Three-body problem with charged particles

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(Received 1 June 1990)

The dynamical treatment of the nonrelativistic three-body problem with Coulomb and short-range interactions is reexamined. The screening technique is abandoned in favor of the general two-potential formalism and a set of well-behaved three-body equations is derived. An approximate Alt-Grassberger-Sandhas-type form of the new equations is also proposed that treats the long-range polarizability forces to first order and preserves post-prior symmetry.

I. INTRODUCTION

Due to the increasing amount of information on both bound and scattering states of the p-d system, lately there has been an increased interest in the nuclear three-body Coulomb problem.[1-3] In the present work we reexamine the dynamical treatment of the three-body problem with charged particles and propose a new approximate form of three-body equations with certain advantageous properties that may make their numerical applications practical.

There exists extensive literature[4,5] on the subject of Coulomb scattering so we restrict ourselves only to a brief summary of the most important developments.

In developing a consistent scattering theory for a nonrelativistic quantum system characterized by a self-adjoint Hamiltonian, one normally starts with the time-dependent approach and establishes the existence of wave operators as well as the S matrix for the system. While these and other properties, such as asymptotic completeness, are of basic importance for a physically satisfactory theory, for the application to real processes one has to construct either the scattering operator of the system or equivalently the resolvent of the total Hamiltonian. Thus a transition to the stationary approach has to be made and proper dynamical equations have to be derived which yield the required quantities in a unique way.[6]

For short-range interactions the stationary approach has been widely used for various applications. However, a full dynamical treatment of a scattering problem is by no means trivial especially if the system has a multichannel structure.[6] The rigorous mathematical theory of the nonrelativistic three-body problem with short-range interactions was developed by Faddeev[7] in terms of exact integral equations. The three-body integral equations were later reformulated by Alt, Grassberger, and Sandhas[8] (AGS) so that they yield immediately the physically relevant transition operators. If the short-range interactions are represented by nonlocal separable expansions, the dimension of the three-body AGS equations is reduced, and a set of multichannel Lippmann-Schwinger-type equations is obtained whose numerical solution is feasible for both bound and scattering states of the system.

The situation is greatly complicated, however, when two or more of the particles are charged. In the presence of long-range Coulomb interactions the usual scattering boundary conditions do not hold so that conventional scattering theory for short-range interactions has to be modified. This is reflected in the Faddeev formalism in that the kernel of the Faddeev or AGS equations becomes noncompact if Coulomb interactions are present.

The development of a time-dependent scattering theory for systems with Coulomb interactions has been pioneered by Dollard,[9] who introduced an asymptotic boundary condition in which the free motion of the particles is modified to more closely approximate the hyperbolic orbits of the classical trajectories. With this new asymptotic boundary condition Dollard could develop a physically and mathematically consistent time-dependent scattering theory for charged-particle scattering. Another interesting, but equivalent, formulation was

derived for two particles by Mulherin and Zinnes[10] and extended to N particles by Chandler and Gibson.[11] Yet another approach in terms of asymptotic observables was developed by Amrein, Martin, and Misra.[12]

The stationary approach to Coulomb scattering has been investigated by Chandler and Gibson,[13] by Zorbas,[14] and more recently by Merkuriev,[15–17] Enns,[18] and Sasakawa and Sawada.[19] However, even though some important steps have been taken in the direction of solving the pure Coulomb problem, at the moment there is no direct practical stationary approach to the few-body Coulomb problem.

Another way of attacking the Coulomb problem is to screen the Coulomb interaction so that the normal short-range theory can be applied and then to seek a limiting procedure which leads to physically meaningful quantities in the zero screening limit. The renormalization technique developed by Gorshkov[20] works in the two-body problem and can also be used to show that general two-cluster to two-cluster transition operators can be defined this way.[21] The screening technique was used by Prugovečki and Zorbas[22] to derive modified Lippmann-Schwinger equations for Coulomb-like potentials. The three-body problem has been studied by Veselova[23] who investigated the Faddeev equations with screened Coulomb interactions. After isolating the two-body terms which generate the well-known Coulomb singularities in the zero screening limit, the corresponding part of the Faddeev kernel can be explicitly inverted. This method results in a set of three-body equations whose kernel is well behaved, at least below the breakup threshold.

The first formally exact approach to the nuclear three-body problem with Coulomb interactions was proposed by Noble.[24] He included the Coulomb interactions in the "free" Green's function and derived integral equations for modified three-body operators from which the pure Coulomb scattering has been removed. These integral equations are mathematically well behaved and in the absence of Coulomb interactions reduce to the standard three-body equations. Noble's method is, in fact, a two-potential formalism and for the complete solution it requires the knowledge of the solution to the pure Coulomb problem. If only two of the particles are charged, this solution is known explicitly so that Noble's method yields the exact solution.

For the general case of three charged particles Noble's equations contain unpleasant three-body operators which arise from the solution of the pure Coulomb problem. However, as was shown by Bencze,[25] if one profits from the experience gained in distorted-wave-Born-approximation (DWBA) nuclear reaction calculations and introduces the channel-distortion approximation (CDA), a manageable set of AGS-type equations can be derived for the "distorted-wave" transition operators. It turns out that the CDA is in fact equivalent to neglecting the long-range electric polarization forces, which in various transfer reactions play only a minor role.

So far the most successful practical integral equation approach based on the use of the screening technique has been developed by Alt, Sandhas, and Ziegelmann[26] (ASZ) and further generalized by Alt and Sandhas.[27] It relies heavily on the widely used quasiparticle approach, so that the three-body equations are transformed into a set of multichannel two-body equations for finite values of the screening radius. The zero screening limit is obtained by the renormalization procedure.[20]

It is important to emphasize that in practical applications all the approaches to the nuclear three-body Coulomb problem discussed above, with the exception of the case of only two charged particles, are essentially of approximate nature. Even if formally exact integral equations are obtained in a mathematically sound way, the kernel and driving terms can be constructed only if the solution to the pure Coulomb problem is known.

