# Algebraic approach to cluster states in odd-mass nuclei. I. Energy spectrum

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An extension of the U(3) limit of the vibron model is proposed to incorporate fermionic degrees of freedom in the algebraic model of nuclear clusterization. This approach enables us to treat explicitly nucleons or holes occupying single-particle states of an oscillator shell in one of the clusters. The dynamical symmetries of the vibron-fermion model built on the U(3) boson dynamical symmetry are explored, and the coupled vibron-fermion bases are constructed. Closed expressions are obtained for the energy spectrum in each case. We suggest that the $\alpha$-cluster states of the $^{19}$F nucleus may be good examples for the application of the SU(3)×U(2) dynamical symmetry. This phenomenologic algebraic model shows some similarity with the local potential cluster model of Buck *et al.*

## I. INTRODUCTION

In the description of the quadrupole collectivity of nuclei the interacting boson model (IBM) proved to be remarkably successful [1]. More recently, Iachello suggested a similar approach, called vibron model, to dipole collectivity [2,3]. Dipole degrees of freedom appear in the rotational-vibrational motion of chemical molecules, as well as in nuclear molecular states. (The position vector of the relative distance, for example, has dipole character.) A few applications of the vibron model have been published so far both in molecular [4,5] and in nuclear physics [6–8]. The U(4) group structure of the vibron model requires somewhat simpler algebraic procedures than the U(6) structure of the IBM.

Later the fermionic extension of the IBM to odd-even nuclei, known as the interacting boson-fermion model (IBFM), was introduced and it was able to handle the interplay between collective and single-particle degrees of freedom. The possible dynamical symmetries of the IBFM have been studied systematically [9–13]. As for the vibron model only a part of this job has been done. This model has two dynamical symmetries, and in Ref. [14] the authors study the fermion (electron) coupling to one of them, which corresponds to a rigid molecule. This limit, called O(4) dynamical symmetry, has practical importance in molecular physics. Here we present the fermion coupling to the other dynamical symmetry, labeled by the U(3) group, corresponding to a soft vibrator. This limit is more useful in applications to nuclear cluster states. Similarly to the nuclear vibron model [15], it is able to describe the internal excitation of one of the clusters.

From the viewpoint of the mathematical description the boson-fermion dynamical symmetries originate from the coupling of the bosonic and fermionic group structures. The fermionic group structure depends on the single-particle states taken into account in the model. The decomposition of the angular momenta into orbital and spin parts is an essential point of the boson-fermion coupling. In certain cases it can be done on the basis of physical orbital and spin momenta; in other cases the concepts of pseudo-orbital and pseudo-spin momenta are used. This latter method allows the application of dynamical symmetries in a wide range.

When the fermions are nucleons and they are allowed to occupy states of an oscillator shell (not necessarily a physical one), the fermionic group chain contains $SU_l^F(3)$ in the (pseudo-)orbital part, so the bosonic and fermionic sectors can be coupled on the SU(3) level. In connection with the IBM this dynamical symmetry has been worked out by Bijker and Kota [12], and here we concentrate on this limit of the vibron-fermion model. It is worth mentioning that the advantage of the SU(3) basis in cluster studies is related to the Pauli exclusion principle. Recent applications to well-established cluster bands in light nuclei revealed the possibility of simulating the Pauli principle by adding restrictions to the quantum number related to the representation of the SU(3) group [7,8].

Here we attempt to give a general treatment of the vibron-fermion problem and discuss its possible dynamical symmetries. This will be done in Sec. II. In Sec. III we introduce the SU(3)×U(2) limit of the vibron-fermion model as an algebraic approach to cluster states in odd-mass nuclei, present the basis states and Hamiltonian associated with this dynamical symmetry, and briefly discuss the relation between this model and other phenomenologic algebraic models. In Sec. IV we compare our model with other models of cluster structure and propose the $\alpha$-cluster states of the $^{19}$F nucleus as a possible example for the SU(3)×U(2) dynamical symmetry. Finally, we summarize the results in Sec. V.

## II. THE VIBRON-FERMION MODEL

In this section we shall give a general introduction to the vibron-fermion model. We follow the usual way of introducing fermionic group structure and group generators. The fermionic group structure, which is then coupled to the bosonic one, depends on the fermionic single-particle states taken into account. In this respect the decomposition of the full fermionic angular momentum

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into (pseudo-)orbital and (pseudo-)spin parts is also essential. We shall discuss the two main dynamical symmetries of the vibron-fermion model, built on the O(4) and U(3) limits of the vibron model. The former one is the vibron-electron model of Ref. [14], while the later one, the SU(3)×U(2) limit of the vibron-fermion model, is an algebraic approach to cluster states of odd-mass nuclei. The detailed presentation of this new model will be given in Sec. III.

The mathematical formulation of the vibron-fermion model can be done using the analogous formulas of the various limiting cases of the IBFM. The bosonic part of the model contains $N$ interacting bosons (vibrons), occupying single-particle states with $l^{\pi}=0^{+}$ and $1^{-}$ giving account of the dipole-type collectivity [2,3]. The bosonic group structure is $U^{B}(4)$, which is generated by bilinear products of the form

$$B^{(k)}_{\kappa}(l,l')=[b^{\dagger}_{l}\times\tilde{b}_{l'}]^{(k)}_{\kappa}=\sum_{\nu,\nu'}\langle l\nu l'\nu'|k\kappa\rangle b^{\dagger}_{l,\nu}\tilde{b}_{l',\nu'}\ , \quad (2.1)$$

where $b^{\dagger}_{0}=\sigma^{\dagger}$, $b^{\dagger}_{1\nu}=\pi^{\dagger}_{\nu}$, and $\tilde{b}_{1,\nu}=(-1)^{l-\nu}b_{l,-\nu}$, and they satisfy the usual commutation relations of the boson operators. (It can easily be shown that choosing the $b^{\dagger}_{1,\nu}$ boson creation operators as spherical tensors, only the $\tilde{b}_{1,\nu}$ operators transform like spherical tensors under rotations, while the $b_{1,\nu}$ operators do not.)

The single-particle degrees of freedom are accounted for by $M$ fermions (nucleons or electrons) occupying single-particle states with certain spin parities $j_i^{\pi_i}$. The fermionic group structure is $U^{F}(m)$, where $m=\sum_i(2j_i+1)$. The $m^2$ generators are bilinear products of the fermion creation and annihilation operators

$$A^{(k)}_{\kappa}(j,j')=[a^{\dagger}_{j}\times\tilde{a}_{j'}]^{(k)}_{\kappa}=\sum_{\nu,\nu'}\langle j\nu j'\nu'|k\kappa\rangle a^{\dagger}_{j,\nu}\tilde{a}_{j',\nu'}\ , \quad (2.2)$$

where $\tilde{a}_{j,\nu}=(-1)^{j-\nu}a_{j,-\nu}$. These creation and annihilation operators obey the anticommutation relations, but the bilinear products of (2.2) satisfy the same relations as those of (2.1).

These bilinear products can be used to construct the Hamiltonian in rotationally invariant form

$$H=H_B+H_F+V_{BF}\ , \quad (2.3)$$

where $H_B$ and $H_F$ are the Hamiltonians of the bosonic and fermionic parts and $V_{BF}$ represents the boson-fermion interaction term. Usually it is enough to keep only the one- and two-body terms in the phenomenologic algebraic models, so we shall follow this treatment in the case of the vibron-fermion model as well:

$$\begin{aligned}
H_B&=E_B+\sum_{l}\epsilon_l B^{(0)}_{0}(l,l)+\sum_{k}\sum_{l_1l_2l_3l_4}[u^{(k)}_{l_1l_2l_3l_4}B^{(k)}(l_1,l_2)\cdot B^{(k)}(l_3,l_4)+\text{H.c.}]\ ,\\
H_F&=E_F+\sum_{j}\eta_j A^{(0)}_{0}(j,j)+\sum_{k}\sum_{j_1j_2j_3j_4}[v^{(k)}_{j_1j_2j_3j_4}A^{(k)}(j_1,j_2)\cdot A^{(k)}(j_3,j_4)+\text{H.c.}]\ ,\\
V_{BF}&=\sum_{k}\sum_{l_1l_2j_1j_2}w^{(k)}_{l_1l_2j_1j_2}[B^{(k)}(l_1,l_2)\cdot A^{(k)}(j_1,j_2)+\text{H.c.}]\ .
\end{aligned} \quad (2.4)$$

Here the dot denotes the scalar product:

$$\begin{aligned}
a^{(l)}\cdot c^{(l)}&=(-1)^{l}\sqrt{2l+1}[a^{(l)}\times c^{(l)}]^{(0)}_{0}\\
&=\sum_{\nu}(-1)^{\nu}a^{(l)}_{\nu}c^{(l)}_{-\nu}\ .
\end{aligned} \quad (2.5)$$

These expressions were taken from Ref. [12] as the most general expression for the IBFM Hamiltonian containing one- and two-body rotationally invariant terms. They can be applied in the vibron-fermion model too, with a slight modification, namely, taking $l_i=0$ or 1 instead of 0 or 2. Certain physical circumstances may lead to simplifications in these expressions; for example, if there is only $M=1$ fermion in the system, we can drop the two-body terms in $H_F$.

