local oscillations or wiggles in M_{1^+}, such as might be generated by rescattering in the $l=1$ channel, could alter our conclusions. Our own rough estimates using a K-matrix approach produced negligible modifications to M_{1^+}. Note that an overall renormalization of f_0 caused by FSI would not alter our conclusions since the data still require $f_0=7.7-8.2$.

Because the pion momentum q appears in the denominator of Eq. (2.3), an energy calibration error in the Mainz experiment could cause a vertical shift in the data points in Fig. 2, and modify the extrapolation to threshold. However, according to the authors, the uncertainty in the tagged-photon energies is about 220 keV. An energy shift of ±220 keV would have the following conse-

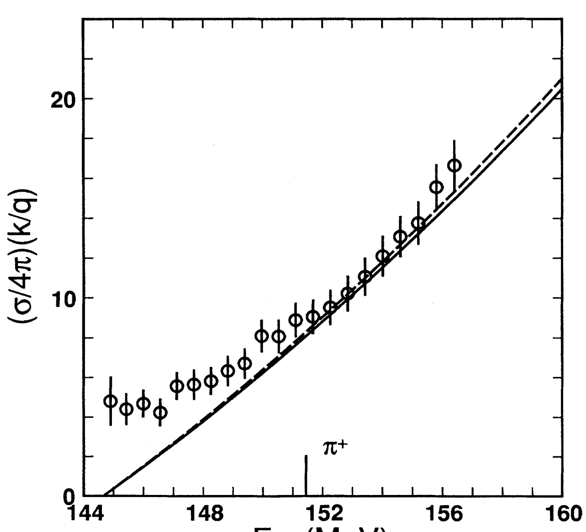

FIG. 1. The Mainz total cross sections, with a kinematic factor removed to emphasize the low energy behavior, in units of $10^{-6}/m_\pi^2$. The solid and dashed curves are the P-wave contributions for $f_0=7.9$ and 8.0, respectively. The threshold for π^+ production ($E_\gamma=151.44$ MeV) is indicated by the marker. The π^0 threshold is $E_\gamma=144.67$ MeV.

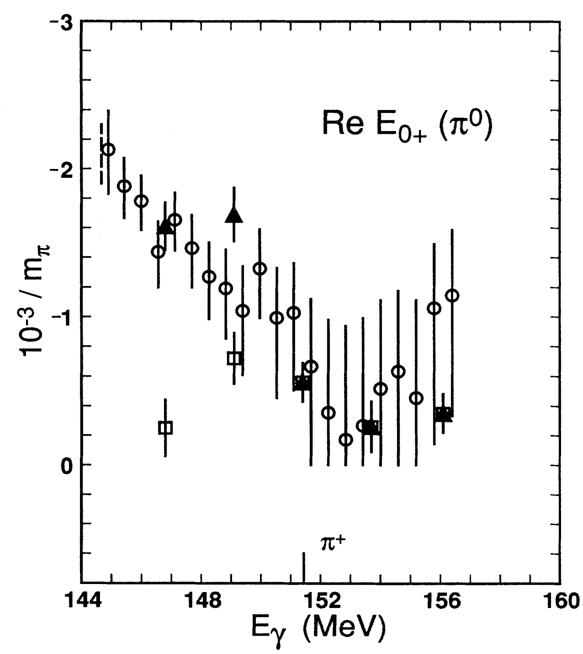

FIG. 2. The amplitudes $\mathrm{Re}E_{0^+}(\pi^0)$ as deduced directly from the Mainz total cross sections using Eq. (2.3) of the text. The P-wave amplitude is taken as $f_0=8.0$ and the imaginary part of E_{0^+} has been ignored. The errors in some cases terminate at 0, since Eq. (2.3) is positive definite. Extrapolation to π^0 threshold by eye gives $\mathrm{Re}E_{0^+}|_{\mathrm{thr}}=-(2.1\pm0.2)\times10^{-3}/m_\pi$, indicated by the dashed line. Also shown are the amplitudes from Beck *et al.* (Ref. [2]), solution 1 (squares), and solution 2 (triangles).

quences for the data shown in Fig. 2. The lowest energy point would either disappear (it would fall below threshold) or would drop to about -1.8. The second lowest point would shift roughly to the extremes of the error bar shown in the figure. The shifts of the other points decreases very rapidly. Thus, if we choose to reject the point closest to threshold, the estimate given by Eq. (2.4) is not seriously compromised.

To conclude, an extrapolation by eye to the π^0 threshold, using the Mainz total cross-section data with quite reasonable assumptions, gives

$$\mathrm{Re}E_{0^+}(\pi^0)|_{\mathrm{thr}} \approx -(2.1\pm0.2)\times10^{-3}/m_\pi \ , \tag{2.5}$$

to be compared with the LET value

$$E_{0^+}(\pi^0) = -2.27\times10^{-3}/m_\pi \ . \tag{2.6}$$

Equation (2.5) thus represents an 8%±8% difference from the LET prediction. *Therefore, there is no significant discrepancy between the low energy theorem and the threshold amplitude as deduced from the experimental data.*

Now let us turn to the general features of E_{0^+} as displayed by Fig. 2. It is clear that the amplitude is decreasing rapidly as one moves away from π^0 threshold, and it goes through a minimum just above the π^+ threshold, after which it seems to increase. That apparent increase contains contributions from $\mathrm{Im}E_{0^+}$, but we will see later that $\mathrm{Re}E_{0^+}$ is indeed rising and, at higher energy, links up with values deduced years ago from the multipole analyses of photoproduction data.

If we assume that rescattering effects are responsible for the behavior of $\mathrm{Re}E_{0^+}$ as a function of energy, then we conclude that they are largest near the π^+ threshold and smallest at the π^0 threshold. As we alluded to in the Introduction, the usual K-matrix approach does not produce such a behavior. (The reason is that the rescattering correction involving charge exchange is proportional to the on-shell momentum of the π^+, which vanishes at π^+ threshold.) Furthermore, the LET value is applied at π^+ threshold, which is certainly questionable, and $\mathrm{Re}E_{0^+}$ remains relatively constant several MeV above π^+ threshold. Therefore, in the following sections we will try to describe the behavior of $\mathrm{Re}E_{0^+}$ using the integral-equation method.

III. OUTLINE OF THE PROCEDURE

The preceding conclusions have derived from somewhat qualitative arguments. We will now build a more quantitative case based on physical principles and show how the $p(\gamma,\pi^0)$ cross section can be interpreted through a rescattering correction to the so-called Born terms. The rescattering involves the two-stage process $\gamma p\to\pi^+ n\to\pi^0 p$ and competes strongly with the direct reaction $\gamma p\to\pi^0 p$ since the S-wave π^+ photoproduction amplitude is an order of magnitude larger than the corresponding π^0 amplitude. The direct and rescattering channels interfere and yield the energy-dependent behavior seen in Fig. 2.

In the calculation we rely on available experimental data whenever possible to constrain certain amplitudes before introducing the phenomenological aspects. However, because phenomenology does play a role, our final conclusions should be regarded as plausibility arguments rather than firm deductions.

Let us outline here the procedure to be adopted in the remainder of this paper. From Eqs. (2.1) and (2.2) we have

$$\sigma_{\gamma\pi^0} = 4\pi\left[\frac{q_0}{k}\right][|E_{0^+}|^2 + 2f_0^2(q_0k)^2] \ , \tag{3.1}$$

where q_0 is the c.m. momentum of the final π^0. The electric dipole amplitude splits into real and imaginary parts:

$$|E_{0^+}|^2 = (\mathrm{Re}E_{0^+})^2 + (\mathrm{Im}E_{0^+})^2 \ , \tag{3.2}$$

and the real part subdivides into a Born part, $E^B_{0^+}$, and the *explicit* rescattering part ΔE_{0^+}:

$$\mathrm{Re}E_{0^+} = E^B_{0^+} + \Delta E_{0^+} \ . \tag{3.3}$$

As stated before, ΔE_{0^+} must vanish at threshold at least through order μ^2 in the limit of equal pion and equal nucleon masses. We will construct ΔE_{0^+} with this in mind using two free phenomenological parameters, combine Eqs. (3.1)–(3.3), and perform a least-squares fit to the Mainz data. With the parameters now determined, we return to the π^0 threshold and evaluate ΔE_{0^+}. This quantity, we claim, represents the discrepancy between the threshold amplitude and the LET prediction.

Some clarification is necessary concerning the quantity $E^B_{0^+}$. Consider the limit of equal pion and equal nucleon masses. Then at threshold we have the well-known equality (up to and including the model-independent orders of μ)

$$E_{0^+}\ (\mathrm{PCAC}) = \text{pole terms (PS } \pi NN \text{ coupling)} + \text{dispersion relation (cuts, etc.)} \ ,$$

where the dispersion relation implicitly contains the various FSI. However, we also know that (to the same orders of μ)

$$E_{0^+}\ (\mathrm{PCAC}) = E^{\mathrm{Born}}_{0^+}\ (\text{PV } \pi NN \text{ coupling}) = E_{0^+}\ (\mathrm{LET}) \ . \tag{3.4}$$

The quantity $E^B_{0^+}$ in Eq. (3.3) is therefore to be identified with $E^{\mathrm{Born}}_{0^+}$ (PV) and will be calculated accordingly, since by Eq. (3.4) it reduces to the LET value at threshold at the appropriate order of μ.

