PHYSICAL REVIEW C VOLUME 44, NUMBER 4 OCTOBER 1991

Symmetry group of point transformations for the time-dependent Schrödinger equation: Harmonic interactions among nucleons

P. Rudra
Department of Physics, University of Kalyani, Kalyani, West Bengal, 741-235, India
(Received 27 February 1990)

We have used Lie's method of extended group to obtain explicit forms of the generators and the structure of the maximal symmetry group of point transformations of the time-dependent Schrödinger equation for motions of nucleons interacting with two-body harmonic potential. The generators of the symmetry group correspond to different states of motion of the system. The maximal symmetry group is found to be a semidirect product of an infinite parameter Abelian invariant subgroup and a proper subgroup. For Z protons and N neutrons, this proper subgroup is a Lie group with $\frac{1}{2}[9Z(Z-1)+9N(N-1)+40]$ generators. Different nuclear modes of excitations have been assigned to the different generators. In particular the giant resonance mode and other collective modes of motion are shown to be consequences of the symmetry of the system.

I. INTRODUCTION

Calculations of nuclear structures depend heavily on the model potential for nucleonic interaction and the harmonic potential has singular success as a useful model. Through another approximation all the nucleons are supposed [1] to be moving in an average potential. The symmetry group and the characterization of states of motion in terms of their symmetry properties have been extensively treated [2,3]. Though the average potential model is mostly followed, the translational invariant Hamiltonian where the particles interact with mutual two-body harmonic potential has also been extensively investigated [4,5]. All these analyses utilize the fact that the Hamiltonian in its second quantized form has the symmetry of the unitary unimodular group [6]. Recently, in an interesting work, Vanagas [7] has given a group-theoretical connection between the time-independent Schrödinger equation and the clear distinction between collective and internal degrees of freedom.

Here we have undertaken an investigation of the symmetry group of the time-dependent Schrödinger equation for nucleons with two-body harmonic interaction. The motivation of such an investigation is the following. The full symmetry group of the Schrödinger equation of the system consists of a complete set of generators. These generators are associated with different states of motion of the system. Some of these states of motion can of course be predicted intuitively or from general physical principles. But only a complete mathematical analysis will give all the states of motion. We have obtained a very rich group structure for the system. The linear homogeneous Schrödinger equation has of course an infinite-parameter Abelian invariant subgroup corresponding to the fact that any linear combination of linearly independent wave functions is again a wave function. The factor group with respect to this invariant subgroup has $\frac{1}{2}[9Z(Z-1)+9N(N-1)+40]$ generators, where Z and N are the numbers of protons and neutrons. Of these generators, time translation, scaling of the wave function, rigid translation, Galilean transformation, rigid rotation and rotation of the center-of-mass supply 14 generators. Our analysis shows that besides these well-known states of motion there are others also. We have assigned some of the other generators to different modes of nuclear excitations, the giant resonance mode, and a rotational-type collective degree of freedom among others. We do not think that the giant resonance mode has previously been shown to be a consequence of the symmetry of the system.

The method used in this analysis is Lie's method of extended group [8,9]. It is to be pointed out that what we have obtained here is the group of point transformations, and not the more general group of contact transformations [10], necessary for studying the linear integrals of the system.

II. EXTENDED GROUP AND SCHRÖDINGER EQUATION

The standard procedure [8,9] for obtaining the symmetry group of a partial differential equation of the rth order $\Delta(q,\Psi;r)=0$, where Ψ is the dependent variable, q^i, $i=1,\ldots,n$ are the independent variables, with the generators

$$X=\sum_i \xi^i(q,\Psi)\partial/\partial q^i+\Phi(q,\Psi)\partial/\partial\Psi \ ,$$

is through the action of $X^{(r)}$, the rth extension of X, on Δ. We separately equate to zero the coefficients of each monomials in the partial derivatives of Ψ occurring in $X^{(r)}\Delta=0$, and obtain a set of partial differential equations in the velocity vectors Φ and ξ^i. We have earlier [9] considered the time-dependent Schrödinger equation

$$\Delta\equiv i\Psi_t+\sum_s a_s \sum_\sigma \Psi_{s\sigma,s\sigma}-v\Psi=0 \ , \tag{1}$$

with generators of the form

$$X=\xi^t(q,t,\Psi)\partial/\partial t+\sum_{s\sigma}\xi^{s\sigma}(q,t,\Psi)\partial/\partial q^{s\sigma}+\Phi(q,t,\Psi)\partial/\partial\Psi \ . \tag{2}$$

Here σ denotes the Cartesian components of the space coordinates, s denotes either the particle or other identifications of the space coordinate, t denotes time, Ψ denotes the wave function, and the potential v is a function of q only. Here and later t and $s\sigma$ as the subscripts to Ψ,Φ, and ξ will mean the corresponding partial derivatives. We have shown in Ref. [9] that

$$\begin{aligned}\xi^{s\sigma}&=b_0^{s\sigma}(t)+f(t)'q^{s\sigma}+\sum_{\mu\lambda}e_{\sigma\mu\lambda}b_1^{s\lambda}q^{s\mu}\\&\quad+a_s\sum_{n<s}\sum_{\mu}b_{n\mu}^{s\sigma}q^{n\mu}-a_s\sum_{n>s}\sum_{\mu}b_{s\sigma}^{n\mu}q^{n\mu}\ ,\\ \xi^t&=2f(t)\ ,\\ \Phi&=\Phi^{(0)}(q,t)+\Psi\Phi^{(1)}(q,t)\ ,\end{aligned} \tag{3}$$

where

$$\Phi^{(1)}=C_1(t)+\sum_{s\sigma}\frac{i}{2a_s}\left[b_0^{s\sigma}(t)'q^{s\sigma}+\frac{f(t)''(q^{s\sigma})^2}{2}\right]\ ,$$

$$\frac{i\,\partial\Phi^{(1)}}{\partial t}-2f(t)'v-\sum_{s\sigma}\xi^{s\sigma}\frac{\partial v}{\partial q^{s\sigma}}+\frac{3i}{2}f(t)''=0\ , \tag{4}$$

and

$$i\Phi_t^{(0)}+\sum_{s\sigma}a_s\Phi_{s\sigma,s\sigma}^{(0)}-v\Phi^{(0)}=0\ . \tag{5}$$

Here b's are constants and the prime on a function of t denotes derivative with respect to t. $\Phi^{(0)}$ will give rise to an infinite dimensional Abelian invariant subgroup of the full symmetry group. This subgroup originates because any linear combination of linearly independent solutions of the homogeneous equation (1) is again a solution. We shall consider the factor group with respect to this invariant subgroup, with the other generators and call this factor group G_0 as the symmetry group.