Dynamical boson-fermion symmetries correspond to specific choices of the parameters in $H$ in the sense that we can write $H$ in terms of the Casimir invariants of groups appearing in some group chain of $U^{B}(4)\times U^{F}(m)$. They are built on the dynamical symmetries of the separate bosonic and fermionic system. There are two bosonic dynamical symmetries of this model, the O(4) and the U(3) limits [14,16]. In order to couple the bosonic and fermionic group structure, $U^{F}(m)$ has to contain fermionic subgroups like $U^{F}(4)$, $O^{F}(4)$, $U^{F}(3)$ [$SU^{F}(3)$], or $O^{F}(3)$. (These groups are expected to describe the orbital part of the fermionic structure.) In the vibron-electron model [14] fermionic states are taken from degenerate hydrogenic levels. In this case the boson-fermion coupling can be established on the level of the O(4) groups. [If the hydrogenic levels are those with $n=2$, the $U^{F}_{l}(4)$ group can also be coupled to the $U^{B}(4)$ group of the vibron model.] If we take the fermionic single-particle states from an oscillator shell (not necessarily a nuclear one), the coupling of the bosonic and fermionic structures can be done on the SU(3) level, similarly to the SU(3)×U(2) limit of the IBFM [12].

## III. THE SU(3)×U(2) DYNAMICAL SYMMETRY

In this section we give a detailed study of the group structure of the SU(3)×U(2) limit of the vibron-fermion model, discuss the question of labeling the basis states, introduce the coupled wave functions of the boson-fermion system, and determine the energy spectrum associated with this dynamical symmetry.

Here we shall follow the notation and presentation of the SU(3)×U(2) limit of the IBFM [12], since from the

mathematical point of view it is closely related to the analogous limit of the vibron-fermion model.

### A. Group structure and generators

In this limit the fermions are allowed to occupy the states of an oscillator shell characterized with $n$ oscillator quanta. In this case the possible fermionic single-particle states are those with $j=\frac{1}{2},\frac{3}{2},\ldots,n+\frac{1}{2}$, with parity $(-1)^n$. The (pseudo-)orbital angular momentum is restricted to the values $l=n,n-2,\ldots,1$ or 0, and the (pseudo-)spin is $s=\frac{1}{2}$ (Ref. [12]). Making use of the decomposition of the angular momentum into (pseudo-)orbital and (pseudo-)spin part, the fermionic group structure is

$$\mathrm{U}^F(m)\supset U_l^F(m/2)\times\mathrm{U}_s^F(2)\supset\mathrm{SU}_l^F(3)\times\mathrm{U}_s^F(2)\supset\mathrm{O}_l^F(3)\times\mathrm{SU}_s^F(2)\supset\mathrm{Spin}^F(3)\supset\mathrm{Spin}^F(2)\ , \tag{3.1}$$

where $m=(n+1)(n+2)$. This group chain can be combined with the

$$\mathrm{U}^B(4)\supset\mathrm{U}^B(3)\supset\mathrm{SU}^B(3)\supset\mathrm{O}^B(3)\supset\mathrm{O}^B(2) \tag{3.2}$$

group chain of the vibron model resulting in the group structure

$$\begin{aligned}\mathrm{U}^B(4)\times\mathrm{U}^F(m)&\supset\mathrm{U}^B(4)\times\mathrm{U}_l^F(m/2)\times\mathrm{U}_s^F(2)\\&\supset\mathrm{U}^B(3)\times\mathrm{SU}_l^F(3)\times\mathrm{U}_s^F(2)\\&\supset\mathrm{SU}^B(3)\times\mathrm{SU}_l^F(3)\times\mathrm{U}_s^F(2)\\&\supset\mathrm{SU}(3)\times\mathrm{U}_s^F(2)\\&\supset\mathrm{O}(3)\times\mathrm{SU}_s^F(2)\\&\supset\mathrm{Spin}(3)\supset\mathrm{Spin}(2)\ .\end{aligned} \tag{3.3}$$

The generator sets of $\mathrm{U}_l^F(m/2)$ and $\mathrm{U}_s^F(2)$ can be obtained after changing the coupling scheme of the angular momenta using the transformation brackets $\langle((ll')J,(ss')0,J|(ls)j,(l's')j',J\rangle$ and $\langle((ll')0,(ss')J,J|(ls)j,(l's')j',J\rangle$:

$$\begin{aligned}K_M^{(J)}(l,l')&=-\sum_s\sum_{j,j'}\sqrt{(2j+1)(2j'+1)}(-1)^{j'+J+l+s}\begin{Bmatrix}j & j' & J\\ l' & l & s\end{Bmatrix}A_M^{(J)}(j,j')\ ,\\ S_M^{(J)}(s,s')&=\sum_l\sum_{j,j'}\sqrt{(2j+1)(2j'+1)}(-1)^{s'+J+j+l}\begin{Bmatrix}s & s' & J\\ j' & j & l\end{Bmatrix}A_M^{(J)}(j,j')\ .\end{aligned} \tag{3.4}$$

These operators satisfy the same commutation relations as the bilinear products of the boson operators, $B_\mu^{(\lambda)}(l,l')$. In order to get the generators of the $\mathrm{SU}_l^F(3)$ group the following combinations have to be taken [12,17]:

$$\begin{aligned}L_{F,\mu}^{(1)}&=\sum_l\sqrt{l(l+1)(2l+1)/3}K_\mu^{(1)}(l,l)\ ,\\ Q_{F,\mu}^{(2)}&=\sum_{l,l'}q_{l,l'}(n)K_\mu^{(2)}(l,l')\ ,\end{aligned} \tag{3.5}$$

where

$$\begin{aligned}q_{l,l}(n)&=-(2n+3)\left[\frac{l(l+1)(2l+1)}{40(2l-1)(2l+3)}\right]^{1/2},\\ q_{l,l+2}(n)&=q_{l+2,l}(n)\\&=\left[\frac{3(l+1)(l+2)(n-1)(n+l+3)}{20(l+3)}\right]^{1/2}.\end{aligned} \tag{3.6}$$

The generators of the $\mathrm{SU}^B(3)$ group are known from the vibron model:

$$\begin{aligned}L_{B,\mu}^{(l)}&=\sqrt{2}B_\mu^{(1)}(1,1)=\sqrt{2}[\pi^\dagger\times\tilde{\pi}]_\mu^{(1)}\ ,\\ Q_{B,\mu}^{(2)}&=\frac{\sqrt{3}}{2}B_\mu^{(2)}(1,1)=\frac{\sqrt{3}}{2}[\pi^\dagger\times\tilde{\pi}]_\mu^{(2)}\ .\end{aligned} \tag{3.7}$$

The factor of $\sqrt{3}/2$ differs from the usual convention used in the vibron model, and it is needed to reproduce the correct $\mathrm{SU}^B(3)$ structure constants. The combined SU(3) generators are the same as in the case of the $\mathrm{SU}(3)\times\mathrm{U}(2)$ limit of the IBFM [12]:

$$\begin{aligned}L_\mu^{(1)}&=L_{B,\mu}^{(1)}+L_{F,\mu}^{(1)}\ ,\\ Q_\mu^{(2)}&=Q_{B,\mu}^{(2)}\pm Q_{F,\mu}^{(2)}\ .\end{aligned} \tag{3.8}$$

The + and − signs of $Q_{F,\mu}^{(2)}$ correspond to cases in which the fermions are particle-like and hole-like, respectively [11,12]. The generators of the Spin(3) and Spin(2) groups are again the same as in the corresponding limit of the IBFM:

$$J_\mu^{(1)}=L_\mu^{(1)}-\frac{1}{\sqrt{2}}S_\mu^{(1)}(\tfrac{1}{2},\tfrac{1}{2})\ . \tag{3.9}$$

We have a special case for $n=1$, which gives rise to the coupling of the fermion and boson groups on the U(3) level:

$$\begin{aligned}
&U^B(4)\times U^F(6)\supset U^B(4)\times U_l^F(3)\times U_s^F(2)\\
&\quad\supset U^B(3)\times U_l^F(3)\times U_s^F(2)\\
&\quad\supset U(3)\times U_s^F(2)\\
&\quad\supset SU(3)\times U_s^F(2)\\
&\quad\supset O(3)\times SU_s^F(2)\\
&\quad\supset \mathrm{Spin}(3)\supset \mathrm{Spin}(2)\ . \qquad (3.10)
\end{aligned}$$

This coupling scheme represents only a minor modification of the SU(3)×U(2) limit of the vibron-fermion model. As we shall see later, there is no difference in the practical applications of these two limits (with $n=1$), in spite of the different labeling.

The generators of U(3) are the eight generators of the SU(3) group, together with the scalar operator

$$\sqrt{3}G_0^{(0)}(1,1)=\sqrt{3}B_0^{(0)}(1,1)\pm\sqrt{3}K_0^{(0)}(1,1)\ , \qquad (3.11)$$

in the case of particle-like and hole-like fermions, respectively.