IV. THE INTEGRAL EQUATION FOR THE RESCATTERING CORRECTION

We start with the integral relation for the transition operator [13]

$$T = V + T\frac{1}{E - H_0 + i\varepsilon}V \ , \tag{4.1}$$

where V is the transition potential, H_0 is the sum of the free-particle Hamiltonian operators, and E is identified as the total energy of the final $\pi^0 p$ system in the c.m. frame. Thus, $E=W_0$, where W_0 is the invariant mass of the final state. In the first Born approximation $T^{\rm Born}=V$. Following the usual procedure, we insert the completeness condition for angular momentum eigenstates [13],

$$\frac{2}{\pi}\sum_{lm}\int \frac{q^2}{N_q^2}dq\,|qlm\rangle\langle qlm|=1\ ,$$

to the left of V in the second term of Eq. (4.1), then form transition matrix elements between the initial γp state (i) and the final $\pi^0 p$ state (f) to obtain

$$T_{fi}=V_{fi}+\frac{2}{\pi}\int \frac{q^2dq}{N_q^2}\frac{T_{fq}V_{qi}}{W_0-W+i\varepsilon}\ . \tag{4.2}$$

Here, W is the invariant mass of the intermediate $\pi^+ n$ system, q is the c.m. momentum of the neutron and π^+, and T_{fq} represent the charge exchange

$$(\pi^+ n)_q \rightarrow (\pi^0 p)_{q_0}\ ,$$

where q_0 is the final-state momentum of the proton and π^0. We have set $l=m=0$ in Eq. (4.2) since T_{fq} conserves angular momentum and we are considering S-wave production. For a two-body intermediate state with masses m_{π^+} and M_n, the normalization constant N_q is given by

$$N_q^2=2q\frac{E_\pi E_N}{W}\ , \tag{4.3}$$

where

$$E_\pi=(m_{\pi^+}^2+q^2)^{1/2},\quad E_N=(M_n^2+q^2)^{1/2}\ ,$$

and $W=E_\pi+E_N$.

Equation (4.2) is developed further by employing the formal identity

$$\frac{1}{W_0-W+i\varepsilon}=\mathrm{P}\frac{1}{W_0-W}-i\pi\delta(W_0-W)\ , \tag{4.4}$$

where P denotes the Cauchy principle value. The δ function is converted to one in momentum giving

$$\delta(W_0-W)=\frac{1}{q}\frac{E_\pi E_N}{W}\delta(\bar{q}-q)\ , \tag{4.5}$$

where $\bar{q}$ is the on-shell momentum defined by

$$W_0=(m_{\pi^+}^2+\bar{q}^{\,2})^{1/2}+(M_n^2+\bar{q}^{\,2})^{1/2}\ . \tag{4.6}$$

For future reference note that since q in Eq. (4.5) is real, $\bar{q}$ must be real as well.

Combining Eqs. (4.2)–(4.5) and integrating over the δ function gives

$$T_{fi}=V_{fi}-iT_{f\bar{q}}V_{\bar{q}i}+\mathrm{P}\int_0^\infty \frac{q\,dq}{\pi}\frac{W}{E_\pi E_N}\frac{T_{fq}V_{qi}}{W_0-W}\ . \tag{4.7}$$

The potential matrix elements are related to the electric dipole amplitudes by

$$\begin{aligned}V_{fi}&=(q_0k)^{1/2}E_{0^+}^B(\pi^0,q_0)\ ,\\ V_{qi}&=(qk)^{1/2}E_{0^+}^B(\pi^+,q)\ ,\end{aligned} \tag{4.8a}$$

etc. A q symbol has been included in the above amplitudes to remind us of the momentum of the pion. The charge-exchange matrix elements are related to the respective scattering amplitudes by

$$T_{fq}=-(q_0q)^{1/2}F_{\rm cx}(q_0,q), \tag{4.8b}$$

etc., where the minus sign has been introduced to conform with the sign convention for scattering. Introducing Eqs. (4.8) into (4.7) yields the final result:

$$\begin{aligned}E_{0^+}(\pi^0,q_0)=&E_{0^+}^B(\pi^0,q_0)+i\bar{q}F_{\rm cx}(q_0,\bar{q})E_{0^+}^B(\pi^+,\bar{q})\\ &-\mathrm{P}\int_0^\infty \frac{q^2dq}{\pi}\frac{W}{E_\pi E_N}\frac{F_{\rm cx}(q_0,q)E_{0^+}^B(\pi^+,q)}{W_0-W}\ .\end{aligned} \tag{4.9}$$

If $F_{\rm cx}$ is real, the above can be separated into real and imaginary parts, giving

$$\mathrm{Re}E_{0^+}(\pi^0,q_0)=E_{0^+}^B(\pi^0,q_0)+\Delta E_{0^+}\ , \tag{4.10a}$$

$$\mathrm{Im}E_{0^+}(\pi^0,q_0)=\bar{q}F_{\rm cx}(q_0,\bar{q})E_{0^+}^B(\pi^+,\bar{q})\ , \tag{4.10b}$$

with the rescattering correction

$$\Delta E_{0^+}=-\mathrm{P}\int_0^\infty \frac{q\,dq}{\pi}\frac{W}{E_\pi E_N}\frac{qF_{\rm cx}(q_0,q)E_{0^+}^B(\pi^+,q)}{W_0-W}\ . \tag{4.11}$$

We have seen by virtue of Eq. (4.5) that $\bar{q}$ must be real, but according to Eq. (4.6) if W_0 is below the π^+ threshold, i.e., $W_0<(m_{\pi^+}+M_n)$, $\bar{q}$ would become purely imaginary. Therefore, $\mathrm{Im}E_{0^+}$ as given by Eq. (4.10b) must vanish below π^+ threshold, as previously noted by Nozawa *et al.* [5].

If one uses a model for T_{fq} in Eq. (4.7) [i.e., $F_{\rm cx}(q_0,q)$ in Eq. (4.11)], the unitarity of the T matrix cannot be guaranteed. This problem can be avoided in principle by employing the so-called Heitler damping equation [13],

$$T=K-i\pi T\delta(W_0-H_0)K\ , \tag{4.12}$$

in which the model dependence is transferred to the K matrix. Inserting the eigenstate completeness condition (including the $\pi^0 p$ channel) into Eq. (4.12) and using Eq. (4.5) gives, after some manipulation,

$$\begin{aligned}T_{fq}=&e^{i\delta(\pi^0p)}\cos\delta(\pi^0p)\\ &\times\left[K(q_0,q)-i\theta\frac{K(q_0,\bar{q})\tilde{K}(\bar{q},q)}{1+i\theta\tilde{K}(\bar{q},\bar{q})}\right],\end{aligned} \tag{4.13}$$

where $\theta=\theta(W_0-m_{\pi^+}-M_n)$ originates from the reality of $\bar{q}$, K describes $\pi^+n\rightarrow\pi^0p$, $\tilde{K}$ describes $\pi^+n\rightarrow\pi^+n$, and $\delta(\pi^0p)$ is the π^0p S-wave phase shift evaluated on shell at W_0. The presence of the complex phase factor in Eq. (4.13) ensures that the rescattering correction satisfies

Watson's theorem. The Born term $E^B_{0^+}(\pi^0)$ in Eq. (4.9) would acquire an identical factor had we included $\pi^0 p$ rescattering in the development leading to Eq. (4.9). However, since $\delta(\pi^0 p)$ is small in the energy range of interest [e.g., $|\delta(\pi^0 p)|<4°$ for $E_\gamma \le 250$ MeV by our estimation], we can set $e^{i\delta(\pi^0 p)}\cos\delta(\pi^0 p)=1$ everywhere with little loss of generality. Then, according to Eq. (4.13), T_{fq} will be real below π^+ threshold since the K matrix is Hermitian. A rough estimate of the relative magnitudes of the two terms in Eq. (4.13) based on the πN scattering lengths indicates that for $E_\gamma \le 250$ MeV the second term contributes about 10% or less, and is negligible at low energy. Therefore, to a good approximation we can assume $F_{cx}(q_0,q)$ is real, and Eqs. (4.10) and (4.11) stand as they are.

Equations (4.10) and (4.11) form the starting point for our analysis. Three ingredients are required to evaluate them, namely, the Born amplitudes $E^B_{0^+}(\pi^0)$ and $E^B_{0^+}(\pi^+)$, and the charge-exchange scattering amplitude $F_{cx}(q_0,q)$. Because of the integral over q, the half-off-shell behavior of the latter two amplitudes must be specified, where by off shell we mean $W \neq W_0$. As it happens, the *final* form of the integrand of Eq. (4.11) converges rapidly for photon energies near and below the π^+ threshold, so in that region at least, we do not have to go far off shell.

One notices that the numerator of $W-W_0$ in Eq. (4.11) is equivalent to $\mathrm{Im}E_{0^+}(\pi^0)$ from Eq. (4.10b). By comparing $\mathrm{Im}E_{0^+}(\pi^0)$ with the available data, we can be sure that the on-shell calibration of the numerator is correct.

In the following sections we will discuss the Born and charge-exchange amplitudes, including our off-energy-shell assumptions.

V. THE BORN AMPLITUDES

A. Pseudovector coupling and vector-meson exchanges

The photoproduction amplitudes decompose in isotopic spin space into isoscalar ($A^{(0)}$) and isovector ($A^{(+)}, A^{(-)}$) components as follows:

$$E^B_{0^+}(\pi^+)=\sqrt{2}(A^{(0)}+A^{(-)}) , \tag{5.1a}$$

$$E^B_{0^+}(\pi^0)=A^{(0)}+A^{(+)} . \tag{5.1b}$$

Convenient expressions for the isospin amplitudes, derived in pseudoscalar (PS) coupling, are given by Berends *et al.* [14] [their Eq. (8.13)]. They are easily modified for pseudovector (PV) coupling according to the prescription of Olsson and Osypowski [15] by including their "current algebra term." Therefore we write

$$A^{(i)}_{\mathrm{PV}}=A^{(i)}_{\mathrm{PS}}+\Delta A^{(i)} , \tag{5.2}$$

where for S waves

$$\Delta A^{(i)}=\left[\frac{Z}{\mu}\right]\left[\frac{1}{2}(1+\xi)\kappa^{(i)}\frac{W_0^2-M^2}{2MW_0}\right] . \tag{5.3}$$

Here $\xi=+1$ for $i=(+,0)$ and -1 otherwise; $\kappa^{(i)}=\kappa_p-\kappa_n$ for $i=(+,-)$ and $\kappa_p+\kappa_n$ otherwise; $\kappa_p=1.79$ and $\kappa_n=-1.51$, and the quantity (Z/μ) is defined by Eq. (8.18) of Berends *et al.* [14]. Equation (5.3) has a small effect on $E^B_{0^+}(\pi^+)$ but is crucial for $E^B_{0^+}(\pi^0)$.