The present study has been initiated by some nagging theoretical questions connected with the screening technique. Also, in the practical applications of the ASZ method the limit of zero screening is taken numerically, thus raising the question of whether the observed convergence is a real one. Moreover, the numerical calculations for the proton-deuteron scattering have (so far) always involved the quasi-Born approximation of the Coulomb t matrix, which has been a source of uneasiness among practitioners.[28]

In the present work we propose an approach for systems with repulsive Coulomb interactions, which avoids the use of screening and thus involves no numerical zero screening limits. By making use of the most general form of two-potential formalism, a set of three-body integral equations is derived, which reduce to the AGS form in the absence of Coulomb interactions. In addition, an approximation scheme is proposed that treats the long-range polarization forces to first order in a consistent way. As a result, the unpleasant post-prior asymmetry inherent in the channel distortion approximation (CDA) is removed.

The organization of the paper is as follows. In Sec. II, for the sake of comparison, the screening technique is briefly described. The general two-potential formalism and the derivation of a new set of modified three-body integral equations is found in Sec. III. In Sec. IV an approximation is proposed for the effective interactions appearing in the three-body equations. Finally, the conclusions are summarized in Sec. V.

II. THE SCREENING APPROACH

In order to facilitate the comparison of our approach with that based on the use of screening, we present here a brief summary of the most important steps involved in the screening technique, following closely the treatment of Alt and Sandhas.[27] In the present section it will be assumed that the Coulomb interactions are screened and that the normal Faddeev or AGS procedures can be used.[7,8]

Let us consider a three-body system with short-range nuclear and screened Coulomb interactions. The Hamiltonian can be written as

$$H^R = H_0 + V + U^R, \tag{2.1}$$

where H_0 denotes the kinetic-energy operator, R the screening radius, and V and U^R the sums of short-range and screened Coulomb interactions, respectively. Thus,

$$V = \sum_\alpha V_\alpha, \qquad U^R = \sum_\alpha U_\alpha^R, \tag{2.2}$$

where the subscript α refers to the particle that is not involved in the (pair) interaction. In addition, the following assortment of resolvent operators can be introduced:

$$G^R(z) = (z - H^R)^{-1}, \tag{2.3}$$

$$G_0(z) = (z - H_0)^{-1}, \tag{2.4}$$

$$G_c^R(z) = (z - H_0 - U^R)^{-1}, \tag{2.5}$$

$$G_\alpha^R(z) = (z - H_0 - V_\alpha - U_\alpha^R)^{-1}, \tag{2.6}$$

$$G_{\alpha c}^R(z) = (z - H_0 - U_\alpha^R)^{-1}. \tag{2.7}$$

The transition operators $U_{\beta\alpha}$ are defined in the usual way,

$$U_{\beta\alpha}^R \equiv G_\beta^{R\,-1}(G^R - \delta_{\beta\alpha}G_\alpha^R)G_\alpha^{R\,-1}, \tag{2.8}$$

and the AGS equations that determine them are the usual ones,[8]

$$U_{\beta\alpha}^R = \bar{\delta}_{\beta\alpha}G_0^{-1} + \sum_\gamma \bar{\delta}_{\beta\gamma}t_\gamma^R G_0 U_{\gamma\alpha}^R, \tag{2.9}$$

with $\bar{\delta}_{\beta\alpha} \equiv 1 - \delta_{\beta\alpha}$.

The two-body transition operators t_γ^R are split up into the sum of the pure Coulomb t operator $t_\gamma^{c,R}$ and the Coulomb-modified short-range part $t_\gamma^{sc,R}$,

$$t_\gamma^R = t_\gamma^{c,R} + (1 + U_\gamma^R G_{\gamma c}^R)t_\gamma^{sc,R}(1 + G_{\gamma c}^R U_\gamma^R). \tag{2.10}$$

Here $t_\gamma^{c,R}$ and $t_\gamma^{sc,R}$ satisfy the following Lippmann-Schwinger equations:

$$t_\gamma^{c,R} = U_\gamma^R + U_\gamma^R G_0 t_\gamma^{c,R}, \tag{2.11}$$

$$t_\gamma^{sc,R} = V_\gamma + V_\gamma G_{\gamma c}^R t_\gamma^{sc,R}. \tag{2.12}$$

After substituting Eq. (2.10) into Eq. (2.9), the part of the kernel that contains the pure Coulomb t operator can be explicitly inverted, yielding the following set of equations:

$$U_{\beta\alpha}^R = U_{\beta\alpha}^{c,R} + \sum_\gamma U_{\beta\gamma}^{c,R} G_{\gamma c}^R t_\gamma^{sc,R} G_{\gamma c}^R U_{\gamma\alpha}^R, \tag{2.13}$$

where

$$U_{\beta\alpha}^{c,R} \equiv G_{\beta c}^{R\,-1}(G_c^R - \delta_{\beta\alpha}G_{\alpha c}^R)G_{\alpha c}^{R\,-1}. \tag{2.14}$$

It is clear that the difficulties are buried in the pure Coulomb quantities $U_{\beta\gamma}^{c,R}$ which normally diverge in the zero screening limit. However, by means of the renormalization technique the divergent phases can be removed and a physically meaningful limit can be shown to exist.[27] All the other operators appearing in Eq. (2.13) have well-defined values in the zero screening limit.