Further dynamical symmetries arise if we couple the group chains (3.1) and (3.2) on the level of angular momenta. There are several possible coupling schemes in this weak-coupling limit. If we couple the fermionic orbital angular momentum to the bosonic angular momentum first, the corresponding chain is

$$\begin{aligned}
&U^B(4)\times U^F(m)\supset U^B(3)\times SU_l^F(m/2)\times U_s^F(2)\\
&\quad\supset U^B(3)\times SU_l^F(3)\times U_s^F(2)\\
&\quad\supset SU^B(3)\times SU_l^F(3)\times U_s^F(2)\\
&\quad\supset O^B(3)\times O_l^F(3)\times SU_s^F(2)\\
&\quad\supset O(3)\times SU_s^F(2)\\
&\quad\supset \mathrm{Spin}(3)\supset \mathrm{Spin}(2)\ . \qquad (3.12)
\end{aligned}$$

The generators of $O^B(3)$ and $O_l^F(3)$ are the angular momentum operators $L_{B,\mu}^{(1)}$ and $L_{F,\mu}^{(1)}$. Another possibility is to couple the full fermionic angular momentum to the bosonic angular momentum. We can get the corresponding group chain replacing the last two lines of (3.12) with

$$\begin{aligned}
&\supset O^B(3)\times \mathrm{Spin}^F(3)\ ,\\
&\supset \mathrm{Spin}(3)\supset \mathrm{Spin}(2)\ .
\end{aligned} \qquad (3.13)$$

The generators of $\mathrm{Spin}^F(3)$ are

$$J_{F,\mu}^{(1)}=L_{F,\mu}^{(1)}-\frac{1}{\sqrt{2}}S_\mu^{(1)}\left(\tfrac{1}{2},\tfrac{1}{2}\right)\ . \qquad (3.14)$$

If we use the first common boson-fermion group of the corresponding group chains to distinguish the dynamical symmetries, we can call these limiting cases the O(3)×SU(2) and the Spin(3) limits. In practical calculations these weak-coupling limits can be used instead of any limits with coupling on a higher level. It is the nature of the actual physical problem which may favor the usage of a particular coupling scheme.

### B. The coupled vibron-fermion basis states

Besides identifying the possible dynamical symmetries, group chains like (3.3) provide a convenient way to find basis states in which the most general Hamiltonian can be diagonalized. These basis states can be labeled using the irreducible representations of groups appearing in the corresponding group chain. This labeling problem amounts to solving the problem of decomposing the irreducible representations (irreps) of the groups belonging to the group chain, which (in favorable cases) is a straightforward group-theoretical task.

First we present the labeling procedure for the bosonic and fermionic states separately, then turn our attention to the coupled boson-fermion basis. The bosonic rotational-vibrational states are of the type [3]

$$\left|\begin{matrix}U^B(4)\supset U^B(3)\supset SU^B(3)\supset O^B(3)\supset O^B(2)\\ [N]\quad [n_\pi]\quad (n_\pi,0)\quad R\quad M_R\end{matrix}\right\rangle \qquad (3.15)$$

where $[N]$ stands for the abbreviation of $[N,0,0,0]$, and similarly $[n_\pi]$ denotes $[n_\pi,0,0]$. The relation of the quantum numbers is the following:

$$\begin{aligned}
&n_\pi=N,N-1,\ldots,1,0\ ,\\
&R=n_\pi,n_\pi-2,\ldots,1\text{ or }0\ ,\\
&-R\le M_R\le R\ .
\end{aligned} \qquad (3.16)$$

Here $R$ stands for the eigenvalues of the operator $L_B$ in Eq. (3.7).

The irreducible representation $[N]$ of $U^B(4)$ is totally symmetric, since it describes the set of $N$ bosons. The number of the basis states is determined by $N$. Therefore it should be chosen in such a way that it produces a large enough model spectrum. In certain applications of the U(3) limit of the vibron model a basis truncation is required [7,8], and only states with large enough $n_\pi$ are considered. This will be discussed in Sec. IV.

The fermionic states can be labeled as

$$\left|\begin{matrix}U^F(m)\supset U_l^F(m/2)\times U_s^F(2)\supset SU_l^F(3)\times U_s^F(2)\supset O_l^F(3)\times SU_s^F(2)\supset \mathrm{Spin}^F(3)\supset \mathrm{Sin}^F(2)\\ \{M\}\quad [f]\quad (\lambda_F,\mu_F)\quad l\quad s\quad j\quad M_j\end{matrix}\right\rangle\ . \qquad (3.17)$$

Here $\{M\}$ standing for $M$ fermions must be a totally antisymmetric representation of $U^F(m)$. Following the notation of Bijker and Kota [12], we denote the representation of $U^{(F)}(m)$ with $\{M\}=\{1^M\}$ and $\{\bar{M}\}=\{1^{m-M}\}$ for cases with

particle-like and hole-like fermions, respectively. Now $[f]=[f_1,\ldots,f_{m/2}]$ stands for the general representation of $U_I^F(m/2)$. If we take only one odd nucleon (or hole) on the oscillator shell characterized with $n$ oscillator quanta, these quantum numbers are

$$\begin{aligned}
&\{M\}=\{1\}\ \ (\text{or } \{\bar{1}\})\ ,\\
&[f]=[1]\ \ (\text{or } [\bar{1}])\ ,\\
&(\lambda_f,\mu_F)=(n,0)\ \ [\text{or } (0,n)]\ ,\\
&l=n,n-2,\ldots,1 \text{ or } 0\ ,\\
&s=\tfrac{1}{2}\ ,\\
&j=l\pm\tfrac{1}{2}\ ,\\
&-j\le M_j\le j\ ,
\end{aligned} \tag{3.18}$$

where $i$ and $j$ are the eigenvalues of the operators $L_F$ and $J_F$ introduced in Eqs. (3.5) and (3.14).

These basis states can be used to generate the coupled boson-fermion basis associated with the group chain (3.3):

$$\left|\begin{array}{cccccccccc} U^B(4), & U^B(3), & U^F(m), & U_I^F(m/2), & SU^B(3), & SU_I^F(3), & SU(3) & O(3), & SU_s^F(2), & Spin(3), & Spin(2) \\ [N] & [n_\pi] & \{M\} & [f] & (n_\pi,0) & (\lambda_F,\mu_F) & (\lambda,\mu)\kappa_L & L & s & J & M_J \end{array}\right\rangle . \tag{3.19}$$

This is the most general form of the basis states in the SU(3)×U(2) limit of the vibron-fermion model. If we take $M=0$, the U(3) limit of the vibron model emerges as a special case. For the sake of simplicity, we shall restrict ourselves to $M=1$ which means the coupling of only one nucleon or a hole to the dipole-type collectivity, i.e., to the rotational-vibrational motion of two clusters.

The coupling between the basis states (3.15) and (3.17) is established on the SU(3) level. This procedure requires the introduction of the $SU^B(3)$ group, the representations $(n_\pi,0)$ of which are trivially determined by the totally symmetric representation $[n_\pi]$ of $U^B(3)$. The irreducible representations (irreps) $(\lambda,\mu)$ of SU(3) are obtained by taking the outer product [17] $(n_\pi,0)\times(\lambda_F,\mu_F)$:

$$(n_\pi,0)\times(n,0)=\sum_r (n_\pi+n-2r,r) \tag{3.20a}$$

and

$$(n_\pi,0)\times(0,n)=\sum_r (n_\pi-r,n-r)\ , \tag{3.20b}$$

with $0\le r\le\min(n_\pi,n)$ in both cases. Due to the simple nature of $(n_\pi,0)$ and $(\lambda_F,\mu_F)$ each representation $(\lambda,\mu)$ is obtained only once in the outer product.

The angular momentum content of the $(\lambda,\mu)$ representations is given by [17]

$$\begin{aligned}
&L=K_L,K_L+1,\ldots,K_L+\max\{\lambda,\mu\}\ ,\\
&K_L=\min\{\lambda,\mu\},\min\{\lambda,\mu\}-2,\ldots,1 \text{ or } 0\ ,
\end{aligned} \tag{3.21}$$

with the exception of $K_L=0$, for which

$$L=\max\{\lambda,\mu\},\max\{\lambda,\mu\}-2,\ldots,1 \text{ or } 0\ . \tag{3.22}$$

Instead of the Elliott basis, which is not orthogonal, the orthogonal Vergados basis [18] is used in most applications. The switch to the Vergados basis means the introduction of another sequence of quantum numbers, namely, $\kappa_L$ is used instead of $K_L$. In fact, $\kappa_L$ and $K_L$ have the same values, but there is a difference in the $L$ values contained in them. The construction of the Vergados basis is such that if a given $L$ occurs in a representation once, twice, thrice, etc., it belongs to the one, two, three, etc., lowest possible values of $\kappa_L$. The only exception is $\kappa_L=0$, for which the allowed $L$ values are restricted to be even or odd, for $\lambda+\mu$ even or odd, respectively. $J$ and $M_J$ are determined by the usual angular momentum coupling rules: $J=L\pm\frac{1}{2}$ (except for $L=0$, when $J=\frac{1}{2}$) and $M_J=-J,\ldots,J$. Finally, the parity of the basis states is determined by the parities assigned to the bosonic and fermionic basis states, which are $(-1)^{n_\pi}$ and $(-1)^n$, so the parity assignment to the basis state (3.19) is $(-1)^{n_\pi+n}$.

