# Vector coherent state theory in new perspective: A hybrid mapping from fermion to fermion-boson space

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Based on a hybrid mapping, the vector coherent state theory is revisited. The hybrid mapping of a fermion state with $N$ coherent (collective) fermion pairs and $u$ unpaired (noncollective) fermions leads to a hybrid state with $N$ bosons and $u$ fermions. The hybrid mapping is *exact* by which the formidable problem of finding the matrix elements of a fermion operator for a $(2N+u)$-fermion system is reduced to that for a $u$-fermion system. The Pauli effects are totally taken into account by the overlap matrix (the $K$ matrix) for the fermion states. The application to a fermion system with the Sp(6)$\supset$U(3) dynamical symmetry is discussed where it is vividly seen that the occurrence of the dynamical Pauli factors is related to the fermion pair$\leftrightarrow$boson transformation. The intricate properties of the intrinsic operators and the $K$ operator are discussed in detail.

## I. INTRODUCTION

The vector coherent state (VCS) theory developed by Rowe[1-3] and in slightly different form by Deenen and Quesne[4] is a very powerful method for computing the reduced Wigner coefficients as well as the matrix elements of the generators of both noncompact groups, such as Sp$(6,R)$,[5] Sp$(4,R)$,[6] and compact groups, such as Sp(4),[7] SO(8),[8] SO(7),[8] Sp(6),[9] and SU(3).[10] Recently, this technique has been extended to the calculation of the matrix elements of the operators lying outside the Lie algebras of the relevant groups.[11,12] This has greatly extended the regions of applicability of the VCS theory and will thus be named as the generalized VCS (GVCS) theory. However the exploration of the physical implications of the VCS has not kept up with the development of the technique. Besides, there remain several puzzling questions which make this elegant theory rather elusive. For example, the VCS theory in Refs. 1 and 10 is based on the unitary mapping $\gamma(O)=K^{-1}\Gamma(O)K$, while in Ref. 4 is based on the dual basis. However both $K^{-1}$ and the dual basis do not exist when the basis vectors are linearly dependent. How to derive the VCS theory in these general cases? What is the essence of the intrinsic operators? In the GVCS theory,[11,12] the intrinsic operator is defined through its left action on an intrinsic state. Is the left-action definition necessary? Can the GVCS theory be presented in a simple and physically oriented way which makes contact with the popular fermion-pair$\rightarrow$boson mapping?

In this paper we will introduce a hybrid mapping $\mathbb{H}$ which can be applied to a system whose states can be described by a certain kind of coherent pairs and a few unpaired fermions, which correspond physically to a certain kind of collective excitation superimposed on some single-particle excitations. The hybrid mapping $\mathbb{H}$ maps a fermion state with $N$ coherent fermion pairs and $u$ unpaired fermions into a hybrid state with $N$ bosons and $u$ fermions, and maps a fermion operator $O$ into a hybrid operator $\mathbb{H}(O)$ consisting of both boson and the pseudofermion operators (in the VCS language, the intrinsic operators). In general, the pseudofermion operator differs from the true fermion operator in that it involves a fermion-boson coupling, i.e., the coupling between the collective and noncollective motions. Only when this coupling term is zero or negligible does the pseudofermion operator become a true fermion operator and commute with boson operators. In light of this hybrid mapping, the VCS theory is rederived.

The Pauli effects are crucial for a fermion system. One of the major advantages of the fermion dynamical symmetry model (FDSM) (Ref. 13) over the interacting boson model (IBM) (Ref. 14) is that the Pauli effects in the former can fully be taken into account, while for the latter they are either totally neglected or taken into account only partially by a certain kind of mapping procedure. One of the Pauli effects is called the dynamical Pauli effect reflecting the disappearance of certain irreducible representations of the dynamical group due to the Pauli principle.[15,16] This was first recognized by Ginocchio in the development of the Sp(6) $S,D$-pair model.[17]

 

We will show in this paper how the VCS theory is tailored for attacking the seemingly formidable problem of computing the Pauli effects in the FDSM in an analytic way. Therefore $\mathbb{H}$ provides an ideal mapping from the FDSM to the IBM or IBFM (Ref. 18) (interacting boson and fermion models). The advantage of the mapping is that it is *exact*, unlike other approximate mappings.[19,20] The calculation of the matrix elements of a fermion operator is carried out in three steps, and in each step the essence of the physics is vividly reflected in the formula. The three steps are as follows: I. Switch from the fermion space to the hybrid space by transforming the $N$ fermion pairs into $N$ bosons. II. Calculate the matrix elements of the hybrid operator $\mathbb{H}(O)$ in the hybrid space, which can easily be factorized into the boson matrix elements and the "fermion" matrix elements between the pairless states. III. Finally switch back to the $u$-fermion space for computing the "fermion" matrix element. In this way the calculation of the matrix elements for a $(2N+u)$-fermion system is reduced to that for a $u$-fermion system. The dynamical Pauli factors occur at the first and third stages where fermion-pair$\leftrightarrow$boson transformations take place. There is no dynamical Pauli factor in the second stage where the boson and "fermion" operators are independent of one another. When the shell degeneracy $\Omega$ becomes infinite, the "pseudofermion" becomes the real fermion and all the dynamical Pauli factors become constants, $\sqrt{\Omega^N}$, and a system with $N$ coherent fermion pairs and $u$ unpaired fermions behaves exactly as one with $N$ bosons and $u$ fermions. That the FDSM is reduced to the IBM (for $u=0$) or IBFM (for $u>0$) serves as a special example.

The paper is organized in the following way. In Sec. II we review the transformation from a linearly dependent basis to an orthonormal basis for a fermion system with $N$ coherent pairs and $u$ unpaired fermions. Here it is pointed out that only the left inverse of the $K$ matrix exists. In Sec. III a hybrid mapping $\mathbb{H}$ is proposed to circumvent the difficulty imposed by the Pauli principle in computing fermion matrix elements. Section IV shows the relation between the hybrid mapping $\mathbb{H}$ and the VCS mapping $\Gamma$. The master equation of the VCS theory is rederived via the hybrid mapping. Section V emphasizes the importance of the $K$ matrix, which takes care of all the Pauli effects and ensures that the hybrid mapping is free of the spurious-state problem. Section VI deals with the calculation of the matrix elements of the hybrid operators, $\Gamma(O)$, by the use of Racah algebra, and shows that the appearance of the $K$ matrix is related to the fermion-pair$\leftrightarrow$boson transformation. Section VII is devoted to an understanding of the intricate properties of the intrinsic, or "pseudofermion" operators. In Sec. VIII we discuss the relation between the nonunitary mapping approach and the conventional unitary mapping approach to the VCS theory. The mapping from the FDSM to the IBFM, and the difference between the hybrid mapping and other mappings are discussed. The distinction between the operator $\hat{K}$ and the matrix $K$ is carefully made. Finally, the new perspective presented by the hybrid mapping is exploited in a proposed new approach to the spectroscopy of real nuclei.

## II. THE $K$ MATRIX AND ITS LEFT INVERSE

The theory for the hybrid mapping to be discussed in this paper is general and does not depend on any group structure the fermion system may have. For easier accessibility, in presenting the general theory we often refer to the Sp(6)$\supset$U(3) case as illustration. We therefore begin by introducing the generators for the Sp(6) group. In line with the Ginocchio model[17] and the FDSM,[13] the shell model single-particle angular momentum **j** is decomposed into the pseudo-orbital angular momentum **k** and pseudospin **i**, $\mathbf{j}=\mathbf{k}+\mathbf{i}$, and the $k=1$ case leads to the Sp(6) dynamical symmetry for the system. The same notation as in Ref. 12 will be used here without explanation except that the creation or annihilation operator in the $k$-$i$ basis[13] is changed from $b^\dagger_{km_k im_i}\to a^\dagger_{km_k im_i}$, $b_{km_k im_i}\to a_{km_k im_i}$. The generators of Sp(6) in the coupled and uncoupled bases are given below.[21]

Generators of $\mathrm{Sp}_6$ (coupled basis) $\{A^{L\dagger}_M, A^L_M, P^L_M\}$:

$$A^{L\dagger}_M=\sum_i \sqrt{\Omega_i/2}[a^\dagger_{1i}a^\dagger_{1i}]^{L0}_{M0},\quad L=0,2\ , \tag{2.1a}$$

$$P^L_M=\sum_i \sqrt{\Omega_i/2}[a^\dagger_{1i}\tilde{a}_{1i}]^{L0}_{M0},\quad L=0,1,2\ . \tag{2.1b}$$

Generators of Sp(6) (uncoupled basis): $\{A^\dagger_{mm'}, A_{mm'}, C_{mm'}\}$

$$A^\dagger_{mm'}=\sum_{im_i}(-)^{i-m_i}a^\dagger_{1m,im_i}a^\dagger_{1m',i-m_i}\ ,\qquad m,m'=1,0,-1\ , \tag{2.2a}$$

$$C_{mm'}=\sum_{im_i}a^\dagger_{1m,im_i}a_{1m',im_i}-\frac{\Omega}{3}\delta_{mm'}\ , \tag{2.2b}$$

$$\Omega_i=\frac{3(2i+1)}{2},\quad \Omega=\sum_i\Omega_i\ . \tag{2.2c}$$

The operators $A^{L\dagger}_M$ or $A^\dagger_{mm'}$ will be called collective pair-creation operators, since they are a coherent mixture of two-particle excitations. The following operators are said to be noncollective:

$$a^\dagger_{km_k im_i}\equiv a^\dagger_i\ , \tag{2.3a}$$

$$A^\dagger(i,i')^{\sigma_0}_{LM,I\mu}=[a^\dagger_{1i}a^\dagger_{1i'}]^{\sigma_0}_{LM,I\mu}\equiv A^\dagger(i,i'),\quad I\neq 0\ , \tag{2.3b}$$

where a shorthand notation has been used; $a^\dagger_i$ represents single-particle excitation, while $A^\dagger(i,i')$ is an irreducible tensor of the group $\mathrm{SU(3)}\times\mathrm{SO}^i(3)$ with $\sigma_0$ as the SU(3) irrep label, and is called a noncollective pair. A state with $u$ nucleons which is totally free of coherent pairs is called a pairless state,

$$A_{LM}|\sigma\alpha,I\mu\rangle=0\ ,\qquad \text{for all possible sublevels } \alpha \text{ and } LM\ , \tag{2.4a}$$

where the orthonormal pairless state $|\sigma\alpha,I\mu\rangle$ is classified according to the irreps of the group $\mathrm{U(3)}\times\mathrm{SO}^i(3)$, $\sigma\equiv[\sigma]\equiv[\sigma_1\sigma_2\sigma_3]$ is U(3) partition label for the $u$ unpaired fermions with

$$u=\sigma_1+\sigma_2+\sigma_3 \ , \tag{2.4b}$$

$\alpha$ are the subgroup labels for U(3), while $u$ is called the heritage (or generalized seniority).