The threshold value of $E^B_{0^+}(\pi^0)$ given by the above amplitudes is

$$E^B_{0^+}(\pi^0)|_{\mathrm{thr}}=-2.47\times 10^{-3}/m_\pi . \tag{5.4a}$$

It differs from the usual LET value, Eq. (1.3), mainly through the presence of a phase-space factor $(1+\mu)^{-3/2}$, which occurs expanded and truncated in the derivation of Eq. (1.1). However, it seems more reasonble to keep this model-independent factor intact, in which case the modified LET becomes

$$E_{0^+}=\frac{ef}{m_\pi}\frac{1}{(1+\mu)^{3/2}}\left[-\mu+\frac{\mu^2}{2}\kappa_p\right] \tag{5.4b}$$

and is numerically equal to Eq. (5.4a). We will use this modified version in later comparisons with experiment.

As a separate check on our PV amplitudes, we have confirmed that at threshold they reproduce the LET's up through the model-independent orders of (m_π/M) as required by Eq. (3.4).

In calculating the Born amplitudes, we will use the appropriate physical pion and nucleon masses, and take the πNN coupling constant to be

$$f^2=0.0796.$$

We will now break PCAC by including (ρ,ω) vector-meson exchanges in the Born amplitudes, following the approach of Olsson and Osypowski [16]. These additions of course introduce some model dependence into the Born amplitudes. Since ρ exchange uniquely affects $A^{(0)}$ while ω exchange only contributes to $A^{(+)}$, the ρ contributes uniquely to $E^B_{0^+}(\pi^+)$ but both exchanges enter into $E^B_{0^+}(\pi^0)$.

With reference to the Olsson paper [16], the vector-meson exchanges are specified by the photon-meson-pion coupling constants $\lambda_\rho,\lambda_\omega$, the ρNN vector and tensor coupling constants $g_{1\rho}$ and $g_{2\rho}$, and the corresponding ωNN constants $g_{1\omega}$ and $g_{2\omega}$. From the ρ and ω photon decay widths we obtain

$$\lambda_\rho=0.102(4\pi\alpha)^{1/2} , \quad \lambda_\omega=0.325(4\pi\alpha)^{1/2} . \tag{5.5}$$

For the vector coupling constants, we take

$$g_{1\rho}=2.63 \quad (\text{Ref. [17]; note } g_{1\rho}=f_1/2) , \quad g_{1\omega}=3g_{1\rho} \quad (\text{Ref. [16]}) . \tag{5.6}$$

For the ρNN tensor coupling constants, we adopt the mean of f_2/f_1 as found by Höhler *et al.* [17] and Gustafson (quoted in Nagels *et al.* [18]), giving $f_2/f_1=6.3$, or in current notation

$$g_{2\rho}/g_{1\rho}=0.47/m_\pi . \tag{5.7}$$

This is a factor 1.7 greater than expected, based on the vector dominance model (VDM). Little is known about the ωNN tensor coupling, so we will adopt the prediction of the VDM [16] and assume it is renormalized by the same factor of 1.7 as the ρ coupling. This gives

$$g_{2\omega}/g_{1\omega} = -0.015/m_\pi \ . \quad (5.8)$$

At π^0 threshold the vector-meson exchanges induce the following changes in the isospin amplitudes:

$$\Delta A^{(0)}(\rho) = +0.0785\times 10^{-3}/m_\pi \ , \quad \Delta A^{(+)}(\omega) = +0.0790\times 10^{-3}/m_\pi \ . \quad (5.9)$$

It may seem surprising that these changes are nearly identical although the ω coupling constants in Eqs. (5.5)–(5.6) are a factor of 3 larger than those of the ρ. However, the ρ contribution is dominated by its strong tensor coupling, while the ω contribution comes mainly from the vector coupling. Nevertheless, a lot of uncertainty is attached to the above estimate of $\Delta A^{(+)}(\omega)$ because of our poor knowledge of the ω tensor coupling. For example, $\Delta A^{(+)}$ nearly vanishes for $g_{2\omega}/g_{1\omega} = -0.06$, which is the median value found by Olsson and Osypowski [16] in their fits to isospin amplitudes deduced from photoproduction data.

We have not included a contribution to E^B_{0+} from the $\Delta(1232)$ since most calculations suggest it is small, the order of a few percent or less of the nucleon term. Besides, the correct treatment of the $\Delta(1232)$ is till a controversial topic in the literature; even the sign of its contribution to E^B_{0+} seems uncertain. (See, for example, Table I in Ref. [7].)

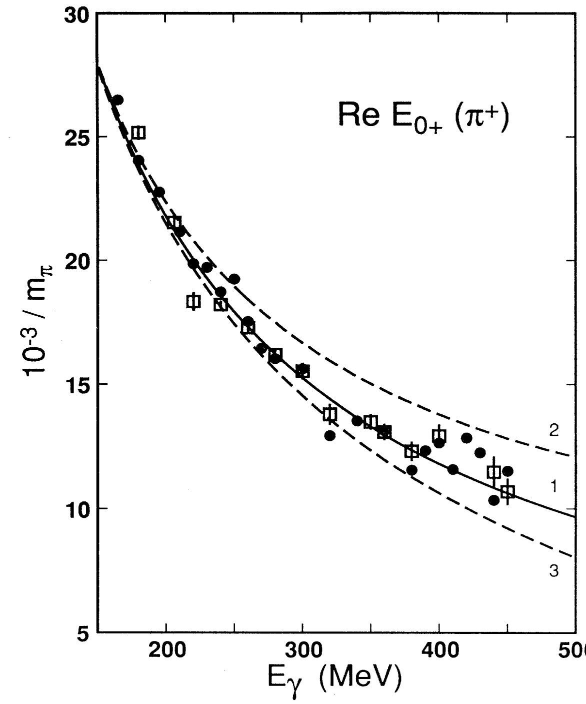

FIG. 3. The Born amplitude $E^B_{0+}(\pi^+)$ including ρ exchange (curve 1) compared with the values of Re$E_{0+}(\pi^+)$ as deduced by Berends and Weaver (Ref. [20]), shown as circles, and Pfeil and Schwela (Ref. [21]), shown as squares. No errors were given in Ref. [20]. The effect of omitting the monopole factor $f_c(q^2)$ is indicated by curve 2, while curve 3 shows the effect of omitting the ρ exchange.

B. The on-shell behavior of $E^B_{0+}(\pi^+)$

The presence of terms like Eq. (5.3) eventually causes the electric dipole amplitudes to diverge with increasing photon energy E_γ. This can be countered to some extent by introducing πNN form factors at the appropriate vertices. A favorite parametrization of πNN form factors used to regularize potentials at short distances is the monopole form, with a cutoff parameter Λ_c lying around 700 MeV/c or so [19]. In order to at least partially mimic the effect of the form factors, we employ the expedient device of simply multiplying the Born amplitudes by a common monopole factor

$$f_c(q^2) = \frac{1}{1+(q/\Lambda_c)^2} \ , \quad (5.10)$$

where q is the pion three-momentum. So far, gauge invariance has not been disturbed.

The cutoff parameter Λ_c was estimated by fitting $E^B_{0+}(\pi^+)$ to the values of Re$E_{0+}(\pi^+)$ up to $E_\gamma = 450$ MeV as deduced by Berends and Weaver [20] and Pfeil and Schwela [21] from their analyses of photoproduction data. We find $\Lambda_c \approx 650$ MeV/c, which is precisely the value determined by Nozawa *et al.* [5] in their fit to the total $\gamma p \to \pi^0 p$ cross section.

As can be seen in Fig. 3, our $E^B_{0+}(\pi^+)$ amplitude gives a good description of the data over the whole energy range. For comparison, we also show the amplitude with $f_c(q^2)$ omitted (curve 2), and then with the ρ exchange omitted (curve 3).

C. The off-shell behavior of $E^B_{0+}(\pi^+)$

There is of course no model-independent way of establishing the off-energy-shell behavior of $E^B_{0+}(\pi^+)$. Nozawa and Lee [22] give a prescription for dealing with the Feynman diagrams for the Born amplitudes which allows them to maintain gauge invariance when the $\pi^+ n$ system is off the energy shell. Our own approach is less sophisticated. In the expressions given by Berends *et al.* [14] terms which depend on the $\pi^+ n$ final-state variables (such as q and E_2) are evaluated at the momentum q corresponding to the off-shell energy W, while those that depend on the initial-state variables (such as k and E_1) are evaluated at the on-shell energy W_0. The common coefficient (Z/μ) contains a factor $1/W$ originating from normalization, and assuming symmetry, we replace $W \to (WW_0)^{1/2}$. The few remaining terms depending on W are evaluated on shell except $f_c(q^2)$.

Admittedly, this rather simplistic approach may not maintain gauge invariance and we have not pursued the matter. However, most of our final conclusions do not require us to go more than few hundred MeV off energy shell from W_0, so hopefully the violation is not serious.

VI. THE CHARGE-EXCHANGE AMPLITUDE

As we have seen, the charge-exchange amplitude $F_{cx}(q_0,q)$ may be considered a real quantity for our application; therefore we will take it to be the real part of the S-wave charge-exchange amplitude as deduced from the experimental πN phase shifts and introduce the necessary phenomenology for going half off shell. Thus we write

$$F_{\rm cx}(q_0,q_0)=\frac{\sqrt{2}}{3}{\rm Re}(f^{(1)}-f^{(3)}) \,, \tag{6.1}$$

where the scattering amplitude for isospin t is

$$f^{(2t)}=\frac{1}{2iq_0}(\eta_{2t}e^{i\delta_{2t}}-1). \tag{6.2}$$

Here, η_{2t} is the inelasticity parameter and δ_{2t} is the real part of the πN S-wave phase shift. In the following we will set $\eta_{2t}=1$, which is a good approximation at least for T_π (lab) ≤ 500 MeV. At low energy Eq. (6.1) reduces to the familiar expression

$$F_{\rm cx}(q_0,q_0)=\frac{\sqrt{2}}{3}(a_1-a_3) \,, \tag{6.3}$$

where the $t=\frac{1}{2}$ and $t=\frac{3}{2}$ scattering lengths are given by [23]

$$a_1=0.175/m_\pi, \quad a_3=-0.100/m_\pi \,. \tag{6.4}$$

In seeking algebraic parametrizations of the amplitudes $f^{(2t)}$ that go off shell in a plausible manner, we will be guided by the separable-potential approach to scattering where the amplitudes decompose symmetrically into initial- and final-state factors. Actually, we will parametrize the phase shifts themselves in a separable form to simplify the numerical analysis. This is almost equivalent to expressing $F_{\rm cx}(q_0,q)$ in a separable form, as is easily seen. From Eq. (6.1) and (6.2) we get

$$F_{\rm cx}(q_0,q_0)=\frac{\sqrt{2}}{3}\frac{1}{2q_0}(\sin 2\delta_1-\sin 2\delta_3) \,. \tag{6.5}$$

Now, with the formulation of the phase shifts described below, we find $\sin 2\delta_{2t}\approx 2\delta_{2t}$ for on-shell momenta q_0 corresponding to $E_\gamma \lesssim 250$ MeV and arbitrary off-shell q. Therefore, a small-angle expansion of $\sin 2\delta_{2t}$ can be employed in Eq. (6.5) to good approximation, and $F_{\rm cx}(q_0,q)$ in effect takes on a separable format.