III. MUTUAL HARMONIC INTERACTION

We consider two types of particles, the first of mass M_1 and N_1 in numbers situated at $\mathbf{q}^{1m}$ and the second of mass M_2 and N_2 in numbers situated at $\mathbf{q}^{2m}$. We assume that w_1 and w_2 are the characteristic frequencies of the intra-type harmonic interaction for the two types and w_{12} is that for the inter-type interaction, so that in Eq. (1)

$$\begin{aligned}&a_1=\hbar/2M_1\ ,\quad a_2=\hbar/2M_2\ ,\\ &v=\sum_{m\neq n}\sum_{\alpha}\frac{w_1^2}{4a_1}(q^{1m\alpha}-q^{1n\alpha})^2\\&\quad+\sum_{m\neq n}\sum_{\alpha}\frac{w_2^2}{4a_2}(q^{2m\alpha}-q^{2n\alpha})^2\\&\quad+\sum_{m,n}\sum_{\alpha}\frac{w_{12}^2}{4\sqrt{a_1a_2}}(q^{1m\alpha}-q^{2n\alpha})^2\ .\end{aligned} \tag{6}$$

In this case we write the velocity vectors of the generators as

$$\begin{aligned}\xi^t&=2f(t)\ ,\\ \xi^{1m\alpha}&=b_0^{1m\alpha}(t)+f(t)'q^{1m\alpha}+\sum_{\beta\gamma}e_{\alpha\beta\gamma}b_1^{1m\gamma}q^{1m\beta}+a_1\sum_{n=1}^{m-1}\sum_{\beta}b_{1n\beta}^{1m\alpha}q^{1n\beta}-a_1\sum_{n=m+1}^{N_1}\sum_{\beta}b_{1m\alpha}^{1n\beta}q^{1n\beta}-a_1\sum_{n=1}^{N_2}\sum_{\beta}b_{1m\alpha}^{2n\beta}q^{2n\beta}\ ,\\ \xi^{2m\alpha}&=b_0^{2m\alpha}(t)+f(t)'q^{2m\alpha}+\sum_{\beta\gamma}e_{\alpha\beta\gamma}b_1^{2m\gamma}q^{2m\beta}+a_2\sum_{n=1}^{N_1}\sum_{\beta}b_{1n\beta}^{2m\alpha}q^{1n\beta}+a_2\sum_{n=1}^{m-1}\sum_{\beta}b_{2n\beta}^{2m\alpha}q^{2n\beta}-a_2\sum_{n=m+1}^{N_2}\sum_{\beta}b_{2m\alpha}^{2n\beta}q^{2n\beta}\ ,\\ \Phi^{(1)}&=C_1(t)+\frac{i}{2a_1}\sum_{m=1}^{N_1}\sum_{\alpha}b_0^{1m\alpha}(t)'q^{1m\alpha}+\frac{i}{4a_1}f(t)''\sum_{m=1}^{N_1}\sum_{\alpha}(q^{1m\alpha})^2\\&\quad+\frac{i}{2a_2}\sum_{m=1}^{N_2}\sum_{\alpha}b_0^{2m\alpha}(t)'q^{2m\alpha}+\frac{i}{4a_2}f(t)''\sum_{m=1}^{N_2}\sum_{\alpha}(q^{2m\alpha})^2\ .\end{aligned} \tag{7}$$

In Appendix A the details of the calculation along the line stated in Sec. II are given. Here we simply give the result so that the physical content of the analysis is better elucidated.

If we use the symbols

$$\begin{aligned}&N=N_1+N_2\ ,\quad M=N_1M_1+N_2M_2\ ,\quad N\Omega_0^2=w_{12}^2(N_1a_2+N_2a_1)/\sqrt{a_1a_2}\ ,\\ &N\Omega_1^2=N_1w_1^2+N_2w_{12}^2\sqrt{a_1/a_2}\ ,\quad N\Omega_2^2=N_2w_2^2+N_1w_{12}^2\sqrt{a_2/a_1}\ ,\\ &R^\alpha=\left[M_1\sum_{n=1}^{N_1}q^{1n\alpha}+M_2\sum_{n=1}^{N_2}q^{2n\alpha}\right]\Big/M\ ,\end{aligned} \tag{8}$$

then we get $\frac{1}{2}[9N_1(N_1-1)+9N_2(N_2-1)+40]$ generators for the symmetry group G_0:

