PHYSICAL REVIEW C VOLUME 44, NUMBER 4 OCTOBER 1991

Formalism for the systematic solution of the low-energy pion-trinucleon scattering problem

E. Harper
Department of Physics, Center for Nuclear Studies, The George Washington University, Washington, D.C. 20052
(Received 3 April 1991)

We present a systematic representation of the low-energy pion-trinucleon scattering problem derived from a reduction of the exact four-body formulation which is due to Haberzettl and Sandhas. This method removes many of the difficulties usually associated with the assignment of the parametric energy of pion-nucleon scattering in the pion-trinucleon center of mass which occur in the traditional optical potential approach based on multiple-scattering theories. In the present formulation both the pion-nucleon and nucleon-nucleon interactions provide the dynamical input and the quasideuteron mechanism appears naturally as a low-order approximation in our approach.

I. INTRODUCTION

The traditional method for solving the pion-nucleus scattering problem involves the construction of an optical potential obtained from the Watson scattering series [1] or, more usually, from the Kerman-McManus-Thaler (KMT) [2] variation of the Watson formalism. Some of the difficulties connected with the practical implementation of this approach are those associated with the identification of the pion-nucleon center-of-mass energy in the pion-nucleus scattering [3] and ambiguities connected to the inclusion of pion absorption terms in the optical potential [4]. For the case of pion scattering from nuclei with mass number $A>4$ the optical potential method is probably the most practical approach at this time. However, for $A=2$ it has been demonstrated [5] that the exact three-body methods of Faddeev [6] and Lovelace [7] provide a more rigorous and systematic approach to pion-deuteron scattering to the extent that satisfactory agreement with the experimental data is obtained with fewer assumptions and arbitrary parameters than in the case of the method employing optical potentials extracted from the KMT formalism [8].

In the usual application of the optical potential one constructs an effective interaction in the form

$$\langle \mathbf{k}'|V(z)|\mathbf{k}\rangle = \sum_{i=1}^{A} \langle \Psi_A \mathbf{k}'|t_{\pi N_i}(z)|\Psi_A \mathbf{k}\rangle \ , \tag{1.1}$$

where Ψ_A is the nuclear ground-state wave function and $t_{\pi N}(z)$ is a free pion-nucleon t matrix defined in the A nucleon plus single pion $(A+1)$-body space, $\mathbf{k}$ and $\mathbf{k}'$ are the off-shell center-of-mass pion momenta, while z is the on-shell $(A+1)$-body energy parameter defined by the equation

$$z=\omega_\kappa+\frac{\kappa^2}{2m_A}-E_A \ , \qquad \omega_\kappa=\sqrt{\kappa^2+m^2}-m \ . \tag{1.2}$$

In Eq. (1.2) m is the pion mass, M_A and E_A are the nuclear mass and binding energy, respectively, κ is the on-shell pion momentum in the pion-nucleus center of mass, and ω_κ is the pion kinetic energy in the same system. In what follows amplitudes and propagators are to be thought of as depending parametrically on z (with $A=3$) except where otherwise specified. Throughout this paper we adopt the usual convention of setting $c=\hbar=1$.

The optical potential, as represented by Eq. (1.1), is inserted in a Lippmann-Schwinger equation for the pion-nucleus t matrix which is then solved numerically. For many-body nuclei the wave function Ψ_A is frequently obtained from the shell model, although for the case $A=3$ Van Geffen *et al.* [9] have obtained good results using a trinucleon wave function generated from a solution of the Faddeev equations using a local potential to represent the N-N interaction.

The representation of the multiple-scattering optical potential given by Eq. (1.1) is the lowest-order approximation to the KMT theory in which the full optical potential is the solution of the integral equation [4],

$$V(z)=V^{(1)}(z)+V^{(1)}(z)G_A(z)QV(z) \ , \qquad G_A(z)=(z-H_A-K)^{-1} \ , \quad Q=1-P \ , \tag{1.3}$$

where H_A and K are the full nuclear Hamiltonian and kinetic-energy operators, respectively, and P is the nuclear ground-state projection operator. The driving term $V^{(1)}$ for (1.3) is given by the pion-bound-nucleon t matrix τ_i

$$V^{(1)}(z)=(A-1)\tau_i(z) \ , \qquad \tau_i(z)=v+vG_Z(z)Q\tau_i(z) \ , \tag{1.4}$$

where v is the πN interaction potential.

The NN correlations are contained in G_A and Q so that the solution of Eq. (1.4) represents an enormous problem, in general. However, τ_i can be related to the free πN t matrix t_i if we adopt the three-body model of Ref. [10] in which the pion scatters from nucleon i which is bound to the $(A-1)$ core by the potential v_N. In this model τ_i can be obtained from the solution of the equation

$$\mathcal{T}_i=T_i+T_it_N\mathcal{T}_i \ , \qquad \mathcal{T}_i=G_3\tau_iG_3 \ , \quad T_i=G_3t_iG_3 \ , \tag{1.5}$$

where G_3 is the three-body (pion, nucleon, core) Green's function

$$G_3(z)=(z-H_{A-1}-K_i-K_\pi)^{-1}$$

and t_N is the solution to the NN Lippmann-Schwinger equation,

$$t_N=v_N+v_N G_3 t_N \ . \tag{1.6}$$

The set of Eqs. (1.5) and (1.6) represents a procedure for the systematic solution of the πA scattering problem in which the NN correlations are accounted for via t_N. In what follows we will present an alternative procedure which is most useful for application to elastic scattering of pions from few-nucleon systems, but which in principle can be generalized to describe arbitrary pion-A elastic and charge-exchange scattering. This method uses the approach of Haberzettl and Sandhas [11] for writing an effective two-body Lippmann-Schwinger equation for bound-state scattering. In spirit, the present approach is the extension to pion-trinucleon scattering of the Faddeev methods of Afnan and Thomas (and others) [12], for the description of pion-deuteron scattering.

We propose to apply the formalism to be described in this paper to a numerical calculation of π-trinucleon elastic scattering and with this end in view the nucleon-nucleon interaction will be assumed to be nonzero in orbital angular momentum $L=0$ states only and the associated t matrix will be taken to be separable. The pion-nucleon t matrix is assumed to be separable also and for present purposes will be restricted to $L=0$ and 1 waves alone. These partial-wave limitations can easily be removed and are assumed here mainly for simplicity because the number of coupled integral equations obtained under these restrictions leads to a tractable but realistic numerical problem at low energy.

In Sec. II of this paper we will summarize the Haberzettl-Sandhas formalism as applied to the pion-trinucleon scattering problem. Section III contains most of our results and is primarily a reduction of the formalism to a set of equations which can be solved numerically to obtain the pion-trinucleon scattering observables. Section IV contains a comparison of our approach with the traditional optical-model approach and a discussion of possible generalizations and extensions of the method.

II. THE HABERZETTL-SANDHAS FORMALISM

Haberzettl and Sandhas [11] have shown that the Alt-Grassberger-Sandhas (AGS) equations [13] for the four-body transition amplitudes $U^{\sigma\rho}_{\beta\alpha}$ may be written in the form

$$\begin{aligned} U^{\sigma\rho}_{\beta\alpha}=&\bar{\delta}_{\sigma\rho}\delta_{\beta\alpha}G_0^{-1}T_\alpha^{-1}G_0^{-1} \\ &+\sum_a U^a_{\beta\alpha}+\sum_{\tau\gamma}\bar{\delta}_{\sigma\tau}U^\tau_{\beta\gamma}G_0T_\gamma G_0U^{\tau\rho}_{\gamma\alpha} \\ &+\sum_{\tau a}\sum_{\mu\gamma}U^a_{\beta\mu}G_0T_\mu G_0U^\tau_{\mu\gamma}G_0T_\gamma G_0U^{\tau\rho}_{\gamma\alpha} \ , \end{aligned} \tag{2.1}$$

where the complementary Kronecker delta is defined by

$$\bar{\delta}_{\alpha\beta}=1-\delta_{\alpha\beta} \ .$$

In Eq. (2.1) the superscript greek letters denote the four possible $(3+1)$ partitions of the four-body system, latin letters denote the three possible $(2+2)$ partitions, and the subscript greek notation specifies the two-body system contained in the corresponding partition. As already mentioned in the Introduction the four-body transition operators $U^{\sigma\rho}_{\beta\alpha}$, the three-body transition operators $U^{\sigma,a}_{\beta\alpha}$, as well as the two-body transition operators T_β, and Green's functions G_0, all depend on the four-body on-shell energy parameter z. Clearly, the above equations have decoupled the $(3+1)$ and $(2+2)$ partition channels of the four-body problem. Moreover, Haberzettl and Sandhas have shown [11] that the use of Eq. (2.1) removes the necessity for finding a separable representation for the $(2+2)$ subsystem amplitudes $U^a_{\beta\alpha}$. Thus, only the $(3+1)$ subamplitudes $U^\sigma_{\beta\alpha}$ will be needed in separable form in order to obtain a coupled set of effective two-body equations. The reader is referred to Ref. [11] for a more complete description of the significance of the formalism and detailed explanation of the notation.