A nonorthonormal fermion state with $N$ collective pairs and $u$ unpaired nucleons can be constructed as

$$|\phi_n\rangle \equiv [Z^n(A^\dagger)\times|\sigma\rangle]_\alpha^{[\omega]\rho}$$
$$\equiv|(\mathbf{n}\times\sigma)[\omega]\rho,\alpha\rangle \ , \tag{2.5a}$$

where the square brackets denote U(3) coupling [a right to left coupling order will be used henceforth unless otherwise stated, as in (4.14) and (6.7)], $Z^{\mathbf{n}}(\mathbf{A}^\dagger)$ is a homogeneous polynomial of $A_M^{L\dagger}$ of order $N$ belonging to the U(3) irrep $\mathbf{n}\equiv[\mathbf{n}]\equiv[n_1n_2n_3]$ with

$$N=\frac{n}{2}, \quad n=n_1+n_2+n_3 \ , \tag{2.5b}$$

and $\rho$ is the multiplicity for $\sigma\times\mathbf{n}\rightarrow\omega$. In (2.5a) we omit the quantum numbers $I\mu$ in the pairless state and use the shorthand notation $|\phi_n\rangle$. We use a single index $n$ to denote $([\mathbf{n}],\rho)$ and assume that there are altogether $\mathbb{N}_b$ sets of quantum numbers $([\mathbf{n}],\rho)$,

$$n\equiv([\mathbf{n}],\rho)=1,2,\ldots,\mathbb{N}_b \ .$$

Now we turn to the general case but keep the Sp(6) concrete case in our mind. Suppose we have a fermion system with $N$ coherent fermion pairs $A_{ab}^\dagger$ defined in some way and $u$ unpaired fermions. The quantum numbers $ab$ can be understood either as the quantum numbers $mm'$ in (2.2), or $LM$ in (2.1). The states for such a system are denoted by

$$|\phi_n\rangle=f_n(A^\dagger)|\sigma\rangle, \quad n=1,2,\ldots,\mathbb{N}_b \ , \tag{2.6a}$$

where $f_n(A^\dagger)$ is a homogeneous polynomial of the pair creation operators $A_{ab}^\dagger$ of order $N$ and $|\sigma\rangle$ are the pairless states,

$$A_{ab}|\sigma\rangle=0 \ ,$$
$$\text{for all possible sublabels } \sigma \text{ and } ab \ , \tag{2.6b}$$

where the quantum number $\sigma$ is now understood as merely an index enumerating all possible states for the $u$ unpaired fermions. The orthonormal states $|\sigma\rangle$ are called the intrinsic states in the VCS theory.

The $\mathbb{N}_b$ basis vectors $|\phi_n\rangle$ are not only nonorthogonal but also may be linearly dependent due to the Pauli principle. Their overlap is defined by

$$\langle\phi_m|\phi_n\rangle=(KK^\dagger)_{mn}, \quad m,n=1,2,\ldots,\mathbb{N}_b \ . \tag{2.7}$$

Let $\mathbf{V}^{(j)}=(V_1^{(j)},V_2^{(j)},\ldots,V_{\mathbb{N}_b}^{(j)})_{\rm col}$ be the eigenvectors of the Hermitian matrix $KK^\dagger$,

$$(KK^\dagger)\mathbf{V}^{(j)}=\lambda_j\mathbf{V}^{(j)} \ , \tag{2.8a}$$

and suppose that there are $\mathbb{N}_f$ nonzero eigenvalues,

$$\lambda_j\neq0, \quad \text{for } j=1,2,\ldots,\mathbb{N}_f \ . \tag{2.8b}$$

The integer $\mathbb{N}_f$ gives the number of linearly independent *fermion* states, evidently, $\mathbb{N}_f\le\mathbb{N}_b$. The $\mathbb{N}_b$ vectors $\mathbf{V}^{(j)}$, $j=1,2,\ldots,\mathbb{N}_b$, form an orthonormal complete set,

$$\sum_{n=1}^{\mathbb{N}_b}V_n^{(j)}V_n^{(j')*}=\delta_{jj'}, \quad \sum_{j=1}^{\mathbb{N}_b}V_m^{(j)}V_n^{(j)*}=\delta_{mn} \ . \tag{2.9a}$$

The existence of $\mathbb{N}_b-\mathbb{N}_f$ zero eigenvalues signifies that there are $\mathbb{N}_b-\mathbb{N}_f$ spurious states, or $\mathbb{N}_b-\mathbb{N}_f$ linear relations among the $\mathbb{N}_b$ basis vectors,

$$\sum_{n=1}^{\mathbb{N}_b}V_n^{(j)}|\phi_n\rangle=0, \quad j=\mathbb{N}_f+1,\ldots,\mathbb{N}_b \ . \tag{2.9b}$$

Let U be the unitary matrix formed by the $\mathbb{N}_b$ column vectors $\mathbf{V}^{(j)}$, i.e., $\mathrm{U}_{n,j}=V_n^{(j)}$. Then Eq. (2.8a) can be rewritten as $\mathrm{U}^\dagger KK^\dagger\mathrm{U}=\Lambda$ or $\mathrm{U}^\dagger K(\mathrm{U}^\dagger K)^\dagger=\Lambda$, where $\Lambda$ is a diagonal matrix, $\Lambda=\mathrm{diag}(\lambda_1,\ldots,\lambda_{\mathbb{N}_f},0,\ldots,0)$. The solution $K$ is not unique and can be chosen in such a way so that $\mathrm{U}^\dagger K$ is real. Therefore $\mathrm{U}^\dagger K=\Lambda^{1/2}$, or $K=\mathrm{U}\Lambda^{1/2}$. Deleting the null columns we get

$$K_{n,j}=\sqrt{\lambda_j}V_n^{(j)} \ ,$$
$$j=1,2,\ldots,\mathbb{N}_f, \quad n=1,\ldots,\mathbb{N}_b \ . \tag{2.10a}$$

$K$ is an $\mathbb{N}_b\times\mathbb{N}_f$ rectangular matrix and the inverse of $K$ does not exist. However, the left inverse of $K$, designated as $K_L^{-1}$, exists,

$$(K_L^{-1})_{j,n}=\frac{1}{\sqrt{\lambda_j}}V_n^{(j)*} \ ,$$
$$j=1,2,\ldots,\mathbb{N}_f, \quad n=1,\ldots,\mathbb{N}_b \ . \tag{2.10b}$$

From (2.10a) and (2.9) we see that

$$(KK^\dagger)_{mn}=\sum_{j=1}^{\mathbb{N}_f}\lambda_jV_m^{(j)}V_n^{(j)*}, \quad K_L^{-1}K=1 \ . \tag{2.10c}$$

However

$$KK_L^{-1}\neq1, \quad \text{unless } \mathbb{N}_f=\mathbb{N}_b \ , \tag{2.10d}$$

i.e., $K_L^{-1}$ is the inverse of $K$ only when it is on the left, but is in general not true when it is on the right. (In all previous publications[1–12] on the VCS theory, it has not been emphasized that $K^{-1}$ in general does not exist. The matrix element $K^{-1}$ defined in[9–12] should be understood as $K_L^{-1}$; for further discussion, see Sec. VIII.)

It is easy to verify that the orthonormalized states can be expressed as

$$|\psi_j\rangle=\sum_{n=1}^{\mathbb{N}_b}(\kappa_L^{-1})_{j,n}|\phi_n\rangle$$
$$=\frac{1}{\sqrt{\lambda_j}}\sum_{n=1}^{\mathbb{N}_b}V_n^{(j)}|\phi_n\rangle, \quad j=1,2,\ldots,\mathbb{N}_f \ . \tag{2.11a}$$

Notice that here $\kappa=K^*$ is introduced for avoiding an extensive use of $K^*$ in later equations. For most applications $K$ is real and the distinction between $K$ and $\kappa$ vanishes.

In the fermion space, the $\mathbb{N}_f$ states $|\psi_j\rangle$ form an orthonormal complete set

$$\langle \psi_{j'}|\psi_j\rangle=\delta_{j'j}, \quad \sum_{j=1}^{\mathbb{N}_f} |\psi_j\rangle\langle\psi_j|=1 \ . \tag{2.11b}$$

By multiplying (2.11a) with $\sqrt{\lambda_j}V_m^{(j)*}$, summing over $j$ from 1 to $\mathbb{N}_b$, and using (2.9a) we obtain the inverse of (2.11a),

$$|\phi_n\rangle=\sum_{j=1}^{\mathbb{N}_f}\kappa_{n,j}|\psi_j\rangle, \quad n=1,2,\ldots,\mathbb{N}_b \ . \tag{2.11c}$$

From (2.11c) we have

$$K_{nj}=(\kappa_{n,j})^*=\langle\phi_n|\psi_j\rangle \tag{2.11d}$$

Notice that from (2.11a) and (2.11c) we have

$$|\phi_n\rangle=\sum_{j=1}^{\mathbb{N}_f}\kappa_{n,j}(\kappa_L^{-1})_{j,m}|\phi_m\rangle \ . \tag{2.11e}$$

However from (2.11e) we cannot infer $KK_L^{-1}=1$ or

$$\sum_{j=1}^{\mathbb{N}_f}\kappa_{n,j}(\kappa_L^{-1})_{j,m}=\delta_{mn} \ ,$$

unless $\mathbb{N}_f=\mathbb{N}_b$, since $|\phi_m\rangle$ is linearly dependent if $\mathbb{N}_f<\mathbb{N}_b$. Notice that if there are no collective pairs at all, one has

$$|\psi_j\rangle=|\phi_n\rangle=|\sigma\rangle, \quad K_{nj}=1, \quad j=n=1 \ . \tag{2.12}$$