$$X_S=(M/\hbar)\Psi\partial/\partial\Psi\ ,\quad X^t=i\partial/\partial t\ ,$$

$$
\begin{aligned}
&X_T^\alpha=-i\sum_{p=1,2}\sum_{n=1}^{N_p}\frac{\partial}{\partial q^{pn\alpha}}\ ,\quad X_G^\alpha=tX_T^\alpha+R^\alpha X_S\ ,\\
&X_R^\alpha=-i\sum_{p=1,2}\sum_{\beta\gamma}e_{\alpha\beta\gamma}q^{pn\beta}\frac{\partial}{\partial q^{pn\gamma}}\ ,\quad X_R^{c\alpha}=\sum_{\beta\gamma}e_{\alpha\beta\gamma}R^\beta X_T^\gamma\ ,\\
&X_\pm^\alpha=e^{\pm i\sqrt{N}\Omega_0 t}\left[\left[\frac{1}{N_1M_1}\sum_{n=1}^{N_1}\frac{\partial}{\partial q^{1n\alpha}}-\frac{1}{N_2M_2}\sum_{n=1}^{N_2}\frac{\partial}{\partial q^{2n\alpha}}\right]\mp\frac{\sqrt{N}\,\Omega_0}{M}\left[\sum_{n=1}^{N_1}\frac{q^{1n\alpha}}{N_1}-\sum_{n=1}^{N_2}\frac{q^{2n\alpha}}{N_2}\right]X_S\right]\ ,\\
&X_\pm^{pn\alpha}=e^{\pm i\sqrt{N}\Omega_p t}\left[\left[\frac{\partial}{\partial q^{pn\alpha}}-\frac{\partial}{\partial q^{pN_p\alpha}}\right]\mp(\sqrt{N}M_p\Omega_p/M)(q^{pn\alpha}-q^{pN_p\alpha})X_S\right]\quad \text{for } n=1,\dots,N_p-1\ ,\ p=1,2\ ,\\
&X_R^{pn\alpha}=-i\sum_{\beta\gamma}e_{\alpha\beta\gamma}(q^{pn\beta}-q^{pN_p\beta})\left[\frac{\partial}{\partial q^{pn\gamma}}-\frac{\partial}{\partial q^{pN_p\gamma}}\right]\quad\text{for } n=1,\dots,N_p-1,\ p=1,2\ ,\\
&X_{pk\beta}^{pn\alpha}=-i\left[(q^{pk\beta}-q^{pN_p\beta})\left[\frac{\partial}{\partial q^{pn\alpha}}-\frac{\partial}{\partial q^{pN_p\alpha}}\right]-(q^{pn\alpha}-q^{pN_p\alpha})\left[\frac{\partial}{\partial q^{pk\beta}}-\frac{\partial}{\partial q^{pN_p\beta}}\right]\right]\\
&\qquad\qquad\text{for } k<n=2,\dots,N_p-1,\ p=1,2 \text{ and all } \alpha,\beta\ .
\end{aligned}
\tag{9}
$$

The commutation relations of these generators are given in Appendix B. These generators have been obtained under the condition that

$$
\begin{aligned}
&\Omega_1\neq\Omega_2\ ,\\
&\Omega_1\ ,\ \Omega_2\neq\Omega_0\ ,\\
&w_1^2\sqrt{a_1/a_2}\ ,\ w_2^2\sqrt{a_2/a_1}\neq w_{12}^2\ .
\end{aligned}
\tag{10}
$$

IV. DISCUSSION

We first give a physical interpretation [8] to the generators obtained in Eq. (9). If X is a generator of G_0 and Ψ is a solution of the Schrödinger equation (1), then $(\exp bX)\Psi$, where b is a constant, is also a solution of Eq. (1). Thus corresponding to the generator X_S we see that if Ψ is a state of motion then $(\exp bM/\hbar)\Psi$ is also a state of motion. Thus X_S denotes the scaling of the wave function. In this way X^t denotes a change in the origin of time t, X_T^α denotes a rigid translation of the whole system, X_G^α denotes the Galilean transformation from one inertial frame to another, $X_R^{c\alpha}$ denotes rotation of the center-of-mass of the system, and X_R^α denotes the rigid rotation of the whole system.

The above-mentioned states are those that can be surmised intuitively from physical principles also. But there are other states of motion that arise in this detailed analysis, and cannot be arrived at intuitively. The first term in the first square bracket in the expression for $X_\pm^\alpha$ in Eq. (9) generates displacement of the particles of the first type and the second term denotes displacement of the particles of the second type in a direction opposite to the former. The exponential factor $\exp\pm i\sqrt{N}\,\Omega_0 t$ denotes a harmonic time dependence of these displacements with frequency $\sqrt{N}\,\Omega_0$. Thus $X_\pm^\alpha$ denotes vibration of the system of particles of the two types with respect to each other with frequency $\sqrt{N}\,\Omega_0$. Similarly $X_\pm^{pn\alpha}$ denotes vibration of the nth particle of type p with respect to the N_pth particle of the same type with frequency $\sqrt{N}\,\Omega_p$. Again $X_R^{pn\alpha}$ denotes rotation of the nth particle of type p about the N_pth particle of the same type. The other generators $X_{pk\beta}^{pn\alpha}$ denote rotation-type modes involving the nth and the kth particle of type p, but cannot be given any simple physical meaning.

In order to understand the operation of $X_{pk\beta}^{pn\alpha}$ in more detail we give here the actual orthogonal point transformations corresponding to this class of generators:

$$
\begin{bmatrix} q^{pk\alpha}\\ q^{pn\alpha}\\ q^{pN_p\alpha}\end{bmatrix}\rightarrow\frac{1}{3}\begin{bmatrix}1+2\cos\Phi & 1-\cos\Phi-\sqrt{3}\sin\Phi & 1-\cos\Phi+\sqrt{3}\sin\Phi\\ 1-\cos\Phi+\sqrt{3}\sin\Phi & 1+2\cos\Phi & 1-\cos\Phi-\sqrt{3}\sin\Phi\\ 1-\cos\Phi-\sqrt{3}\sin\Phi & 1-\cos\Phi+\sqrt{3}\sin\Phi & 1+2\cos\Phi\end{bmatrix}\begin{bmatrix} q^{pk\alpha}\\ q^{pn\alpha}\\ q^{pN_p\alpha}\end{bmatrix}\quad\text{for } \alpha=\beta
\tag{11}
$$

and

$$
\begin{bmatrix}[q^{pk\beta}-q^{pN_p\beta}]\\ [q^{pn\alpha}-q^{pN_p\alpha}]\end{bmatrix}\rightarrow\begin{bmatrix}\cos\Phi & -\sin\Phi\\ \sin\Phi & \cos\Phi\end{bmatrix}\begin{bmatrix}[q^{pk\beta}-q^{pN_p\beta}]\\ [q^{pn\alpha}-q^{pN_p\alpha}]\end{bmatrix}\quad\text{for } \alpha\neq\beta\ ,
\tag{12}
$$

where Φ is the parameter of the transformation; all the other variables remain unchanged. It should be noted that this class of generator, is present only if the number of particles of type p is greater than 2. Thus ${}^{6}_{3}\mathrm{Li}_3$ is the lightest stable nuclide for which this symmetry generator is present, actually one each for proton and neutron for each choice of α and β.