The off-energy-shell amplitudes corresponding to the scattering of particle ρ from the bound state of the remaining particles leading to particle σ free are written as [13]

$$A^{\sigma\rho}=\sum_{\beta\alpha}\langle\Gamma^\sigma_\beta|G_0T_\beta G_0U^{\sigma\rho}_{\beta\alpha}G_0T_\alpha G_0|\Gamma^\rho_\alpha\rangle \ , \tag{2.2}$$

where the three-body form factor $|\Gamma^\sigma_\beta\rangle$ is a solution of the homogeneous AGS three-body equation

$$|\Gamma^\sigma_\beta\rangle=\sum_\gamma\bar{\delta}_{\beta\gamma}T_\gamma(E_\sigma)G_0(E_\sigma)|\Gamma^\sigma_\gamma\rangle \ , \tag{2.3}$$

with E_σ being the three-body bound-state energy for system σ.

For our purposes we find it convenient to write the two-body t matrix in the separable form

$$T_\alpha(z)=|g_\alpha\rangle t_\alpha(z)\langle g_\alpha| \ . \tag{2.4}$$

Multiplying Eq. (2.1) from the left-hand side by $\langle\Gamma^\sigma_\beta|G_0T_\beta$ and from the right-hand side by $T_\alpha G_0|\Gamma^\rho_\alpha\rangle$ and using Eq. (2.4) we can write Eq. (2.1) in the compact form

$$\begin{aligned} X^{\sigma\rho}_{\beta\alpha}=&\bar{\delta}_{\sigma\rho}\delta_{\beta\alpha}t_\alpha^{-1}+\sum_a T^a_{\beta\alpha}+\sum_{\tau\gamma}\bar{\delta}_{\sigma\tau}T^\tau_{\beta\gamma}t_\gamma X^{\tau\rho}_{\gamma\alpha} \\ &+\sum_{\tau a}\sum_{\gamma\mu}T^a_{\beta\mu}t_\mu T^\tau_{\mu\gamma}t_\gamma X^{\tau\rho}_{\gamma\alpha} \ , \end{aligned} \tag{2.5}$$

where the various quantities are defined as follows:

$$\begin{aligned} X^{\sigma\rho}_{\beta\alpha}&=\langle g_\beta|G_0U^{\sigma\rho}_{\beta\alpha}G_0|g_\alpha\rangle \ , \\ T^{a,\tau}_{\beta\alpha}&=\langle g_\beta|G_0U^{a,\tau}_{\beta\alpha}G_0|g_\alpha\rangle \ . \end{aligned} \tag{2.6}$$

Clearly Eq. (2.5) can be written in the form of a multichannel Lippmann-Schwinger equation,

$$X^{\sigma\rho}_{\beta\alpha}=V^{\sigma\rho}_{\beta\alpha}+\sum_{\gamma\mu\tau}V^{\sigma\tau}_{\beta\mu}t_\mu T^\tau_{\mu\gamma}t_\gamma X^{\tau\rho}_{\gamma\alpha} \ , \tag{2.7}$$

where the potential $V^{\sigma\rho}_{\beta\mu}$ can be expressed as

$$V^{\sigma\rho}_{\beta\mu}=\bar{\delta}_{\sigma\rho}\delta_{\beta\mu}t_\beta^{-1}+\sum_a T^a_{\beta\mu} \ . \tag{2.8}$$

We now approximate the three-body amplitude by the single-term separable form [11]

$$T^{\tau}_{\mu\gamma}(z)=|F^{\tau}_{\mu}\rangle t^{\tau}(z)\langle F^{\tau}_{\gamma}| \ , \tag{2.9}$$

where the $|F^{\tau}_{\mu}\rangle$ satisfy the homogeneous AGS equations

$$|F^{\tau}_{\mu}\rangle=\sum_{\gamma}\bar{\delta}_{\mu\gamma}\langle g_{\mu}|G_0(E_{\tau})|g_{\gamma}\rangle t_{\gamma}(E_{\tau})|F^{\tau}_{\gamma}\rangle \ , \tag{2.10}$$

and we will return to the question of the representation of the three-body propagator t^{τ} below. It should be noticed that, unlike the solutions $|\Gamma^{\sigma}_{\beta}\rangle$ of Eq. (2.3), the $|F^{\tau}_{\mu}\rangle$ which solve Eq. (2.10) are functions of a single vector variable.

From Eqs. (2.3) and (2.10) we can easily deduce the relationship

$$\langle g_{\beta}|G_0(z)|\Gamma^{\sigma}_{\beta}\rangle=\sum_{\gamma}\bar{\delta}_{\beta\gamma}\langle g_{\beta}|G_0(z)|g_{\gamma}\rangle t_{\gamma}(E_{\sigma})|F^{\sigma}_{\gamma}\rangle \tag{2.11}$$

from which it immediately follows, using also Eq. (2.10), that

$$|F^{\sigma}_{\beta}\rangle=\langle g_{\beta}|G_0(E_{\sigma})|\Gamma^{\sigma}_{\beta}\rangle \ . \tag{2.12}$$

Using Eq. (2.11) in (2.2) we readily see that the on-shell (physical) scattering-amplitude components assume the form

$$\langle \boldsymbol{\kappa}'|A^{\sigma\rho}_{\beta\alpha}(z)|\boldsymbol{\kappa}\rangle=\langle \boldsymbol{\kappa}'F^{\sigma}_{\beta}|t_{\beta}(z)X^{\sigma\rho}_{\beta\alpha}(z)t_{\alpha}(z)|F^{\rho}_{\alpha}\boldsymbol{\kappa}\rangle \ , \tag{2.13}$$

where $\boldsymbol{\kappa}'$ and $\boldsymbol{\kappa}$ are the final and initial on-shell pion-trinucleon momenta. For the case of elastic scattering, $\kappa=\kappa'$.

Inserting Eq. (2.9) into (2.7) and multiplying from the left-hand side by $\langle F^{\sigma}_{\beta}|t_{\beta}(z)$ and from the right-hand side by $t_{\alpha}(z)|F^{\tau}_{\alpha}\rangle$ we arrive at the coupled set of equations

$$\langle F^{\sigma}_{\beta}|t_{\beta}X^{\sigma\rho}_{\beta\alpha}t_{\alpha}|F^{\rho}_{\alpha}\rangle=\langle F^{\sigma}_{\beta}|t_{\beta}V^{\sigma\rho}_{\beta\alpha}t_{\alpha}|F^{\rho}_{\alpha}\rangle+\sum_{\tau,\mu\gamma}\langle F^{\sigma}_{\beta}|t_{\beta}V^{\sigma\tau}_{\beta\mu}t_{\mu}|F^{\tau}_{\mu}\rangle t^{\tau}\langle F^{\tau}_{\gamma}|t_{\gamma}X^{\tau\rho}_{\gamma\alpha}t_{\alpha}|F^{\rho}_{\alpha}\rangle \ . \tag{2.14}$$

The off-shell scattering amplitudes are

$$A^{\sigma\rho}=\sum_{\beta\alpha}A^{\sigma\rho}_{\beta\alpha} \tag{2.15}$$

and satisfy the equations

$$A^{\sigma\rho}=B^{\sigma\rho}+\sum_{\tau}B^{\sigma\tau}t^{\tau}A^{\tau\rho} \ , \tag{2.16}$$

where the driving term $B^{\sigma\rho}$ is given by the expression

$$B^{\sigma\rho}=\bar{\delta}_{\sigma\rho}\sum_{\beta\alpha}\delta_{\beta\alpha}\langle F^{\sigma}_{\beta}|t_{\beta}|F^{\rho}_{\alpha}\rangle + \sum_{\beta\alpha,a}\langle F^{\sigma}_{\beta}|t_{\beta}\langle g_{\beta}|G_0U^{a}_{\beta\alpha}G_0|g_{\alpha}\rangle t_{\alpha}|F^{\rho}_{\alpha}\rangle \ . \tag{2.17}$$

For a system consisting of a distinguishable particle [the partition $(\pi,3N)$ is denoted by the superfix π] scattering from a bound state of three identical particles comprising the trinucleon, Eq. (2.16) can be written as a coupled set of equations:

$$\begin{aligned}A^{\pi\pi}&=B^{\pi\pi}+B^{\pi\pi}t^{\pi}A^{\pi\pi}+\sum_{\sigma}\bar{\delta}_{\pi\sigma}B^{\pi\sigma}t^{\sigma}A^{\sigma\pi} \ ,\\ A^{\sigma\pi}&=B^{\sigma\pi}+B^{\sigma\pi}t^{\pi}A^{\pi\pi}+\sum_{\tau}\bar{\delta}_{\pi\tau}B^{\sigma\tau}t^{\tau}A^{\tau\pi} \ .\end{aligned} \tag{2.18}$$

By inserting the second of the above equations in the first recursively, we obtain the alternative set of equations (this is entirely equivalent to the Feshbach projector formalism),

$$A^{\pi\pi}=Z^{\pi\pi}+Z^{\pi\pi}t^{\pi}A^{\pi\pi} \ , \tag{2.19a}$$

with the driving term $Z^{\pi\pi}$ obtained from the subsidiary set of equations,

$$Z^{\rho\pi}=B^{\rho\pi}+\sum_{\sigma}\bar{\delta}_{\pi\sigma}B^{\rho\sigma}t^{\sigma}Z^{\sigma\pi} \ . \tag{2.19b}$$

The graphical representation of Eq. (2.19b) is given in Fig. 1. Equation (2.19a) is now a single-channel Lippmann-Schwinger equation whose driving term is obtained from the solution of Eq. (2.19b). Diagrams corresponding to pion absorption are seen to be legitimately included with the (2,2) channels in the driving terms and correspond to the nucleon pole term in the pion-nucleon propagator $t_{\pi N}$. Clearly, the approximation $Z^{\pi\pi}=B^{\pi\pi}$ gives the well-known single-nucleon contribution to the optical potential. Note that $B^{\pi\pi}$ obtained from Eq. (2.17) contains only the second term since the first term is not topologically possible. A single iteration of the subsidiary equation generates the quasideuteron contribution in addition to others terms. However, a numerical calculation need not be restricted to the single-nucleon and quasideuteron terms as it will become clear that the solution of the full set of equations presents a tractable numerical problem.