Equation (2.13) can be conveniently rewritten in a matrix Lippmann-Schwinger form,

$$T_{\beta\alpha}^R = K_{\beta\alpha}^R + \sum_\gamma K_{\beta\gamma}^R t_\gamma^{sc,R} T_{\gamma\alpha}^R, \tag{2.15}$$

where

$$T_{\beta\alpha}^R = G_{\beta c}^R U_{\beta\alpha}^R G_{\alpha c}^R, \tag{2.16}$$

$$K_{\beta\alpha}^R = G_{\beta c}^R U_{\beta\alpha}^{c,R} G_{\alpha c}^R. \tag{2.17}$$

Let us now assume that the short-range interactions can be represented by a separable nonlocal expansion. If for the sake of simplicity only one term is taken in each channel, one can write

$$V_\gamma = |\chi_\gamma\rangle\lambda_\gamma\langle\chi_\gamma|, \tag{2.18}$$

and, as a consequence of Eq. (2.12), also

$$t_\gamma^{sc,R} = |\chi_\gamma\rangle\tau_\gamma^R(z)\langle\chi_\gamma|, \tag{2.19}$$

$$\tau_\gamma^R(z) \equiv [\lambda^{-1} - \langle\chi_\gamma|G_{\gamma c}^R(z)|\chi_\gamma\rangle]^{-1}. \tag{2.20}$$

By substituting Eq. (2.19) into Eq. (2.15), the matrix equations for the transition operators are transformed into a set of multichannel effective two-body equations of Lippmann-Schwinger form

$$\mathcal{T}_{\beta\alpha}^R = \mathcal{V}_{\beta\alpha}^R + \sum_\gamma \mathcal{V}_{\beta\gamma}^R \tau_\gamma^R(z)\mathcal{T}_{\gamma\alpha}^R, \tag{2.21}$$

where

$$\mathcal{T}_{\beta\alpha}^R \equiv \langle\chi_\beta|T_{\beta\alpha}^R|\chi_\alpha\rangle, \quad \mathcal{V}_{\beta\alpha}^R \equiv \langle\chi_\beta|K_{\beta\alpha}^R|\chi_\alpha\rangle. \tag{2.22}$$

It can be shown[27] that the physical amplitude of a rearrangement process is given by the on-shell value of the matrix element of the corresponding transition operator,

$$\mathcal{T}_{\beta\alpha}^R(\mathbf{q}_\beta, \mathbf{q}_\alpha) \equiv \langle\mathbf{q}_\beta|T_{\beta\alpha}^R(E + i0)|\mathbf{q}_\alpha\rangle. \tag{2.23}$$

The derivation of Eq. (2.21) can be summarized as follows. Due to the screening of the Coulomb interactions the Faddeev-AGS formalism can be used without any modification. The partial inversion technique makes it possible to separate the pure Coulomb interactions so that the difficult Coulomb operators appear as "effective interactions" in the set of equations for the Coulomb-modified transition operators. The nonlocal separable representation of the short-range interactions in addition reduces the dimension of the equations so that a set of effective two-body equations results,[27]

$$\mathcal{T}^R = \mathcal{V}^R + \mathcal{V}\mathcal{G}_0^R\mathcal{T}^R, \tag{2.24}$$

where $\mathcal{G}_0^R$ is the matrix of operators defined by

$$(\mathcal{G}_0^R)_{\beta\alpha} \equiv \delta_{\beta\alpha}\tau_\alpha^R(z), \tag{2.25}$$

and $\mathcal{T}$ and $\mathcal{V}$ are the matrices with elements defined in Eqs. (2.22).

Let us now turn off the screening. In the zero screening limit the diagonal terms $\mathcal{V}_{\alpha\alpha}^R$ in the effective interaction have a characteristic Coulomb tail, so that in the momentum representation the well-known Coulomb singularity appears. The obvious way to circumvent this difficulty is to employ the subtraction technique, which is closely related to but not exactly the same as the two-potential formalism. Following a long-established precedent, which is reviewed in Ref. 5, we introduce the (screened) channel Coulomb interaction,

$$\mathcal{V}_{\beta\alpha}^{c,R} \equiv \delta_{\beta\alpha}W^{\alpha,R}, \tag{2.26}$$

in which the spectator particle of the channel interacts with the charge of the other particles as if it was all located at their center of mass. Associated with the matrix $\mathcal{V}^{c,R}$ is a (diagonal) matrix t operator which is defined as the solution to the Lippmann-Schwinger equation

$$\mathcal{T}^{c,R} = \mathcal{V}^{c,R} + \mathcal{V}^{c,R}g_0^R\mathcal{T}^{c,R}, \tag{2.27}$$

where g_0^R is the multichannel two-body propagator with matrix elements defined by

$$(g_0^R)_{\beta\alpha}(z) \equiv \delta_{\beta\alpha}g_{0\alpha}^R(z), \tag{2.28}$$

$$\langle \mathbf{p}|g_{0\alpha}^R(z)|\mathbf{q}\rangle \equiv \delta(\mathbf{p}-\mathbf{q})(z-\hat{E}_\alpha^R - |\mathbf{q}|^2/2M_\alpha)^{-1}. \tag{2.29}$$

Here $\hat{E}_\alpha^R$ is the bound-state energy of the bound pair in channel α, $\mathbf{p}$ is the relative momentum of the free particle relative to the bound pair, and M_α is the appropriate reduced mass. Next we subtract the effects of $\mathcal{V}^{c,R}$ and $\mathcal{T}^{c,R}$, introducing the matrix operators

$$\mathcal{V}^{sc,R} \equiv \mathcal{V}^R\mathcal{G}_0^R g_0^{R\,-1} - \mathcal{V}^{c,R}, \tag{2.30}$$

$$\mathcal{T}^{sc,R} \equiv (1+\mathcal{T}^{c,R}g_0^R)^{-1}(\mathcal{T}^R\mathcal{G}_0^R g_0^{R\,-1} - \mathcal{T}^{c,R}) \times(1+g_0^R\mathcal{T}^{c,R})^{-1}. \tag{2.31}$$