For nucleons interacting with a mutual harmonic interaction we can take the protons and the neutrons as the two types of particles with $N_1 = Z$ and $N_2 = A - Z$, where Z is the atomic number and A is the mass number, the number of neutrons [not to be confused with N in Eq. (8)] being $A - Z$. For a single type of nucleon, A in number, Vanagas [7] showed that the orthogonal group $O(A-1)$ provides the collective degree of freedom. Our generators $X_R^{pn\alpha}$ give rise to these modes. It seems natural to associate $X_\pm^\alpha$ to the giant resonance modes in nuclei [11] which is the oscillation of the proton and the neutron fluids with respect to each other. This mode was introduced in nuclear physics from a phenomenological description of the interaction of the charged nuclear fluid with incident electromagnetic radiation. We now see that this mode is a natural consequence of the symmetry of the system. In this connection we mention that we have also made an investigation of the symmetry group for Z protons and N neutrons moving in an average harmonic field. In this case the generators $X_\pm^\alpha$ corresponding to the giant resonance modes do not appear. The oscillations described by the generators $X_\pm^{pn\alpha}$ will be another type of collective mode. The three basic frequencies w_1, w_2, and w_{12} supply all the characteristic frequencies of the different modes.

We find in Eq. (9) that time t explicitly occurs only in the generators $X_G^\alpha, X_\pm^\alpha, X_\pm^{pn\alpha}$. A symmetry analysis of the time-independent Schrödinger equation where t does not appear as an independent variable will thus fail to give these generators. Only an analysis of the time-dependent Schrödinger equation, where time appears explicitly as an independent variable, could show the existence of these modes of nuclear excitations.

We now mention the difference of the symmetry group of point transformations, as has been obtained here, and the symmetry group of contact transformations. The generators arising in the symmetry group of contact transformations will give the invariants of the system, whereas the generators of the symmetry group of point transformations give the modes or states of motion of the system.

We now mention a point about the forms of the generators obtained in Eq. (9). The velocity vectors Φ and ξ-s have been obtained by solving the determining partial differential equations of Lie. Actually any linearly independent combination of a complete set of generators with constant coefficients will also give another equally suitable complete set of generators. The choice made in Eq. (9) is such that the nature of the states of motion has a well-known physical interpretation.

As a final point we give an interesting subgroup structure of the symmetry group G_0. The symmetry group G_0 can be written as a semidirect product $G_0 = N \circledS H$ of the invariant subgroup $N = \{X_S, X^t, X_T^\alpha, X_G^\alpha, X_\pm^\alpha, X_\pm^{pn\alpha}\}$ and the proper subgroup $H = \{X_R^{c\alpha}, X_R^\alpha, X_R^{pn\alpha}, X_{pk\beta}^{pn\alpha}\}$.

APPENDIX A

In this appendix we show how the different unknown quantities in Eqs. (3) and (4) are obtained. If we put the expressions of $v, \xi^{pn\alpha}, \xi^t$ and $\Phi^{(1)}$ in the last part of Eq. (4), then we get on the left-hand side different monomials in $q^{pn\alpha}$. We now equate the coefficients of each such monomial separately to zero. If we use the symbols given in Eq. (8), then we get the following equations:

$$C_1(t)' + \tfrac{3}{2} f(t)'' = 0 \,, \tag{A1}$$

$$b_0^{1m\alpha}(t)'' + N\Omega_1^2 b_0^{1m\alpha}(t) - w_1^2 \sum_{k=1}^{N_1} b_0^{1k\alpha}(t) - w_{12}^2 \sqrt{a_1/a_2} \sum_{k=1}^{N_2} b_0^{2k\alpha}(t) = 0 \quad \text{for all } \alpha \text{ and } m = 1, \ldots, N_1 \,, \tag{A2}$$

$$b_0^{2m\alpha}(t)'' + N\Omega_2^2 b_0^{2m\alpha}(t) - w_2^2 \sum_{k=1}^{N_2} b_0^{2k\alpha}(t) - w_{12}^2 \sqrt{a_2/a_1} \sum_{k=1}^{N_1} b_0^{1k\alpha}(t) = 0 \quad \text{for all } \alpha \text{ and } m = 1, \ldots, N_2 \,, \tag{A3}$$

$$f(t)''' + 4(N\Omega_1^2 - w_1^2) f(t)' - 2a_1 w_1^2 \left[\sum_{k>m} b_{1m\alpha}^{1k\alpha} - \sum_{k<m} b_{1k\alpha}^{1m\alpha} \right] - 2w_{12}^2 \sqrt{a_1 a_2} \sum_{k=1}^{N_2} b_{1m\alpha}^{2k\alpha} = 0$$
$$\text{for all } \alpha \text{ and } m = 1, \ldots, N_1 \,, \tag{A4}$$

$$f(t)''' + 4(N\Omega_2^2 - w_2^2) f(t)' - 2a_2 w_2^2 \left[\sum_{k>m} b_{2m\alpha}^{2k\alpha} - \sum_{k<m} b_{2k\alpha}^{2m\alpha} \right] + 2w_{12}^2 \sqrt{a_1 a_2} \sum_{k=1}^{N_1} b_{1k\alpha}^{2m\alpha} = 0$$
$$\text{for all } \alpha \text{ and } m = 1, \ldots, N_2 \,, \tag{A5}$$