The authors of Ref. [11] have suggested a form for the three-body propagator which they describe as optimal to the extent that it satisfies full off-shell unitarity in the subspace spanned by the Faddeev wave-function component Ψ^{τ}_{β} and which represent the leading term in a rigorous separable expansion of the exact three-body amplitude $T^{\tau}_{\beta\alpha}$. This representation is given by Eq. (2.9), and depicted graphically in Fig. 2, with $t^{\tau}(z)$ taking the form

$$t^{\tau}(z)=\frac{1}{\Lambda^{-1}_{\tau}(z)-\sum_{\alpha}\langle F^{\tau}_{\alpha}|t_{\alpha}(z)|F^{\tau}_{\alpha}\rangle} \ , \tag{2.20}$$

with

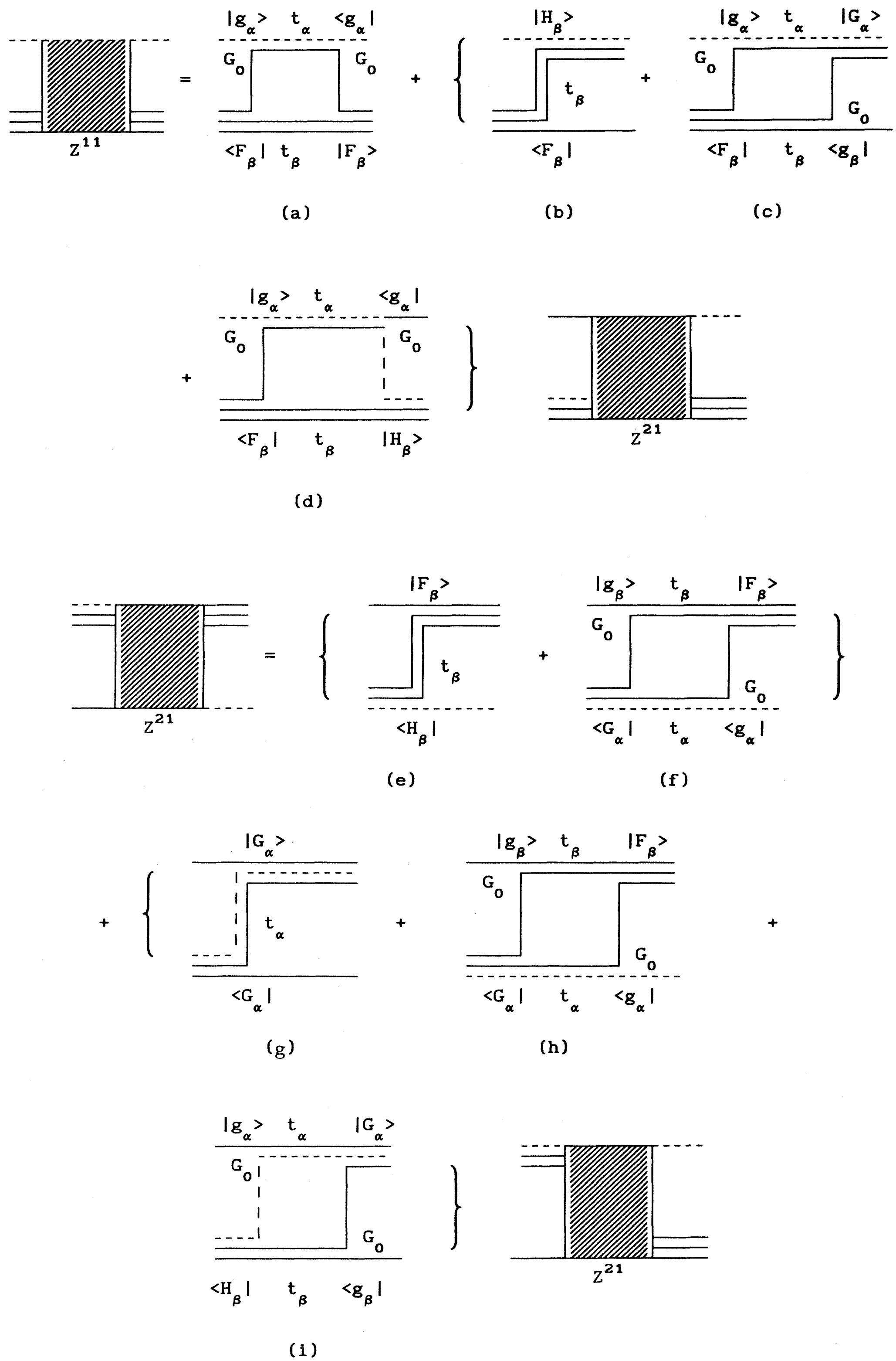

FIG. 1. Graphical representation of the auxiliary equations for the pion-trinucleon effective interaction. Vertex functions and propagators from Eqs. (2.19a) and (2.19b) which comprise the potentials are explicitly displayed. The dotted line is the pion and the solid lines are the nucleons. Vertices corresponding to the three-particle partitions (NN,N), (NN,π), and $(N\pi,N)$ are denoted by F_β, H_β, and G_α, respectively. The NN and $N\pi$ two-body vertices are denoted by g_β and g_α, respectively, and the corresponding propagators by t_β and t_α.

$$\Lambda_\tau(z)=\lambda_\tau^2\sum_{\alpha\gamma}\bar{\delta}_{\alpha\gamma}\langle F_\alpha^\tau|t_\alpha(E_\tau)\langle g_\alpha|G_0(z)|g_\gamma\rangle t_\gamma(E_\tau)|F_\gamma^\tau\rangle \qquad (2.21)$$

and

$$\lambda_\tau^{-1}=\sum_\alpha\langle F_\alpha^\tau|t_\alpha(E_\tau)|F_\alpha^\tau\rangle\ , \qquad (2.22)$$

where E_τ is the bound-state energy of the three-body subsystem characterized by the index τ. We again refer the reader to Ref. [11] for a discussion of the motivation for choosing the particular form of the three-body propagator given by Eq. (2.21) and for a detailed description of its structure.

Equations (2.19a), (2.19b), and (2.20)–(2.22) are the principle results of Ref. [11] which we now intend to apply to the problem of pion-trinucleon scattering. Although not all these equations are contained explicitly in Ref. [11], they are implicit in the formalism contained therein.

In the following section the equations will be reduced to a tractable form suitable for numerical solution.

III. REDUCTION OF THE EQUATIONS

Ignoring spin and isospin for the moment, it can be shown [11] that the contribution to the interaction represented by diagram (a) in Fig. 1 may be written explicitly in the form

$$\begin{aligned}\langle\mathbf{k}'|Z^{\pi\pi}(a)|\mathbf{k}\rangle=3\langle g_\alpha|\mathbf{k}'\rangle\langle\mathbf{k}|g_\alpha\rangle\int d\mathbf{q}\Bigg[&\frac{R_\beta t_\alpha(z-E_\beta-3q^2/4m)}{(z-E_\beta-3q^2/4M-E_{k'})(z-E_\beta-3q^2/4M-E_k)}\\&-\frac{1}{\pi}\int_0^\infty dx\frac{\mathrm{Im}t_\beta(x+i0)t_\alpha(z-x-3q^2/4M)}{(z-x-3q^2/4M-E_{k'})(z-x-3q^2/4M-E_k)}\Bigg]\\&\times\langle F_\beta|\mathbf{q}+\tfrac{2}{3}\mathbf{k}'\rangle\langle\mathbf{q}+\tfrac{2}{3}\mathbf{k}|F_\beta\rangle\ ,\end{aligned} \qquad (3.1)$$

where we have made the static limit approximation of setting the pion mass m equal to zero in the definition of the various Jacobi momenta for the four-body system. Analogous expressions hold for the other direct-box diagrams represented by Figs. 1(d) and 1(h).