The Coulomb-modified transition operator $\mathcal{T}^{sc,R}$ then satisfies the following Lippmann-Schwinger equation:

$$\mathcal{T}^{sc,R} = \mathcal{V}^{sc,R} + \mathcal{V}^{sc,R}g^{c,R}\mathcal{T}^{sc,R}, \tag{2.32}$$

where $g^{c,R}$ is the (diagonal) matrix operator

$$g^{c,R} \equiv g_0^R + g_0^R\mathcal{T}^{c,R}g_0^R. \tag{2.33}$$

Alt and Sandhas have shown[27] that the zero screening limit $R \to \infty$ can now be performed by means of the renormalization technique and Eq. (2.32) has a well-defined limit. The physical reaction amplitudes will then be determined by the on-shell matrix elements of the transition operators in the Coulomb wave representation,[27] $\langle \mathbf{q}_\beta^{(-)}|\mathcal{T}_{\beta\alpha}^{sc,\infty}|\mathbf{q}_\alpha^{(+)}\rangle$, where $|\mathbf{q}^{(\pm)}\rangle$ denotes the Coulomb wave function with the appropriate boundary condition and the corresponding momentum $\mathbf{q}$.

It should be emphasized that the construction of the effective interactions defined by Eq. (2.30) is by no means trivial, since it involves the solution of the pure Coulomb three-body problem. If there are only two charged particles in the system, the Coulomb transition operators can be analytically constructed.[29] However, even in this case the numerical evaluation of the corresponding matrix elements is extremely difficult so that one has to resort to various simplifying approximations.[26]

III. THE TWO-POTENTIAL APPROACH

In the previous section the application of the screening technique made it possible for one to start with the Faddeev-AGS stationary formalism. The resulting three-body equations then had to be shown to yield amplitudes that possess a physically meaningful limit as the screening radius tends to infinity.

In the present section, in order to develop a physically and mathematically more transparent theory, we avoid the use of screening. Instead, we invoke the very powerful two-potential formalism,[30,31] which makes it possible to define transition operators which can immediately be seen to possess well-behaved stationary representations. Once the stationary form of the modified transition operators is introduced, the three-body equations can be derived by purely algebraic manipulations.

Let us then start with the Hamiltonian

$$H = H_0 + V + U, \tag{3.1}$$

$$V = \sum_\alpha V_\alpha, \qquad U = \sum_\alpha U_\alpha, \tag{3.2}$$

where the notation is the same as in Sec. II with the important difference that the Coulomb interactions are not screened. We also introduce various resolvent operators, which involve only unscreened quantities:

$$G(z) = (z-H)^{-1}, \tag{3.3}$$

$$G_0(z) = (z-H_0)^{-1}, \tag{3.4}$$

$$G_c(z) = (z-H_0-U)^{-1}, \tag{3.5}$$

$$G_\alpha(z) = (z-H_0-V_\alpha-U_\alpha)^{-1}, \tag{3.6}$$

$$G_{\alpha c}(z) = (z-H_0-U_\alpha)^{-1}, \tag{3.7}$$

$$\tilde{G}_\alpha(z) = (z-H_0-V_\alpha-U_\alpha-W^\alpha)^{-1}, \tag{3.8}$$

$$\tilde{G}_{\alpha c}(z) = (z-H_0-U_\alpha-W^\alpha)^{-1}. \tag{3.9}$$

Here, in parallel with Eq. (2.26), the channel Coulomb

interaction W^α represents the (unscreened) interaction of the spectator particle α of the channel with the charge of the other two particles as if it was all concentrated at their center of mass.

In the spirit of the two-potential formalism we introduce the (two-body) channel Coulomb wave operator,

$$\omega_\alpha^\pm \equiv s - \lim_{t\to\pm\infty} e^{i(H_\alpha+W^\alpha)t}\, U_{D\alpha}(-t) e^{-iH_\alpha t}, \tag{3.10}$$

where $H_\alpha = H_0 + V_\alpha + U_\alpha$ is the channel Hamiltonian and $U_{D\alpha}(-t)$ denotes the Dollard operator for Coulomb scattering.[9] The two-potential formalism[30,31] then yields

$$S_{\beta\alpha} = \delta_{\beta\alpha}\omega_\alpha^{+*}\omega_\alpha^- - 2\pi i\delta(E_f - E_i)T_{\beta\alpha}. \tag{3.11}$$

Here, $S_{\beta\alpha}$ is the scattering operator, the asterisk denotes adjoint, E_f and E_i are the final and initial energies, and $T_{\beta\alpha}$ is the on-shell matrix element of a transition operator $U_{\beta\alpha}$ defined by

$$U_{\beta\alpha} = \omega_\beta^{+*}\tilde{U}_{\beta\alpha}\omega_\alpha^-, \tag{3.12}$$

$$\tilde{U}_{\beta\alpha} \equiv \tilde{G}_\beta^{-1}(G - \delta_{\beta\alpha}\tilde{G}_\alpha)\tilde{G}_\alpha^{-1}. \tag{3.13}$$

Equation (3.13) is the symmetric form of the transition operator introduced in this context by Alt and Sandhas;[5,32] the post form was used by Bencze and Zankel[33] and the prior form by Chandler and Gibson.[34]

The essence of the two-potential formalism is that the asymptotic states are redefined and no longer correspond to freely moving clusters. Since this can be done in various ways, the formalism is very flexible. If, for example, one replaces in Eq. (3.10) the channel Coulomb interaction W^α by $U - U_\alpha$, the $\omega_\alpha^\pm$ are in fact three-body Coulomb wave operators and Noble's formalism[24] is recovered. One could also imagine replacing W^α by a folding potential created from $U - U_\alpha$. Our considerations in this paper would apply equally well to this latter replacement, though not to the former (because of the three-body nature of the Coulomb wave operators in Noble's formalism).