$$w_1^2 \left[\sum_{k>m} (b_{1m\beta}^{1k\alpha} + b_{lm\alpha}^{lk\beta}) - \sum_{k<m} (b_{1k\alpha}^{1m\beta} + b_{1k\beta}^{1m\alpha}) \right] + w_{12}^2 \sqrt{a_2/a_1} \sum_{k=1}^{N_2} (b_{1m\beta}^{2k\alpha} + b_{1m\alpha}^{2k\beta}) = 0$$
$$\text{for } \alpha \neq \beta \text{ and } m = 1, \ldots, N_1 \,, \tag{A6}$$

$$w_2^2\left[\sum_{k>m}(b_{2m\beta}^{2k\alpha}+b_{2m\alpha}^{2k\beta})-\sum_{k<m}(b_{2k\alpha}^{2m\beta}+b_{2k\beta}^{2m\alpha})\right]-w_{12}^2\sqrt{a_1/a_2}\sum_{k=1}^{N_1}(b_{1k\alpha}^{2m\beta}+b_{1k\beta}^{2m\alpha})=0$$

for $\alpha\neq\beta$ and $m=1,\ldots,N_2$, (A7)

$$4w_1^2f(t)'+a_1w_1^2\left[\left[\sum_{k>n}b_{1n\alpha}^{1k\alpha}-\sum_{k<n}b_{1k\alpha}^{1n\alpha}\right]+\left[\sum_{k>m}b_{1m\alpha}^{1k\alpha}-\sum_{k<m}b_{1k\alpha}^{1m\alpha}\right]\right]+w_{12}^2\sqrt{a_1a_2}\sum_{k=1}^{N_2}(b_{1n\alpha}^{2k\alpha}+b_{1m\alpha}^{2k\alpha})=0$$

for all α and $n<m=2,\ldots,N_1$, (A8)

$$4w_2^2f(t)'+a_2w_2^2\left[\left[\sum_{k>n}b_{2n\alpha}^{2k\alpha}-\sum_{k<n}b_{2k\alpha}^{2n\alpha}\right]+\left[\sum_{k>m}b_{2m\alpha}^{2k\alpha}-\sum_{k<m}b_{2k\alpha}^{2m\alpha}\right]\right]-w_{12}^2\sqrt{a_1a_2}\sum_{k=1}^{N_1}(b_{1k\alpha}^{2n\alpha}+b_{1k\alpha}^{2m\alpha})=0$$

for all α and $n<m=2,\ldots,N_2$, (A9)

$$4w_{12}^2f(t)'+N[\Omega_1^2-\Omega_2^2]\sqrt{a_1a_2}\,b_{1m\alpha}^{2n\alpha}+w_2^2\sqrt{a_1a_2}\sum_{k=1}^{N_2}b_{1m\alpha}^{2k\alpha}-w_1^2\sqrt{a_1a_2}\sum_{k=1}^{N_1}b_{1k\alpha}^{2n\alpha}$$
$$+w_{12}^2\left[a_1\left[\sum_{k>m}b_{1m\alpha}^{1k\alpha}-\sum_{k<m}b_{1k\alpha}^{1m\alpha}\right]+a_2\left[\sum_{k>n}b_{2n\alpha}^{2k\alpha}-\sum_{k<n}b_{2k\alpha}^{2n\alpha}\right]\right]=0$$

for all α and $m=1,\ldots,N_1$, $n=1,\ldots,N_2$, (A10)

$$w_1^2\sum_\gamma e_{\alpha\beta\gamma}(b_1^{1n\gamma}-b_1^{1m\gamma})+w_{12}^2\sqrt{a_1a_2}\sum_{k=1}^{N_2}(b_{1n\beta}^{2k\alpha}+b_{1m\alpha}^{2k\beta})$$
$$+a_1w_1^2\left[\left[\sum_{k>n}b_{1n\beta}^{1k\alpha}-\sum_{k<n}b_{1k\alpha}^{1n\beta}\right]+\left[\sum_{k>m}b_{1m\alpha}^{1k\beta}-\sum_{k<m}b_{1k\beta}^{1m\alpha}\right]\right]=0$$

for $\alpha\neq\beta$ and $n<m=2,\ldots,N_1$, (A11)

$$w_2^2\sum_\gamma e_{\alpha\beta\gamma}(b_1^{2n\gamma}-b_1^{2m\gamma})-w_{12}^2\sqrt{a_1a_2}\sum_{k=1}^{N_1}(b_{1k\alpha}^{2n\beta}+b_{1k\beta}^{2m\alpha})$$
$$+a_2w_2^2\left[\left[\sum_{k>n}b_{2n\beta}^{2k\alpha}-\sum_{k<n}b_{2k\alpha}^{2n\beta}\right]+\left[\sum_{k>m}b_{2m\alpha}^{2k\beta}-\sum_{k<m}b_{2k\beta}^{2m\alpha}\right]\right]=0$$

for $\alpha\neq\beta$ and $n<m=2,\ldots,N_2$, (A12)

$$w_{12}^2\sum_\gamma e_{\alpha\beta\gamma}(b_1^{2n\gamma}-b_1^{1m\gamma})+N[\Omega_1^2-\Omega_2^2]\sqrt{a_1a_2}\,b_{1m\alpha}^{2n\beta}+w_2^2\sqrt{a_1a_2}\sum_{k=1}^{N_2}b_{1m\alpha}^{2k\beta}$$
$$-w_1^2\sqrt{a_1a_2}\sum_{k=1}^{N_1}b_{1k\alpha}^{2n\beta}+w_{12}^2\left[a_1\left[\sum_{k>m}b_{1m\alpha}^{1k\beta}-\sum_{k<m}b_{1k\beta}^{1m\alpha}\right]+a_2\left[\sum_{k>n}b_{2n\beta}^{2k\alpha}-\sum_{k<n}b_{2k\alpha}^{2n\beta}\right]\right]=0$$

for $\alpha\neq\beta$ and $m=1,\ldots,N_1$, $n=1,\ldots,N_2$. (A13)