In Eq. (3.1) $\mathbf{k}$ and $\mathbf{k}'$ are the initial and final relative momenta of the pion-trinucleon, respectively, and M is the mass of the nucleon. Here β refers to the N-N system and α to the π-N system. R_β is the residue of the two-nucleon propa-

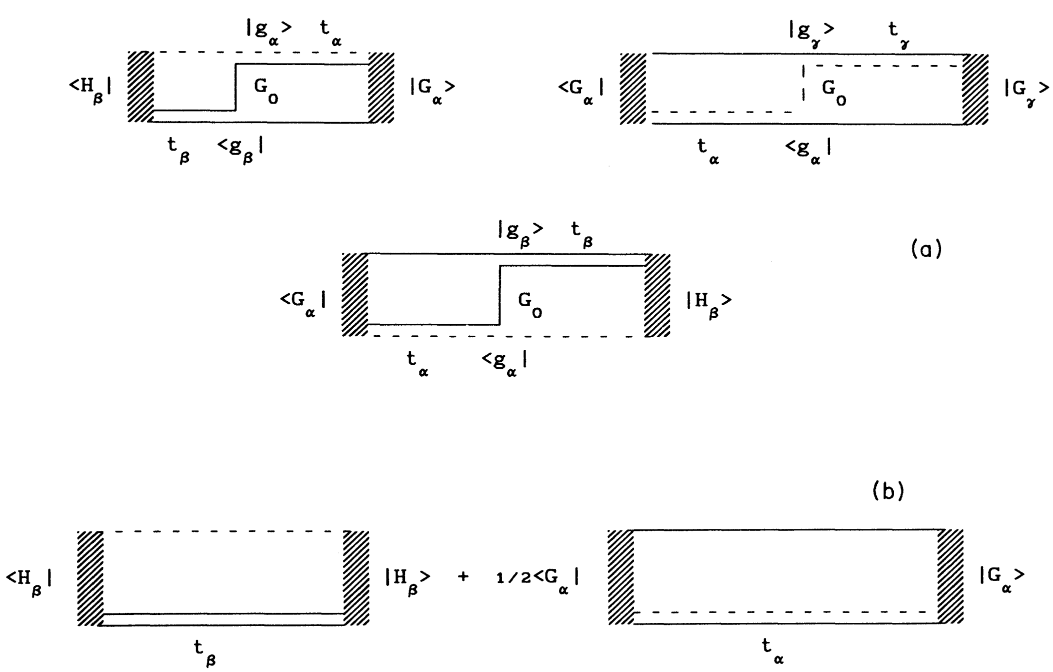

FIG. 2. (a) Graphical representation of the terms contributing to the energy-dependent pion-trinucleon interaction strength, $\Lambda_\tau(z)$. (b) Graphical representation of terms contributing to the energy-independent inverse potential strength, $1/\lambda_\tau$.

gator t_β at the bound-state pole which occurs at the two-body energy $E=E_B$. $\mathrm{Im}t_\beta$ is the imaginary part of the NN propagator evaluated along the scattering cut. $|F_\beta\rangle$ is any of the three β components of the trinucleon vertex function, $\langle g_\alpha|\mathbf{k}\rangle$ is the π-N vertex function, and E_k is the pion-trinucleon relative kinetic energy which is given by the semirelativistic expression

$$E_k=\omega_k+\frac{k^2}{2M} . \tag{3.2}$$

Clearly, for the definition of the energy E_k given by Eq. (3.2) the static limit $m=0$ has been suspended.

The first iteration of Eq. (2.19b), which is represented graphically by the product of diagrams (b) and (e) in Fig. 1 and which we shall call the quasideuteron term, has the momentum-space representation in the static limit (q.v.)

$$\begin{aligned}\langle\mathbf{k}'|Z^{\pi\pi}(b,e)|\mathbf{k}\rangle=3\langle\mathbf{k}'|H_\beta\rangle\langle H_\beta|\mathbf{k}\rangle \int d\mathbf{q}\langle F_\beta|\mathbf{q}+\tfrac{1}{3}\mathbf{k}'\rangle t_\beta(z-E_{k'}-3q^2/4M)\\ \times t^\tau(z-3q^2/4M)t_\beta(z-E_k-3q^2/4M)\langle\mathbf{q}+\tfrac{1}{3}\mathbf{k}|F_\beta\rangle ,\end{aligned} \tag{3.3}$$

where $|H_\beta\rangle$ is any of the β components of the (π,NN) vertex function obtained from the solution of Eq. (3.18) below, and t^τ is the three-body propagator corresponding to the πNN system.

From Eq. (2.19a) of the previous section we can see that the structure of the equations to be solved takes the Lippmann-Schwinger form

$$\langle\mathbf{k}'|A(z)|\mathbf{k}\rangle=\langle\mathbf{k}'|Z(z)|\mathbf{k}\rangle+\int d\mathbf{k}''\langle\mathbf{k}'|Z(z)|\mathbf{k}''\rangle\mathsf{T}_N(k''^2)\langle\mathbf{k}''|A(z)|\mathbf{k}\rangle . \tag{3.4}$$

The approximation of the driving term $Z(z)$ of Eq. (3.4) by the sum of Eqs. (3.1) and (3.3) represents a frequently used point of departure but in our formalism it is by no means a necessary one, though it is probably useful to make a calculation using this approximation in order to compare with similar approximations in other works [9]. In Eq. (3.4) $\mathsf{T}_N(k''^2)$ is the trinucleon propagator expressed by Eq. (2.20) and which, of course, corresponds to three-nucleon propagation. From Eqs. (2.20)–(2.22) we can write this propagator as

$$\mathsf{T}_N(k^2)=\left[\Lambda^{-1}(z-E_k)-3\int d\mathbf{q}\langle F_\beta|\mathbf{q}\rangle t_\beta(z-E_k-3q^2/4M)\langle\mathbf{q}|F_\beta\rangle\right]^{-1} , \tag{3.5}$$

with $\Lambda(z-E_k)$ given by

$$\begin{aligned}\Lambda(z-E_k)=6\lambda^2\int d\mathbf{q}\langle F_\beta|\mathbf{q}\rangle t_\beta(z-E_k-3q^2/4M)\int d\mathbf{q}' t_\beta(z-E_k-3q'^2/4M)\\ \times\frac{\langle g_\beta|\mathbf{q}'+\frac{1}{2}\mathbf{q}\rangle\langle\mathbf{q}+\frac{1}{2}\mathbf{q}'|g_\beta\rangle}{z-E_k-(\mathbf{q}'+\frac{1}{2}\mathbf{q})^2/M-3q^2/4M}\langle\mathbf{q}'|F_\beta\rangle .\end{aligned} \tag{3.6}$$

Finally, λ is given by the reciprocal of the integral term in Eq. (3.5) with k taking its on-shell value κ. Thus,

$$\lambda^{-1}=3\int d\mathbf{q}\langle F_\beta|\mathbf{q}\rangle t_\beta(-E_3-3q^2/4M)\langle\mathbf{q}|F_\beta\rangle , \tag{3.7}$$

where E_3 is the trinucleon binding energy obtained from the solution of Eq. (2.10). It should be clear from Eqs. (3.5) and (3.6) that, on the energy shell, where $k=\kappa$, $\Lambda(-E_3)=\lambda$, so that $\mathsf{T}_N(k^2)$ has a pole at this point.

To obtain a solution of the auxiliary Eq. (2.19b) one needs expressions for pole terms such as those represented by Figs. 1(b), 1(e), and 1(g), and crossed-box terms such as those represented by Figs. 1(c), 1(f), and 1(i). Again neglecting spin and isospin Fig. 1(b) can be expressed by

$$\langle\mathbf{k}'|Z_p^{\pi p}|\mathbf{k}\rangle=3\langle F_\beta|\mathbf{k}+\tfrac{1}{3}\mathbf{k}'\rangle t_\beta(z-(\mathbf{k}+\tfrac{1}{3}\mathbf{k}')^2/4M)\langle\mathbf{k}'|H_\beta\rangle . \tag{3.8}$$

The other pole diagrams have an analogous representation.

The crossed-box term depicted by Fig. 1(c) has the explicit representation

$$\begin{aligned}\langle\mathbf{k}'|Z_{cb}^{\pi p}|\mathbf{k}\rangle=3\langle g_\alpha|\mathbf{k}'\rangle\int d\mathbf{q}\langle F_\beta|\mathbf{q}+\mathbf{k}'/3\rangle\Bigg[\frac{R_\beta t_\alpha(z-E_\beta-3q^2/4M)}{(z-E_\beta-3q^2/4M-E_{k'})[E_\beta-(\mathbf{q}-\mathbf{k}/2)^2/M]}\\ \times\frac{t_\beta((\mathbf{q}-\mathbf{k}/2)^2-i0)t_\alpha(z-3q^2/4M-(\mathbf{q}-\mathbf{k}/2)^2)}{z-3q^2/4M-(\mathbf{q}-\mathbf{k}/2)^2/M-E_{k'}}\\ -\frac{1}{\pi}\int_0^\infty dx\frac{\mathrm{Im}t_\beta(x+i0)t_\alpha(z-3q^2/4M-x)}{(z-x-3q^2/4M-E_{k'})[x-(\mathbf{q}-\mathbf{k}/2)^2/M+i0]}\Bigg]\\ \times\langle\mathbf{k}-\mathbf{q}/2|g_\beta\rangle\langle\mathbf{q}+\mathbf{k}/2|G_\alpha\rangle\end{aligned} \tag{3.9}$$

and analogous representations for the other crossed-box diagrams, Figs. 1(f) and 1(i). In Eq. (3.9) $|G_\alpha\rangle$ is the $N\pi,N$ vertex function which is obtained from the solution of Eq. (3.18) below.