We now proceed to derive dynamical equations for the "distorted" transition operator $\tilde{U}_{\beta\alpha}$. We begin with Eq. (5.39) of Ref. 5,

$$\tilde{U}_{\beta\alpha} = \bar{\delta}_{\beta\alpha}\tilde{G}_\alpha^{-1} + \mathcal{V}_{\beta\alpha} + \sum_\gamma \mathcal{V}_{\beta\gamma}\tilde{G}_\gamma\tilde{U}_{\gamma\alpha}, \tag{3.14}$$

and make a particular choice of $\mathcal{V}_{\beta\alpha}$,

$$\mathcal{V}_{\beta\alpha} \equiv \bar{\delta}_{\beta\alpha}V_\alpha + \delta_{\beta\alpha}U_\alpha^{\rm pol}, \tag{3.15}$$

$$U_\alpha^{\rm pol} \equiv U - U_\alpha - W^\alpha. \tag{3.16}$$

Substitution of Eq. (3.15) into Eq. (3.14) yields

$$\tilde{U}_{\beta\alpha} = \bar{\delta}_{\beta\alpha}\tilde{G}_{\alpha c}^{-1} + \delta_{\beta\alpha}U_\beta^{\rm pol} + U_\beta^{\rm pol}\tilde{G}_\beta\tilde{U}_{\beta\alpha} + \sum_\gamma \bar{\delta}_{\beta\gamma}V_\gamma\tilde{G}_\gamma\tilde{U}_{\gamma\alpha}. \tag{3.17}$$

When Eq. (3.17) is considered as a set of coupled equations for the $\tilde{U}_{\beta\alpha}$, it is troublesome that the kernel of the third term on the right-hand side of the equation is proportional to the long-range potential $U_\beta^{\rm pol}$, rather than to a short-range potential. This leads us to substitute the resolvent relation

$$\tilde{G}_\beta = \tilde{G}_{\beta c} + \tilde{G}_{\beta c}V_\beta\tilde{G}_\beta \tag{3.18}$$

into Eq. (3.17) and perform some simple algebraic manipulations, obtaining

$$[1 - (U^\beta - W^\beta)\tilde{G}_{\beta c}]\tilde{U}_{\beta\alpha} = [\bar{\delta}_{\beta\alpha}\tilde{G}_{\alpha c}^{-1} + \delta_{\beta\alpha}(U^\beta - W^\beta)] + \sum_\gamma [\bar{\delta}_{\beta\gamma}\tilde{G}_{\gamma c}^{-1} + \delta_{\beta\gamma}(U^\beta - W^\beta)]\tilde{G}_{\gamma c}V_\gamma\tilde{G}_\gamma\tilde{U}_{\gamma\alpha}. \tag{3.19}$$

Because

$$1 - (U^\beta - W^\beta)\tilde{G}_{\beta c} = G_c^{-1}\tilde{G}_{\beta c}, \tag{3.20}$$

multiplying Eq. (3.19) by $\tilde{G}_{\beta c}^{-1}G_c$ yields

$$\tilde{U}_{\beta\alpha} = \tilde{G}_{\beta c}^{-1}K_{\beta\alpha}\tilde{G}_{\alpha c}^{-1} + \sum_\gamma \tilde{G}_{\beta c}^{-1}K_{\beta\gamma}\tilde{G}_{\gamma c}^{-1}\tilde{G}_{\gamma c}V_\gamma\tilde{G}_\gamma\,\tilde{U}_{\gamma\alpha}, \tag{3.21}$$

where

$$K_{\beta\alpha}(z) = G_c(z) - \delta_{\beta\alpha}\tilde{G}_{\alpha c}(z). \tag{3.22}$$

Finally, since $V_\gamma\tilde{G}_\gamma = \tilde{t}_\gamma^{sc}\tilde{G}_{\gamma c}$, with

$$\tilde{t}_\gamma^{sc} \equiv V_\gamma + V_\gamma\tilde{G}_\gamma V_\gamma, \tag{3.23}$$

Eq. (3.21) can be rewritten in the form

$$\tilde{U}_{\beta\alpha} = \tilde{G}_{\beta c}^{-1}K_{\beta\alpha}\tilde{G}_{\alpha c}^{-1} + \sum_\gamma \tilde{G}_{\beta c}^{-1}K_{\beta\gamma}\,\tilde{G}_{\gamma c}^{-1}\tilde{G}_{\gamma c}\,\tilde{t}_\gamma^{sc}\tilde{G}_{\gamma c}\,\tilde{U}_{\gamma\alpha}. \tag{3.24}$$

Unlike Eq. (3.17), the kernels of Eq. (3.24) are all proportional to some short-range quantity (viz., $\tilde{t}_\gamma^{sc}$).

The set of equations Eq. (3.24) determines the "distorted" transition operators $\tilde{U}_{\beta\alpha}$, whose matrix elements must be taken between the "distorted" asymptotic states. In the case of separable short-range interactions Eq. (3.24) can be transformed into a set of effective two-body equations in the standard way.[8] If the polarization forces are neglected, Eq. (3.24) is finally reduced to the channel distortion approximation form of Bencze.[25] It also should be noted that in obtaining Eq. (3.24) no restricting conditions on the Coulomb interactions have been imposed, so that they are valid even if the Coulomb interactions are attractive.