Taking only one fermion, the coupled boson-fermion wave function is

$$
\begin{aligned}
&|[N](n_\pi,0),\{M\}[f](\lambda_F,\mu_F);(\lambda,\mu)\kappa_L L\tfrac{1}{2};JM_J\rangle \\
&\quad=\sum_{Rl}\langle(n_\pi,0)R,(\lambda_F,\mu_F)l\|(\lambda,\mu)\kappa_L L\rangle|[N](n_\pi,0)R,\{M\}[f](\lambda_F,\mu_F)l;L\tfrac{1}{2};JM_J\rangle \\
&\quad=\sum_{Rl}\langle(n_\pi,0)R,(\lambda_F,\mu_F)l\|(\lambda,\mu)\kappa_L L\rangle\sum_{M_R M_l}\sum_{M_L M_s}\langle RM_R lM_l|LM_L\rangle \\
&\qquad\qquad\times\langle LM_L\tfrac{1}{2}M_S|JM_J\rangle|[N](n_\pi,0)R,M_R\rangle|\{M\}[f](\lambda_F,\mu_F)lM_l;\tfrac{1}{2}M_S\rangle,
\end{aligned}
\tag{3.23}
$$

where $\{M\}[f](\lambda_F,\mu_F)=\{1\}[1](n,0)$ for particle coupling and $\{\bar{1}\}[\bar{1}](0,n)$ for hole coupling. The double barred symbols are SU(3)⊃O(3) isoscalar factors (ISF's) or Wigner coefficients. They have been tabulated for some simple cases in Ref. [18], and a computer code has been written to determine them in more complicated cases [19]. Note that there is no need for the additional quantum numbers $\kappa_R$ and $\kappa_l$ at the $SU^B(3)\supset O^B(3)$ and the $SU_l^F(3)\supset O_l^F(3)$ decomposition, since each $R$ and $l$ is unique within the $(n_\pi,0)$ and $(\lambda_F,\mu_F)=(n,0)$ or $(0,n)$ representations.

As we have already mentioned, a special coupling scheme arises in the $n=1$ case,since now the coupling of the bosonic and fermionic states can be carried out on the U(3) level. This special limit, associated with the group chain (3,10), can be considered the analogue of the U(5)×U(2) limit of the IBFM.

If we take one fermion on the $p$ shell, the fermionic state can be labeled as

$$
\left|\begin{matrix}U^F(6)\supset U_l^F(3)\times U_s^F(2)\supset SU_l^F(3)\times U_s^F(2)\supset O_l^F(3)\times SU_s^F(2)\supset \mathrm{Spin}^F(3)\supset \mathrm{Spin}^F(2)\\ \{M\}\quad [f]\qquad\qquad (\lambda_F,\mu_F)\qquad\quad l\qquad s\qquad j\qquad M_j\end{matrix}\right\rangle . \tag{3.24}
$$

where $\{M\}=\{1\}$ or $\{\bar{1}\}=\{1^5\}$ and $[f]=[1,0,0]$ or $[1,1,0]$ for particle coupling and hole coupling, respectively. $[f]$ determines the $SU_l^F(3)$ representations uniquely: $(\lambda_F,\mu_F)=(1,0)$ or $(0,1)$ in the two cases.

The coupled boson-fermion basis states in this limit can be written as

$$
\left|\begin{matrix}U^B(4),\ U^F(6),\ U^B(3),\ U_l^F(3),\ \ U(3),\ \ SU(3),\ O(3),\ SU_s^F(2),\ \mathrm{Spin}(3),\ \mathrm{Spin}(2)\\ [N]\quad \{M\}\quad [n_\pi]\quad [f]\quad [N_1,N_2,N_3]\ (\lambda,\mu)\kappa_L\quad L\qquad s\qquad J\qquad M_J\end{matrix}\right\rangle . \tag{3.25}
$$

It is easy to check that the U(3) representations $[N_1,N_2,N_3]$ given by the $[n_\pi]\times[f]$ outer product are

$$[n_\pi]\times[1]=[n_\pi+1]\oplus[n_\pi,1] \tag{3.26a}$$

for particle coupling, and

$$[n_\pi]\times[1,1]=[n_\pi+1,1]\oplus[n_\pi,1,1] \tag{3.26b}$$

for hole coupling. The trivial U(3)⊃SU(3) decomposition yields the SU(3) representations $(n_\pi+1,0),(n_\pi-1,1)$ and $(n_\pi,1),(n_\pi-1,0)$, which are identical with the representations obtained from Eqs. (3.20a) and (3.20b) with $n=1$. Therefore we conclude that the basis states of this special limit are identical with the basis states of the SU(3)×U(2) limit for $n=1$. (This result holds for $M>1$ as well.)

The basis states belonging to the weak-coupling [O(3)×SU(2) and Spin (3)] limits can be obtained from the bosonic and fermionic basis states [Eqs. (3.15) and (3.16)] by ordinary angular momentum coupling.

We remark that the boson-fermion bases obtained from the coupling of the bosonic (3.15) and fermionic (3.17) basis states seem to be convenient for numerical diagonalization of Hamiltonians with bosonic terms breaking the $U^B(3)$ dynamical symmetry, since most of the terms in the general bosonic Hamiltonian are diagonal in this basis [3]. We do not consider symmetry-breaking fermionic terms at present.

### C. The energy eigenvalues

Having constructed a basis we can now calculate the matrix elements of operators; in particular, we can diagonalize the Hamiltonian $H=H_B+H_F+V_{BF}$. If the Hamiltonian can be written as the linear combination of Casimir invariants of groups appearing in the group chains (3.3), (3.10), (3.12), or (3.13), i.e., in the case of dynamical boson-fermion symmetry, the eigenvalues can be obtained in closed analytic form. We only have to calculate the expectation values of the Casimir invariants in the corresponding basis. If we consider only one- and two-body terms in the Hamiltonian, we have to take only linear and quadratic Casimir invariants.

First we consider the SU(3)×U(2) limit associated with the group chain (3.3). The most general one- and two-body Hamiltonian which is diagonal in this basis can be written as

$$
\begin{aligned}
H=&E_0+\alpha_B C_1(U^B(3))+\gamma_B C_2(SU^B(3))+\gamma C_2(SU(3))\\
&+\delta C_2(O(3))+\epsilon C_2(\mathrm{Spin}(3)) .
\end{aligned}
\tag{3.27}
$$

The Casimir operators of $U^B(4)$, $U^F(m)$, $U_l^F(m/2)$, $SU_l^F(3)$, $U_s^F(2)$, and $SU_s^F(2)$ contribute equally to $H$ from every state, so they do not split the energy spectrum. These terms are absorbed in $E_0$. We dropped $C_2(U^B(3))$ since including it would only mean the redefinition of $\alpha_B$ and $\gamma_B$. We have displayed the relevant Casimir invari-

TABLE I. Casimir operators and their eigenvalues.

| $C_i(G)$ | Casimir operator | Labels | Eigenvalues |
|---|---|---|---|
| $C_1(\mathrm{U}(3))$ | $-\sqrt{3}G_0^{(0)}(1,1)$ | $[N_1,N_2N_3]$ | $N_1+N_2+N_3$ |
| $C_2(\mathrm{U}(3))$ | $\sum_k G^{(k)}(1,1)\cdot G^{(k)}(1,1)$ | $[N_1,N_2,N_3]$ | $N_1(N_1+2)+N_2^2+N_3(N_3-2)$ |
| $C_2(\mathrm{SU}(3))$ | $2Q^{(2)}\cdot Q^{(2)}+\frac{3}{4}L^{(1)}\cdot L^{(1)}$ | $(\lambda,\mu)$ | $\lambda+\mu^2+\lambda\mu+3\lambda+3\mu$ |
| $C_2(\mathrm{O}(3))$ | $L^{(1)}\cdot L^{(1)}$ | $L$ | $L(L+1)$ |
| $C_2(\mathrm{Spin}(3))$ | $J^{(1)}\cdot J^{(1)}$ | $J$ | $J(J+1)$ |

ants and their expectation values in Table I.

In Figs. 1(a) and 1(b) we have presented typical energy spectra with SU(3)×U(2) dynamical symmetry for particle and hole coupling, respectively. In order to calculate the energy eigenvalues we used the expression

$$E(n_\pi,(\lambda,\mu)L,J)=E_0+\alpha_B n_\pi+\gamma_B n_\pi(n_\pi+3)+\gamma(\lambda^2+\mu^2+\lambda\mu+3\lambda+3\mu)+\delta L(L+1)+\epsilon J(J+1)\ . \tag{3.28}$$