Suppose that under a fermion operator $O$ the nonorthonormal basis is transformed as

$$O|\phi_n\rangle=\sum_{n'}O_{n'n}|\phi_{n'}\rangle \ . \tag{2.13}$$

Notice that here $n$ and $n'$ may range over a different set of quantum numbers. In contrast to (2.6a) we have

$$|\phi_{n'}\rangle=f_{n'}(A^\dagger)|\sigma'\rangle, \quad n'=1,2,\ldots,\mathbb{N}_{b'} \ .$$

Now let us calculate the matrix elements of the operator $O$ in the orthonormal basis $|\psi_j\rangle$. Using (2.11a) and (2.13) we have

$$O|\psi_j\rangle=\sum_{nn'}(\kappa_L^{-1})_{j,m}O_{n'n}|\phi_{n'}\rangle \ . \tag{2.14a}$$

Using (2.11d) we have

$$\langle\psi_{j'}|O|\psi_j\rangle=\sum_{nn'}(\kappa_L^{-1})_{j,n}O_{n'n}\kappa_{n',j'} \ . \tag{2.14b}$$

Specified to the Sp(6) case, Eqs. (2.7) and (2.10a) become

$$\langle(\mathbf{n}'\times\sigma)[\omega]\rho',\alpha|(\mathbf{n}\times\sigma)[\omega]\rho,\alpha\rangle =[KK^\dagger(\sigma\omega)]_{\mathbf{n}'\rho',\mathbf{n}\rho} \ , \tag{2.15}$$

$$\begin{aligned}&\kappa_{n,j}\to\kappa(\sigma\omega)_{\mathbf{n}\rho,j} \ ,\\ &(\kappa_L^{-1})_{j,n}\to(\kappa_L^{-1}(\sigma\omega))_{j,\mathbf{n}\rho}\equiv\kappa_L^{-1}(\sigma\omega)_{j,\mathbf{n}\rho} \ .\end{aligned} \tag{2.16}$$

The orthonormal fermion state (2.11a) is

$$|\sigma\omega j,\alpha\rangle=\sum_{\mathbf{n}\rho}\kappa_L^{-1}(\sigma\omega)_{j,\mathbf{n}\rho}|(\mathbf{n}\times\sigma)[\omega]\rho,\alpha\rangle \ , \tag{2.17}$$

where $j=1,2,\ldots,\mathbb{N}_f$ is the inner multiplicity for the subduction

$$\left[\frac{\Omega}{2}-\sigma_3,\frac{\Omega}{2}-\sigma_2\frac{\Omega}{2}-\sigma_1\right]\downarrow[\omega]$$

of Sp(6)↓U(3), where

$$\left[\frac{\Omega}{2}-\sigma_3,\frac{\Omega}{2}-\sigma_2\frac{\Omega}{2}-\sigma_1\right]$$

is the Cartan-Weyl irrep label for $\mathrm{Sp}_6$, and $\mathbb{N}_f$ gives the number of Pauli-allowed states for given $[\sigma]$, $[\omega]$, and $\alpha$.

## III. THE MOTIVATION FOR A NONUNITARY MAPPING

The next question is how to calculate the matrix elements $O_{n'n}$. Suppose that $\mathbb{N}_{f'}=\mathbb{N}_{b'}$, then we can introduce the dual basis

$$\langle\bar{\phi}_{m'}|=\sum_{n'}(KK^\dagger)^{-1}_{n'm'}\langle\phi_{n'}| \ ,$$

which satisfies $\langle\bar{\phi}_{m'}|\phi_{n'}\rangle=\delta_{m'n'}$ and $O_{n'n}$ can be expressed as

$$O_{n'n}=\langle\bar{\phi}_{n'}|O|\phi_n\rangle \ .$$

In the general cases with $\mathbb{N}_{f'}<\mathbb{N}_{b'}$, the dual basis does not exist and thus we even do not know how to express $O_{n'n}$ let alone how to calculate it. Therefore the matrix element $O_{n'n}$ is exceedingly difficult to compute in the fermion space. To avoid this difficulty we will map the fermion space to a boson+fermion space, or a *hybrid* space. The required mapping is denoted by $\mathbb{H}$ which maps a fermion operator $O$ into a hybrid operator $\mathbb{H}(O)$,

$$O\to\mathbb{H}(O) \ . \tag{3.1}$$

The mapping is required to satisfy the following conditions.

1. The fermion pair creation operators are mapped into boson creation operators

$$\mathbb{H}(A^\dagger_{ab})=\mathbf{b}^\dagger_{ab} \ . \tag{3.2a}$$

2. $\mathbb{H}$ is a linear and isomorphic mapping,

$$\begin{aligned}&\mathbb{H}(c_1O_1+c_2O_2)=c_1\mathbb{H}(O_1)+c_2\mathbb{H}(O_2) \ ,\\ &\mathbb{H}(O_1O_2)=\mathbb{H}(O_1)\mathbb{H}(O_2) \ .\end{aligned} \tag{3.2b}$$

3. The fermion vacuum state $|0\rangle$ is mapped into the fermion-boson vacuum state $|0)$, while the pairless state $|\sigma\rangle$ remains the same but will be denoted by $|\sigma)$,

$$|0\rangle\to|0), \quad |\sigma\rangle\to|\sigma)=|\sigma\rangle \ . \tag{3.3}$$

Due to the isomorphism (3.2b) we have

$$\begin{aligned}&\mathbb{H}(f_n(A^\dagger))=f_n(\mathbb{H}(A^\dagger))=f_n(b^\dagger) \ ,\\ &\mathbb{H}(Of_n(A^\dagger))=\mathbb{H}(O)f_n(b^\dagger) \ .\end{aligned}$$

Therefore under the mapping $\mathbb{H}$, the fermion state (2.6a) goes over to a hybrid state,

$$|\phi_n\rangle = f_n(A^\dagger)|\sigma\rangle \to |\phi_n) = f_n(b^\dagger)|\sigma) ,\quad n = 1,2,\ldots,\mathbb{N}_b , \qquad (3.4a)$$

while Eq. (2.13) is mapped into

$$O|\phi_n\rangle = \sum_{n'} O_{n'n}|\phi_{n'}\rangle \to \mathbb{H}(O)|\phi_n) = \sum_{n'} O_{n'n}|\phi_{n'}) . \qquad (3.4b)$$

Notice that in this paper we always use angular (round) brackets to represent fermion (boson, or boson+fermion) states. The pairless states $|\sigma')$ are already assumed to be orthonormal, whereas the boson wave function $f_{n'}(b^\dagger)|0)$ can be easily normalized (it is tacitly understood that we are using the usual boson inner product, i.e., the Bargmann inner product). Therefore the $\mathbb{N}_{b'}$ fermion-boson, or hybrid states $|\phi_{n'})$ can be assumed to be orthonormal and form an orthonormal complete set,

$$(\phi_{m'}|\phi_{n'}) = \delta_{m'n'},\quad \sum_{n'=1}^{\mathbb{N}_{b'}} |\phi_{n'})(\phi_{n'}| = 1 . \qquad (3.4c)$$

Similarly for the states $|\phi_n)$. From (3.4b) and (3.4c) we then have

$$O_{n'n} = (\phi_{n'}|\mathbb{H}(O)|\phi_n) , \qquad (3.5)$$

namely, the calculation of the matrix elements of a fermion operator between the nonorthogonal and *linearly dependent* fermion bases $|\phi_n\rangle$ and $|\phi_{n'}\rangle$, $n(n') = 1,2,\ldots,\mathbb{N}_b(\mathbb{N}_{b'})$, is converted to a much easier one of computing the matrix elements of the operator $\mathbb{H}(O)$ between the orthonormal (and thus necessarily *linearly independent*) boson-fermion bases $|\phi_n)$ and $|\phi_{n'})$, $n(n') = 1,2,\ldots,\mathbb{N}_b(\mathbb{N}_{b'})$. This of course has great significance for practical applications, since there is no Pauli principle among bosons and between bosons and fermions in the hybrid space.

Obviously, the mapping $\mathbb{H}$ from the *nonorthonormal basis* $\{|\phi_n\rangle|\}$ to the *orthonormal basis* $\{|\phi_n)\}$ is certainly not unitary with respect to the Bargmann measure. In the following we will show that the mapping $\mathbb{H}$ is an outgrowth of the VCS mapping.

Specialized to the Sp(6) case, Eqs. (3.4a) and (3.4c) become

$$[Z^{\mathbf{n}}(A^\dagger)\times|\sigma\rangle]_\alpha^{[\omega]\rho} \equiv |(\mathbf{n}\times\sigma)[\omega]\rho,\alpha\rangle \to [(Z^{\mathbf{n}}(b^\dagger)\times|\sigma)]_\alpha^{[\omega]\rho} \equiv |(\mathbf{n}\times\sigma)[\omega]\rho,\alpha) , \qquad (3.6a)$$

$$((\mathbf{n}'\times\sigma)[\omega]\rho',\alpha|(\mathbf{n}\times\sigma)[\omega]\rho,\alpha) = \delta_{\mathbf{n}'\rho',\mathbf{n}\rho} . \qquad (3.6b)$$

It is worth mentioning that the polynomial $Z^{\mathbf{n}}(\mathbf{b}^\dagger)|0)$ is just the IBM wave function for even nuclei in the SU(3) limit.[22]

## IV. THE VECTOR COHERENT STATE THEORY REVISITED VIA THE HYBRID MAPPING H

Now we are going to show that the required hybrid mapping $\mathbb{H}$ is related to the VCS mapping[12] $\Gamma$ by

$$O \to \mathbb{H}(O) = (\Gamma(O^\dagger))^\dagger . \qquad (4.1)$$

The hybrid operator $\Gamma(O)$ which is to act on the boson-fermion space is obtained from the $z$-dependent fermion operator $\hat{\Gamma}(O)$,

$$\hat{\Gamma}(O) \equiv \exp(\mathbf{z}\cdot\mathbf{A})O\exp(-\mathbf{z}\cdot\mathbf{A}) = O + [\mathbf{z}\cdot\mathbf{A},O] + \frac{1}{2!}[\mathbf{z}\cdot\mathbf{A},[\mathbf{z}\cdot\mathbf{A},O]] + \cdots \qquad (4.2)$$

by the following three steps.[12]

1. Delete $A_{ab}^\dagger$ (since $\langle\sigma|A_{ab}^\dagger = 0$).
2. Make the substitutions $A_{ab} \to \nabla_{ab} \to b_{ab}$ and $z_{ab} \to b_{ab}^\dagger$, where $\nabla_{ab}$ is the differentiation operator with respect to $z_{ab}$, while $b_{ab}$ and $b_{ab}^\dagger$ are boson creation and annihilation operators.
3. Shift all the remaining fermion operators $O_f$ in $\hat{\Gamma}(O)$ to the left (a reminiscence of the fact that the fermion operators commute with $\nabla_{ab}$ and $z_{ab}$, for further discussion, see Sec. VIII) and change them into the *intrinsic* operators denoted by the shadowed symbols $\mathbb{O}_f$, e.g.,

$$a_i^\dagger \to \mathbb{a}_i^\dagger,\quad a_i \to \mathbb{a},\quad A(i,i') \to \mathbb{A}(i,i') . \qquad (4.3)$$

The intrinsic operator will be called pseudofermion, or "*fermion*" operator, which acts on the intrinsic (or pairless) state $|\sigma)$. The commutation relations between the pseudofermion operators $(\mathbb{a}_i^\dagger,\mathbb{a}_i)$ and the true fermion operators $(a_i^\dagger,a_i)$ are unknown. The essence of the pseudofermions will be discussed in Sec. VII.