In order to derive equations for the operators $U_{\beta\alpha}$, however, one needs to insert into Eq. (3.24) the identity

$$1 = \omega_\gamma^+\omega_\gamma^{+*} \tag{3.25}$$

just to the left of $\tilde{U}_{\gamma\alpha}$. For this identity to be true there must exist no Coulomb bound states between the particle γ and the pair (α,β), considered as a single particle. Therefore, from this point on, our treatment is restricted to systems with only repulsive Coulomb interactions.

We sandwich Eq. (3.24) between the appropriate wave operators as indicated in Eq. (3.12) and insert $\omega_\gamma^+\omega_\gamma^{+*}$ between $\tilde{G}_{\gamma c}$ and $\tilde{U}_{\gamma\alpha}$. The result is a set of equations for $U_{\beta\alpha}$,

$$U_{\beta\alpha} = \omega_\beta^{+*}(\tilde{G}_{\beta c}^{-1}K_{\beta\alpha}\tilde{G}_{\alpha c}^{-1})\omega_\alpha^- + \sum_\gamma \omega_\beta^{+*}\tilde{G}_{\beta c}^{-1}K_{\beta\gamma}\tilde{t}_\gamma^{sc}\tilde{G}_{\gamma c}\omega_\gamma^+ U_{\gamma\alpha}. \tag{3.26}$$

Because V_γ and ω_γ^+ act on different coordinates, they commute:

$$V_\gamma\omega_\gamma^\pm = \omega_\gamma^\pm V_\gamma. \tag{3.27}$$

In addition, the following intertwining relations are well known:

$$G_{\beta c}^{-1}\omega_\beta^{\pm *} = \omega_\beta^{\pm *}\tilde{G}_{\beta c}^{-1}, \qquad \omega_\beta^\pm G_{\beta c}^{-1} = \tilde{G}_{\beta c}^{-1}\omega_\beta^\pm, \tag{3.28}$$

$$G_\beta^{-1}\omega_\beta^{\pm *} = \omega_\beta^{\pm *}\tilde{G}_\beta^{-1}, \qquad \omega_\beta^\pm G_\beta^{-1} = \tilde{G}_\beta^{-1}\omega_\beta^\pm. \tag{3.29}$$

Combining Eqs. (3.26)–(3.29) yields

$$U_{\beta\alpha} = (G_{\beta c}^{-1}\omega_\beta^{+*}K_{\beta\alpha}\,\omega_\alpha^- G_{\beta c}^{-1}) + \sum_\gamma (G_{\beta c}^{-1}\omega_\beta^{+*}K_{\beta\gamma}\,\omega_\gamma^+ G_{\gamma c}^{-1})G_{\gamma c}t_\gamma^{sc}G_{\gamma c}U_{\gamma\alpha}, \tag{3.30}$$

where

$$t_\gamma^{sc} \equiv V_\gamma + V_\gamma G_\gamma V_\gamma. \tag{3.31}$$

It is important to note that in Eq. (3.31) the only Coulomb interaction included is internal to the pair (α,β) and, hence, that the operator t_γ^{sc} treats the particle γ as a spectator and is short range in the relative coordinates of the pair (α,β).

Let us assume now that the short-range interactions can be adequately represented by a nonlocal separable form

$$V_\gamma = \sum_{n_\gamma,n'_\gamma} \lambda_{n_\gamma n'_\gamma}|g_{n_\gamma}(z)\rangle\langle g_{n'_\gamma}(z)|. \tag{3.32}$$

As a consequence the transition operators t_γ^{sc} are also separable and have the form

$$t_\gamma^{sc}(z) = \sum_{n_\gamma,n'_\gamma}^{N} |g_{n_\gamma}(z)\rangle\tau_{n_\gamma n'_\gamma}(z)\langle g_{n'_\gamma}(z)|. \tag{3.33}$$

While the form factor $|g_{n_\gamma}(z)\rangle$ may have a rather general energy dependence, we do assume that they have been chosen so that every bound state $|\varphi_{n_\gamma}\rangle$ is related to one of the form factors through the equation[27]

$$|g_{n_\gamma}(E_{n_\gamma})\rangle = V_\gamma|\varphi_{n_\gamma}\rangle. \tag{3.34}$$

Here E_{n_γ} is the bound-state energy and n_γ denotes the set of quantum numbers needed to uniquely specify the bound state. Equation (3.34) implies that

$$G_{\gamma c}(E_{n_\gamma})|g_{n_\gamma}(E_{n_\gamma})\rangle = |\varphi_{n_\gamma}\rangle. \tag{3.35}$$

We can now rewrite Eq. (3.30) in an effective two-particle form,

$$T_{\beta\alpha}^{n_\beta n_\alpha}(z) = Z_{\beta\alpha}^{n_\beta n_\alpha}(z)\,\omega_\alpha^{+*}\omega_\alpha^- + \sum_{\gamma,n_\gamma,n'_\gamma} Z_{\beta\gamma}^{n_\beta n_\gamma}\tau_{n_\gamma n'_\gamma}T_{\gamma\alpha}^{n'_\gamma n_\alpha}. \tag{3.36}$$

Here,

$$T_{\beta\alpha}^{n_\beta n_\alpha}(z) \equiv \langle g_{n_\beta}(z)|G_{\beta c}(z)U_{\beta\alpha}(z)G_{\alpha c}(z)|g_{n_\alpha}(z)\rangle, \tag{3.37}$$

and

$$Z_{\beta\alpha}^{n_\beta n_\alpha}(z) \equiv \langle g_{n_\beta}(z)|Z_{\beta\alpha}(z)|g_{n_\alpha}(z)\rangle, \tag{3.38}$$

where

$$Z_{\beta\alpha}(z) \equiv \omega_\beta^{+*}K_{\beta\alpha}(z)\,\omega_\alpha^+. \tag{3.39}$$