The on-shell phase shifts are parametrized by

$$\delta_1=a_1q_0\left[\frac{1}{[1+(q_0/\Lambda_{1a})^2]^{n_1}}+\frac{(q_0/\Lambda_{1c})^{n_3}}{[1+(q_0/\Lambda_{1b})^2]^{n_2}}\right] \tag{6.6a}$$

and

$$\delta_3=a_3q_0\left[\frac{1}{[1+(q_0/\Lambda_{3a})^2]^{n_4}}+\frac{(q_0/\Lambda_{3c})^{n_6}}{[1+(q_0/\Lambda_{3b})^2]^{n_5}}\right] \,, \tag{6.6b}$$

where the n_i are even integers and the Λ_i are constants, all to be determined from the experimental data. The forms of these expressions ensure that Eq. (6.3) is satisfied as $q_0\to 0$. The set of integers n_i is constrained by the requirement that δ_{2t} varies as $1/q_0^3$ as $q_0\to\infty$, this being the asymptotic behavior which derives from separable potentials of the form

$$V_0(r,r')=g\frac{e^{-\Lambda r}}{r}\frac{e^{-\Lambda r'}}{r'}$$

as used by some authors [24,25].

The procedure for taking Eqs. (6.6) half off shell is clear: the overall factor q_0 becomes $(q_0q)^{1/2}$ [also in Eq. (6.5)] and each interior term factors symmetrically, such as

$$[1+(q_0/\Lambda_{1a})^2]^{n_1/2}[1+(q/\Lambda_{1a})^2]^{n_1/2} \,.$$

The functional forms given by Eqs. (6.6) are too simplistic to reproduce in detail the available phase shifts up to $W_0=2200$ MeV, so we quantitatively fit the low energy regions, $T_\pi\leq 500$ MeV for δ_1 and $T_\pi\leq 300$ MeV for δ_3, and at the same time tried to reproduce the general qualitative features at high energy. The optimal parameters so obtained are, for $t=\frac{1}{2}$,

$$\begin{aligned} n_1&=2, \quad \Lambda_{1a}=350 \text{ MeV}/c \,,\\ n_2&=6, \quad \Lambda_{1b}=1113 \text{ MeV}/c \,,\\ n_3&=4, \quad \Lambda_{1c}=356 \text{ MeV}/c \,, \end{aligned} \tag{6.7a}$$

and for $t=\frac{3}{2}$,

$$\begin{aligned} n_4&=2, \quad \Lambda_{3a}=450 \text{ MeV}/c \,,\\ n_5&=4, \quad \Lambda_{3b}=965 \text{ MeV}/c \,,\\ n_6&=2, \quad \Lambda_{3c}=221 \text{ MeV}/c \,. \end{aligned} \tag{6.7b}$$

The resulting phase shifts are compared with the data in Fig. 4, where the rapid rise of δ_1 is seen to be especially difficult to model.

The experimental data presented in Fig. 4 have been drawn from several sources and were smoothed by eye, so to speak. They do not necessarily represent current "accepted values." For example, our compilation on δ_1 below $T_\pi=300$ MeV closely follows the results of Carter *et al.* [26] and Arndt as quoted in Ref. [25].

The fit to δ_3 is not very sensitive to the value of Λ_{3a} and acceptable results are obtained for values as large as 800 MeV/c. For comparison, from their fits to the low energy $t=\frac{3}{2}$ phase shifts, Siegel *et al.* [25] obtained 600 MeV/c while Desgrolard *et al.* [24] obtained 690 MeV/c for the corresponding separable-potential range parameter.

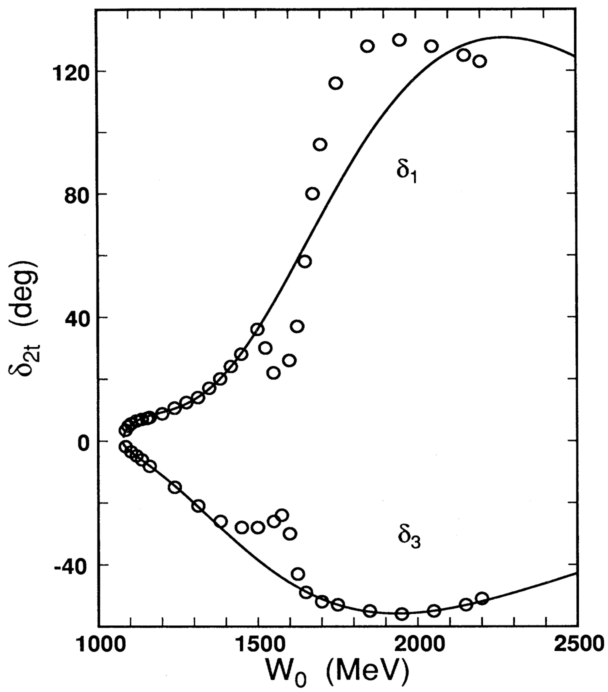

FIG. 4. The $t=\frac{1}{2}$ and $t=\frac{3}{2}$ πN phase shifts, as parametrized by Eqs. (6.6) and (6.7) of the text, compared with the data as a function of the total c.m. energy W_0. No attempt was made to separately parametrize the resonances $N(1535)$ and $\Delta(1620)$, evident in the figure.

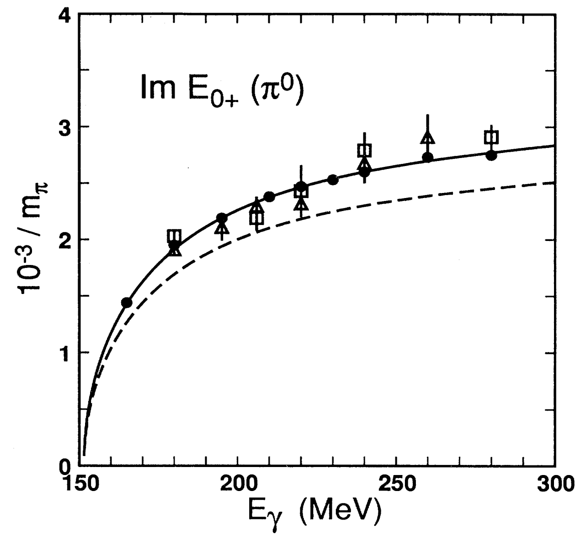

FIG. 5. The imaginary amplitude $\mathrm{Im}E_{0^+}(\pi^0)$ for π^0 production (dashed curve) compared with the reference data shown as circles. These data derive from the amplitudes of Berends and Weaver (Ref. [20]) as described in the text. Also shown are results from Pfeil *et al.* (squares, Ref. [21]) and Noelle *et al.* (triangles, Ref. [27]). The solid line represents a simple renormalization of the theory by a factor of 1.13.

VII. THE IMAGINARY AMPLITUDE $\mathrm{Im}E_{0^+}(\pi^0)$

Since the imaginary amplitude $\mathrm{Im}E_{0^+}(\pi^0)$ forms part of the kernel in the rescattering integral, Eq. (4.11), it is useful to compare the corresponding on-shell amplitude as given by Eq. (4.10b), with the available experimental data. Using the Born amplitude $E^B_{0^+}(\pi^+,\bar{q})$ and $F_{cx}(q_0,\bar{q})$ as derived above, and $\bar{q}$ as defined by Eq. (4.6), the prediction for $\mathrm{Im}E_{0^+}(\pi^0)$ is given by the dashed curve in Fig. 5. The reference data, shown as circles in the figure, have been derived from the amplitudes of Berends and Weaver [20] as described below. Also shown are data selected from Pfeil *et al.* [21] and Noelle *et al.* [27].

The theoretical amplitude falls short of the data although the energy dependence appears correct. However, a simple upward renormalization by a factor of 1.13 yields a very good description of the data up to $E_\gamma=300$ MeV, as illustrated by the solid curve in Fig. 5.

The source of the 13% discrepancy is unclear. With reference to Eq. (4.10b), it probably does not originate with $E^B_{0^+}(\pi^+,\bar{q})$, considering the good agreement seen in Fig. 3. The phase-shift parametrization is not likely at fault, since the charge-exchange amplitude does not deviate appreciably from Eq. (6.3) for the energies of interest here. Finally, the scattering length combination $a_1-a_3=0.275$ from Ref. [23] agrees with others [28], although a recent analysis by Siegel and Gibbs [25] prefers a slightly larger value, $a_1-a_3=0.290$, which is in the right direction to reduce the discrepancy.

Whatever its origin, we will assume the same factor of 1.13 also applies to the half-off-shell amplitude appearing in Eq. (4.11).

The reference data in Fig. 5 were derived from a smoothed version of the $E^{3/2}_{0^+}$ amplitudes taken from Table 3 of Ref. [20], together with the unmodified $E^{1/2}_{0^+}$ amplitudes from the same source. The $E^{3/2}_{0^+}$ amplitudes were smoothed by fitting them to a linear function of E_γ (MeV) which gives a very good fit over the range $E_\gamma=150$–450 MeV. Specifically, we find

$$E^{3/2}_{0^+}=(aE_\gamma+b)\times10^{-3}/m_\pi\ ,$$

where $a=5.611\times10^{-2}$ and $b=-29.224$. Finally, the phases of the amplitudes were taken from Ref. [29], interpolating when necessary.