From (4.1) and (4.2) it is seen that the mappings for annihilation operators are always very simple,

$$\hat{\Gamma}(O) = O,\quad \text{for } O = A_{ab}, a_i, A(i,i') ; \qquad (4.4a)$$

$$\Gamma(A_{ab}) = b_{ab},\quad \Gamma(a_i) = \mathbb{a}_i ,\quad \Gamma(A(i,i')) = \mathbb{A}(i,i') ; \qquad (4.4b)$$

$$\mathbb{H}(A_{ab}^\dagger) = b_{ab}^\dagger,\quad \mathbb{H}(a_i^\dagger) = (\mathbb{a}_i)^\dagger ,\quad \mathbb{H}(A^\dagger(i,i')) = (\mathbb{A}(i,i'))^\dagger , \qquad (4.4c)$$

independent of the concrete structure of the collective pairs. Equation (4.4c) satisfies the requirement (3.2a).

One of the important features of the VCS mapping $\Gamma$ is that it is a linear and ismorphic mapping,[1,2]

$$\Gamma(c_1O_1 + c_2O_2) = c_1\Gamma(O_1) + c_2\Gamma(O_2) ,\quad \Gamma(O_1O_2) = \Gamma(O_1)\Gamma(O_2) . \qquad (4.5)$$

Using the definition (4.1), from (4.5) we get the requirement (3.2b). Therefore the mapping $\mathbb{H}$ (4.1) is what we are looking for.

From (2.14b) and (3.5) we have

$$\langle\psi_{j'}|O|\psi_j\rangle = \sum_{nn'} (\kappa_L^{-1})_{j,n}(\phi_{n'}|\mathbb{H}(O)|\phi_n)\kappa_{n'j'} . \qquad (4.6a)$$

The hybrid mapping $\mathbb{H}$ is introduced for implementing the mapping (3.2) and getting (3.5). Once these have been done we switch back to the more familiar VCS mapping $\Gamma$. According to (4.1)

$$O_{n'n} = (\phi_{n'}|\mathbb{H}(O)|\phi_n) = (\phi_n|\Gamma(O^\dagger)|\phi_{n'})^* . \qquad (4.6b)$$

By letting $O \rightarrow O^\dagger$ in (4.6), remembering $\kappa^* = K$ and interchanging $j \leftrightarrow j'$, $n \leftrightarrow n'$, we get the familiar equation in the VCS theory,

$$\langle \psi_{j'}|O|\psi_j\rangle = \sum_{nn'} (K_L^{-1})_{j',n'}(\phi_{n'}|\Gamma(O)|\phi_n)K_{nj} \ . \tag{4.7a}$$

If there are no collective pairs in both the initial and final states, due to (2.12) Eq. (4.7a) collapses to

$$\langle \sigma'|O|\sigma\rangle = (\sigma'|\Gamma(O)|\sigma) \ . \tag{4.7b}$$

It is convenient to rewrite the master equation (4.7a) as a matrix equation,

$$\mathbf{O} = K_L^{-1}\Gamma^{(\text{hyb})}(O)K \ , \tag{4.7c}$$

where $\mathbf{O}$ and $\Gamma^{(\text{hyb})}(O)$ are $\mathbb{N}_{f'}\times\mathbb{N}_f$ and $\mathbb{N}_{b'}\times\mathbb{N}_b$ matrices, respectively. It should be stressed that Eq. (4.7c) cannot be inverted,

$$\Gamma^{(\text{hyb})}(O) \neq K\mathbf{O}K_L^{-1} \ , \tag{4.8a}$$

due to (2.10d). The reason is obvious: the elements of an $\mathbb{N}_{f'}\times\mathbb{N}_f$ matrix can be expressed in terms of the elements of an $\mathbb{N}_{b'}\times\mathbb{N}_b$ matrix if $\mathbb{N}_{f'} < \mathbb{N}_{b'}$ and $\mathbb{N}_f < \mathbb{N}_b$, but not the reverse. The existence of $K_L^{-1}$ instead of $K^{-1}$ finds its origin in the *oneway mapping* from the fermion space to the hybrid space. Only when $\mathbb{N}_{f'} = \mathbb{N}_{b'}$ *and* $\mathbb{N}_f = \mathbb{N}_b$ can we have the inverse of (4.7c),

$$\Gamma^{(\text{hyb})}(O) = K\mathbf{O}K^{-1} \ . \tag{4.8b}$$

Equation (4.7) enables us to transfer back and forth between the fermion space and the hybrid space for calculating the matrix elements, and is called the *master equation*.

From the matrix equation (4.7c) we have

$$K_L^{-1}\Gamma^{(\text{hyb})}(O^\dagger)K = K^\dagger\Gamma^{(\text{hyb})}(O)^\dagger(K_L^{-1})^\dagger \ , \tag{4.9a}$$

which is equivalent to the following:

$$\sum_{mn}^{\mathbb{N}_b} \Gamma^{(\text{hyb})}(O^\dagger)_{mn}\lambda_j V_m^{(i)*}V_n^{(j)} = \sum_{mn}^{\mathbb{N}_b} (\Gamma^{(\text{hyb})}(O^\dagger))_{mn}\lambda_i V_m^{(i)*}V_n^{(j)} \ . \tag{4.9b}$$

Multiplying (4.9b) with $V_{m'}^{(i)}$, summing over $i$ from 1 to $\mathbb{N}_b$, and using (2.9a), then multiplying the result with $V_{n'}^{(j)*}$ and summing over $j$ from 1 to $\mathbb{N}_b$, and using (2.9a) again, we get

$$\Gamma^{(\text{hyb})}(O^\dagger)KK^\dagger = KK^\dagger\Gamma^{(\text{hyb})}(O)^\dagger \ . \tag{4.9c}$$

Equation (4.9c) is a key equation in the VCS theory[2,10] for calculating the $K$ matrix and was derived through the introduction of the unitary mapping $\gamma(O)$ (see Sec. VIII).

Letting $O = A_{ab}$ in (4.7c) and (4.9c) we get

$$\mathbf{A}_{ab} = K_L^{-1}\mathbf{b}_{ab}K, \quad \Gamma^{(\text{hyb})}(A_{ab}^\dagger)KK^\dagger = KK^\dagger\mathbf{b}_{ab}^\dagger \ . \tag{4.10}$$

Notice that Eq. (4.7c) only shows the relation between the representation matrix of a fermion operator in the fermion space and that of the operator $\Gamma(O)$ in the hybrid space, but does not in the least mean that a fermion operator $O$ can be expressed in terms of its corresponding hybrid operator $\Gamma(O)$.

Specialized to the Sp(6) case,[9,12] we have

$$\hat{\Gamma}(A_{ab}^\dagger) = A_{ab}^\dagger - (C\mathbf{z})_{ab} - (C\mathbf{z})_{ba} - \sum_{cd} z_{ac}z_{bd}A_{cd} \ , \tag{4.11a}$$

$$\hat{\Gamma}(C_{ab}) = C_{ab} + \sum_c \mathbf{z}_{ac}A_{cb} \ , \tag{4.11b}$$

$$\hat{\Gamma}(a_i^\dagger) = a_i^\dagger - 2[\tilde{a}_i^{(01)}\times\mathbf{z}^{(20)}]^{(10)} \ . \tag{4.11c}$$

From (4.4b) and (4.11) we immediately obtain the following hybrid operators:

$$\Gamma(A_{ab}) = b_{ab}, \quad \Gamma(a_i) = \mathbb{a}_i \ , \quad \Gamma(A(i,i')) = \mathbb{A}(i,i') \ , \tag{4.12a}$$

$$\Gamma(A_{ab}^\dagger) = -(\mathbb{C}\mathbf{b}^\dagger)_{ab} - (\mathbb{C}\mathbf{b}^\dagger)_{ba} - \sum_{cd} \mathbf{b}_{ac}^\dagger\mathbf{b}_{bd}^\dagger\mathbf{b}_{cd} \ , \tag{4.12b}$$

$$\Gamma(C_{ab}) = \mathbb{C}_{ab} + \sum_c \mathbf{b}_{ac}^\dagger\mathbf{b}_{cb} \ , \tag{4.12c}$$

$$\Gamma(a_i^\dagger) = \mathbb{a}_i^\dagger - 2[\tilde{\mathbb{a}}_i^{(01)}\times\mathbf{b}^{(20)\dagger}]^{(10)} \ . \tag{4.12d}$$

More examples are given in Eq. (12) of Ref. 12 (notice the notation difference: $z \rightarrow b^\dagger$, $\mathbb{b}_{km_k im_i}^\dagger \rightarrow \mathbb{a}_{km_k im_i}^\dagger$). From (4.12) it is seen that except for the U(3) generators $C_{ab}$, the mapping $\Gamma$ is not unitary.