The unitary two-body pure Coulomb scattering operator $\omega_\alpha^{+*}\omega_\alpha^-$ is present in Eq. (3.36) because of the way $Z_{\beta\alpha}$ is defined. It can be removed and Eq. (3.36) immediately recast in a Lippmann-Schwinger form,

$$X_{\beta\alpha}^{n_\beta n_\alpha}(z) = Z_{\beta\alpha}^{n_\beta n_\alpha}(z) + \sum_{\gamma n_\gamma,n'_\gamma} Z_{\beta\gamma}^{n_\beta n_\gamma}(z)\tau_{n_\gamma n'_\gamma}(z)X_{\gamma\alpha}^{n'_\gamma n_\alpha}(z), \tag{3.40}$$

by introducing the amplitudes

$$X_{\beta\alpha}^{n_\beta n_\alpha}(z) \equiv T_{\beta\alpha}^{n_\beta n_\alpha}(z)\,\omega_\alpha^{-*}\omega_\alpha^+. \tag{3.41}$$

Note that if the Coulomb interactions are set equal to zero in Eq. (3.40), the AGS quasiparticle equations are immediately recovered.

The transition operator for the breakup process is

$$U_{0\alpha} = G_0^{-1}\omega_0^{+*}G\omega_\alpha^- G_\alpha^{-1}, \tag{3.42}$$

where the wave operator ω_0^+ is the "free" three-particle Coulomb wave operator [cf., Eq. (3.10)]. The usual manipulations[8] lead from Eq. (3.42) to

$$U_{0\alpha} = \omega_0^{+*}\omega_\alpha^- G_\alpha^{-1} + \omega_0^{+*}\omega_\alpha^- V_\alpha + \sum_\gamma \omega_0^{+*}\omega_\gamma^+ t_\gamma G_{\gamma c}U_{\gamma\alpha}. \tag{3.43}$$

The breakup matrix element

$$T_{0\alpha}^{n_\alpha}(z) \equiv \langle \mathbf{pq}|U_{0\alpha}(z)G_{\alpha c}(z)|g_{n_\alpha}(z)\rangle, \tag{3.44}$$

where $\langle \mathbf{pq}|$ denotes a free breakup state with Jacobi momenta $\mathbf{p}$ and $\mathbf{q}$, is then obtained from Eq. (3.43) in a straightforward way:

$$T_{0\alpha}^{n_\alpha}(z) = \langle \mathbf{pq}|\omega_0^{+*}\omega_\alpha^-|g_{n_\alpha}(z)\rangle + \sum_{\gamma\, n_\gamma, n'_\gamma} \langle \mathbf{pq}|\omega_0^{+*}\omega_\gamma^+|g_{n_\gamma}(z)\rangle \tau_{n_\gamma n'_\gamma}(z) T_{\gamma\alpha}^{n'_\gamma n_\alpha}(z). \tag{3.45}$$

Note that the driving term in Eq. (3.45) corresponds to a pure Coulomb breakup process caused by the polarization forces, arising from the potential $U_\alpha^{\rm pol}$, that are included in the breakup channel wave operator ω_0^+.

Merkuriev[16,17] has pointed out that the kernel of Eq. (3.40) is compact below but not above the breakup threshold. Our preliminary analysis of the kernels $Z_{\beta\alpha}^{n_\beta n_\alpha}$ leads us to believe that this lack of compactness manifests itself in the appearance of integrable singularities. Should this belief be sustained by more complete analysis, careful use of singularity subtraction techniques[35] should be adequate for the numerical solution of the equations.

IV. APPROXIMATION OF THE EFFECTIVE INTERACTION

The matrix elements of the effective interactions $Z_{\beta\alpha}$ are in general very complicated. While it is possible that they might be calculable for special cases by taking advantage of closed forms that have already been obtained,[29] in most practical applications approximations have to be invoked.

The most obvious simplification is the total neglect of polarization forces, i.e., only the channel Coulomb distortion is taken into account (CDA).[25] This approximation is generally accepted in nuclear reaction theory. In fact, DWBA calculations for cluster transfer reactions at energies near and above the Coulomb barrier indicate that the role of long-range interactions in both the distortion of the relative motion of the fragments and the coupling between the channels with different fragmentation is negligible. The situation might be different, however, in inelastic-scattering processes, where Coulomb excitation as well as excitation by nuclear forces may interfere.[36] In breakup processes, due to the interplay of short-range and long-range forces, there might be more pronounced effects, so that electric polarization forces may have to be fully included. Finally, at very low energies the interplay of Coulomb and other long-range electric polarization forces is known to produce singular behavior.[37]

It is therefore desirable that the approximation of the effective interaction is carried beyond the channel Coulomb distortion approximation. We start with the resolvent relations

$$G_c = \tilde{G}_{\beta c} + \tilde{G}_{\beta c} U_\beta^{\rm pol} G_c = \tilde{G}_{\alpha c} + G_c U_\alpha^{\rm pol} \tilde{G}_{\alpha c}, \tag{4.1}$$

and combine them to obtain the following two equivalent expressions:

$$G_c = \tilde{G}_{\alpha c} + \tilde{G}_{\beta c} U_\alpha^{\rm pol} \tilde{G}_{\alpha c} + \tilde{G}_{\beta c} U_\beta^{\rm pol} G_c U_\alpha^{\rm pol} \tilde{G}_{\alpha c} \tag{4.2}$$

$$= \tilde{G}_{\beta c} + \tilde{G}_{\beta c} U_\beta^{\rm pol} \tilde{G}_{\alpha c} + \tilde{G}_{\beta c} U_\beta^{\rm pol} G_c U_\alpha^{\rm pol} \tilde{G}_{\alpha c}. \tag{4.3}$$