VIII. THE OFF-ENERGY-SHELL FORM FACTOR

The explicit rescattering correction ΔE_{0^+} must vanish at threshold, at least through terms of order μ^2, in the limit of equal pion and equal nucleon masses as required by the LET's, but this condition is not met by Eq. (4.11) as presently constituted. Apparently there is some half-off-shell characteristic of the charge-exchange amplitude we have neglected so far, which we will now ascribe to a hypothetical off-energy-shell form factor $f(W_0,W)$. The existence of such a form factor which forces the LET requirement would imply a fundamental connection between the underlying physics governing the LET's and the half-off-shell behavior of the πN charge-exchange amplitude in the physically inaccessible region below the π^+ threshold.

With only a few propositions to guide us, we have tried to construct a purely phenomenological form factor that satisfies the degenerate-mass LET constraint, yet does not conflict with known observables. As we shall see in the next section, this form factor with two free parameters gives a very good accounting of the Mainz data, including a predicted "dip" in $E_{0^+}(\pi^0)$ just above π^+ threshold, and even has some predictive power up to $E_\gamma = 300$ MeV. The phenomenological nature of our model means of course it is hardly unique; however, it is far more successful than other versions we have explored, with less symmetry and more parameters.

The structure of $f(W_0,W)$ is constrained by the following basic premises. First, we require $f(W_0,W)=1$ when $W=W_0$ in order to keep the on-shell amplitudes intact. Second, we will assume ΔE_{0^+} vanishes identically at threshold in the absence of mass splittings. Third, ΔE_{0^+} may or may not vanish at π^0 threshold when the mass degeneracy is lifted. Fourth and last, we assume symmetry between the initial- and final-state parameters W and W_0: $f(W,W_0)=f(W_0,W)$.

The form factor should depend on q_0^2, q^2, and perhaps $(q_0-q)^2$, but the first premise is accommodated more succinctly when mass splittings are considered by using $(W_0-W)^2$ instead of $(q_0-q)^2$. Also, we will treat the invariant masses W_0 and W as the fundamental quantities entering the form factor (since it is an energy-shell form factor), and derive the momenta from them using the appropriate masses. This has particular significance when imposing the fourth premise on $f(W_0,W)$ when the isospin symmetry is lifted.

An exponential function satisfies all the requirements in the degenerate-mass limit, and our model takes the form

$$f(W_0,W)=e^{-g(W_0,W)}, \tag{8.1a}$$

with

$$g(W_0,W)=\frac{\Lambda^2(W-W_0)^2}{(q_0^2+\delta^2)(q^2+\delta^2)}. \tag{8.1b}$$

The momenta q_i in the above are given in terms of W_i by

$$q_i^2=\frac{1}{4S_i}(S_i-M_+^2)(S_i-M_-^2)\theta(W_i-M_+), \tag{8.2}$$

where $S_i=W_i^2$, $M_+=M+m$, $M_-=M-m$, and M,m are the nucleon and pion masses in the absence of isospin splitting. The quantity Λ in Eq. (8.1b) is a free parameter with units of mass, and $\hbar=c=1$ as usual.

In order to satisfy the second, and later the third of our premises, the quantity δ in Eq. (8.1b) must depend on the mass splitting of the πN systems, so we will assume

$$\delta=p(M_n+m_{\pi^+}-M_p-m_{\pi^0}), \tag{8.3}$$

where p is a dimensionless free parameter. In the degenerate-mass limit $\delta\to 0$; otherwise $\delta=p$ (5.9 MeV).

Let us see how $f(W_0,W)$ modifies the rescattering correction at threshold with no mass splittings. Mathematical ambiguities which can arise in $g(W_0,W)$ in this case are avoided by taking the limit $\delta\to 0$ after the integration in Eq. (4.11) is performed. A good approximation to the integral is obtained by noting that for small δ the integrand peaks at small q, so W can be expanded in terms of $q^2/2\mu_r$, where μ_r is the πN reduced mass. With this and other simplifications one obtains

$$\Delta E_{0^+}|_{\rm thr}=\lim_{\delta\to 0}\left[\frac{2}{\sqrt{\pi}}F_{\rm cx}(0,0)E^B_{0^+}(\pi^+,0)\delta\frac{\mu_r}{\Lambda}\right]. \tag{8.4}$$

Therefore ΔE_{0^+} vanishes at threshold as required, but not above threshold (i.e., $q_0>0$). Immediately above threshold, ΔE_{0^+} is proportional to q_0.

The quantity Λ in $g(W_0,W)$ must be proportional to some characteristic mass of the πN system, and it is reasonable to assume it is symmetric between the two particles. An obvious candidate is the πN reduced mass μ_r, so we will write

$$\Lambda=\lambda\left[\frac{\mu_r}{\pi}\right], \tag{8.5}$$

where λ is a dimensionless constant and π is included for future convenience.

One might object to Eq. (8.5) because in the soft pion limit ($m_\pi\to 0$) it implies $f(W_0,W)\to 1$, and as a result ΔE_{0^+} would no longer appear to vanish at threshold. However, according to the Weinberg–Tomozawa relations for the S-wave scattering lengths [30], the combination a_1-a_3 [recall Eq. (6.3)] is proportional to μ_r to lowest order, and so vanishes as $m_\pi\to 0$. It follows, then, that ΔE_{0^+} also vanishes in the soft pion limit.

The Weinberg–Tomozawa relations together with Eq. (8.5) as applied to Eq. (8.4) would yield a rescattering correction at threshold that is proportional to m_π to lowest order. This is also the leading order of m_π in the model-independent regime of the LET for $p(\gamma,\pi^0)$. Consequently ΔE_{0^+} *must* vanish at threshold; i.e., δ *must* approach 0 in the degenerate-mass limit, as we have already assumed through Eq. (8.3). Of course, we have not yet excluded the possibility that $p=0$.

For the final step of our phenomenological construction, we will generalize $f(W_0,W)$ to accommodate the mass splittings, and reimpose the symmetry condition $f(W,W_0)=f(W_0,W)$. Admittedly this condition is somewhat arbitrary and our only justification for it is (1) it certainly applies in the degenerate-mass case, and (2) it seems necessary, *a posteriori*.

The expressions analogous to Eq. (8.2) for the "usual" $\pi^0 p$ and $\pi^+ n$ momenta now become

$$q_0^2=\frac{1}{4S_0}[S_0-M_+^2(p)][S_0-M_-^2(p)]\theta(W_0-M_+(p)),$$
$$q^2=\frac{1}{4S}[S-M_+^2(n)][S-M_-^2(n)]\theta(W-M_+(n)), \tag{8.6}$$

where $M_+(p)=M_p+m_{\pi^0}$, $M_+(n)=M_n+m_{\pi^+}$, $M_-(p)=M_p-m_{\pi^0}$, $M_-(n)=M_n-m_{\pi^+}$, $S_0=W_0^2$, and $S=W^2$. In order to symmetrize $f(W_0,W)$, we introduce two new momentumlike quantities:

$$\hat{q}_0^2=\frac{1}{4S_0}[S_0-M_+^2(n)][S_0-M_-^2(n)]\theta(W_0-M_+(n)) ,$$
$$\hat{q}^2=\frac{1}{4S}[S-M_+^2(p)][S-M_-^2(p)]\theta(W-M_+(p)) . \tag{8.7}$$

Thus, under the exchange $S_0\leftrightarrow S$, one has $q_0^2\leftrightarrow\hat{q}^2$ and $q^2\leftrightarrow\hat{q}_0^2$. The function $g(W_0,W)$ is symmetrized by $g(W_0,W)\rightarrow\frac{1}{2}[g(W_0,W)+g(W,W_0)]$, and using Eqs. (8.1) and (8.5) we obtain the final expression for the phenomenological off-energy-shell form factor,

$$f(W_0,W)=e^{-g(W_0,W)} , \tag{8.8a}$$

where

$$g(W_0,W)=\lambda^2\left[\frac{\mu_r^2}{2\pi^2}\right](W_0-W)^2 \times\left[\frac{1}{(q_0^2+\delta^2)(q^2+\delta^2)}+\frac{1}{(\hat{q}_0^2+\delta^2)(\hat{q}^2+\delta^2)}\right] \tag{8.8b}$$

and

$$\delta=p[M_+(n)-M_+(p)] ,$$
$$\mu_r^2=\mu_r(p\pi^0)\mu_r(n\pi^+) . \tag{8.8c}$$

The final form of the rescattering correction equation (4.11), including $f(W_0,W)$ and the "missing-physics" factor of 1.13 from Sec. VII, is

$$\Delta E_{0^+}=-\mathrm{P}\int_0^\infty\frac{q\,dq}{\pi}\frac{W}{E_\pi E_N}\frac{1.13q[f(W_0,W)F_{\mathrm{cx}}(q_0,q)]E_{0^+}^B(\pi^+,q)}{W_0-W} . \tag{8.9}$$

We are now ready to compare with the experimental $p(\gamma,\pi^0)$ data

IX. COMPARISON WITH THE $p(\gamma,\pi^0)$ DATA

The total $p(\gamma,\pi^0)$ cross section is given by Eqs. (3.1)–(3.3). The unknown quantities are the two phenomenological constants p and λ of the explicit rescattering correction and the P-wave amplitude f_0 defined by Eq. (2.2). As we have seen, one expects the latter to lie in the range $f_0=7.7-8.2$. These parameters will be determined by least-squares fits to selected subsets of the experimental data. Because of our poor knowledge of the ω-meson tensor coupling, two separate analyses have been performed, one using $\Delta A^{(+)}(\omega)$ as given by Eq. (5.9) and the other with $\Delta A^+(\omega)=0$. These alternatives have a larger impact then their relative contributions at π^0 threshold would suggest, primarily because of the rapid drop in $E_{0^+}(\pi^0)$ with E_γ as seen in Fig. 2. The experimental data, shown in Fig. 1, is comprised of 21 points, 12 points below π^+ threshold and 9 points above.

A. Determination of p

The first parameter to be determined is p, using only the data below π^+ threshold. This restriction makes the results completely insensitive to the imaginary amplitude $\mathrm{Im}E_{0^+}(\pi^0)$. It also reduces the sensitivity to f_0 by avoiding the "dip" region in Fig. 2 near $E_\gamma=153$ MeV. Finally, the rescattering integral converges quickly below threshold so the far off-energy-shell behavior of $F_{\mathrm{cx}}(q_0,q)$ and $E_{0^+}^B(\pi^+,q)$ is of minimal concern. In this part of the analysis, λ is allowed to float while f_0 is held at a series of values.