For Sp(6), the master equation (4.7a) becomes

$$\langle \sigma'\omega'j'\|O\|\sigma\omega j\rangle = \sum_{\mathbf{n}\rho\mathbf{n}'\rho'} K_L^{-1}(\sigma'\omega')_{j',\mathbf{n}'\rho'}((\mathbf{n}'\times\sigma')[\omega']\rho'\|\Gamma^{(\text{hyb})}(O)\|(\mathbf{n}\times\sigma)[\omega]\rho)K(\sigma\omega)_{\mathbf{n}\rho,j} \ , \tag{4.13}$$

where the double lined matrix elements are the SU(3) reduced matrix elements defined by

$$\langle\omega'\|T^{\omega_0}\|\omega\rangle = \langle\omega'\alpha'|[T^{\omega_0}\times\omega\rangle]_{\alpha'}^{\omega'} = \left[\frac{\dim(\omega)}{\dim(\omega')}\right]^{1/2}\langle[T^{\bar{\omega}_0}\times\omega']_{\text{L}\rightarrow\text{R}}{}_{\alpha}^{\omega}|\omega\alpha\rangle \ , \tag{4.14}$$

with the subscript L$\rightarrow$R indicating a left to right coupling which is exceptional to our otherwise right to left coupling order convention, $\bar{\omega}_0$ being the contragredient representation of $\omega_0$, and $\dim(\omega)$ being the dimension for the SU(3) irrep $\omega$.

Since $A^\dagger$ and $Z^{\mathbf{n}}(A^\dagger)$ are $\text{SO}^i(3)$ scalars, in calculating the reduced matrix elements of an SU(3) irreducible tensor, we can ignore all the $i$-spin labels. The matrix elements of an $\text{SU(3)}\times\text{SO}^i(3)$ irreducible tensor can be obtained from the SU(3) reduced matrix element by simply adding the labels $I$, $I'$, and $I_0$ to the initial states, final states, and irreducible tensor, respectively, and by replacing the double-bar matrix element "$\| \;\|$" with the triple-bar matrix element "$|||\;|||$," as in Ref. 12.

## V. THE PAULI DYNAMICAL FACTORS

The restriction of the possible irreps of a dynamical group due to the Pauli principle has been called the Pauli dynamical effect.[15,16] For the Sp(6)⊃U(3) case, it is known that the U(3) Young diagram for nucleons in the pseudo-orbital space can have at most $2\Omega/3$ columns. In this section we are going to show that all the Pauli effects are automatically reflected in the $K$ factors, which are then named dynamical Pauli factors.

Hecht[9] has given all the $K$ matrices for the Sp(6) case with $u=0,1,2$. In order to convey some feelings about these $K$ matrices, in Table I we give some examples which belong to the case of simple states[9] and thus are associated with diagonal $K$ matrices.

From Table I we see that

$$K(\sigma\omega)_{[\mathbf{n}][\mathbf{n}]}\rightarrow(\bar{\Omega})^{n/2}, \quad \text{when } \bar{\Omega}\equiv\frac{2\Omega}{3}\gg n\ . \tag{5.1}$$

On the other hand from (2.12) we know that if a state is entirely free of collective pairs, its $K$ matrix is equal to 1, $K(\sigma\sigma)_{[0][0]}=1$. Therefore, the magnitude of $K(\sigma\omega)_{[\mathbf{n}][\mathbf{n}]}$ can serve as a measure of the collectivity of the fermion state $|\sigma\omega j=[\mathbf{n}],\alpha\rangle$.

From Eq. (4.13) and Table I we see clearly how the $K$ matrix restricts the possible irreps of U(3). As an example, using (4.13) and Table I, we have[16]

$$\langle[0](n,0)L\|S^{\dagger}S\|[0](n,0)L\rangle=\frac{3[(2\Omega/3)-n+2]}{2}((n,0)L\|s^{\dagger}s\|(n,0)L)_{\text{IBM}}$$
$$\rightarrow 0, \quad \text{when } n=\frac{2\Omega}{3}+2\ , \tag{5.2}$$

where $((n,0)L\|s^{\dagger}s\|(n,0)L)_{\text{IBM}}$, the matrix elements of $s^{\dagger}s$ in the hybrid space (now a pure boson space), is precisely the expectation value of $s^{\dagger}s$ in the IBM. From Eq. (5.2) we see that although there is no restriction on $n$ in the boson matrix elements $((n,0)L\|s^{\dagger}s\|(n,0)L)_{\text{IBM}}$, the dynamical Pauli factor $[3((2\Omega/3)-n+2)]/2$ automatically restricts the maximum possible columns for a U(3) Young diagram of the original fermion system to be $2\Omega/3$. We thus see that the hybrid mapping is free of the spurious-state problem, which haunted most of the other boson mapping procedures.

## VI. THE THREE STEPS FOR COMPUTING THE MATRIX ELEMENTS

Equation (4.13) shows that the evaluation of the matrix elements of an operator between two many-fermion states can be carried out in three steps.

*Step 1.* Switch from the fermion space to the hybrid space by mapping the $N'(N)$ collective fermion pairs in the final (initial) state into $N'(N)$ bosons. The Pauli factors $K_L^{-1}(\sigma'\omega')_{j',\mathbf{n}'\rho'}$ and $K(\sigma\omega)_{\mathbf{n}\rho,j}$ are related to the fermion pair→boson transformation. The disparity of the two Pauli factors for initial and final states stems from the nonunitarity of the mapping $\mathbb{H}$.

*Step 2.* In the hybrid space compute the matrix elements

$$M=((\mathbf{n}'\times\sigma')[\omega']\|\Gamma(O)\|(\mathbf{n}\times\sigma)[\omega])\ , \tag{6.1}$$

where for simplicity we ignore the multiplicity label $\rho$. The strategy used here is to eliminate all the bosons (the images of the collective pairs) in the bra vector $((\mathbf{n}'\times\sigma')[\omega']|$ so that the intrinsic operator in $\Gamma(O)$ sits adjacent to the intrinsic state $(\sigma'|$. The most general form of the hybrid operator can be written as an SU(3) coupled tensor,

$$\Gamma(O)=[\mathbb{O}_f^{\sigma_0'}\times Z^{\mathbf{n}_0}(b^{\dagger})]^{\omega_0}_{\mathrm{L}\rightarrow\mathrm{R}}\ . \tag{6.2}$$

There are two possibilities:

$$\text{Case 1.} \quad (\sigma'\|\mathbb{O}_f^{\sigma_0}\|\sigma)\neq 0\ . \tag{6.3}$$

For this case the matrix elements of $\Gamma(O)$ can be simply calculated by using standard Racah algebra,

TABLE I. Some simple examples of the $K$ matrix $K^2(\sigma\omega)_{[n][n]}$.[a]

| $[\sigma][\omega]$ | $[0][n]$ | $[1][n+1]$ | $[2][n+2]$ | $[2][n+1,1]$ | $[11][n+1,1]$ |
|---|---|---|---|---|---|
| $K^2(\sigma\omega)_{[n][n]}$ | $\frac{\bar{\Omega}!!}{(\bar{\Omega}-n)!!}$ | $\frac{(\bar{\Omega}-2)!!}{(\bar{\Omega}-n-2)!!}$ | $\frac{(\bar{\Omega}-4)!!}{(\bar{\Omega}-n-4)!!}$ | $\frac{\bar{\Omega}(\bar{\Omega}-4)!!}{(\bar{\Omega}-n-2)!!}$ | $\frac{(\bar{\Omega}-2)!!}{(\bar{\Omega}-n-2)!!}$ |

[a] $\bar{\Omega}=2\Omega/3$.

$$((\mathbf{n}'\times\sigma')[\omega']\rho'\|\Gamma(O)\|(\mathbf{n}\times\sigma)[\omega]\rho)=\begin{bmatrix}[\sigma] & [\mathbf{n}] & [\omega]\\ [\sigma_0] & [\mathbf{n}_0] & [\omega_0]\\ [\sigma'] & [\mathbf{n}'] & [\omega']\end{bmatrix}(\sigma'\|\mathbb{O}_f^{\sigma_0}\|\sigma)(\mathbf{n}'\|Z^{\mathbf{n}_0}(b^\dagger)\|\mathbf{n})\ , \tag{6.4}$$

where the first factor is the normalized SU(3) 9-$j$ coefficient. The reduced matrix elements of the boson operators $(\mathbf{n}'\|Z^{\mathbf{n}_0}(b^\dagger)\|\mathbf{n})$ can be easily calculated,[12] while those of the intrinsic operators are calculated in step 3:

$$\text{Case 2.}\quad (\sigma'\|\mathbb{O}_f^{\sigma_0}\|\sigma)=0\ . \tag{6.5a}$$

For example,

$$(\sigma' u\|\mathbb{A}(i,i')^{\sigma_0}\|\sigma u)=0\ . \tag{6.5b}$$

Since the pseudofermion operator $\mathbb{A}(i,i')^{\sigma_p}$ does not connect two pure intrinsic states in this case,

$$((\mathbf{n}'\times\sigma')[\omega'],u\|\mathbb{A}(i,i')^{\sigma_0}\|(\mathbf{n}\times\sigma)[\omega],u)\neq\delta_{\mathbf{n}\mathbf{n}'}(-)^{\hat\omega-\hat\omega'-\hat\sigma+\hat\sigma'}U(\sigma_0\sigma\omega'\mathbf{n};\sigma'\omega)(\sigma' u\|\mathbb{A}(i,i')^{\sigma_0}\|\sigma u)=0\ , \tag{6.6}$$

where $\hat\omega=\lambda_\omega+\mu_\omega$. Instead, the noncollective pair-annihilation operator $\mathbb{A}(i,i')^{\sigma_0}$ has to annihilate a boson in the polynomial $Z^{n}(b^\dagger)$ contained in the ket vector $|(\mathbf{n}\times\sigma)[\omega],u)$. To do this, one has to detach a boson from $Z^{\mathbf{n}}(b^\dagger)$ which is accomplished by inserting a set of intermediate states. Using (4.14) we have

$$\begin{aligned}&((\mathbf{n}'\times\sigma')[\omega'],u\|\mathbb{A}(i,i')^{\sigma_0}\|(\mathbf{n}\times\sigma)[\omega],u)\\ &\qquad=\sum_{\sigma''}((\mathbf{n}'\times\sigma')\omega'\|\mathbb{A}(i,i')^{\sigma_0}\|[\mathbf{n}'\times([2]\times\sigma)\sigma'']\omega)([\mathbf{n}'\times([2]\times\sigma)\sigma'']^{\omega}_{\alpha}|[\mathbf{n}\times\sigma]^{\omega}_{\alpha})\ .\end{aligned} \tag{6.7a}$$