We can now introduce the spin and isospin of the nucleons by projecting Eq. (2.10) in the spin/isospin basis. Since, in this work, we consider S-wave N-N interactions only, the trinucleon form factor thereby obtained has two components, corresponding to two-body isospin $T=0$ and 1, respectively. These satisfy the well-known set of coupled equations,

$$F_T(q)=-(M/2)\sum_{T'}\int_0^\infty dq'q'^2K_{TT'}(q,q')F_{T'}(q')\ , \tag{3.10a}$$

with

$$K_{TT'}(q,q')=R_{TT'}t_{T'}(-E_3-3q'^2/4M)\int_{-1}^{1}dx\frac{f_T(p^2)f_{T'}(p'^2)}{ME_3+q^2+q'^2+qq'x}\ , \tag{3.10b}$$

and

$$p^2=q'^2+\tfrac{1}{4}q^2+qq'x\ ,\quad p'^2=q^2+\tfrac{1}{4}q'^2+qq'x\ , \tag{3.10c}$$

$$R_{TT'}=\begin{bmatrix}1 & -3\\ -3 & 1\end{bmatrix},\quad T,T'=0,1\ , \tag{3.10d}$$

where $f_T(p)$ is the NN vertex function corresponding to the isospin quantum number T and $t_T(E)$ is the propagator at the two-body relative energy E in the same channel.

Equation (3.4) takes the form, after partial-wave projection in states of definite angular momentum and isospin,

$$\langle k'\alpha'|A_\lambda^{ij}(z)|k\alpha\rangle=\langle k'\alpha'|Z_\lambda^{ij}(z)|k\alpha\rangle+\sum_{\alpha''}\int_0^\infty dk''k''^2\langle k'\alpha'|Z_\lambda^{ij}(z)|k''\alpha''\rangle\mathsf{T}_N^{\alpha''}(k''^2)\langle k''\alpha''|A_\lambda^{ij}(z)|k\alpha\rangle\ , \tag{3.4$'$}$$

where the total angular momentum and isospin of the pion-trinucleon system is denoted by j and i, respectively, and $\lambda=j\pm\frac{1}{2}$ is the conserved orbital angular momentum of the pion-trinucleon relative motion. The designation α means the total quantum numbers of the three-nucleon system, i.e., $\{\mathcal{J},\mathcal{T}\}$, so that $\alpha=\alpha'=\{\frac{1}{2},\frac{1}{2}\}$. As a result of our use of the single term expansion in Eq. (2.9) we also have $\alpha''=\{\frac{1}{2},\frac{1}{2}\}$ as the sole contribution to the sum in Eq. (3.4$'$), and with this understanding we will drop the specification $\{\alpha\}$ in what follows.

The expression for the trinucleon propagator T_N^{α} in the partial-wave basis is easily obtained from Eq. (3.5), viz.

$$\mathsf{T}_N(k^2)=\left[\Lambda^{-1}(z-E_k)-3\sum_T\int_0^\infty dqq^2F_T^2(q)t_T(z-E_k-3q^2/4M)\right]^{-1}\ , \tag{3.5$'$}$$

where we have dropped the α designation and F_T is obtained as the solution of the bound-state equation (3.8). $\Lambda(z-E_k)$ is defined by Eq. (3.6), and reduces to

$$\Lambda(z-E_k)=6\lambda^2\sum_{TT'}R_{TT'}\int_0^\infty dqq^2F_T(q)t_T(z-E_k-3q^2/4M)\int_0^\infty dq'q'^2t_{T'}(z-E_k-3q'^2/4M)$$
$$\times F_{T'}(q')\int_{-1}^{1}dx\frac{f_T(p^2)f_{T'}(p'^2)}{z-E_k-p^2/M-3q^2/4M}\ . \tag{3.6$'$}$$

The spin-isospin recoupling matrix $R_{TT'}$ is given by Eq. (3.10d) and the momenta p^2, p'^2 can be found from Eq. (3.10c).

It remains for us to express the effective interaction, $\langle k'\alpha'|Z(z)|k\alpha\rangle$ which appears in Eq. (3.4$'$), in a partial-wave basis. These terms are obtained as the solution of the auxiliary Eq. (2.19b) whose graphical representation is given in Fig. 1. The driving terms of these equations contain pole diagrams, Figs. 1(b), 1(e), and 1(g), contain direct-box diagrams, Figs. 1(a), 1(d), and 1(h) and crossed-box diagrams, Figs. 1(c), 1(f), and 1(i) examples of which are given in Eqs. (3.8), (3.1), and (3.9), respectively. A single iteration of the auxiliary equation yields the quasideuteron term given by Eq. (3.3) as described above. In this approximation the driving term for Eq. (2.19a) has two contributions, $Z(a)$ arising from the pion scattering from a single nucleon according to Fig. (1a) and $Z(b,e)$ which represents the contribution of the pion scattering from a correlated pair of nucleons, often referred to as the quasideuteron. In what follows we will give the partial-wave angular momentum and isospin reduction for these two terms because (i) they are representative of the complications contained in any other term and (ii) they form a convenient starting point for a systematic comparison of the numerical results of the present formalism with those of others [9].

The angular momentum/isospin reduction for both of these terms is quite similar. For $Z(a)$ this projection is given by the angular momentum/isospin sum

$$\langle k'|Z^{ij}_{\lambda}(a)|k\rangle$$
$$=\frac{3}{\hat{j}^2}\sum_{\{\mathcal{J}\}}\sum_{\{m\}}\langle \mathcal{J}'m_{\mathcal{J}'}\lambda'm_{\lambda'}|j'm_{j'}\rangle\langle l'm_{l'}\mathscr{s}'m_{\mathscr{s}'}|\mathscr{j}'m_{\mathscr{j}'}\rangle\langle J'm_{J'}\mathscr{j}'m_{\mathscr{j}'}|jm_j\rangle$$
$$\times\langle \Lambda'm_{\Lambda'}\mathscr{s}'m_{\mathscr{s}'}|J''M_{J''}\rangle\langle \Lambda m_{\Lambda}\mathscr{s}m_{\mathscr{s}}|J''m_{J''}\rangle\langle Jm_J\mathscr{j}m_{\mathscr{j}}|\mathcal{J}m_{\mathcal{J}}\rangle\langle lm_l\mathscr{s}m_{\mathscr{s}}|\mathscr{j}m_{\mathscr{j}}\rangle\langle \mathcal{J}m_{\mathcal{J}}\lambda m_{\lambda}|jm_j\rangle$$
$$\times\delta_{l0}\delta_{m_l0}\delta_{l'0}\delta_{m_l'0}\frac{1}{\hat{i}^2}\sum_{\{\mathcal{T}\}}\sum_{\{\mathcal{T}_z\}}\langle T'T'_zt't'_z|\mathcal{T}'\mathcal{T}'_z\rangle\langle \tau'\tau'_zt't'_z|T''T''_z\rangle\langle \tau'\tau'_z\mathcal{T}'\mathcal{T}'_z|ii_z\rangle$$
$$\times\langle TT_ztt_z|\mathcal{T}\mathcal{T}_z\rangle\langle \tau\tau_ztt_z|T''T''_z\rangle\langle \tau\tau_z\mathcal{T}\mathcal{T}_z|ii_z\rangle\delta_{\tau\tau'}\delta_{\tau_z\tau'_z}\delta_{\mathcal{T}\mathcal{T}'}\delta_{\mathcal{T}_z\mathcal{T}'_z}$$
$$\times\frac{1}{4\pi}\int d\hat{k}d\hat{k}'Y_{\lambda'm_{\lambda'}}(\hat{k}')Y^*_{\lambda m_{\lambda}}(\hat{k})\int_0^{\infty}dqq^2\int d\hat{q}F_{T'}(Q')F_T(Q)$$
$$\times\left[\frac{R_Tt^{\Lambda'}_{J''T''}(E_2-3q^2/4M)}{(z-E_2-3q^2/4M-E_{k'})(z-E_2-3q^2/4M-E_k)}-\frac{1}{\pi}\int_{-1}^{1}dx\frac{\mathrm{Im}t_T(x+i0)t^{\Lambda}_{J''T''}(z-x-3q^2/4M)}{(z-x-3q^2/4M-E_{k'})(z-x-3q^2/4M-E_k)}\right]$$
$$\times g^{\Lambda}_{J''T''}(k)g^{\Lambda'}_{J''T''}(k')Y^*_{\Lambda'm'_{\Lambda}}(\hat{k}')Y_{\Lambda m_{\Lambda}}(\hat{k})\;, \tag{3.11}$$