The set of p values corresponding to the χ^2 minima fell in the range $p=1.8-2.2$, with an average $p=1.96$, and were only mildly sensitive to the two ω-exchange options. The reduced χ^2's were typically $\chi_\nu^2=0.35$ for 9 (12−3) degrees of freedom. This is rather low, and more will be said about it later. The average value $p=1.96$ is sufficiently close to the integral value $p=2$ that we will use the latter for the remainder of the analysis.

At this stage we believe we can rule out $p=0$, which would have implied no explicit rescattering correction at π^0 threshold within the framework of our model. The corresponding χ_ν^2 is twice the minimum value for $p=2$, and visually the fit is clearly inferior.

Therefore we conclude

$$p=2 , \tag{9.1}$$

with an uncertainty of about 20%, judging from χ_ν^2.

B. Determination of f_0

In order to gain more sensitivity to f_0, the data set is now expanded to include six points above π^+ threshold. The highest energy pair has been excluded because of an apparent inconsistency with the value of $\mathrm{Re}E_{0^+}$, as given by Beck *et al.* [2], near $E_\gamma=156$ MeV (see Fig. 2). As before, λ is allowed to float, but we set $p=2$. The resulting values for f_0 were quite insensitive to the two options for the ω exchange (this is not true for λ, however), and our best estimate based solely on χ^2 is

$$f_0=7.90\times10^{-3}/m_\pi^3 . \tag{9.2}$$

The reduced χ^2 for 16 (18−2) degrees of freedom is $\chi_\nu^2=0.21$. It is difficult to place a standard deviation on f_0 (as with p also) because of the nonlinear nature of the functions, but a *rough* estimate, based on several factors in the final analysis, is $f_0=7.90\pm0.04$.

Assuming f_0 is dominated by the M_{1^+} amplitude, the present estimate is lower but not inconsistent with other determinations. For example, Beck [12] finds $f_0=7.95\pm0.4$, while Mazzucato *et al.* [1] report $f_0=8.0\pm0.3$ in units of $10^{-3}/m_\pi^3$.

C. Determination of λ

Once again we will use only the data below π^+ threshold, for the reasons specified in the determination of p. We fix $p=2$ and $f_0=7.90$ and seek a χ^2 minimum with λ. With the full ω contribution $\Delta A^+(\omega)$ included, the fit yields

$$\lambda=1.074\pm0.040\ , \tag{9.3a}$$

and with $\Delta A^+(\omega)=0$ we get

$$\lambda=0.991\pm0.036\ . \tag{9.3b}$$

The reduced χ^2's are both about $\chi_\nu^2=0.29$ for 11 (12−1) degrees of freedom, but the error estimates were calculated as if $\chi_\nu^2=1$, which is more realistic. The fact that $\lambda\approx1$ is interesting, but not independent of the factor π introduced in Eq. (8.5).

Finally, the reduced χ^2's for the complete set of 21 data points with $p=2$ and $f_0=7.90$ is $\chi_\nu^2=0.19$ and 0.18 for $\lambda=1.074$ and 0.991, respectively. The small χ^2's encountered here and throughout this analysis suggest the errors associated with the Mainz cross sections [12], assuming they are statistical, are about a factor of 2 too large.

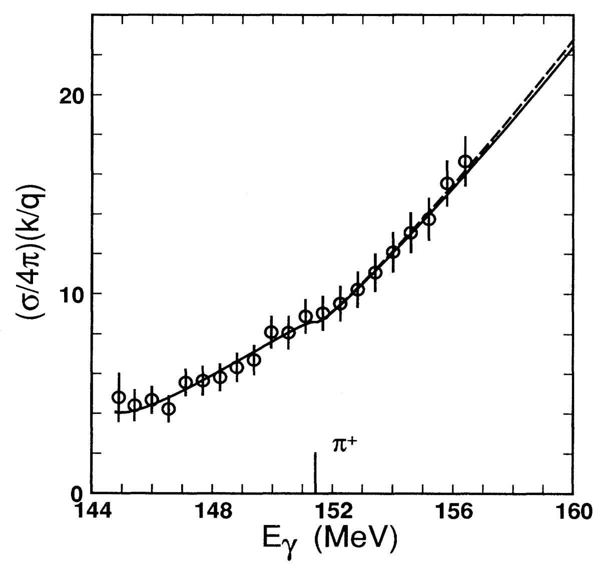

FIG. 6. Description of the Mainz total cross sections of Fig. 1 by the rescattering formalism developed in the text. Only data below π^+ threshold were used to determine the off-energy-shell form-factor parameters λ and p. The solid and dashed lines correspond to analyses with and without ω-exchange contributions. The reduced χ^2's are about 0.19 for 21 degrees of freedom. Note the hint of a cusplike structure near the π^+ threshold. Units are $10^{-6}/m_\pi^2$.

D. Description of the data

Let us now examine how effectively the present formalism describes the Mainz data, both total and differential cross sections, and test it at higher energies as well.

The total cross section is displayed in Fig. 6 where, as in Fig. 1, the factor q/k has been removed to emphasize the low energy behavior. In this and future illustrations, the solid and dashed curves correspond to $\Delta A^+(\omega)$, as given by our estimate [Eq. (5.9)], and $\Delta A^+(\omega)=0$, respectively [i.e., λ according to Eqs. (9.3)]. Notice there is a hint of a cusplike structure near the π^+ threshold.

The structure is more apparent if we examine $\mathrm{Re}E_{0^+}(\pi^0)$ in Fig. 7. The experimental data shown as circles have been reduced according to Eq. (2.3), using the amplitudes f_0 and $\mathrm{Im}E_{0^+}$ as deduced from the preceding analysis. The triangles are the solution 2 multiples from Beck *et al.* [2] *Keeping in mind that p and λ were determined below π^+ threshold, our results predict a pronounced dip in $\mathrm{Re}E_{0^+}$ above π^+ threshold.* The dip is a direct consequence of our forced symmetrization of $f(W_0,W)$ and reflects the increasing influence of the second term in Eq. (8.8b) above π^+ threshold. At the minimum, $\mathrm{Re}E_{0^+}$ is quenched to a value of about $-0.28\times10^{-3}/m_\pi$.

As seen in Fig. 7, we achieve a very good agreement

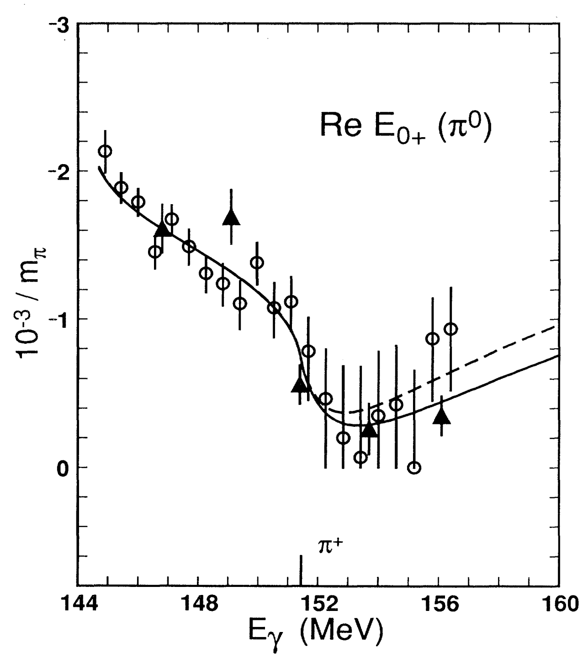

FIG. 7. The amplitude $\mathrm{Re}E_{0^+}(\pi^0)$ as given by the rescattering calculation compared with the values (circles) deduced from the Mainz total cross sections using Eq. (2.3) with $f_0=7.90$ and $\mathrm{Im}E_{0^+}$ as described in the text. The errors in the original cross sections have been reduced by a factor of 2 for display purposes, and because of the small χ_ν^2 obtained in the analyses. The solid and dashed curves represent analyses with and without ω exchange included. The triangles are the solution 2 amplitudes of Beck *et al.* (Ref. [2]).