Notice that since $|([2]\times\sigma)\sigma'')$ is not a pairless state,

$$\begin{aligned}&((\mathbf{n}'\times\sigma')\omega'\|\mathbb{A}(i,i')^{\sigma_0}\|[\mathbf{n}'\times([2]\times\sigma)\sigma'']^{\omega})\\ &\quad=(-)^{\hat\omega-\hat\omega'-\hat\sigma''+\hat\sigma'}U(\sigma_0\sigma''\omega'\mathbf{n};\sigma'\omega)(\sigma'\|\mathbb{A}(i,i')^{\sigma_0}\|([2]\times[\sigma])\sigma'')\ .\end{aligned} \tag{6.7b}$$

To calculate the left-hand side of (6.7b), the intrinsic operator must be shifted to the left where it can act on the intrinsic state $\sigma'$. We therefore use (4.14) to shift the operator $\mathbb{A}(i,i')^{\sigma_0}$ to the left of the intrinsic state $(\sigma'|$ and then recouple the tensors,

$$\begin{aligned}&((\mathbf{n}'\times\sigma')\omega'\|\mathbb{A}(i,i')^{\sigma_0}\|[\mathbf{n}'\times([2]\times\sigma)\sigma'']^{\omega})\\
&=\left[\frac{\dim(\omega)}{\dim(\omega')}\right]^{1/2}([(\mathbb{A}(i,i')^{\sigma_0})^\dagger\times(\sigma'\times\mathbf{n}')_{\mathrm{L}\to\mathrm{R}}\omega']_{\mathrm{L}\to\mathrm{R}\alpha}^{\omega}|[\mathbf{n}'\times([2]\times\sigma)\sigma'']^{\omega}_{\alpha})\\
&=\left[\frac{\dim(\omega)}{\dim(\omega')}\right]^{1/2}\sum_{\omega''}U(\tilde\sigma_0\sigma'\omega\mathbf{n}';\omega''\omega')([[\mathbb{A}(i,i')^{\sigma_0})^\dagger\times\sigma']_{\mathrm{L}\to\mathrm{R}}\omega''\times\mathbf{n}']_{\mathrm{L}\to\mathrm{R}\alpha}^{\omega}|[\mathbf{n}'\times([2]\times\sigma)\sigma'']^{\omega}_{\alpha})\\
&=\left[\frac{\dim(\omega)}{\dim(\omega')}\right]^{1/2}\sum_{\omega''}\delta_{\omega''\sigma''}U(\tilde\sigma_0\sigma'\omega\mathbf{n}';\sigma''\omega')([(\mathbb{A}(i,i')^{\sigma_0})^\dagger\times\sigma']_{\mathrm{L}\to\mathrm{R}}\sigma''\alpha''|([2]\times\sigma)\sigma''\alpha'')\\
&=\left[\frac{\dim(\omega)\dim(\sigma')}{\dim(\omega')\dim(\sigma'')}\right]^{1/2}U(\tilde\sigma_0\sigma'\omega\mathbf{n}';\sigma''\omega')(\sigma'\|\mathbb{A}(i,i')^{\sigma_0}\|([2]\times[\sigma])\sigma'')\ .\end{aligned} \tag{6.7c}$$

The second factor in (6.7a) is easy to calculate and the final result is

$$\begin{aligned}&((\mathbf{n}'\times\sigma')[\omega'],u\|\mathbb{A}(i,i')^{\sigma_0}\|(\mathbf{n}\times\sigma)[\omega],u)\\
&=\sum_{\sigma''}\left[\frac{\dim(\omega)\dim(\sigma')}{\dim(\omega')\dim(\sigma'')}\right]^{1/2}U(\tilde\sigma_0\sigma'\omega\mathbf{n}';\sigma''\omega')U(\sigma[2]\omega\mathbf{n}';\sigma''\mathbf{n})(\sigma' u\|\mathbb{A}(i,i')^{\sigma_0}\|([2]\times[\sigma])\sigma'' u)(\mathbf{n}\|b^\dagger\|\mathbf{n}')\ .\end{aligned} \tag{6.8}$$

From (6.4) and (6.8) it is seen that there is no Pauli factor in the second stage.

*Step 3.* Calculate the matrix element $(\sigma'\|\mathbb{O}_f^{\sigma_0}\|\sigma)$ of the "fermion" operators between the pairless states in (6.4) is simply equal to the matrix element of the corresponding real fermion operator [see Eq. (7.7) in Sec. VII],

$$(\sigma'\|\mathbb{O}_f^{\sigma_0}\|\sigma)=\langle\sigma'\|O_f^{\sigma_0}\|\sigma\rangle \ . \qquad (6.9)$$

In this case there is no Pauli factor, since there is no boson↔fermion transformation occurring.

For case 2 the evaluation of a matrix element such as $(\sigma'u+1\|\mathbb{a}\|(b^\dagger\times[\sigma])\omega u)$, or $(\sigma'u\|\mathbb{A}(i,i')^{\sigma_0}\|(b^\dagger\times[\sigma])\omega u)$, requires the use of the master equation to switch back to the fermion space, since the relation between the operators $(\mathbb{a}_i,\mathbb{a}_i^\dagger)$ and $(a_i,a_i^\dagger)$ are unknown. The calculation of these matrix elements will be discussed in the next section. From Eq. (7.9a) given below, it is seen that a Pauli factor $K^{-2}(\sigma\omega)_{[2][2]}=1/K^2(\sigma\omega)_{[2][2]}$ will appear in conjunction with the boson→fermion-pair transformation.

We thus vividly see that whenever a transformation between bosons and fermion pairs occurs, there appears a Pauli factor, which in an exact manner corrects the errors incurred by shifting from one space to the other. The boson↔fermion transformation occurs in step 1 and sometimes in step 3, but never in step 2.

## VII. THE PSEUDOFERMION OPERATOR

The intrinsic or pseudofermion operator is the key to the VCS theory or hybrid mapping. The discussion of its properties deserves a separate section.

### A. The intrinsic operator $\mathbb{C}_{ab}$

From (4.12) we see that for the generators of Sp(6), the only intrinsic operator is $\mathbb{C}_{ab}$. Let us begin with a study of $\mathbb{C}_{ab}$ which is the simplest intrinsic operator. From (4.12c) we know that in $\Gamma(C_{ab})$ there is no coupling between the intrinsic operator $\mathbb{C}_{ab}$ and the boson operators, or physically no coupling between the noncollective and collective motions. Therefore the pseudofermion operator $\mathbb{C}_{ab}$ commutes with the boson operators,

$$[\mathbb{C}_{ab},b_{cd}^\dagger]=[\mathbb{C}_{ab},b_{cd}]=0 \ . \qquad (7.1)$$

The meaning of the operators $\mathbb{C}_{ab}$ can be seen more clearly when they are written in the coupled form (2.1b). By using

$$P_M^L=\frac{\sqrt{3}}{2}\sum_{ab}\langle 1a,1b|LM\rangle(-)^{1+b}\left[C_{a,-b}+\frac{\Omega}{3}\delta_{a-b}\right],$$

and (4.12c) we have

$$\Gamma(P_M^L)=\underset{\text{intrinsic}}{\mathbb{P}_M^L}+\underset{\text{collective}}{\mathcal{P}_M^L} \ ; \qquad (7.2a)$$

$$\Gamma(P_0^0)=\hat{\mathbb{n}}/2+\hat{N}=u/2+N, \quad \hat{N}=s^\dagger s+d^\dagger\cdot d \ ; \qquad (7.2b)$$

$$\Gamma(P_M^l)=\sqrt{3/8}\mathbb{L}_M+\sqrt{3/8}\mathcal{L}_M \ , \quad \mathcal{L}_M=\sqrt{10}(d^\dagger\times\tilde{d})_M^l \ ; \qquad (7.2c)$$

$$\Gamma(P_M^2)=\mathbb{P}_M^2+\mathcal{P}_M^2 \ , \quad \mathcal{P}_M^2=(d^\dagger\times\tilde{s}+s^\dagger\times\tilde{d})_M^2+\frac{\sqrt{7}}{2}(d^\dagger\times\tilde{d})_M^2 \ , \qquad (7.2d)$$

where $\mathbb{L}_M(\mathcal{L}_M)$ and $\mathbb{P}_M^2(\mathcal{P}_M^2)$ are angular momentum and quadrupole operators, respectively, for the unpaired (paired) fermions. It is interesting to note that $\hat{N}$, $\mathcal{L}_M$, and $\mathcal{P}_M^2$ are exactly the operators used in the IBM.[14,22] Now it is quite clear that $\mathbb{P}_M^L$ are the contributions to the $2^L$-multipole operators due to the noncollective (or intrinsic) excitation, while $\mathcal{P}_M^L$ are the contributions due to the collective excitation. Therefore $\mathbb{P}_M^L$ and $\mathcal{P}_M^L$ can be called the intrinsic and collective operators, respectively, and Eq. (7.2) shows that the particle number, the angular momentum, and quadrupole moment are additive quantities due to the absence of the coupling between the noncollective and collective motions, in contrast to the single-or double-nucleon transfer operators, and multipole transition operator which involve such a coupling [see Eq. (4.12d), as well as Eqs. (12d) and (12e) of Ref. 12].