where $\langle a\alpha b\beta|g\gamma\rangle$ is the Clebsch-Gordan coefficient and the various angular momentum quantum numbers have the following definitions; $\mathcal{J}=\mathcal{J}'$ is the total angular momentum of the trinucleon, $\lambda=\lambda'$ is the orbital angular momentum of the pion relative to the trinucleon, $j=j'$ is the total angular momentum of the pion-trinucleon system, l and l' are the orbital angular momenta of the uncorrelated nucleon at the initial and final vertices, $\mathscr{s}=\mathscr{s}'$ is the spin of these nucleons, $\mathscr{j}$ and $\mathscr{j}'$ are the angular momenta resulting from the coupling of $l,\mathscr{s}$ and $l',\mathscr{s}'$. J and J' are the total angular momenta of the correlated nucleon pairs at the initial and final vertices, $\Lambda=\Lambda'$ is the orbital angular momentum of the π-N system, and J'' is the total angular momentum of this system. The isospin quantum numbers are defined analogously so that $\mathcal{T}=\mathcal{T}'$ is the isospin of the trinucleon, T and T' are the isospins of the correlated nucleon pair at the initial and final vertices, $t=t'$ is the nucleon isospin, $\tau=\tau'$ is the pion isospin, and the notation $\{\mathcal{J},m\}$ means summation over all nonconserved angular momenta and projections internal to the four-body system. The πN vertex functions are denoted by g^{Λ}_{JT} and the πN propagator by t^{Λ}_{JT}. The symbol $\hat{j}$ means

$$\hat{j}=\sqrt{2j+1}\;.$$

Integrating over the angular variables and performing the summations over the magnetic quantum numbers using methods such as those contained in Ref. [14] we arrive at the expression for the single-scattering contribution to the effective interaction:

$$\langle k'|Z^{ij}_{\lambda}(a)|k\rangle=-3\hat{\lambda}^2\sum_{\substack{JT\\J''T''}}\sum_{r\Lambda}\hat{r}^2\hat{J}''^2\hat{T}''^2\hat{\Lambda}^2\begin{pmatrix}\lambda & r & \Lambda\\ 0 & 0 & 0\end{pmatrix}^2\begin{Bmatrix}1 & T'' & \frac{1}{2}\\ T & \frac{1}{2} & i\end{Bmatrix}^2 g^{\Lambda}_{J''T''}(k')g^{\Lambda}_{J''T''}(k)(-)^{S+T''+i}\begin{Bmatrix}J & S & r & j\\ \Lambda & \Lambda & \frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \lambda & \frac{1}{2} & \lambda\end{Bmatrix}$$
$$\times\int_0^{\infty}dqq^2\left[\frac{R_Tt^{\Lambda}_{J''T''}(z-E_2-3q^2/4M)}{(z+E_2-3q^2/4M-E_k)(z+E_2-3q^2/4M-E_{k'})}-\frac{1}{\pi}\int_0^{\infty}dxt^{\Lambda}_{J''T''}(z-3q^2/4M-x)\times\frac{\mathrm{Im}t_T(x+i0)}{(z-x-3q^2/4M-E_{k'})(z-x-3q^2/4M-E_k)}\right]I^r_T(q,k')I^r_T(q,k)\;. \tag{3.12}$$

In Eq. (3.12) the 3-jm, 6-j, and 12-j coefficients are denoted by the symbols in parentheses and curly brackets, respectively. The symbol I^r_T is the integral

$$I^r_T(q,k)=\int_{-1}^{1}dxP_r(x)F_T(Q)\;, \tag{3.13}$$

with

$$Q^2=q^2+\tfrac{4}{9}k^2+\tfrac{4}{3}qkx\;. \tag{3.14}$$

In a similar way we arrive at the following expression for the quasideuteron contribution to the effective interaction:

$$\langle k'|Z_{\lambda}^{ij}(b,e)|k\rangle = -3\hat{\lambda}^2 \sum_{\{\alpha\}} \sum_{r\Lambda} \hat{r}^2 \hat{\mathcal{J}}^2 \hat{\mathcal{T}}^2 \hat{\Lambda}^2 \begin{bmatrix} \lambda & r & \Lambda \\ 0 & 0 & 0 \end{bmatrix}^2 \begin{Bmatrix} 1 & \mathcal{T} & T \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix} \begin{Bmatrix} 1 & \mathcal{T} & T' \\ \frac{1}{2} & \frac{1}{2} & i \end{Bmatrix} \begin{Bmatrix} \mathcal{J} & \frac{1}{2} & r & j \\ \Lambda & \Lambda & \frac{1}{2} & \frac{1}{2} \\ S' & \lambda & S & \lambda \end{Bmatrix} (-)^{i+1/2+S+S'+T+T'+\mathcal{T}}$$
$$\times H_\alpha(k') H_\alpha(k) \int_0^\infty dq q^2 t_{T'}(z-E_{k'}-3q^2/4M) t_T(z-E_k-3q^2/4M) \mathrm{T}_{\pi NN}^{\mathcal{J}\mathcal{T}}(z-3q^2/4M)$$
$$\times I_{T'}^r(q,k') I_T^r(q,k) , \qquad (3.15)$$

where the (π,NN) vertex function with the indicated quantum numbers is denoted by $H_\alpha(k)$, $\mathrm{T}_{\pi NN}^{\mathcal{J}\mathcal{T}}$ is the πNN propagator with angular momentum $\mathcal{J}$ and isospin $\mathcal{T}$, and α is the set of quantum numbers $[(LS)J,(\Lambda s)j,(Jj)\mathcal{J};(Tt)\mathcal{T}]$ of the (π,NN) system which can couple to $\mathcal{J},\mathcal{T}$. In this work we consider π,NN propagation only in the isobaric states $D^{\mathcal{J}\mathcal{T}}$ with $\mathcal{T}=0$, $\mathcal{J}=1$ (the deuteron channel) and $\mathcal{T}=1$, $\mathcal{J}=0$ (the singlet deuteron channel).

The constituents of the three-body πNN propagator in the angular momentum/isospin state $\mathcal{J}$, $\mathcal{T}$, can be obtained from Eqs. (2.19)–(2.21). To do this we first show how Eq. (2.10) for the πNN system may be decoupled into parts whose solution is antisymmetric with respect to exchange of the nucleons (taken to be particles 2 and 3) and a part whose solution is symmetric under this exchange. Explicitly these equations become

$$H = Z_{12'} t_{2'} G^{(-)} , \quad G^{(-)} = -Z_{2'3} t_3 G^{(-)} + 2Z_{2'1} t_1 H , \quad G^{(+)} = Z_{2'3} t_3 G^{(+)} , \qquad (3.16)$$

where 2′ means the state obtained from state 3 by exchange of the particles 2 and 3. The function H is the same as F_1 while $G^{(+)}$ and $G^{(-)}$ are defined by

$$G^{(\pm)} = F_{2'} \pm F_3 \qquad (3.17)$$

and the notation $Z_{\beta\alpha}$ means

$$Z_{\beta\alpha} = \langle g_\beta | G_0(E_\sigma) | g_\alpha \rangle \qquad (3.18)$$

with E_σ being the bound-state energy for the coupled set of equations in (3.16). The last uncoupled equation in this set has no physical significance and must be excluded in the following expressions for the πNN propagators.

The propagator of the πNN system for the state of total angular momentum $\mathcal{J}$ and isospin $\mathcal{T}$ is obtained from Eq. (2.20),

$$\mathrm{T}_{\pi NN}^{\mathcal{J}\mathcal{T}}(z) = [\Lambda_{\mathcal{J}\mathcal{T}}^{-1}(z) - L_{\mathcal{J}\mathcal{T}}(z)]^{-1} \qquad (3.19)$$

where now z means the three-body subenergy, and

$$L_{\mathcal{J}\mathcal{T}}(z) = \sum_\alpha \int_0^\infty dq q^2 [H_\alpha^2(q) t_T(z-q^2/4M-\omega_q) + \tfrac{1}{2} G_\alpha^2(q) t_{JT}^L(z-q^2/M)] \qquad (3.20)$$

and

$$\Lambda_{\mathcal{J}\mathcal{T}}(z) = \lambda_{\mathcal{J}\mathcal{T}}^2 \sum_\alpha \sum_{\alpha'} \int_0^\infty dq q^2 \int_0^\infty dq' q'^2 [H_\alpha(q) t_T(z-\omega_q-q^2/4M)_1 \langle fq\alpha | G_0(z) | g'q'\alpha' \rangle_2 t_{J'T'}^{L'}(z-q'^2/M) G_{\alpha'}(q')$$
$$-\tfrac{1}{2} G_\alpha(q) t_{JT}^L(z-q^2/M)_2 \langle gq\alpha | G_0(z) | fq'\alpha' \rangle_3 t_{J'T'}^{L'}(z-q'^2/M) G_{\alpha'}(q')$$
$$+ G_\alpha(q) t_{JT}^L(z-q^2/M)_2 \langle gq\alpha | G_0(z) | fq'\alpha' \rangle_1 t_{T'}(z-\omega_{q'}-q'^2/4M) H_{\alpha'}(q')],$$
$$(3.21)$$

where the $(N,\pi N)$ vertex function with the quantum numbers $\{\alpha\}$ is represented by function $G_\alpha(k)$ and $\lambda_{\mathcal{J}\mathcal{T}}$ is given by $L_{\mathcal{J}\mathcal{T}}(z)$ at the on-shell point, $z=E_\alpha$. In Eq. (3.20) the minus superscript of G has been dropped, as has the prime in the 2 subscript, since no ambiguity can exist.