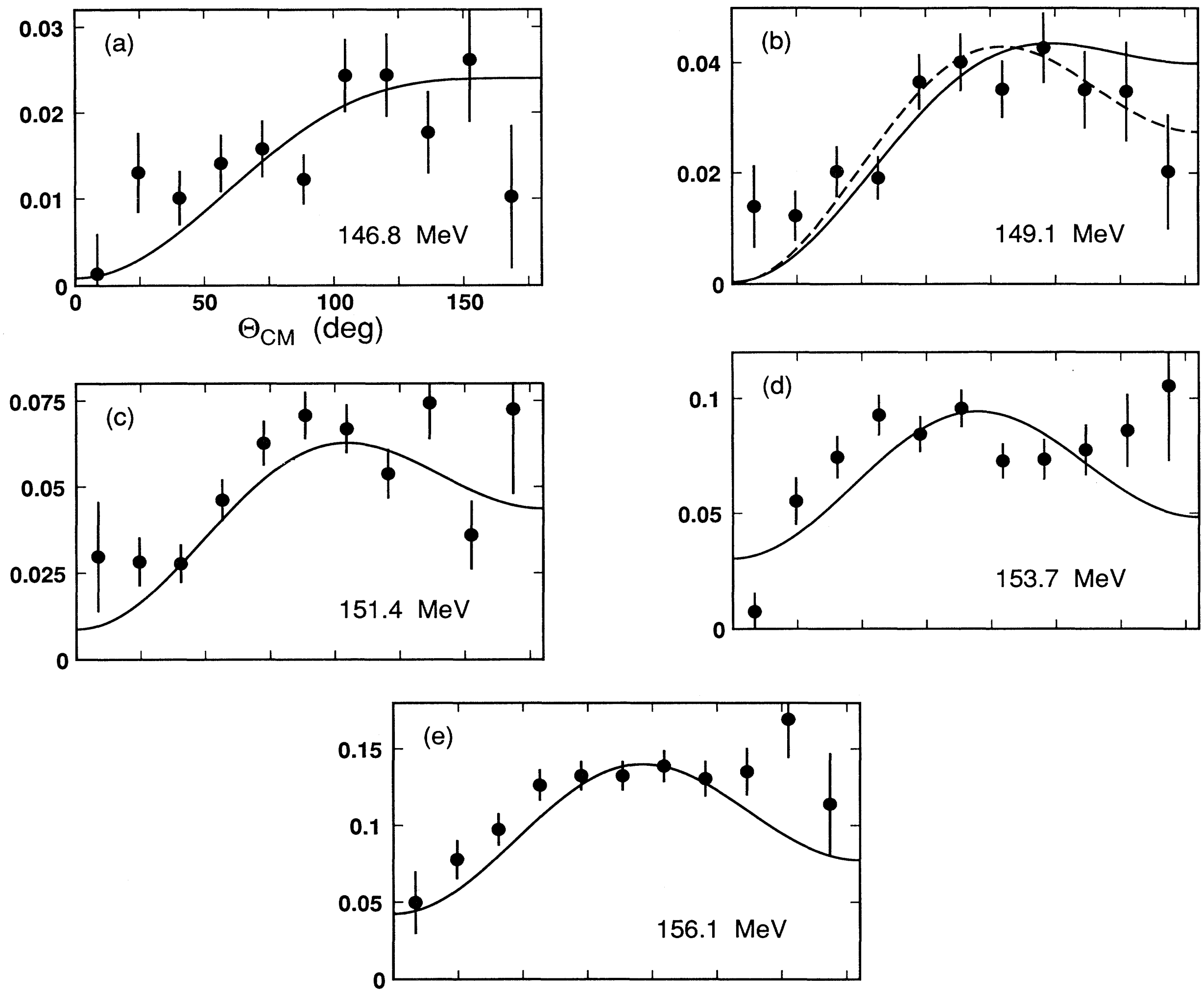

FIG. 8. Differential cross sections in the c.m. frame, assuming the P-wave amplitudes are dominated by M_{1^+}, compared with the Mainz data (Ref. [31]). The dashed curve in Fig. 8(b) includes a small E_{1^+} contribution, $E_{1^+}/M_{1^+}=0.10$, simply to illustrate the sensitivity to the omitted amplitudes. Units are μb/sr.

with the solution 2 multipoles of Beck *et al.*, providing the tensor coupling of the ω meson is small, as in Eq. (5.8). The alternative corresponds to $g_{2\omega}/g_{1\omega} \approx -0.06/m_\pi$, the median value deduced by Olsson and Osypowski [16].

Next, we consider the pion angular distributions in the c.m. frame, i.e., the differential cross sections. To the extent that partial waves $l \geq 2$ are unimportant, the cross sections depend on $\cos\theta_\pi$ up to order 2 and contain interference terms between the multipoles E_{0^+}, M_{1^+}, M_{1^-}, and E_{1^+}. Since E_{1^+} and M_{1^-} are poorly known at low energy and are presumed to be small, we will ignore them and set $M_{1^+}=f_0(qk)$ from Eq. (2.2). The resulting differential cross sections are compared with the Mainz data [31] in Fig. 8, and although some differences exist, on the whole the comparisons are seen to be satisfactory. Some improvement might be gained by including and optimizing E_{1^+} and M_{1^-}, but this is beyond our present scope. However, just to illustrate the degree of sensitivity to these amplitudes, the dashed curve in Fig. 8(b) corresponds to $E_{1^+}/M_{1^+}=0.10$.

Finally, Fig. 9 depicts $\mathrm{Re}E_{0^+}(\pi^0)$ from threshold up to $E_\gamma=300$ MeV. The "data" shown as squares were derived from the isospin amplitudes of Berends and Weaver [20] as described at the end of Sec. VII above. Since no errors were given for those amplitudes, we have included an arbitrary $\pm 10\%$ uncertainty in Fig. 9. Up to $E_\gamma=300$ MeV the general trend of the data is reproduced fairly well, but at higher energies (not shown) the Berends–Weaver amplitude decreases rapidly and changes sign between $E_\gamma=350$ and 400 MeV. The solid curve in Fig. 9 changes sign around 500 MeV.

The overall structure of $\mathrm{Re}E_{0^+}(\pi^0)$ is quite striking, and deviates markedly from the behavior when the rescattering term ΔE_{0^+} is excluded, as shown by the dashed line in Fig. 9.

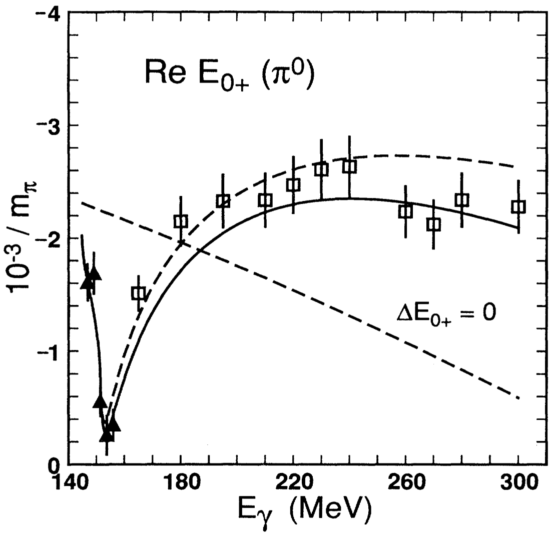

FIG. 9. The amplitude $\mathrm{Re}E_{0^+}(\pi^0)$, from threshold to $E_\gamma = 300$ MeV, compared with the solution 2 values of Beck *et al.* (triangles) and the results derived from Berends *et al.* (squares, Ref. [20]) as described in the text. An arbitrary uncertainty of $\pm 10\%$ has been assigned to the latter values. The two curves represent different ω-exchange options as in Fig. 6. The dashed line represents no rescattering, but includes both vector-meson exchanges.

A comment is in order here concerning the convergence of the rescattering integral, Eq. (8.9), in the various energy domains of Figs. 7 and 9. Near and below π^+ threshold the integral saturates quickly for $q \lesssim 300$ MeV/c; therefore the amplitudes need project only a few hundred MeV half off shell (i.e., $W - W_0$). However, at the higher energies of Fig. 9 the integral does not saturate until q reaches a few thousand MeV/c; hence it must depend to some degree on the asymptotic behavior of the half-off-shell πN scattering amplitudes. The reasonable agreement displayed by Fig. 9 thus lends some support to our off-shell parametrization, Eq. (6.5) *et seq.*

X. AMENDMENT TO THE LOW ENERGY THEOREM

The rescattering calculation has been shown to give a good account of the $p(\gamma,\pi^0)$ data , so let us return to threshold and examine the implications for the low energy theorem.

Three components entered into the formulation of $\mathrm{Re}E_{0^+}$, namely, the Born term $E^B_{0^+}$, the rescattering term ΔE_{0^+}, and the vector-meson exchanges $\Delta E_{0^+}(\rho,\omega)$:

$$\mathrm{Re}E_{0^+} = E^B_{0^+} + \Delta E_{0^+} + \Delta E_{0^+}(\rho,\omega) \ . \tag{10.1}$$

Their threshold values are summarized in Table I for the two ω-exchange options.

The standard LET prediction at threshold, based on Eq. (1.1), is

$$E_{0^+}(\pi^0) = -2.27\times 10^{-3}/m_\pi \ , \tag{10.2}$$

while the modified LET, Eq. (5.4b), gives

$$E_{0^+}(\pi^0) = -2.47\times 10^{-3}/m_\pi \ . \tag{10.3}$$

The above differ numerically for the reason previously discussed. While both are legitimate LET statements, we consider Eq. (10.3) to be the more realistic estimate because it does not involve the expansion of $(1+\mu)^{-3/2}$. In deriving Eq. (1.1), all terms of order μ^3 and higher are rejected even if the respective coefficients contain some model-independent parts [10]. Actually, we are forced to use Eq. (10.3) as our benchmark LET anyway, since $\mathrm{Re}E_{0^+}$ as defined by Eq. (10.1) reduces to it at threshold in the absence of explicit rescattering corrections and meson exchanges.

The low energy theorem is based on the strict validity of PCAC; therefore any comparison with $\mathrm{Re}E_{0^+}(\pi^0)$ should exclude the PCAC-violating part caused by the vector-meson exchanges. The threshold value of ΔE_{0^+} then becomes the measure of the departure from the LET due to the isospin splitting of the masses.

Of the two sets of entries in Table I, the first set ($\lambda = 1.074$) is preferred since it is more compatible with the E_{0^+} amplitudes of Beck *et al.* [2] (triangles, Fig. 7) and furthermore corresponds to a small ω tensor coupling, in contrast to the second set where it is unreasonably large.

We would conclude, then, that the mass splitting of the πN system induces a change in the threshold $E_{0^+}(\pi^0)$ amplitude of about

$$\Delta E_{0^+} \approx +0.29\times 10^{-3}/m_\pi \ , \tag{10.4}$$

which represents a 12% deviation from the LET prediction, Eq. (10.3).

The magnitude of the deviation is too small to describe it as a "violation" of the LET. If one claims that the iso-

TABLE I. Threshold values of the Born term, the rescattering term ΔE_{0^+}, and the vector-meson exchange for the two ω-exchange options employed in the analysis. The corresponding threshold value of $\mathrm{Re}E_{0^+}(\pi^0)$ is given in the final column. Units (except λ) are in $10^{-3}/m_\pi$.

$\Delta A^{(+)}(\omega)$	λ	$E^B_{0^+}$	ΔE_{0^+}	$\Delta E_{0^+}(\rho,\omega)$	$\mathrm{Re}E_{0^+}\vert_{\mathrm{thr}}$
0.079	1.074	−2.468	0.291	0.158	−2.019
0.000	0.991	−2.468	0.357	0.079	−2.032

spin splitting in fact induces a very small change in the LET [32], implying all rescattering corrections are still implicitly accounted for at threshold, then ΔE_{0^+} could be interpreted as a measure of the model-dependent terms of order μ^3 and higher. Instead, we prefer to think of ΔE_{0^+} as an "amendment" to the LET and offer an alternative hypothesis. In our phenomenological formalism, the parameter δ was introduced to insure that as the mass splitting was reduced to 0, the explicit rescattering term vanished at threshold, and the conventional prediction was recovered. This rescattering term was appended to the Born term and the sum is model dependent. Now it seems reasonable to expect that a LET in which the mass splitting was included from the beginning would again be model independent to some order, say, μ^2, and that a correct treatment of Born +rescattering would reduce under expansion to the same LET expression. In our particular model, the rescattering term converges to the expression given by Eq. (8.4) as $\delta \to 0$. From the discussion following Eq. (8.4), we saw that this expression was of order m_π [or m_π^2 if one rejects Eq. (8.5)], i.e., of order μ or μ^2. By our hypotheses, then, the LET might also carry terms that similarly resemble $\mu\delta$ (or $\mu^2\delta$) and therefore vanish when the isospin splitting is lifted. The result expressed by Eq. (10.4) would then be a measure of these terms.