Hence we see that the pseudofermion operators $\mathbb{C}_{ab}$ are just the true fermion operators $C_{ab}$ acting on the pairless or intrinsic state. Since $(C_{ab})^\dagger=C_{ab}^\dagger=C_{ba}$, we have

$$(\mathbb{C}_{ab})^\dagger=\mathbb{C}_{ab}^\dagger=\mathbb{C}_{ba} \ . \qquad (7.3)$$

From (7.3) and (4.12) we conclude that for the generators of the Sp(6),

$$(\Gamma(O))^\dagger=\Gamma(O)^\dagger, \quad \text{for } O=A_{ab}^\dagger, A_{ab}, C_{ab} \ , \qquad (7.4)$$

where $\Gamma(O)^\dagger$ is the operator obtained from $\Gamma(O)$ by the conventional Hermitian conjugation (reversing the order of the operators in a product and interchanging creation and annihilation operators), while $(\Gamma^{(\text{hyb})}(O))^\dagger$ is defined also by the conventional meaning

$$(\phi_m|(\Gamma(O))^\dagger|\phi_n)=(\phi_n|\Gamma(O)|\phi_m)^* \ . \qquad (7.5)$$

It is also worth mentioning that the relation between the operator $\Gamma(O)$ and the operator $\hat{O}_{\alpha'\alpha}$ defined by Deenen and Quesne [Eq. (6.1b) of Ref. 4] is

$$\langle\sigma\alpha'|\Gamma(O)|\sigma\alpha\rangle=\hat{O}_{\alpha'\alpha}, \quad \text{for } O=A_{ab}^\dagger, A_{ab}, C_{ab} \ , \qquad (7.6)$$

where $\langle\sigma\alpha'|\Gamma(O)|\sigma\alpha\rangle=\hat{O}_{\alpha'\alpha}$ is a pure boson operator. Notice that although both $|\sigma\alpha\rangle$ and $|\sigma\alpha)$ are the same pairless state, the matrix elements $\langle\sigma\alpha'|\Gamma(O)|\sigma\alpha\rangle$ and $(\sigma\alpha'|\Gamma(O)|\sigma\alpha)$ are totally different; the former is an integration over the *fermion space* only while the latter is over *both the boson and fermion spaces*, with the result that the former is a function of the boson operators while the latter is a *c-number* equal to $\langle\sigma\alpha'|O|\sigma\alpha\rangle$ [see Eq. (4.7b)]. Notice that if $O$ is not a generator of Sp(6), we do not know how to calculate $\langle\sigma\alpha|\Gamma(O)|\sigma\alpha\rangle$, since the intrinsic operators in $\Gamma(O)$ contains "fermion"↔boson transformation terms instead of the fermion operators acting on the intrinsic state.

### B. The pseudofermion operators $\mathbb{a}_i$, $\mathbb{a}_i^\dagger$, and $\mathbb{A}(i,i')$

(a) Firstly from (4.7b) and (4.12a) we have

$$(\sigma'\alpha'|\mathbb{O}|\sigma\alpha)=\langle\sigma'\alpha'|O|\sigma\alpha\rangle\ ,\qquad \text{for } O=a_i,\ \text{or } A(i,i')\ . \tag{7.7}$$

Therefore the annihilation operator $a_i$ and $A(i,i')$ behave as true fermion operators between pairless states. By adding the heritage quantum number $u$ explicitly, (7.7) yields

$$(\sigma'u-1\|\tilde{\mathbb{a}}_i\|\sigma u)=\langle\sigma'u-1\|\tilde{a}_i\|\sigma u\rangle\neq 0\ , \tag{7.8a}$$

$$(\sigma'u+1\|\tilde{\mathbb{a}}_i\|\sigma u)=\langle\sigma'u+1\|\tilde{a}_i\|\sigma u\rangle=0\ , \tag{7.8b}$$

since the operator $a_i$ acting to the right on a state with heritage $u$ must lower the heritage to $u'=u-1$.

(b) Although $\mathbb{a}_i$ and $\mathbb{A}(i,i')$ behave as genuine fermion operators between the pairless states, it is no longer true if the initial state contains a boson. Now let us look at the matrix element $(\sigma'u'\|\tilde{\mathbb{a}}_i\|(b^\dagger\times[\sigma])\omega u)$. For calculating it we use the master equation (4.13) along with (2.17) to transfer from the hybrid space back to the fermion space,[12]

$$\begin{aligned}(\sigma'u'\|\tilde{\mathbb{a}}_i\|(b^\dagger\times[\sigma])\omega u)&=\frac{1}{K^2(\sigma\omega)_{[2][2]}}\langle\sigma'u'\|\tilde{a}_i\|(A^\dagger\times[\sigma])\omega u\rangle\\&=\frac{1}{K^2(\sigma\omega)_{[2][2]}}\sum_{\sigma_0}U([\sigma][2][\sigma'][11];[\omega][\sigma_0])\langle\sigma'u'\|[\tilde{a}_i,A^\dagger]^{\sigma_0}\|\sigma u\rangle\\&=\frac{2}{K^2(\sigma\omega)_{[2][2]}}U([\sigma][2][\sigma'][11];[\omega][1])\langle\sigma'u'\|a_i^\dagger\|\sigma u\rangle\ .\end{aligned} \tag{7.9a}$$

Letting $u'=u\pm1$ we obtain

$$(\sigma'u+1\|\tilde{\mathbb{a}}_i\|(b^\dagger\times[\sigma])\omega u)=\frac{2}{K^2(\sigma\omega)_{[2][2]}}U([\sigma][2][\sigma'][11];[\omega][1])\langle\sigma'u+1\|a_i^\dagger\|\sigma u\rangle\xrightarrow[\Omega\to\infty]{}0\ , \tag{7.9b}$$

$$(\sigma'u-1\|\tilde{\mathbb{a}}_i\|(b^\dagger\times[\sigma])\omega u)=0\ , \tag{7.9c}$$

where we have used the fact that $\langle\sigma'u-1\|a_i^\dagger\|\sigma u\rangle=0$ due to (7.8b). Note that the partition [2] in the $K$ matrix signifies there is *one* boson with SU(3) symmetry [2] which has been transformed into a fermion pair with the same symmetry.

Ignoring the angular momentum coupling, (7.9a) can be put in shorthand form,

$$\tilde{\mathbb{a}}_i\mathbf{b}^\dagger\to\frac{1}{K^2}\tilde{a}_iA^\dagger\to\frac{2U}{K^2}a_i^\dagger\ . \tag{7.9d}$$

It is seen that $\mathbb{a}_i$ can transform a boson into a fermion. The probability amplitude for this transformation is inversely proportional to $K^2(\sigma\omega)_{[2][2]}$. When $\Omega\to\infty$, the probability$\to$0 due to (5.1). It means that only in this limit does $\mathbb{a}_i$ become a genuine fermion operator and therefore can no longer annihilate a boson.

As we said in Sec. V, the quantity $K(\sigma\omega)_{[2][2]}$ is a measure of the collectivity of a fermion state with one fermion pair. Now from (7.9) we see that $K^{-2}(\sigma\omega)_{[2][2]}(<1)$ can be regarded as the coupling strength between the collective and noncollective motions. When the shell degeneracy goes to infinite, the coupling becomes zero and the collective fermion pair can be treated as a boson.

The matrix element $(\sigma'u\|\mathbb{A}(i,i')^{\sigma_p}\|(b^\dagger\times[\sigma])\sigma''u)$ can be similarly calculated (see Table 2 of Ref. 12), where it is shown that the pseudofermion operator $\mathbb{A}(i,i')^{\sigma_p}$ can transform a boson into a fermionic particle-hole pair $[a_i^\dagger\times\tilde{a}_{i'}]$ with probability amplitude again proportional to $1/K^2(\sigma\omega)_{[2][2]}$.

(c) The pseudofermion operator $\mathbb{a}_i^\dagger$.

From (4.7b), (4.12d), (7.9a), and (4.7b) we have

$$\begin{aligned}(\sigma'u'\|\mathbb{a}_i^\dagger\|\sigma u)&=(\sigma'u'\|\Gamma(a_i^\dagger)+2[\tilde{\mathbb{a}}_i\times b^\dagger]^{[1]}\|\sigma u)\\&=\left[1+\frac{4}{K^2(\sigma\omega)_{[2][2]}}U([\sigma][2][\sigma'][11];[\omega][1])^2\right]\langle\sigma'u'\|a_i^\dagger\|\sigma u\rangle\ .\end{aligned} \tag{7.10a}$$

Letting $u'=u\pm1$,

$$\begin{aligned}(\sigma'u+1\|\mathbb{a}_i^\dagger\|\sigma u)&=\left[1+\frac{4}{K^2(\sigma\omega)_{[2][2]}}U([\sigma][2][\sigma'][11];[\omega][1])^2\right]\langle\sigma'u+1\|a_i^\dagger\|\sigma u\rangle\\&\xrightarrow[\Omega\to\infty]{}\langle\sigma'u+1\|a_i^\dagger\|\sigma u\rangle\ ,\end{aligned} \tag{7.10b}$$

$$(\sigma'u-1\|\mathbb{a}_i^\dagger\|\sigma u)=0\ . \tag{7.10c}$$

Ignoring the coupling in (7.10a),

$$\mathbb{a}_i^\dagger \to a_i^\dagger + 2\tilde{\mathbb{a}}_i \times b^\dagger = a_i^\dagger + \frac{4U^2}{K^2} a_i^\dagger \xrightarrow[\Omega\to\infty]{} a_i^\dagger \ . \quad (7.11)$$

It is seen that acting on an intrinsic state, $\mathbb{a}_i^\dagger$ contains a term which annihilates a "fermion" and creates a boson. This "fermion" annihilation operator can transform the boson back to a real fermion. When $\Omega\to\infty$, the "fermion"$\to$boson transformation term becomes zero and again the pseudofermion$\to$genuine fermion.

From (7.5) and (7.7) we have [noting carefully the parentheses in $(\mathbb{a}_i)^\dagger$],

$$(\sigma' u+1\|(\mathbb{a}_i)^\dagger\|\sigma u) = (\sigma u\|\tilde{\mathbb{a}}_i\|\sigma' u+1)^* = \langle \sigma u\|\tilde{a}_i\|\sigma' u+1\rangle^* = \langle \sigma' u+1\|a_i^\dagger\|\sigma u\rangle \ . \quad (7.12)$$

Comparing it with (7.10b) we discovered a strange feature of the pseudofermion operators $\mathbb{a}_i^\dagger$ and $\mathbb{a}_i$, i.e.,

$$(\mathbb{a}_i)^\dagger \neq \mathbb{a}_i^\dagger, \quad \text{unless } \Omega\to\infty \ , \quad (7.13)$$

due to the extra term in (7.10b) which involves $K^{-2}$, i.e., the "fermion" creation and annihilation operators *are not Hermitian conjugate to one another.* Therefore $\mathbb{a}_i^\dagger$ and $\mathbb{a}_i$ differ from the true fermion operators $a_i^\dagger$ and $a_i$. In general we have

$$[\Gamma^{(\mathrm{hyb})}(O)]^\dagger \neq \Gamma^{(\mathrm{hyb})}(O)^\dagger \ , \quad \text{if } O \text{ is not a generator of Sp(6)} \ . \quad (7.14)$$

Now it is clear that the reason why $\mathbb{C}_{ab}$ are real fermion operators while $\mathbb{a}_i^\dagger$ and $\mathbb{a}_i$ are not is due to the fact that $\mathbb{C}_{ab}$ do not involve "fermion"$\leftrightarrow$boson transformation, while $\mathbb{a}_i^\dagger$ and $\mathbb{a}_i$ do. In other words, there is a coupling between the noncollective and collective degrees of freedom in $\Gamma(a_i^\dagger)$, but not in $\Gamma(O)$ if $O$ is a generator of Sp(6) as is seen in (4.12).