In this strong-coupling limit several bands of the odd nu-

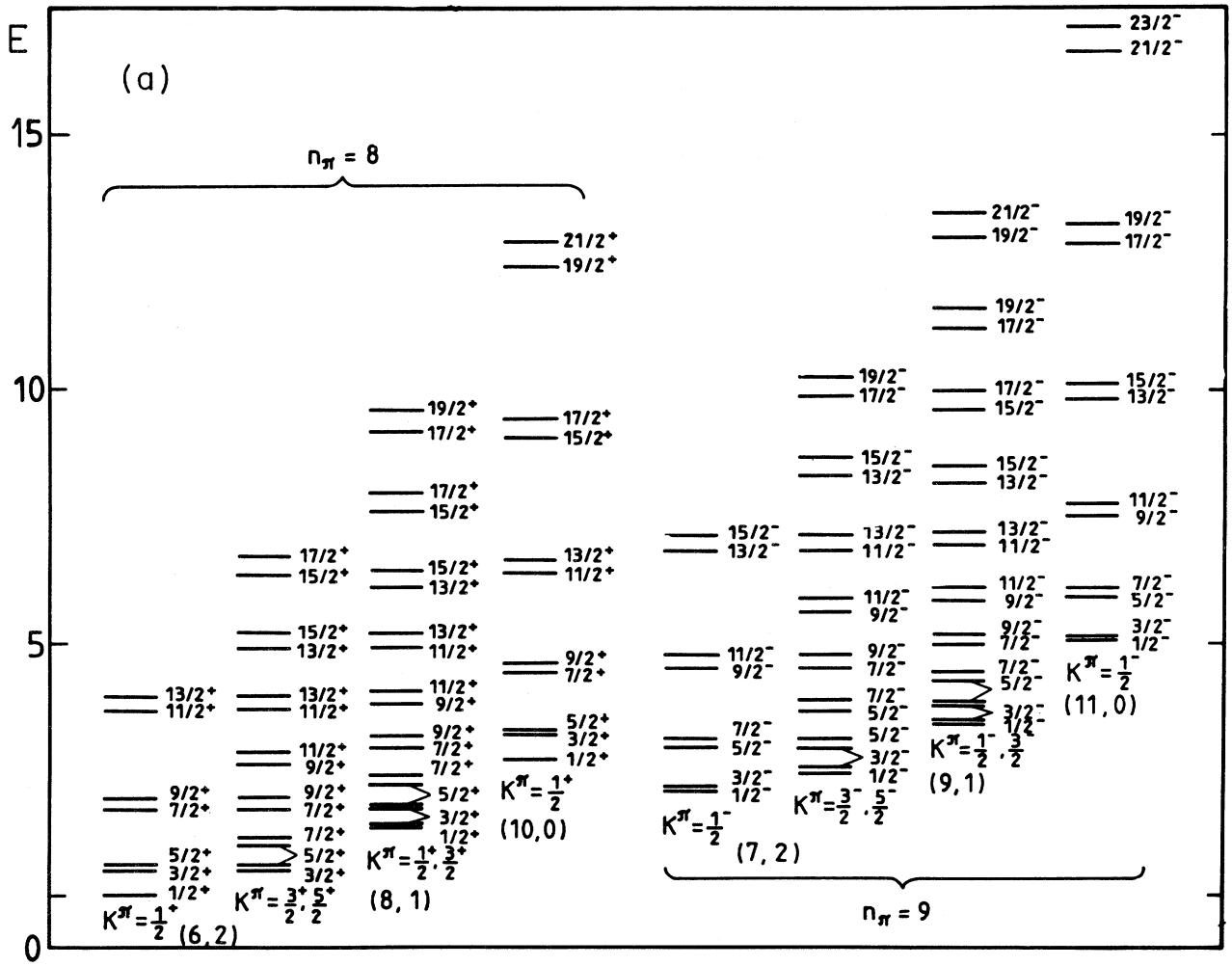


FIG. 1. (a) Part of a model spectrum with SU(3)×U(2) symmetry in the particle coupling case. The fermionic states considered here are from the $n=2$ shell, with (pseudo-)orbital angular momenta $l=0$, 2, and $s=\frac{1}{2}$, ($j^\pi=\frac{1}{2}^+,\frac{3}{2}^+,\frac{5}{2}^+$). Here we considered bosonic excitations with only $n_\pi=8$ and 9. States with lower $n_\pi$ are excluded on the basis of arguments presented in Sec. IV. The parameters used in Eq. (3.28) are $\alpha_B+20\gamma_B=1.0$. $\gamma=0.05$, $\delta=0.07$, and $\epsilon=0.02$. (b) The same as (a), in the hole coupling case.

cleus are built on the same collective (bosonic or cluster) band, and each of them has contributions from several fermionic states, in general. Their mixing is governed by the $SU(3)\supset O(3)$ Wigner coefficients and the $O(3)$ Clebsh-Gordan coefficients.

The Hamiltonian associated with group chain (3.10) can be written as

$$H=E_0+\alpha_B C_1(U^B(3))+\beta_B C_2(U^B(3))+\alpha C_1(U(3))+\beta C_2(U(3))+\gamma C_2(SU(3))+\delta C_2(O(3))+\epsilon C_2(\text{Spin}(3)) . \quad (3.29)$$

Here we again dropped the Casimir operators of groups $U^B(4)$, $U^F(6)$, $U_l^F(3)$, $U_s^F(2)$, and $SU_s^F(2)$, since they do not split the spectrum. It is easy to prove that this Hamiltonian is still redundant, since $C_1(U(3))$ and $C_2(U(3))$ can be expressed in terms of other Casimir operators: $C_2(U(3))=\frac{2}{3}C_2(SU(3))+\frac{1}{3}[C_1(U(3))]^2$ and $C_1(U(3))=C_1(U^B(3))+C_1(U_l^F(3))$, so these two terms can be absorbed into the remaining terms. In addition to this, noting that the eigenvalues of $C_2(U^B(3))$ can be replaced with the eigenvalues of $C_2(SU^B(3))$ and $C_1(U^B(3))$, we can conclude that the energy spectrum belonging to the Hamiltonian (3.28) is identical with that of the Hamiltonian (3.27) with $SU(3)\times U(2)$ dynamical symmetry.

The Hamiltonian in the $O(3)\times SU(2)$ limit is written in terms of the Casimir invariants of group chain (3.12):

$$H=E_0+\alpha_B C_1(U^B(3))+\gamma_B C_2(SU^B(3))+\delta_B C_2(O^B(3))+\delta_F C_2(O_l^F(3))+\delta C_2(O(3))+\epsilon C_2(\text{Spin}(3)) . \quad (3.30)$$

Finally, the Hamiltonian associated with group chain (3.13) of the Spin (3) limit is

$$H=E_0+\alpha_B C_1(U^B(3))+\gamma_B C_2(SU^B(3))+\delta_B C_2(O^B(3))+\delta_F C_2(O_l^F(3))+\epsilon_F C_2(\text{Spin}^F(3))+\epsilon C_2(\text{Spin}(3)) . \quad (3.31)$$

Contrary to the other dynamical symmetries, the energy eigenstates of the Spin(3) limit have contribution from only one fermionic state. It is also worth mentioning that

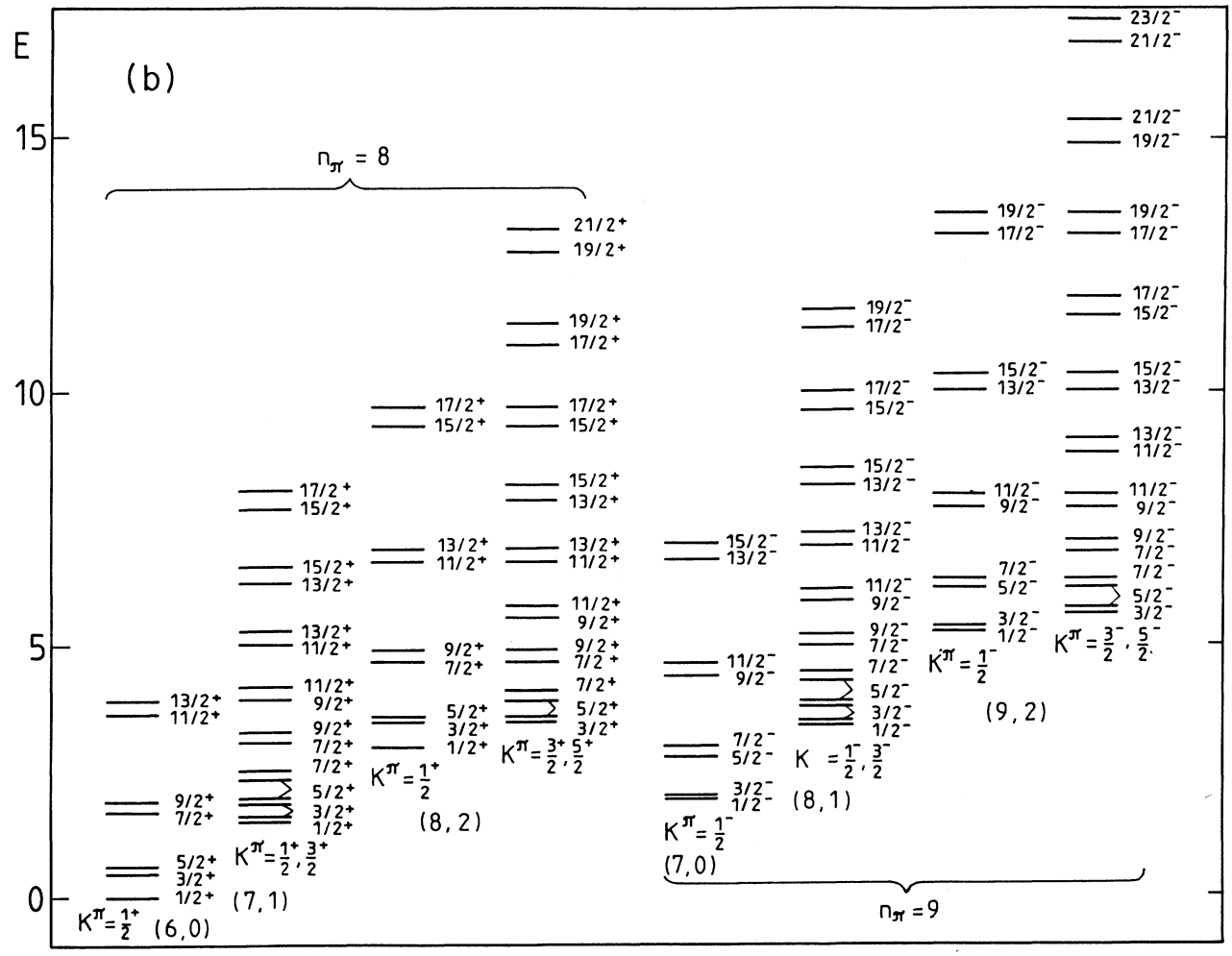


FIG. 1. (*Continued*).

there is no difference between the energy spectra belonging to the particle and hole coupling cases in the weak-coupling O(3)×SU(2) and Spin(3) limits.