The couplings involved in the effective four-body interaction strength $\Lambda_{\mathcal{J}\mathcal{T}}(z)$ are depicted graphically in Fig. 2. The recoupling coefficients ${}_i\langle fq\alpha | G_0(z) | f'q'\alpha' \rangle_j$ can be calculated using the methods of Ref. [15] for the summation over magnetic quantum numbers. The result for the general coefficient is given by

$$ {}_i\langle \mathscr{f} q\alpha|G_0(z)|\mathscr{f}'q'\alpha'\rangle_j=(-)^{2(J'+J+\mathcal{T})+\mathscr{s}+j'+J'+L'+L+t''+t'+S'-T} $$

$$ \times\tfrac{1}{2}\hat{\mathcal{J}}\hat{\mathcal{J}}'\hat{j}\hat{j}'\hat{S}\hat{S}'\hat{L}\hat{L}'\hat{t}\hat{t}'\hat{T}\hat{T}' \begin{Bmatrix} \mathcal{T} & t & T \\ t'' & t' & T' \end{Bmatrix} \sum_{\lambda\lambda' r} \hat{\Lambda}\hat{\Lambda}'\hat{r}^2(-)^r a^\lambda b^\Lambda c^{\lambda'} d^{\Lambda'} $$

$$ \times \begin{bmatrix} 2L+1 \\ 2\lambda \end{bmatrix}^{1/2} \begin{bmatrix} 2L'+1 \\ 2\lambda' \end{bmatrix}^{1/2} q^{\lambda+\lambda'} q'^{\Lambda+\Lambda'} \sum_{ff'f''} \hat{f}^2\hat{f}'^2\hat{f}''^2 \begin{bmatrix} l & r & f \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} l' & r & f' \\ 0 & 0 & 0 \end{bmatrix} $$

$$ \times \begin{bmatrix} \lambda & \lambda' & f \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Lambda & \Lambda' & f' \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} j' & & J & \mathscr{s}' & & S & l' & & L \\ & \mathcal{J} & & & \mathscr{s}'' & & & f'' & \\ J' & & j & S' & & \mathscr{s} & L' & & l \end{Bmatrix} \begin{Bmatrix} f'' & L' & \lambda' & f \\ l' & \Lambda' & \lambda & l \\ r & f' & \Lambda & L \end{Bmatrix} $$

$$ \times \int_{-1}^{1} dx \frac{P_r(x)\mathscr{f}_\alpha(p)\mathscr{f}'_{\alpha'}(p')}{p^L p'^{L'}[z-E(p)-E(q)]} , \tag{3.22} $$

with

$$ \mathbf{p}=a\mathbf{q}+b\mathbf{q}' , \quad \mathbf{p}'=c\mathbf{q}+d\mathbf{q}' , \tag{3.23} $$

and where $\mathscr{f}$, $\mathscr{f}'$ can be a NN vertex f_T, or a πN vertex g_{JT}^L. In practice, only the cases (a) $i=2$, $j=3$, $\mathscr{f}=g_{JT}^L$, $\mathscr{f}'=g_{J'T'}^{L'}$ (b) $i=2$, $j=1$, $\mathscr{f}=g_{JT}^L$, $\mathscr{f}'=f_{T'}$, (c) $i=1$, $j=2$, $\mathscr{f}=f_T$, $\mathscr{f}'=g_{J'T'}^{L'}$ can arise. In the static limit we have for case (a) $a=b=-1$, $c=d=1$, for case (b) $a=0$, $b=1$, $c=1$, $d=\frac{1}{2}$ and for case (c) $a=\frac{1}{2}$, $b=1$, $c=-1$, $d=0$. These assignments can be inferred from an examination of the diagrams displayed in Fig. 2(a). The expressions for the intermediate free energies $E(p)$ and $E(q)$ can also be found by referring to Fig. 2 for the three cases (a), (b), and (c), viz.

$$ E(p)=\omega_p+p^2/2M , \quad E(q)=q^2/M \ [\text{case (a)}] , $$

$$ E(p)=\omega_p+p^2/2M , \quad E(q)=q^2/M \ [\text{case (b)}] , $$

$$ E(p)=p^2/M , \quad E(q)=\omega_q+q^2/4M \ [\text{case (c)}] . $$

The binomial coefficients appearing in Eq. (3.19) are defined by

$$ \begin{bmatrix} 2L+1 \\ 2\lambda \end{bmatrix} = \frac{\hat{L}^2!}{\hat{\Lambda}^2!(2\lambda)!} , \quad \Lambda=L-\lambda . \tag{3.24} $$

The 12-j and 15-j symbols appearing in Eq. (3.19) may be expressed as sums of products of the more readily available 6-j and 9-j symbols [15] by means of the following expressions:

$$ \begin{Bmatrix} j_1 & j_2 & j_3 & j_4 \\ l_1 & l_2 & l_3 & l_4 \\ k_1 & k_2 & k_3 & k_4 \end{Bmatrix} = \sum_x \hat{x}^2(-)^{l_1-l_2-l_3+l_4} \begin{Bmatrix} k_2 & k_1 & x \\ j_1 & j_3 & l_1 \end{Bmatrix} \begin{Bmatrix} j_3 & j_1 & x \\ k_3 & k_4 & l_2 \end{Bmatrix} \begin{Bmatrix} k_2 & k_1 & x \\ j_2 & j_4 & l_3 \end{Bmatrix} \begin{Bmatrix} k_3 & k_4 & x \\ j_4 & j_2 & l_4 \end{Bmatrix} , \tag{3.25} $$

$$ \begin{Bmatrix} k_1 & & k_1' & k & & k' & k_2 & & k_2' \\ & p_1 & & & p & & & p_2 & \\ j_1 & & j_1' & j & & j' & j_2 & & j_2' \end{Bmatrix} = \sum_x \hat{x}^2(-)^{x+p-j-k'} \begin{Bmatrix} k & j & x \\ j' & k' & p \end{Bmatrix} \begin{Bmatrix} k & j & x \\ k_1 & j_1 & p_1 \\ k_2 & j_2 & p_2 \end{Bmatrix} \begin{Bmatrix} k' & j' & x \\ k_1' & j_1' & p_1 \\ k_2' & j_2' & p_2 \end{Bmatrix} . \tag{3.26} $$

The (NN,π) system has only one bound state with $\mathcal{J}=1$, $\mathcal{T}=0$, the deuteron. We will approximate the antibound state with quantum numbers $\mathcal{J}=0$, $\mathcal{T}=1$ by a bound state with zero binding energy. The various configurations H_α and G_α contributing to each of these three-body states are given in Table I. From this table we see that the deuteron D^{01} is obtained from the solution of a coupled set of five equations and the singlet deuteron D^{10} is obtained from the solution of a coupled set of four equations. These equations are the symmetrized partial-wave projections of Eq. (2.10) and may be written in the form [16]

$$ H_\alpha(q)=\sum_{\alpha'} \int_0^\infty dq' q'^2 {}_1\langle fq\alpha|G_0(E_\alpha)|gq'\alpha'\rangle_2 t_{\pi N}^{\alpha'}(E_\alpha-q'^2/M)G_{\alpha'}(q') , $$

$$ G_\alpha(q)=\sum_{\alpha'} \int_0^\infty dq' q'^2 [{}_2\langle gq\alpha|G_0(E_\alpha)|gq'\alpha'\rangle_3 t_{\pi N}^{\alpha'}(E_\alpha-q'^2/M)G_{\alpha'}(q') $$

$$ +2{}_2\langle gq\alpha|G_0(E_\alpha)|fq'\alpha'\rangle_1 t_{NN}^{\alpha'}(E_\alpha-\omega_{q'}-q'^2/4M)H_{\alpha'}(q')] , \tag{3.27} $$

TABLE I. NN,π (H_α) and $N\pi,N$ (G_α) configurations contributing to (a) the deuteron (D^{01}) three-body bound states and (b) the antibound (D^{10}) three-body state.