Since the polarization potentials are generally weak and only their interference with other forces may be important, one may neglect the last terms in Eqs. (4.2) and (4.3):

$$G_c \cong \tilde{G}_{\alpha c} + \tilde{G}_{\beta c} U_\alpha^{\rm pol} \tilde{G}_{\alpha c} = \tilde{G}_{\beta c} + \tilde{G}_{\beta c} U_\beta^{\rm pol} \tilde{G}_{\alpha c}. \tag{4.4}$$

Now if Eqs. (3.22), (3.39), and (4.4) are combined, the following approximate form for the effective interactions emerges:

$$Z_{\beta\alpha} \cong \bar{\delta}_{\beta\alpha}\omega_\beta^{+*}(\tilde{G}_{\alpha c} + \tilde{G}_{\beta c} U_\alpha^{\rm pol} \tilde{G}_{\alpha c})\omega_\alpha^+ + \delta_{\beta\alpha}\omega_\beta^{+*}\tilde{G}_{\beta c}\, U_\alpha^{\rm pol} \tilde{G}_{\alpha c}\, \omega_\alpha^+. \tag{4.5}$$

Finally, if the intertwining property in Eq. (3.29) is employed, one arrives at the form

$$Z_{\beta\alpha} \cong \bar{\delta}_{\beta\alpha}(\omega_\beta^{+*}\omega_\alpha^+ + G_{\beta c}\omega_\beta^{+*} U_\alpha^{\rm pol}\omega_\alpha^+) G_{\alpha c} + \delta_{\beta\alpha} G_{\alpha c}\, \omega_\alpha^{+*} U_\alpha^{\rm pol}\omega_\alpha^+ G_{\alpha c}. \tag{4.6}$$

The approximate forms in Eqs. (4.5) and (4.6) preserve the post-prior symmetry of the effective interaction, a property not present in the channel-distortion approximation. The CDA form of the effective interaction[25] can be immediately recovered if one sets $U_\alpha^{\rm pol}$ to zero in both Eqs. (4.5) and (4.6).

We now turn in more detail to the question of how justified is the approximation of Eq. (4.6). For energies below the breakup threshold, the operators $G_{\alpha c}$ are bounded and $G_{\alpha c}|g_{n_\alpha}\rangle$ behaves like a state in which particle α is free and the other particles are bound. For such energies, in the context of the matrix element $Z_{\beta\alpha}^{n_\beta n_\alpha}$, the polarization potential $U_\alpha^{\rm pol}$ effectively behaves, as one would expect, as a function of the relative coordinate between the free particle and the center of mass of the "bound" pair. Hence, one can argue that the approximation is reasonable since the polarization potential is relatively weak. Above the breakup threshold $G_{\alpha c}$ is no longer bounded (for real energies), and $G_{\alpha c}|g_{n_\alpha}\rangle$ behaves as a scattering state in all variables. For these energies the polarization potential $U_\alpha^{\rm pol}$, even in the context of the matrix elements $Z_{\beta\alpha}^{n_\beta n_\alpha}$, cannot be expected to fall off for large interparticle distances. For these energies, therefore, the accuracy of our approximation remains to be established, much as does the quasi-Born approximation of the Coulomb t matrix in screened Coulomb calculations.[28]

V. CONCLUSIONS

The present work has been devoted to the study of the three-body problem with charged particles. Since the Hamiltonian of the system is assumed to include short-range two-body interactions as well, it may be more conveniently referred to as the nuclear three-body problem. In nuclear physics applications the short-range nuclear

interactions are expected to play a dominant role. However, the Coulomb interactions are by no means negligible and their presence gives rise to serious mathematical difficulties.

We have developed, in Sec. III, an approach to this problem based on a generalized two-potential formalism. Careful consideration of the asymptotic conditions appropriate to a scattering theory with charged particles led us, in a physically and mathematically clean way, to a set of dynamical equations for the stationary Coulomb modified transition operators. These equations are divergence free and involve no screening, and hence also no limit process to remove the screening. The treatment in Sec. III was restricted to repulsive Coulomb interactions.

If the short-range interactions are represented by a nonlocal separable expansion, the three-body equations can be reduced in a well-known way to a set of effective two-particle equations. However, our approach leads to a different set of multichannel two-body equations than that of Alt *et al.*, which were presented in Sec. II. The difference reflects the fact the two sets of equations determine different off-shell extensions of the Coulomb modified transition operators. Our off-shell extension is dictated by our treatment, at the outset, of the asymptotic conditions and our subsequent use of the two-potential formalism, while the off-shell extension of Alt *et al.* is dictated by the way they remove the two-body Coulomb singularities from their equations.

The kernel of our equations is not compact for energies above the breakup threshold, with the consequence that Fredholm theory is not applicable. We anticipate, but have not yet firmly established, that singularity subtraction techniques[35] will still prove to be adequate for the numerical solution of the equations.

In Sec. IV a new approximation scheme is suggested for the three-body equations for the Coulomb-modified transition operators. This scheme goes further than the channel distortion approximation (CDA) and treats electric polarization forces to first order, much as do Alt *et al.* As a result the post-prior asymmetry inherent in the CDA is removed. Throughout the derivation of the approximate form of the equations an effort was made to preserve their AGS form. In this way it is hoped that computer codes developed for the solution of the AGS equations can be modified with reasonable effort to include an approximate treatment of Coulomb effects.

ACKNOWLEDGMENTS

We gratefully acknowledge the support of the Hungarian Academy of Sciences and the U.S. National Science Foundation (Grants PHY8802774 and INT8800898). We also acknowledge with pleasure many interesting and constructive conversations with E. O. Alt, G. H. Berthold, B. J. Fisk, J. L. Friar, B. F. Gibson, Z. C. Kuruoglu, W. Sandhas, and A. J. Waters.

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