These equations are called the determining equations of Lie. The solutions of these equations are

$$f(t)=f_0\ ,\quad C_1(t)=C_1\ , \tag{A14}$$

$$b_{1m\alpha}^{2n\beta}=\frac{-1}{N_1N_2}\sum_\gamma e_{\alpha\beta\gamma}C^\gamma\ , \tag{A15}$$

$$b_0^{1m\alpha}(t)=(A^\alpha+B^\alpha t)/(N_1a_2+N_2a_1)+(A_+^\alpha e^{i\sqrt{N}\Omega_0t}+A_-^\alpha e^{-i\sqrt{N}\Omega_0t})/N_1a_2+(A_+^{1m\alpha}e^{i\sqrt{N}\Omega_1t}+A_-^{1m\alpha}e^{-i\sqrt{N}\Omega_1t})\ , \tag{A16}$$

$$b_0^{2m\alpha}(t)=(A^\alpha+B^\alpha t)/(N_1a_2+N_2a_1)-(A_+^\alpha e^{i\sqrt{N}\Omega_0t}+A_-^\alpha e^{-i\sqrt{N}\Omega_0t})/N_2a_1$$
$$+(A_+^{2m\alpha}e^{i\sqrt{N}\Omega_2t}+A_-^{2m\alpha}e^{-i\sqrt{N}\Omega_2t})\ , \tag{A17}$$

with

$$\sum_{m=1}^{N_p}A_\pm^{pm\alpha}=0\ ,$$

$$b_{1m\alpha}^{1N_1\beta}=\sum_{k=1}^{m-1} b_{1k\beta}^{1m\alpha}-\sum_{k=m+1}^{N_1-1} b_{1m\alpha}^{1k\beta}+\frac{1}{a_1}\sum_{\gamma} e_{\alpha\beta\gamma}\left[b_1^{1m\gamma}+C^{1\gamma}+\sqrt{a_1a_2}\left[\frac{w_{12}^2}{N_1w_1^2}\right]C^\gamma\right] \quad \text{for } m=1,\ldots,N_1-1\ , \tag{A18}$$

$$b_{2n\alpha}^{2N_2\beta}=\sum_{k=1}^{n-1} b_{2k\beta}^{2n\alpha}-\sum_{k=n+1}^{N_2-1} b_{2n\alpha}^{2k\beta}+\frac{1}{a_2}\sum_{\gamma} e_{\alpha\beta\gamma}\left[b_1^{2n\gamma}+C^{1\gamma}+\sqrt{a_1a_2}\frac{w_{12}^2}{N_1w_1^2}C^\gamma-C^\gamma\left[\frac{a_1}{N_1}-\frac{a_2}{N_2}\right]\right]$$

$$\text{for } n=1,\ldots,N_2-1\ , \tag{A19}$$

$$\sum_{\gamma} e_{\alpha\beta\gamma} b_1^{1N_1\gamma}=a_1\sum_{n=1}^{N_1-1}\left[\sum_{k=1}^{n-1} b_{1k\beta}^{1n\alpha}-\sum_{k=n+1}^{N_1-1} b_{1n\alpha}^{1k\beta}\right]+\sum_{\gamma} e_{\alpha\beta\gamma}\left[\sum_{n=1}^{N_1-1} b_1^{1n\gamma}+(N_1-2)\left[C^{1\gamma}+\sqrt{a_1a_2}\frac{w_{12}^2}{N_1w_1^2}C^\gamma\right]\right]\ , \tag{A20}$$

$$\sum_{\gamma} e_{\alpha\beta\gamma} b_1^{2N_2\gamma}=-a_2\sum_{n=1}^{N_2-1}\left[\sum_{k=1}^{n-1} b_{2k\alpha}^{2n\beta}-\sum_{k=n+1}^{N_2-1} b_{2n\beta}^{2k\alpha}\right]$$

$$+\sum_{\gamma} e_{\alpha\beta\gamma}\left\{\sum_{n=1}^{N_2-1} b_1^{2n\gamma}+(N_2-2)\left[C^{1\gamma}+\sqrt{a_1a_2}\frac{w_{12}^2}{N_1w_1^2}C^\gamma-C^\gamma\left[\frac{a_1}{N_1}-\frac{a_2}{N_2}\right]\right]\right\}\ . \tag{A21}$$

Here f_0, C_1, C^γ, $C^{1\gamma}$, A^α, B^α, $A_\pm^\alpha$, $A_\pm^{pm\alpha}$ are constants and $\Omega_1\neq\Omega_2$, $\Omega_1, \Omega_2\neq\Omega_0$, $w_1^2\sqrt{a_1/a_2}$, $w_2^2\sqrt{a_2/a_1}\neq w_{12}^2$. Using Eqs. (2), (7), and (A14)–(A21) we get the generators given in Eq. (9).

We now briefly indicate how these solutions of the determining equations (A1)–(A13) of Lie were obtained. From Eqs. (A2) and (A3) we get

$$\left[a_2\sum_{m=1}^{N_1} b_0^{1m\alpha}(t)+a_1\sum_{n=1}^{N_2} b_0^{2n\alpha}(t)\right]''=0\ , \tag{A22}$$

which leads to

$$\left[a_2\sum_{m=1}^{N_1} b_0^{1m\alpha}(t)+a_1\sum_{n=1}^{N_2} b_0^{2n\alpha}(t)\right]=A^\alpha+B^\alpha t\ , \tag{A23}$$

where A^α and B^α are constants. This relation together with Eq. (A2) will give us the equation

$$\left[\sum_{m=1}^{N_1} b_0^{1m\alpha}(t)\right]''+w_{12}^2(N_2a_1+N_1a_2)\left[\sum_{m=1}^{N_1} b_0^{1m\alpha}(t)\right]\Big/\sqrt{a_1a_2}=(N_1w_{12}^2/\sqrt{a_1a_2})(A^\alpha+B^\alpha t)\ , \tag{A24}$$

and the solution

$$\sum_{m=1}^{N_1} b_0^{1m\alpha}(t)=\frac{N_1(A^\alpha+B^\alpha t)}{N_1a_2+N_2a_1}+\frac{A_+^\alpha \exp i\sqrt{N}\,\Omega_0 t+A_-^\alpha \exp -i\sqrt{N}\,\Omega_0 t}{a_2}\ . \tag{A25}$$

Using these relations in Eq. (A2) we get (A16) as the solution. A similar procedure gives Eq. (A17).