The interaction terms in the Hamiltonians associated with the SU(3)×U(2), O(3)×SU(2), and Spin(3) limits of the vibron-fermion model can be interpreted in a straightforward way, since the Casimir operators can be identified with certain physical operators (see, for example, Table I). In order to give a uniform treatment of these limiting cases we write the three Hamiltonians in terms of the same operators. The Hamiltonians in Eqs. (3.27), (3.30), and (3.31) can be rewritten as

$$\begin{aligned} H(\mathrm{SU}(3)\times\mathrm{U}(2)) = {} & E_0' + \alpha_B n_\pi + 2(\gamma_B+\gamma) Q_B\cdot Q_B \\ & + (\tfrac{3}{4}(\gamma_B+\gamma)+\delta+\epsilon) L_B\cdot L_B \\ & + \delta L_F\cdot L_F + \epsilon J_F\cdot J_F \pm 4\gamma Q_B\cdot Q_F \\ & + (\tfrac{3}{2}\gamma+2\delta) L_B\cdot L_F + 2\epsilon L_B\cdot J_F \ , \end{aligned} \tag{3.32a}$$

$$\begin{aligned} H(\mathrm{O}(3)\times\mathrm{U}(2)) = {} & E_0' + \alpha_B n_\pi + 2\gamma_B Q_B\cdot Q_B \\ & + (\tfrac{3}{4}\gamma_B+\delta_B+\delta+\epsilon) L_B\cdot L_B \\ & + (\delta_F+\delta) L_F\cdot L_F + \epsilon J_F\cdot J_F \\ & + 2\delta L_B\cdot L_F + 2\epsilon L_B\cdot J_F \ , \end{aligned} \tag{3.32b}$$

$$\begin{aligned} H(\mathrm{Spin}(3)) = {} & E_0' + \alpha_B n_\pi + 2\gamma Q_B\cdot Q_B \\ & + (\tfrac{3}{4}\gamma_B+\delta_B+\epsilon) L_B\cdot L_B + \delta_F L_F\cdot L_F \\ & + (\epsilon+\epsilon_F) J_F\cdot J_F + 2\epsilon L_B\cdot J_F \ . \end{aligned} \tag{3.32c}$$

The structures of the bosonic and fermionic parts is the same in all three Hamiltonians; the difference arises only in the boson-fermion interaction terms. The bosonic part is equivalent with the Hamiltonian of the vibron model in the U(3) limit [3] and it can be interpreted as an anharmonic-oscillator system, in which the anharmonicity is represented by the operators $n_\pi^3+3n_\pi=2Q_B\cdot Q_B+\frac{3}{4}L_B\cdot L_B$ and $L_B\cdot L_B$. The number of the oscillator quanta is equal with the number of $\pi$ bosons.

The complexity of the boson-fermion interaction terms in (3.32a)–(3.32c) depends on the coupling scheme associated with the given dynamical symmetry. In the strong-coupling SU(3)×U(2) limit a quadrupole-quadrupole interaction is present, while in the Spin (3) limit the only interaction term is the angular momentum coupling of the relative motion to the spin of the core nucleus, giving rise to the weak-coupling limit.

Although there are terms in the Hamiltonian (3.32a) and (3.32b) which are not diagonal in the corresponding bases, the full Hamiltonians are diagonal. This is due to the restrictions imposed on the parameters by the dynamical symmetry, since the number of terms in these expressions is bigger than the number of independent parameters in the original expressions (3.27) and (3.30). If the parameters of each term in Eq. (3.32a) and (3.32b) were varied independently, the Hamiltonians would cease to be diagonal in the corresponding bases.

In addition to the interaction terms discussed above, we can introduce further symmetry-conserving phenomenologic terms in the Hamiltonians. Certain physical circumstances may require the introduction of third-order terms. Noting that $C_1(\mathrm{U}^B(3))$ is the number operator of the $\pi$ bosons, we can include terms like $n_\pi C_2(G)$ in the Hamiltonian. Since $n_\pi$ is diagonal in any of the bases discussed above, terms like this do not destroy the dynamical symmetries. Basis states characterized by different values of $n_\pi$ correspond to different collective bands of the underlying bosonic configuration. This difference in the nature of the collective bands may influence the coupling to the single-particle (fermionic) degrees of freedom. This mechanism may manifest itself, for example, in rotation-vibration coupling, or $n_\pi$-dependent angular momentum (spin-orbit) coupling. These interactions can be approximated with terms like $n_\pi C_2(\mathrm{O}(3))$ or $n_\pi C_2(\mathrm{Spin}(3))$ in the Hamiltonian. Similarly, phenomenologic parity-dependent interactions, not destroying the dynamical symmetry, can be taken into account by terms like $(-1)^{n_\pi} C_2(G)$.

Dynamical symmetries corresponding to special choices of the physical interactions are only rarely realized in real physical systems. Thus, in order to give a more accurate description, symmetry-breaking terms may be necessary. In the simplest cases these new terms can be appropriate expressions of operators with clear physical interpretation (e.g., multipole operators, etc.), but in general any term from Eq. (2.4), or similar higher-order terms, can be included.

### D. Relation to other phenomenologic algebraic models

The new model introduced in this section and the vibron-electron model [14] can be interpreted as the dynamical symmetries of the general vibron-fermion model. These dynamical symmetries with SU(3)×U(2) and O(4)×U(2) group structure correspond to the fermionic extension of the U(3) and O(4) limits of the vibron model, respectively. Similarly to the vibron model, the two dynamical symmetries of the vibron-fermion model are applicable in two different branches of physics, namely, the SU(3)×U(2) limit seems to be more appropriate in nuclear physical applications, while the O(4)×U(2) limit (i.e., the vibron-electron model) was proposed to describe certain problems in molecular physics [14].

The difference between the possible fields of applicability of these two limits can be viewed as the consequences of the group structure of the vibron-fermion model. The two possible group chains of U(4) group appearing in both the vibron and the vibron-fermion model are just the two well-known groups giving account of the degeneracies of two model problems of fundamental importance, the Coulomb and harmonic-oscillator problems in three dimensions. The orbital degeneracy group of the $H$ atom is the O(4) group, while that of the three-dimensional harmonic oscillator is the SU(3) group [20]. Fermionic single-particle states can be approximated with hydrogenic states and with harmonic-oscillator states in molecular physics and in nuclear physics, respectively.

The discussion of the vibron-electron model in Ref.

[14] was presented for the united atoms limit. Considering the hydrogenic levels of the unified atom with hydrogenic principal quantum number $n$ the electronic group turns out to be $U^F(2n^2)$. (Instead of $F$ the authors in Ref. [14] use the superscript $e$, standing for electron.) This group plays the same role as the $U^F((n+1)(n+2))$ group in the case of the $SU(3)\times U(2)$ limit of the vibron-fermion model. The factor of 2 accounts for the spin of the electrons. The orbital and spin parts can again be decomposed, leading to the reduction $U^F(2n^2) \supset U_l^F(n^2)\times U_s^F(2)$. The generators of these groups can be expressed as bilinear products of the fermion creation and annihilation operators. One possible choice is to use the operators presented in Eq. (2.2). It is only a matter of convenience to label the single-particle states with the orbital angular momentum and spin quantum numbers instead of the full fermionic angular momentum. The vibron-electron model was formulated using the former set of quantum numbers [14].

Since the vibron-fermion model and the IBFM have similar mathematical structure, one can find analogies between the dynamical symmetries of these two models. Here we shall not study this question in detail, only mention some general observations. One of these is that the dynamical boson-fermion symmetries of the IBFM show a bigger variety due to the richer group structure of the corresponding boson model. It also has some dynamical symmetries based on isomorphisms [like $o(5)\simeq sp(4)$ and $o(6)\simeq su(4)$] with no equivalents in the vibron-fermion model. Comparing the structure of the generator sets of the group chains associated with the dynamical symmetries of these two boson-fermion models we can see that the $SU(3)\times U(2)$ limit of the vibron-fermion model [and its special subcase, the $U(3)\times U(2)$ limit] can be related to the $SU(3)\times U(2)$ and $U(5)\times U(2)$ limits of the IBFM (see Refs. [12], and [10]), while the vibron-electron model [14] [with $O(4)\times U(2)$ group structure] can be considered the analogue of the $O(6)\times U(2)$ limit of the IBFM [11].

Another phenomenologic algebraic model which can be related to the $SU(3)\times U(2)$ limit of the vibron-fermion model is the nuclear vibron model [15]. In this model one of the clusters (the core) is assumed to have quadrupole deformation, and the relative motion of the clusters is coupled to this collectivity. The group structure of this model is $U(6)\times U(4)$, and the only dynamical symmetry discussed in this model is the $SU(3)$ one, which is based on the coupling of the $SU(3)$ limit of the IBM to the $U(3)$ limit of the vibron model.

## IV. OTHER CLUSTER MODELS AND POSSIBLE APPLICATIONS

Cluster structure of nuclei can be discussed in terms of microscopic and phenomenologic cluster models. Since the $SU(3)\times U(2)$ limit of the vibron-fermion model can be viewed as a phenomenologic approach to cluster structure of odd-mass nuclei, its comparison with other phenomenologic cluster modes seems necessary. Here we point out some similarities between our model and the local cluster model of Buck *et al.* [21].