α	$(L,S)J$	$(\ell,s)j$	$(J,j)\mathcal{J}$	$(T,t)\mathcal{T}$
		(a)		
H_α	(0,0)0	(1,0)1	(0,1)1	(1,1)0
G_α	$(0,\frac{1}{2})\frac{1}{2}$	$(1,\frac{1}{2})\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2})1$	$(\frac{1}{2},\frac{1}{2}0$
	$(0,\frac{1}{2})\frac{1}{2}$	$(1,\frac{1}{2})\frac{3}{2}$	$(\frac{1}{2},\frac{3}{2})1$	$(\frac{1}{2},\frac{1}{2})0$
	$(1,\frac{1}{2})\frac{1}{2}$	$(0,\frac{1}{2})\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2})1$	$(\frac{1}{2},\frac{1}{2})0$
	$(1,\frac{1}{2})\frac{3}{2}$	$(0,\frac{1}{2})\frac{1}{2}$	$(\frac{3}{2},\frac{1}{2})1$	$(\frac{1}{2},\frac{1}{2})0$
		(b)		
H_α	(0.1)1	(1,0)1	(1,1)0	(0,1)1
G_α	$(0,\frac{1}{2})\frac{1}{2}$	$(1,\frac{1}{2})\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2})0$	$(\frac{3}{2},\frac{1}{2})1$
	$(1,\frac{1}{2})\frac{1}{2}$	$(0,\frac{1}{2})\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2})0$	$(\frac{3}{2},\frac{1}{2})1$
	$(1,\frac{1}{2})\frac{3}{2}$	$(2,\frac{1}{2})\frac{3}{2}$	$(\frac{3}{2},\frac{3}{2})0$	$(\frac{3}{2},\frac{1}{2})1$

where E_α is the bound-state energy in the channel defined by the quantum numbers $\mathcal{J}$ and $\mathcal{T}$, i.e., for $(\mathcal{J}=1,\mathcal{T}=0)E_\alpha=-2.225$ MeV and for $(\mathcal{J}=0,\mathcal{T}=1)E_\alpha=0$. The recoupling coefficients ${}_i\langle q\alpha|G_0(-E_\alpha)|g'q'\alpha'\rangle_j$ are given by Eq. (3.19) with z replaced by $-E_\alpha$ and α defined by the set of quantum numbers $\{(LS)J,(ls)j,(Jj)\mathcal{J}m:(Tt)\mathcal{T}\mathcal{T}_z\}$.

IV. CONCLUSION

In the foregoing sections we have outlined in some detail a formalism for the calculation of the low-energy pion-trinucleon scattering problem. The methods are essentially the extension to the four-body domain of those used by Afnan and Thomas [5] for the solution of the low-energy pion-deuteron scattering problem. The straightforward extension of the approach of Afnan and Thomas to the four-body sector has been considerably facilitated by our adoption of the device of Haberzettl and Sandhas (HS) whereby one obtains effective two-body equations from the exact four-body formalism in which the $(2+2)$ subamplitudes are not required in a separable form but are incorporated in the effective potentials exactly. Thus one needs only the $(3+1)$ subamplitudes in separable form and for these we exploit the suggestion of HS for the representation of the propagators of the associated three-body isobars. In order to simplify our proposed numerical solution of the pion-trinucleon scattering problem we have adopted a separable representation of all two-body t matrices and confine ourselves to S-wave NN interactions and S- and P-wave pion-nucleon interactions. This is probably not an unreasonable approach at low energies and moreover is a restriction that can be systematically eased if the necessity for doing so should arise. For instance, this can be accomplished by writing unitary pole expansions for both the three-body and the two-body t matrices in place of Eqs. (2.9) and (2.4), respectively.

The HS formalism results in a coupled set of effective two-body equations the potentials and propagators of which are expressed in terms of the two-body vertex functions used for the representation of the two-body interactions and also three-body vertices of the systems (NN,N), (NN,π), and $(N\pi,N)$ which must be obtained from the solution of coupled sets of bound-state equations. These are by no means new and in the case of the (NN,N) system are familiar since the seminal paper of Lovelace [7]. The equations in the partial-wave projection have been included here to elucidate the connection to the present problem. With the limitations we have placed on the two-body partial waves we find that the deuteron sector of the πNN problem requires the solution of a coupled set of five equations and the singlet deuteron requires the solution of a coupled set of four equations. This task does not present an insuperable obstacle with the computing facilities available today.

In our representation of the effective potentials extracted from the HS formalism we have obtained a single-channel Lippmann-Schwinger equation with an effective interaction which is obtained as the solution of a subsidiary set of integral equations which contain all the complexities of the NN as well as the πN interaction. A low-order approximation to this effective optical potential results from a single iteration of the auxiliary equations and contains two terms one of which corresponds to the pion scattering from a single nucleon and another corresponding to pion scattering from a correlated pair within the trinucleon, easily identifiable as the quasideuteron mechanism. The remaining terms involve higher orders of multiple scattering and will be included in a future calculation. In addition, we do not consider intermediate trinucleon propagation in states other than $\mathcal{J}=\mathcal{T}=\frac{1}{2}$. This does not mean that we neglect intermediate nuclear excitation entirely, merely that we are restricting the intermediate states to these quantum numbers. This latter approximation can be eased at the cost of a considerable increase in the algebraic complexity of the final equations and a concomitant increase in the computer time and storage required for accurate solutions.

We believe that the present approach may represent an improvement over the application of the optical-potential treatment of multiple-scattering theories to pion-trinucleon scattering to the extent that the exact dynamics are treated more systematically, that it is subject to systematic improvement, and is perhaps even more tractable numerically. The advantages of the present approach can be seen from Eq. (3.4) which is an equation of the Lippman-Schwinger type. The dynamical input is embodied in the effective potentials defined according to equations such as (3.1) and (3.3) and the propagator defined by Eq. (3.5) preserves the exact four-body dynamics. The failure to incorporate the latter requirement is a serious shortcoming of the optical-potential approach but which is perhaps unavoidable for nuclei with $A>4$. The implementation of the program for solution of the πA scattering problem based on multiple scattering and embodied in Eqs. (1.3)–(1.5) plus the solution of the Lippmann-Schwinger equation driven by $V(z)$ is possible in principle, though, in practice, it has been necessary to make physically reasonable approximations [17] such as the neglect of intermediate nuclear excitation. This latter effect is well accounted for in the region of the Δ resonance by other formulations such as the Δ-hole model [18].

Finally, it should be noted that it is possible in principle to extend the present treatment to higher mass nuclei, but the advantages mentioned above for this case appear to be more problematic.

We are applying the formalism described above to the problem of pion elastic scattering from the bound trinuclei in the energy region below $E_\pi = 400$ MeV, and the results of this study will appear in a future paper.

ACKNOWLEDGMENTS

I would like to express my indebtedness to Dr. Helmut Haberzettl whose insight into the many-body problem provided the inspiration for the present work. This research was supported in part by the U.S. Department of Energy Grant No. DE-FG05-86-ER40270.

[1] K. M. Watson, Phys. Rev. **89**, 575 (1953).
[2] A. K. Kerman, H. McManus, and R. M Thaler, Ann. Phys. (N.Y.) **8**, 551 (1959).
[3] R. H. Landau, Ann. Phys. (N.Y.) **205**, 92 (1975).
[4] A. W. Thomas and R. H. Landau, Phys. Rep. **58**, 123 (1980).
[5] I. R. Afnan and A. W. Thomas, Phys. Rev. C **10**, 109 (1974).
[6] L. D. Faddeev, Zh. Eksp. Teor. Fiz. **39**, 1459 (1960) [Sov. Phys. JETP **12**, 1014 (1961)].
[7] C. Lovelace, Phys. Rev. **135**, B1225 (1964).
[8] L. L. Foldy and J. D. Walecka, Ann. Phys. (N.Y.) **54**, 447 (1969); W. R. Gibbs, Phys. Rev. C **3**, 1127 (1971); H. N. Pendelton, Phys. Rev. **131**, 1833 (1963); F. Myhrer and D. S. Koltun, Phys. Lett. B **46**, 32 (1973), Nucl. Phys. B **86**, 441 (1975); R. M. Woloshyn, E. J. Moniz, and R. Aaron, Phys. Rev. C **13**, 286 (1976).
[9] F. M. M. Van Geffen, B. L. G. Bakker, H. J. Boersma, and R. Van Wageningen, Nucl. Phys. A **468**, 683 (1987); I. V. Falomkin *et al.*, Nuovo Cimento A **57**, 111 (1980).
[10] P. C. Tandy, E. F. Redish, and D. Bolle, Phys. Rev. C **16**, 1924 (1977); R. H. Landau and A. W. Thomas, Phys. Lett. B **41**, 361 (1976); Nucl. Phys. A **302**, 461 (1978).
[11] H. Haberzettl and W. Sandhas, Phys. Rev. C **24**, 359 (1981).
[12] I. R. Afnan and A. W. Thomas, in *Modern Three Hadron Physics,* edited by A. W. Thomas (Springer, New York, 1977), Vol. 1.
[13] P. Grassberger and W. Sandhas, Nucl. Phys. B **2**, 181 (1967).
[14] E. P. Harper, Y. E. Kim and A. Tubis, Phys. Rev. C **2**, 877 (1970).
[15] A. P. Yutsis, I. B. Levinson, and V. V. Vanagas, *Theory of Angular Momentum* (Israel Program for Scientific Translations, Jerusalem, 1962); D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum* (World Scientific, Singapore, 1989).
[16] H. Garcilazo and T. Mitzutani, *πNN Systems* (World Scientific, Singapore, 1990).
[17] E. Siciliano and G. E. Walker, Phys. Rev. C **13**, 257 (1970); M. Wakamatsu, Nucl. Phys. A **312**, 427 (1978).
[18] F. Lenz, Ann. Phys. (N.Y.) **95**, 348 (1975); G. E. Brown and W. Weise, Phys. Rep. C **22**, 279 (1975).