From Eq. (A8) we find that $f(t)'=$ constant, i.e, $f(t)=f_0+f_1t$. This relation with Eqs. (A4), (A8), and (A9) will now give $f(t)'=0$, and thus from Eq. (A1) we get Eq. (A14). We also get

$$\sum_{m=1}^{N_1}\sum_{n=1}^{N_2} b_{1m\alpha}^{2n\alpha}=0\ . \tag{A26}$$

Equations (A11) and (A12) together with Eqs. (A8), (A9) give us

$$w_1^2\sum_{\gamma} e_{\alpha\beta\gamma}(b_1^{1m\gamma}+C^{1\gamma})=a_1w_1^2\left[\sum_{k>m} b_{1m\alpha}^{1k\beta}-\sum_{k<m} b_{1k\beta}^{1m\alpha}\right]+w_{12}^2\sqrt{a_1a_2}\sum_{k=1}^{N_2} b_{1m\alpha}^{2k\beta}\ , \tag{A27}$$

$$w_2^2\sum_{\gamma} e_{\alpha\beta\gamma}(b_1^{2n\gamma}+C^{2\gamma})=-a_2w_2^2\left[\sum_{k>n} b_{2n\beta}^{2k\alpha}-\sum_{k<n} b_{2k\alpha}^{2n\beta}\right]+w_{12}^2\sqrt{a_1a_2}\sum_{k=1}^{N_1} b_{1k\alpha}^{2n\beta}\ , \tag{A28}$$

with constants $C^{1\gamma}$ and $C^{2\gamma}$ for all m, n, α, and β. From these two relations follow

$$\sum_{m=1}^{N_1}\sum_{n=1}^{N_2} b_{1m\alpha}^{2n\beta}+\sum_{\gamma} e_{\alpha\beta\gamma}C^\gamma=0\ , \tag{A29}$$

$$C^{2\gamma}-C^{1\gamma}=-(C^\gamma/N_1N_2)[(N_2a_1-N_1a_2)+w_{12}^2\sqrt{a_1a_2}(N_1/w_2^2-N_2/w_1^2)]\ , \tag{A30}$$

where C^γ is constant. Equation (A13) will together with Eq. (A30) give

$$\sum_{k=1}^{N_2} b_{1m\alpha}^{2k\beta} = -\sum_{\gamma} e_{\alpha\beta\gamma} \frac{C^\gamma}{N_1}\ ,\ \sum_{k=1}^{N_1} b_{1k\alpha}^{2n\beta} = -\sum_{\gamma} e_{\alpha\beta\gamma} \frac{C^\gamma}{N_2}\ , \tag{A31}$$

and finally Eq. (A15) when $\Omega_1 \neq \Omega_2$, and $w_1^2\sqrt{a_1/a_2}, w_2^2\sqrt{a_2/a_1} \neq w_{12}^2$. These results together with Eqs. (A6) to (A10) give Eqs. (A18) and (A19). With these forms for all the other parameters we use Eqs. (A27) and (A28) to obtain Eqs. (A20) and (A21).

APPENDIX B

In this appendix we give the commutation relations of the generators given in Eq. (9).