As it was shown in connection with the application of the vibron model to the $^{16}O+\alpha$ system [7,8], the Pauli principle can be taken into account at an approximate level if we exclude states with $n_\pi < q$, where $q$ can be determined on the basis of microscopic considerations by applying the Wildermuth condition, [22] for example. (As discussed in Refs. [7] and [8], it originates from the relation between the shell model and the microscopic cluster model in the harmonic-oscillator limit.) In the shell-model picture, nucleons forming the lighter cluster are placed on orbits above the Fermi level of the core. If we assume that the lighter cluster is not excited, the excitation quanta carried by these nucleons can be viewed as the excitation quanta of the relative motion of the clusters. Thus, $q$ is the number of shell-model oscillator quanta carried by the nucleons of the lighter cluster when it is assumed to be in the lowest orbit. This exclusion procedure is the same as that in the local cluster model of Buck, Dover, and Vary [21] used in this description of two-cluster systems.

The basis states with the same $n_\pi$ form a rotational band and belong to the $(n_\pi,0)$ $SU^B(3)$ representation. This configuration corresponds to the maximal alignment of the orbits of the nucleons forming the lighter cluster. The states of the boson-fermion system are built on this bosonic configuration. Besides the way of handling the Pauli principle the band structure also shows some similarity between the local potential cluster model and the algebraic models of dipole-type collectivity. In the local potential cluster model the energy levels with the same value of $2N+L$ (i.e., the members of a rotational band) follow an almost perfect rotational spacing, proportional with $L(L+1)$, while the energy of the band heads changes monotonously with increasing $2N+L$ [21]. (Here $N$ is the number of nodes in the radial wave function.) This behavior can be summarized in the approximate energy expression

$$E(N,L)=F(2N+L)+CL(L+1)\ . \tag{4.1}$$

This is very similar to the energy spectrum obtained from the vibron model, if we replace $2N+L$ with $n_\pi$ and approximate the function $F(n_\pi)$ with a quadratic expression of $n_\pi$. Therefore we expect the $U(3)$ limit of the vibron model and its fermionic extensions to lie close to the local potential cluster model of Buck, Dover, and Vary [21], which is a successful model of cluster structure of light nuclei with even and odd mass alike.

Although it is not our aim to give detailed study of any nucleus in terms of the new model presented in this paper, before closing this section we briefly refer to nuclear cluster systems which can be the subject of such investigations. Here we introduced the $SU(3)\times U(2)$ limit of the vibron-fermion model as an extension of the $U(3)$ limit of the vibron model. This fact helps us to identify its most promising field of application. Originally the $U(3)$ limit of the vibron model was applied to nuclear cluster systems in which the clusters have no internal structure. This requirement allows only closed shell nuclei (like $^4$He, $^{16}$O, or $^{40}$Ca) as clusters, so the most obvious example for this dynamical symmetry is the $^{16}O+\alpha$ system,

which is also a textbook example of other cluster models [22,23]. We expect the neighboring light nuclei to be candidates for the SU(3)×U(2) dynamical symmetry. The most obvious examples are the $A=19$ and $A=21$ nuclei ($^{19}$F, $^{19}$Ne, $^{21}$Ne, $^{21}$Na) for hole coupling and particle coupling. Besides these odd-mass nuclei the model can handle other cluster systems as well, with $M>1$ (i.e., with more than one hole or nucleon on a shell). A whole series of nuclei are known to have marked $\alpha$-cluster character in this mass region [23,24]. Some other examples of $\alpha$ clustering (such as the $^{40}$Ca+$\alpha$, $^{39}$K+$\alpha$ systems, etc.) are also known near the closure of the *sd* shell.

Among the examples mentioned above, the $^{19}$F nucleus is the most well studied cluster system. In addition to this, the simplicity of the mathematical formulation (due to the $M=1$ choice) also suggests this nucleus to be studied first in terms of our model. Many low-lying states of $^{19}$F have been identified as $^{15}$N+$\alpha$ or $^{16}$O+$t$ cluster states [25–27]. Several cluster bands have been identified, some of which have equivalents in the $^{16}$O+$\alpha$ system. Later, microscopic studies showed [27] that satisfactory results can be obtained taking only the $^{15}$N+$\alpha$ configuration. At the same time the importance of the excited state of the $^{15}$N core with $J^\pi=\frac{3}{2}^-$ has been emphasized. The coupling of the configurations $^{15}$N($\frac{1}{2}^-$)+$\alpha$ and $^{15}$N($\frac{3}{2}^-$)+$\alpha$ has been discussed in terms of several cluster models [27,28]. In our model, this system corresponds to the coupling of the relative motion of the clusters (as bosonic structure) to a hole in the $p$ shell (as fermionic structure).

In the left side of Fig. 2 we displayed the known $\alpha$-cluster states of the $^{19}$F nucleus. These are classified into six cluster bands [27]. The $K^\pi=\frac{1}{21}^-$, $\frac{1}{22}^+$, and $\frac{1}{22}^-$ bands have well developed $^{15}$N+$\alpha$ cluster character and are known as the equivalents of the $K^\pi=0_1^+$, $0^-$, and $0_4^+$ cluster bands of the $^{20}$Ne nucleus [25,27,29], while the members of the $K^\pi=\frac{1}{23}^-$ and $\frac{3}{22}^-$ bands can be interpreted as states with $^{15}$N($\frac{3}{2}^-$)+$\alpha$ configuration [27]. The $K^\pi=\frac{1}{21}^+$ ground-state band has often been identified as a $^{16}$O+$t$ cluster band [25]; nevertheless, recent microscopic investigations showed that it can also be interpreted as a $^{15}$N+$\alpha$ band [27]. [In addition to the $\alpha$-cluster bands, there are two more cluster bands in the spectrum of the $^{19}$F nucleus (denoted by $\frac{3}{21}^+$ and $\frac{3}{21}^-$), but these are interpreted [27] as examples for other cluster structures, like $^7$Li+$^{12}$C.] In the right hand side of Fig. 2 we presented a model spectrum with SU(2)×U(2) dynamical symmetry, obtained from a fitting procedure in which the energy expression of Eq. (3.28) was used. Although the ordering of

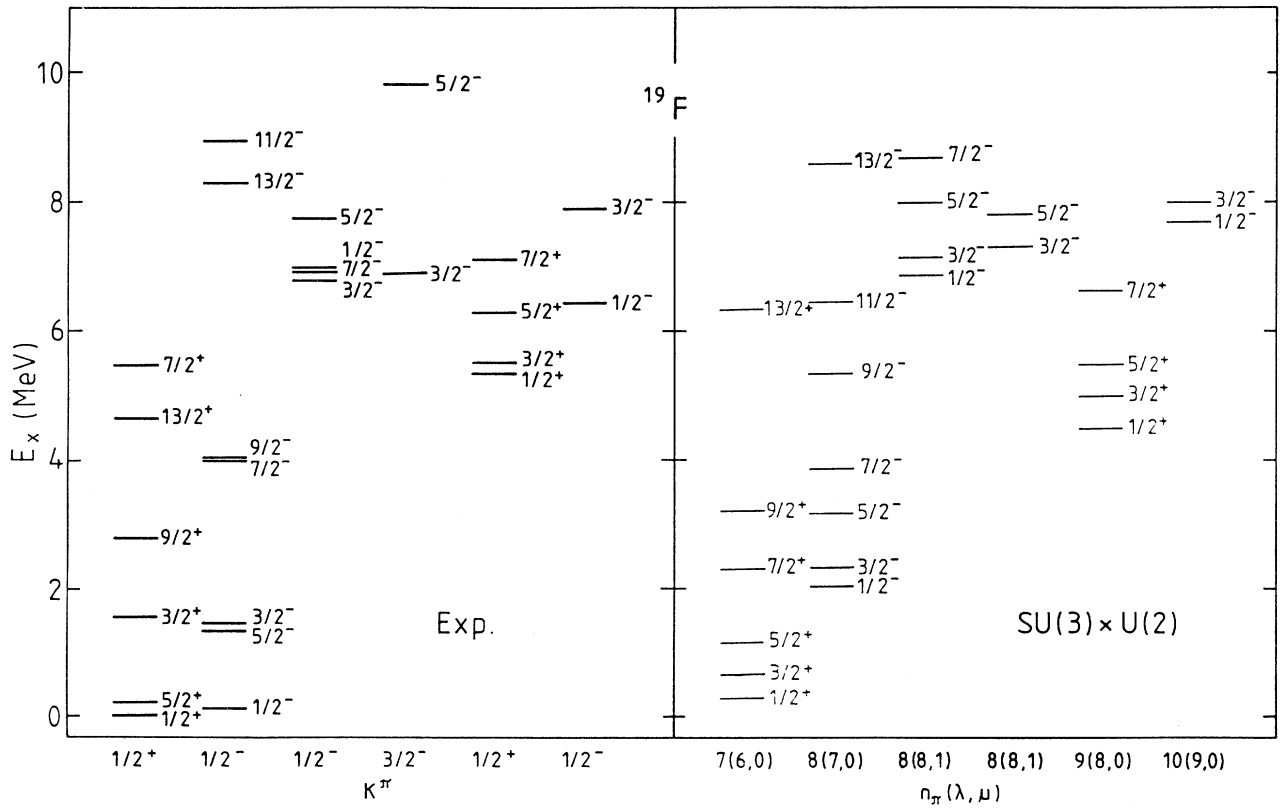


FIG. 2. Experimental energy spectrum of the $\alpha$-cluster states of the $^{19}$F nucleus (left panel) and the corresponding model states with SU(3)×U(2) dynamical symmetries (right panel). The assignment of states to cluster bands in the experimental spectrum was done following Ref. [27]. The parameters of the model spectrum (in MeV) are as follows: $E_0=6.398$, $\alpha_B=-3.850$, $\gamma_B=0.172$, $\gamma=0.159$, $\delta=0.032$, and $\epsilon=0.100$ [see Eq. (3.28)]. The lowest allowed value of $n_\pi$ is $q=7$ (see text for details).