XI. CONCLUDING REMARKS

We have demonstrated that the recent Mainz $p(\gamma,\pi^0)$ measurements are not in serious conflict with the photopion low energy theorem, and such disagreement as exists, about 12%, can be attributed to an explicit rescattering effect. Above threshold, the rescattering induces a substantial quenching of $\mathrm{Re}E_{0^+}(\pi^0)$, but at higher energies the amplitude recovers until numerically it becomes roughly commensurate with the threshold value.

We suggest that the rescattering correction at π^0 threshold ($\Delta E_{0^+} \approx 0.29\times 10^{-3}/m_\pi$) constitutes an amendment to the *conventional* LET due to the isospin splitting of the pion and nucleon masses. Within the context of our semiphenomenological description, such amendments would lie within the model-independent domain (i.e., be of order of μ or μ^2, where $\mu = m_\pi/M$), and would depend explicitly on the $\pi^0 p$-$\pi^+ n$ mass splitting. The correction is about 12% and therefore not unreasonable. We find it difficult to subscribe to the claim that such effects are automatically contained within the conventional LET because the mass splittings are not a factor in the original derivation. Since the Born amplitude $E_{0^+}^B$ yields the LET at threshold, we presume that it too is devoid of any rescattering effects deriving from the isospin splittings.

The rescattering formalism developed here, as in Ref. [5], is based on the integral equation for the transition operator and therefore depends on the half-off-energy-shell behavior of certain amplitudes. Wherever possible, the on-shell behavior of these quantities, such as $\mathrm{Re}E_{0^+}(\pi^+)$, $\mathrm{Im}E_{0^+}(\pi^0)$, and the πN phase shifts, have been constrained by their respective experimental values. The only free parameters in the analysis, aside from the P-wave photoproduction amplitude f_0, occur in a phenomenological form factor $f(W_0,W)$ representing the off-energy-shell behavior of the $\pi^+ n \leftrightarrow \pi^0 p$ charge-exchange amplitude. This form factor provides a mechanism for continuing the amplitude into the physically inaccessible region between the π^0 and π^+ thresholds. In the usual K-matrix approach to rescattering, in effect one assumes $f(W_0,W)=0$ in the off-shell region and employs the on-shell amplitude everywhere. We make no such assumption, and instead use the Mainz total cross sections below π^+ threshold to provide some hints as to how $f(W_0,W)$ *might* behave. Naturally this requires some sort of parametrization, and our particular choice was constructed around a few basic premises. With the parameters (λ and p) determined, we predict a strong depresssion of $\mathrm{Re}E_{0^+}(\pi^0)$ just above the π^+ threshold and then a gradual increase with increasing photon energy. The results are in good agreement with the amplitudes deduced by Beck *et al.* [2] (solution 2), and in reasonable agreement with others up to $E_\gamma = 300$ MeV.

Our ansatz for the form factor goes off shell rather slowly since $df/dW|_{W=W_0}=0$, but in the degenerate-mass limit ($\delta \to 0$) the transition becomes more rapid as one approaches threshold. However, we have employed phenomenology to promote plausibility arguments, not necessarily to reach firm deductions concerning off-shell behavior. (For example, we have not considered the question of how the off-shell unitarity of the πN S matrix can be maintained in the presence of off-shell form factors in general.) Our main conclusion here, simply stated, is that most of the structure seen in $\mathrm{Re}E_{0^+}(\pi^0)$ derives from some kind of off-shell behavior which we postulate is associated with the charge-exchange amplitude, and in this conclusion we differ with the K-matrix approach.

Actually, the present formalism reduces to the usual K-matrix expression for rescattering in the limit where $f(W_0,W)$ is unity on shell and vanishes otherwise. This is equivalent to letting $\lambda \to \infty$ in the form factor, Eqs. (8.8), and as a result the integrand of the rescattering correction, Eq. (8.9), becomes highly localized around the pole. By converting to an integral over W, one easily verifies that the principal value integral vanishes as $\lambda \to \infty$. If we now turn to Eq. (4.9), employ Eq. (6.3) for the charge-exchange amplitude, and allow $\bar{q}$ to be complex below π^+ threshold, we recover the familiar K-matrix result to lowest order.

We conclude with a brief statement concerning the πNN coupling constant. Throughout the present analysis the traditional value $f^2 = 0.0796$ has been employed, but two recent works [33,34] have seriously questioned this result, both proposing somewhat smaller values. As it pertains to the present work, a change in coupling constant has no effect on the integrand of ΔE_{0^+} [recall Eq. (8.9)], since the numerator was normalized against the experimental values of $\mathrm{Im}E_{0^+}(\pi^0)$. However, it does alter $\mathrm{Re}E_{0^+}(\pi^0)$ because the Born amplitude in Eq. (4.10a) gets modified slightly. Therefore, we have repeated the data analysis according to the procedure of Sec. IX using $f^2(\pi^0 NN)=0.0749$ (Ref. [33]), and make the following observations. The values of p and f_0 remain essentially unchanged, while λ increases to $\lambda=1.146$ compared with the previous value of $\lambda=1.074$. Below π^+ threshold $\mathrm{Re}E_{0^+}(\pi^0)$ is indistinguishable from the solid curve in Fig. 7, but above threshold we now achieve perfect agreement with the amplitudes of Beck *et al.* [2], shown as the triangles. The discrepancy with the LET at threshold is reduced to about 10%, while the LET value itself is reduced by about 3%.

Finally, as this paper was nearing completion we became aware of a preprint by Bernstein and Holstein [35] in which the Mainz total cross sections were analyzed in a manner similar to Sec. II above, with similar conclusions regarding the threshold amplitude. There is sufficient difference in approach, however, that we have retained our discussion, also keeping in mind its relevance to the subsequent analysis.

ACKNOWLEDGMENTS

We are grateful to R. Beck for communicating his data to us prior to publication, and for helpful discussions concerning details of his analysis. We also thank J.-M. Laget and E. L. Tomusiak for clarifying some details regarding the LET's. Many discussions with J. L. Friar on related questions are gratefully acknowledged.

[1] E. Mazzucato *et al.,* Phys. Rev. Lett. **57**, 3144 (1986).
[2] R. Beck *et al.,* Phys. Rev. Lett. **65**, 1841 (1990).
[3] T. Schäfer and W. Weise, Phys. Lett. B **250**, 6 (1990).
[4] G. M. Radutsky, Yad. Fiz. **51**, 777 (1990) [Sov. J. Nucl. Phys. **51**, 495 (1990)].
[5] S. Nozawa, T.-S. H. Lee, and B. Blankleider, Phys. Rev. C **41**, 213 (1990).
[6] A. N. Kamal, Phys. Rev. Lett. **63**, 2346 (1989).
[7] L. M. Nath and S. K. Singh, Phys. Rev. C **39**, 1207 (1989).
[8] M. Araki, Phys. Lett. B **219**, 135 (1989).
[9] A. A. Chumbalov, S. S. Kamalov, and T. V. Tetereva, Phys. Lett. B **213**, 255 (1988).
[10] P. de Baenst, Nucl. Phys. **B24**, 633 (1970).
[11] H. W. L. Naus, J. H. Koch, and J. L. Friar, Phys. Rev. C **41**, 2852 (1990).
[12] R. Beck, Ph.D. thesis, University of Mainz, 1989.
[13] L. S. Rodberg and R. M. Thaler, *Quantum Theory of Scattering* (Academic, New York, 1967).
[14] F. A. Berends, A. Donnachie, and D. L. Weaver, Nucl. Phys. **B4**, 1 (1967).
[15] M. G. Olsson and E. T. Osypowski, Nucl. Phys. **B87**, 399 (1975).
[16] M. G. Olsson and E. T. Osypowski, Phys. Rev. D **17**, 174 (1978).
[17] G. Höhler and E. Pietarinen, Nucl. Phys. **B95**, 210 (1975).
[18] M. M. Nagels *et al.,* Nucl. Phys. **B147**, 189 (1979).
[19] J.-F. Mathiot, Phys. Rep. **173**, 63 (1989).
[20] F. A. Berends and D. L. Weaver, Nucl. Phys. **B30**, 575 (1971).
[21] W. Pfeil and D. Schwela, Nucl. Phys. **B45**, 379 (1972).
[22] S. Nozawa and T.-S. H. Lee, Nucl. Phys. **A513**, 511 (1990).
[23] R. Koch, Nucl. Phys. **A448**, 707 (1986).
[24] P. Desgrolard and T. F. Hammann, Phys. Rev. C **6**, 482 (1972).
[25] P. B. Siegel and W. R. Gibbs, Phys. Rev. C **33**, 1407 (1986).
[26] J. R. Carter, D. V. Bugg, and A. A. Carter, Nucl. Phys. **B58**, 378 (1973).
[27] P. Noelle, W. Pfeil, and D. Schwela, Nucl. Phys. **B26**, 461 (1971).
[28] J. Duclos *et al.,* Phys. Lett. **43B**, 245 (1973).
[29] F. A. Berends, A. Donnachie, and D. L. Weaver, Nucl. Phys. **B4**, 54 (1967).
[30] S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966); Y. Tomozawa, Nuovo Cimento A **46**, 707 (1966).
[31] R. Beck, private communication. These data represent a coarser binning of the cross sections than those of Ref. [2].
[32] H. W. L. Naus, Phys. Rev. C **43**, R365 (1991).
[33] J. R. Bergervoet *et al.,* Phys. Rev. C **41**, 1435 (1990).
[34] R. A. Arndt, Z. Li, L. D. Roper, and R. L. Workman, Phys. Rev. Lett. **65**, 157 (1990).
[35] A. M. Bernstein and B. R. Holstein, Comments Nucl. Part. Phys. (in press).