$$\begin{aligned}
&[X^t, X_G^\alpha] = iX_T^\alpha, [X^t, X_\pm^\alpha] = \mp\sqrt{N}\,\Omega_0 X_\pm^\alpha, [X^t, X_\pm^{pn\alpha}] = \mp\sqrt{N}\,\Omega_p X_\pm^{pn\alpha}\ ,\\
&[X_T^\alpha, X_G^\beta] = -i\delta_{\alpha\beta} X_S, [X_T^\alpha, X_R^{c\beta}] = i\sum_\gamma e_{\alpha\beta\gamma} X_T^\gamma, [X_T^\alpha, X_R^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} X_T^\gamma\ ,\\
&[X_G^\alpha, X_R^{c\beta}] = i\sum_\gamma e_{\alpha\beta\gamma} X_G^\gamma, [X_G^\alpha, X_R^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} X_G^\gamma, [X_R^{c\alpha}, X_R^{c\beta}] = i\sum_\gamma e_{\alpha\beta\gamma} X_R^{c\gamma}\ ,\\
&[X_R^{c\alpha}, X_R^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} X_R^{c\gamma}, [X_R^\alpha, X_R^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} X_R^\gamma, [X_R^\alpha, X_\pm^\beta] = i\sum_\gamma e_{\alpha\beta\gamma} X_\pm^\gamma\ ,\\
&[X_R^\alpha, X_\pm^{pn\beta}] = i\sum_\gamma e_{\alpha\beta\gamma} X_\pm^{pn\gamma}, [X_R^\alpha, X_R^{pn\beta}] = i\sum_\gamma e_{\alpha\beta\gamma} X_R^{pn\gamma}\ ,\\
&[X_R^\alpha, X_{pn\gamma}^{pm\beta}] = i\sum_\lambda e_{\alpha\beta\lambda} X_{pn\gamma}^{pm\lambda} + i\sum_\lambda e_{\alpha\gamma\lambda} X_{pn\lambda}^{pm\beta}, [X_+^\alpha, X_-^\beta] = 2\delta_{\alpha\beta}(\sqrt{N}\,\Omega_0/N_1M_1N_2M_2)X_S\ ,\\
&[X_+^{pm\alpha}, X^{pn\beta}] = 2\delta_{\alpha\beta}(1+\delta_{mn})(\sqrt{N}\,\Omega_p M_p/M)X_S\ ,\\
&[X_\pm^{pm\alpha}, X_R^{pn\beta}] = i(1+\delta_{mn})\sum_\gamma e_{\alpha\beta\gamma} X_\pm^{pm\gamma}\ ,\\
&[X_\pm^{pm\gamma}, X_{pn\beta}^{pk\alpha}] = i\delta_{\gamma\alpha}(1+\delta_{mk})X_\pm^{pn\beta} - i\delta_{\gamma\beta}(1+\delta_{mn})X_\pm^{pk\alpha}\ ,\\
&[X_R^{pm\alpha}, X_R^{pn\beta}] = 2i\delta_{mn}\sum_\gamma e_{\alpha\beta\gamma} X_R^{pm\gamma} + i(1-\delta_{mn})\left[\delta_{\alpha\beta}\sum_\gamma (X_{pm\gamma}^{pn\gamma} - X_{pn\gamma}^{pm\gamma}) + (X_{pn\alpha}^{pm\beta} - X_{pm\beta}^{pn\alpha})\right]\ ,\\
&[X_R^{pk\alpha}, X_{pn\gamma}^{pm\beta}] = i\delta_{km}\left[2\sum_\lambda e_{\alpha\beta\lambda} X_{pn\gamma}^{pm\lambda} + (\delta_{\alpha\beta} X_R^{pm\gamma} - \delta_{\beta\gamma} X_R^{pm\alpha})\right] + i\delta_{kn}\left[2\sum_\lambda e_{\alpha\gamma\lambda} X_{pn\lambda}^{pm\beta} - (\delta_{\alpha\gamma} X_R^{pn\beta} - \delta_{\beta\gamma} X_R^{pn\alpha})\right]\\
&\qquad + i(1-\delta_{km})(1-\delta_{kn})\left[\sum_\lambda e_{\alpha\beta\lambda}(X_{pn\gamma}^{pk\lambda} - X_{pk\lambda}^{pn\gamma}) + \sum_\lambda e_{\alpha\gamma\lambda}(X_{pk\lambda}^{pm\beta} - X_{pm\beta}^{pk\lambda})\right]\ ,\\
&[X_{pr\beta}^{pk\alpha},\ X_{pn\lambda}^{pm\mu}] = i\delta_{km}\delta_{rn}\left[\delta_{\alpha\mu}\sum_\gamma e_{\beta\lambda\gamma} X_R^{pr\gamma} + \delta_{\beta\lambda}\sum_\gamma e_{\alpha\mu\gamma} X_R^{pk\gamma}\right] + i\delta_{rm}\delta_{\beta\mu} X_{pn\lambda}^{pk\alpha} - i\delta_{kn}\delta_{\alpha\lambda} X_{pr\beta}^{pm\mu} + i\delta_{rm}\delta_{\alpha\lambda}\sum_\gamma e_{\mu\beta\gamma} X_R^{pr\gamma}\\
&\qquad + i\delta_{kn}\delta_{\beta\mu}\sum_\gamma e_{\lambda\alpha\gamma} X_R^{pk\gamma} + i\delta_{rn}\delta_{\alpha\mu}\sum_\gamma e_{\beta\lambda\gamma} X_R^{pr\gamma} + i\delta_{km}\delta_{\beta\lambda}\sum_\gamma e_{\alpha\mu\gamma} X_R^{pk\gamma}\\
&\qquad + i(1-\delta_{rm})\delta_{\alpha\lambda}(X_{pm\mu}^{pr\beta} - X_{pr\beta}^{pm\mu}) + i(1-\delta_{kn})\delta_{\beta\mu}(X_{pn\lambda}^{pk\alpha} - X_{pk\alpha}^{pn\lambda})\\
&\qquad + i(1-\delta_{rn})\delta_{\alpha\mu}(X_{pr\beta}^{pn\lambda} - X_{pn\lambda}^{pr\beta}) + i(1-\delta_{km})\delta_{\beta\lambda}(X_{pk\alpha}^{pm\mu} - X_{pm\mu}^{pk\alpha})\\
&\qquad + i(1-\delta_{rn})\delta_{km}\delta_{\alpha\mu}(X_{pr\beta}^{pn\lambda} - X_{pn\lambda}^{pr\beta}) + i(1-\delta_{km})\delta_{rn}\delta_{\beta\lambda}(X_{pk\alpha}^{pm\mu} - X_{pm\mu}^{pk\alpha})\ .
\end{aligned} \tag{B1}$$

If in any $X_{pn\lambda}^{pm\mu}$ occurring on the right-hand side of the commutators $m < n$, then that particular term is absent.

[1] P. Kramer and M. Moshinsky, in *Group Theory and Its Applications,* edited by E. M. Loebl (Academic, New York, 1968), Vol. I, p. 339.

[2] M. Kretschmar, Z. Phys. **157**, 433 (1960); **158**, 284 (1960).

[3] V. Bargmann and M. Moshinsky, Nucl. Phys. **18**, 697 (1960); **23**, 177 (1961).

[4] M. Moshinsky, Nucl. Phys. **31**, 384 (1962).

[5] M. Resnikoff, Nucl. Phys. **83**, 632 (1966).

[6] G. A. Baker, Jr., Phys. Rev. **102**, 1119 (1956).

[7] V. Vanagas, in *Group Theoretical Methods in Physics,* edited by K. B. Wolf, Lecture Notes in Physics No. 135 (Springer, Berlin, 1980), p. 408.

[8] L. V. Ovsjannikov, *Group Properties of Differential Equations,* English translation by G. Bluman (University of British Columbia, Vancouver, B. C., 1967).

[9] P. Rudra, Phys. Rev. A **36**, 5 (1987); **39**, 1570 (1989).

[10] R. L. Anderson, S. Kumei, and C. E. Wulfman, Rev. Mex. Fis. **21**, 1, 35 (1972).

[11] J. M. Eisenberg and W. Greiner, *Nuclear Theory,* Vol. I of *Nuclear Models, Collective and Single Particle Phenomena* (North-Holland, Amsterdam, 1970).