the levels within a band is not always satisfactory, the basic trends of the experimental spectrum are reproduced by this simple fit. The results can considerably be improved if we introduce symmetry-conserving third-order terms (discussed earlier in Sec. III C) in the Hamiltonian. These new terms can account for the band dependence of the spin-orbit coupling, for example. In Fig. 3 we present a spectrum which was obtained from the Hamiltonian

$$H=E_0+\alpha_B n_\pi+\gamma_B(2Q_B^{(2)}\cdot Q_B^{(2)}+\tfrac{3}{4}L_B^{(1)}\cdot L_B^{(1)}) +\gamma(2Q^{(2)}\cdot Q^{(2)}+\tfrac{3}{4}L^{(1)}\cdot L^{(1)})+\delta L^{(1)}\cdot L^{(1)} +\epsilon J^{(1)}\cdot J^{(1)}+[\tau n_\pi+\kappa(-1)^{n_\pi}]L^{(1)}\cdot S^{(1)}\,, \quad (4.2)$$

which differs from the Hamiltonian in Eq. (3.27) in the last term describing parity- and band-dependent spin-orbit coupling. [Here we used Table I to express the Casimir operators in Eq. (3.27) in terms of physical operators.] More detailed investigations of this cluster system in terms of the vibron-fermion model concerning the electromagnetic properties will be given in a forthcoming publication [30].

The algebraic cluster models are unable at present to describe cluster spectroscopic factors, but work is in progress to overcome this problem [31].

Further applications of the model are possible using pseudoshells. In this case the fermionic structure would consist of fermionic single-particle states assigned to an arbitrary, non-nuclear shell, similarly to the IBFM. In this case the whole procedure could be viewed as a mathematical tool that helps to formulate the physical problem. We expect this approach to become important in the case of heavier nuclei.

## V. SUMMARY AND CONCLUSIONS

In this paper we have introduced an algebraic approach to cluster states of a class of nuclei in which nucleonic degrees of freedom also play an important role besides the relative motion of the clusters. This new model is a natural extension of the U(3) limit of the vibron model [2,3] and it is able to handle the interplay between collective (bosonic) and single-particle (fermionic) degrees of freedom. The bosonic part accounts for the relative motion of the clusters, while the fermionic part consists of nucleons (or holes) occupying single-particle states with $j=\frac{1}{2},\frac{3}{2},..,n-\frac{1}{2},n+\frac{1}{2}$ and with the same parity. The single-particle states can be assigned to a shell with $n$ oscillator quanta and their angular momentum can be decomposed into a (pseudo-)orbital part with $l=n$, $n-2,\ldots,1$, or 0 and a (pseudo-)spin part with $s=\frac{1}{2}$. (This oscillator shell need not be a physical one.) The group structure of the fermionic part contains the $SU_l^F(3)$ group, so the coupling of the bosonic and fermionic sec-

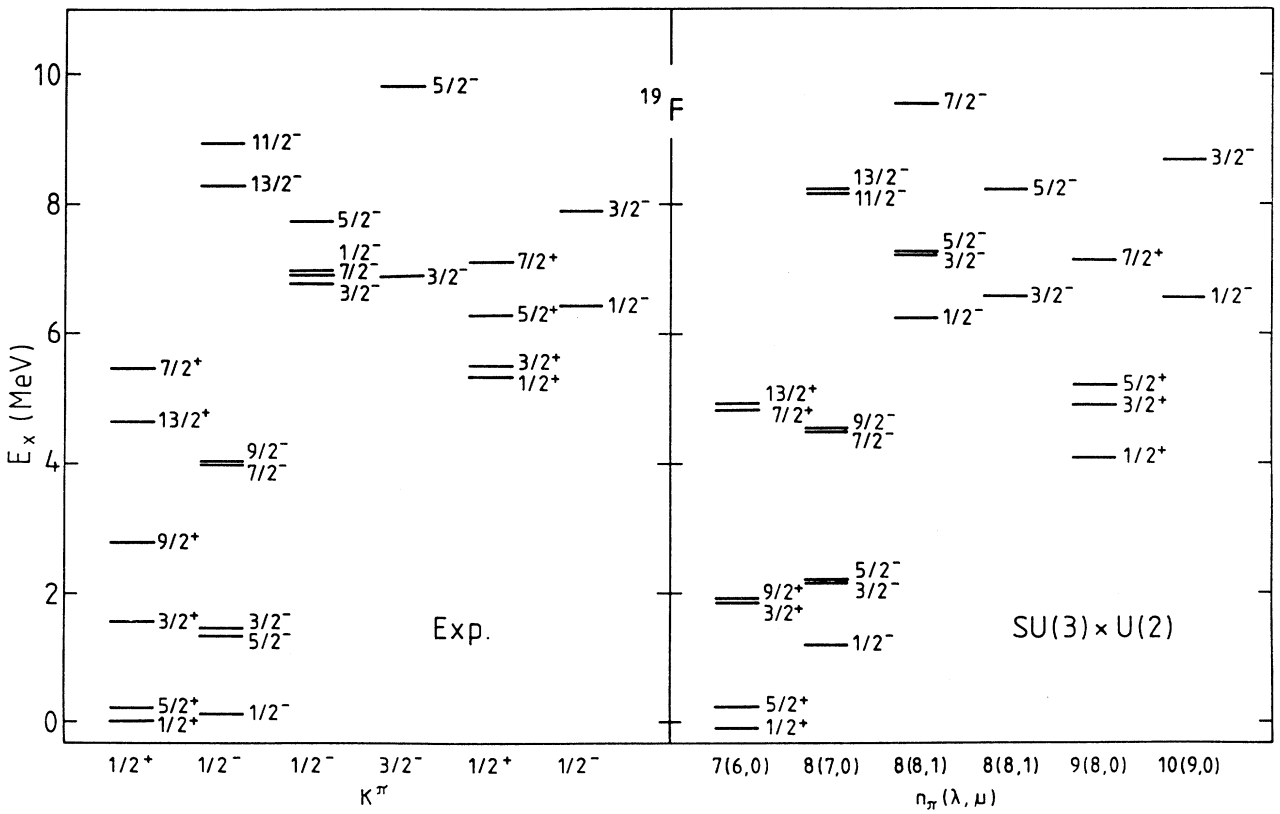


FIG. 3. The same as Fig. 2, with model states obtained from Hamiltonian (4.2) including two three-body terms describing the parity and band dependence of the spin-orbit coupling. The parameters of Hamiltonian (4.2) (in MeV) are as follows: $E_0=16.844$, $\alpha_B=-6.692$ $\gamma_B=0.312$, $\gamma=0.169$, $\delta=1.610$, $\epsilon=-1.443$, $\tau=0.461$, and $\kappa=0.385$.

tors can be established on the level of the SU(3) groups, giving rise to the SU(3)×U(2) limit of the vibron-fermion model. The mathematical formulation of the model becomes more complicated as $M$, the number of fermions (nucleons or holes) is increased. In the $M=0$ case the U(3) limit of the vibron model emerges as a special case.

The relation of this model to other algebraic models can be discussed from various viewpoints. Its relation to the vibron model has already been mentioned. Similarly to the nuclear vibron model [15] (which allows quadrupole collective excitations of the constituent nuclei), it takes into account the excitations of one of the clusters. The basic idea of the model is the same as that of the vibron-electron model introduced recently as an algebraic approach to molecular electronic spectra [14]. In this latter model the other dynamical symmetry of the vibron model [with O(4) rather than U(3) group structure] is used, and the fermionic (electronic) single-particle states taken into account are hydrogenic levels with a given principal quantum number. The role of the SU(3) and O(4) groups as the degeneracy groups of the harmonic oscillator and Coulomb problem in three dimensions helps to explain why the field of application of the two dynamical symmetries of the vibron model are so different.

From the mathematical point of view the SU(3)×U(2) limit of the vibron-fermion model and the vibron-electron model [with O(4)×U(2) group structure] can be related to various dynamical symmetries of the IBFM [9–13]. Here we followed the presentation of the corresponding SU(3)×(2) limit of the IBFM, using the similar mathematical structure given by Bijker and Kota [12]. Similarities between the dynamical symmetries of the vibron-fermion problem and the IBFM originate from the similar group structure of the vibron model and the IBM-1.

We have examined the coupled boson-fermion basis states associated with the dynamical symmetries of the model and determined the structure of the corresponding energy spectra. The study of other physical quantities (such as electromagnetic transitions, etc.) in terms of this model will be done in the following paper [30].

We proposed certain light nuclear cluster systems as possible subjects of investigations in terms of the SU(3)×U(2) limit of the vibron-fermion model. Among these examples the $\alpha$-cluster states of the $^{19}$F nucleus (with both positive and negative parity) seem to be the most promising. Our investigations show that three-body operators are required to give a realistic description of this system.

We discussed the physical interpretation of this phenomenologic model and also studied its relation to a few existing models of cluster structure of light nuclei. It turned out that our model shows some similarity with the local potential cluster model of Buck *et al.* [21].

Having established our model, we can proceed further in several directions. We can investigate light cluster systems in terms of the vibron-fermion model in either the case of dynamical symmetries or introducing symmetry-breaking-interaction terms in the Hamiltonian.


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