Notice that the use of the operator $(\Gamma(O))^\dagger$ can always be avoided by using (7.5). As is seen from (4.12), of the two operators $\Gamma(O^\dagger)$ and $\Gamma(O)$, the one associated with the annihilation operator is always very simple and in most cases we only need to use the simpler one.

In the GVCS theory,[11,12] the intrinsic operator is defined by its left action on an intrinsic state. This definition is not necessary and violates the associativity law of the action of operators $(\sigma'|\mathbb{O}|\psi) = ((\sigma'|\mathbb{O}|\psi) = (\sigma'|(\mathbb{O}|\psi))$. For example, Eq. (7.9b) can be interpreted either as a right action: the intrinsic operator $\mathbb{a}_i$ annihilates a boson to its right and creates a fermion, or as a left action: the left action of $\mathbb{a}_i$ on an intrinsic state can lower its heritage by one accompanied by creating a boson (or "a symplectic excitation" in the language of Ref. 12).

In Sec. IV, we give a rule for obtaining the intrinsic operator $\mathbb{O}_f$ from the fermion operator $O_f$ in the $z$-dependent operator $\hat{\Gamma}(O)$, i.e., shift the operator $O_f$ all the way to the left and then change it to $\mathbb{O}_f$. Now suppose we were to abandon this rule and let the intrinsic operator sit to the right, then Eq. (7.10a) would have become

$$(\sigma' u'\|\mathbb{a}_i^\dagger\|\sigma u) = (\sigma' u'\|\Gamma(a_i^\dagger) + 2[b^\dagger \times \tilde{\mathbb{a}}_i]^{[1]}\|\sigma u) \ . \quad (7.15)$$

Now we would get into trouble, since the master equation (4.13) cannot be used to calculate the matrix element of the operator $[b^\dagger \times \tilde{\mathbb{a}}_i]^{[1]}$. This is the reason for the above-mentioned left-ward shift rule. In addition, in a full matrix element, such as that illustrated by Eqs. (6.7), the intrinsic operator $\mathbb{O}_f$ must be shifted to the left, see the second step of Eq. (6.7c), so that its matrix element is put in the form $(\sigma'|\mathbb{O}_f|\psi)$. This could be calculated either through a left action as in Eq. (6.7c) or a right action. The vital point, however, is that the intrinsic operator $\mathbb{O}_f$ must be contiguous to the left intrinsic state to have a meaning. Hence a left-ward shift is required; and it is in this sense only that the "left-action" of Refs. 11 and 12 is to be interpreted.

In summary, the pseudofermion is different from the true fermion in that the "fermion" operator involves the *"fermion"$\leftrightarrow$boson* transformation and consequently does not commute with the boson operators, while the true fermion operator cannot induce the fermion$\leftrightarrow$boson transformation and thus commutes with the boson operators. However in the following three cases, the pseudofermion operator behaves as a true fermion operator and thus commutes with the boson operators: 1. The pseudofermion operators are $\mathbb{C}_{ab}$. 2. When no *"fermion"$\leftrightarrow$boson* transformation is involved, as in Eq. (6.9) or (7.7). 3. When $\Omega\to\infty$, the *"fermion"$\leftrightarrow$boson* term approaches zero.

## VIII. DISCUSSIONS

### A. The unitary mapping versus nonunitary mapping

The VCS theory was derived either by introducing the dual basis[4] or by using the unitary mapping approach.[1,10] It is clear that $(KK^\dagger)$ has an inverse only when $\mathbb{N}_f = \mathbb{N}_b$. In this case the dual basis can be introduced. Therefore the dual basis approach to the VCS theory applies only when $\mathbb{N}_f = \mathbb{N}_b$ [e.g., the Sp$(6,R)$ case].

Now let us turn to the unitary mapping approach. We first define a set of orthonormal states $|\psi_j)$ in the hybrid space by

$$|\psi_j) = \sum_{n=1}^{\mathbb{N}_b} V_n^{(j)}|\phi_n), \quad j=1,2,\ldots,\mathbb{N}_b \quad [\text{cf. } (2.11\text{a})] \ . \quad (8.1)$$

According to (2.9a) they form an orthonormal complete set in the hybrid space,

$$(\psi_{j'}|\psi_j) = \delta_{jj'}, \quad \sum_{j=1}^{\mathbb{N}_b} |\psi_j)(\psi_j| = 1 \ . \quad (8.2\text{a})$$

However,

$$\sum_{j=1}^{\mathbb{N}_f} |\psi_j)(\psi_j| \neq 1 \ . \quad (8.2\text{b})$$

The difference between (8.2) and (2.11b) should be noted.

The subspace spanned by the $\mathbb{N}_f$, $(\mathbb{N}_b-\mathbb{N}_f)$, states $|\psi_j)$ with nonzero (zero) eigenvalues will be called the physical (unphysical) subspace.

Up to this point, the $K$ matrix is defined only in the fermion space via Eqs. (2.7) and (2.11d). Now let us introduce the self-adjoint operators $\hat{K}^{\pm 1}$ and the projection operator $P$ which are to act in the hybrid space by

$$\hat{K}^{\pm 1} \equiv \sum_{j=1}^{\mathbb{N}_f} (\sqrt{\lambda_j})^{\pm 1} |\psi_j)(\psi_j| , \quad \text{with } \hat{K}^{+1} \equiv \hat{K} = \hat{K}^{\dagger} , \tag{8.3a}$$

$$P = \hat{K}^{-1}\hat{K} = \hat{K}\hat{K}^{-1} = \sum_{j=1}^{\mathbb{N}_f} |\psi_j)(\psi_j| . \tag{8.3b}$$

Notice that due to the fact that $\hat{K}^{-1}\hat{K}=\hat{K}\hat{K}^{-1}$, there is no need to differentiate $\hat{K}_L^{-1}$ from $\hat{K}_R^{-1}$. The operator $P$ is a projection operator from the full hybrid space to the physical subspace. For the Sp(6) case, (8.3a) becomes

$$\hat{K}^{\pm 1}(\sigma\omega) \equiv \sum_{j=1}^{\mathbb{N}_f} (\sqrt{\lambda_j})^{\pm 1} \sum_{\alpha} |\sigma\omega j,\alpha)(\sigma\omega j,\alpha| . \tag{8.3c}$$

From (8.3c) it is evident that the operators $\hat{K}^{\pm 1}(\sigma\omega)$ are SU(3) scalars which only depend on the quantum number $\sigma$ and $\omega$, but not on $\alpha$. There are three representations for the operators $\hat{K}^{\pm 1}$ and $P$.

1. The diagonal representation (or the $j$ representation for short). In the basis $|\psi_j)$, $j=1,2,\ldots,\mathbb{N}_b$,

$$(\hat{K}^{\pm 1})_{j'j} = \delta_{j'j} \begin{cases} (\sqrt{\lambda_j})^{\pm 1}, & j=1,2,\ldots,\mathbb{N}_f , \\ 0, & j=\mathbb{N}_f+1,\ldots,\mathbb{N}_b , \end{cases} \tag{8.4a}$$

$$P = \hat{K}^{-1}\hat{K} = \hat{K}\hat{K}^{-1} = K_L^{-1}K \oplus \mathbf{0} = \mathbf{I} \oplus \mathbf{0} . \tag{8.4b}$$

Equation (8.4b) should be understood as a matrix equation, where $\mathbf{I}$ and $\mathbf{0}$ are a unit matrix and a null matrix of dimensions $\mathbb{N}_f$ and $\mathbb{N}_b-\mathbb{N}_f$, respectively.

2. The nondiagonal representation (or the $n$ representation for short)

$$(\hat{K}^{\pm 1})_{mn} = \sum_{j=1}^{\mathbb{N}_f} (\sqrt{\lambda_j})^{\pm 1} V_m^{(j)} V_n^{(j)*} , \tag{8.5a}$$

$$(\hat{K}\hat{K}^{\dagger})_{mn} = (\hat{K}^2)_{mn} = (KK^{\dagger})_{mn} = \sum_{j=1}^{\mathbb{N}_f} \lambda_j V_m^{(j)} V_n^{(j)*} , \tag{8.5b}$$

$$P_{mn} = (\hat{K}^{-1}\hat{K})_{mn} = (K_L^{-1}K)_{mn} = \sum_{j=1}^{\mathbb{N}_f} V_m^{(j)} V_n^{(j)*} \neq \delta_{mn} \quad [\text{cf. (2.9a)}] . \tag{8.5c}$$

It is interesting to compare the representation matrices $(\hat{K})_{mn}$, $(KK^{\dagger})_{mn}$, and $P_{mn}$ given in (8.5). They all have $\mathbf{V}^{(j)}$ [see (2.8a)] as eigenvectors with eigenvalues $\sqrt{\lambda_j}$, $\lambda_j$, and 1, respectively, in the physical subspace, and with eigenvalue zero in the unphysical subspace.

3. The generalized representations (or the $n$-$j$ and $j$-$n$ representations for short)

$$(\phi_n|\hat{K}|\psi_j) = K_{n,j}, \quad (\psi_j|\hat{K}^{-1}|\phi_n) = (K_L^{-1})_{j,n} , \quad j=1,2,\ldots,\mathbb{N}_f , \tag{8.6a}$$

$$(\phi_n|\hat{K}|\psi_j) = (\psi_j|\hat{K}^{-1}|\phi_n) = 0 , \quad j=\mathbb{N}_f+1,\ldots,\mathbb{N}_b . \tag{8.6b}$$

Using (8.6), Eq. (4.7a) can be rewritten as

$$\langle \psi_{j'}|O|\psi_j \rangle = (\psi_{j'}|\gamma(O)|\psi_j) , \quad j',j=1,2,\ldots,\mathbb{N}_f , \tag{8.7a}$$

where $\gamma$ is a mapping from the fermion operator $O$ to the hybrid operator $\gamma(O)$,

$$\gamma(O) = \hat{K}^{-1}\Gamma(O)\hat{K} . \tag{8.7b}$$

In the $j$ representation,

$$\gamma(O) = K_L^{-1}\Gamma^{(\text{hyb})}(O)K \oplus 0 \quad [\text{cf. (4.7c)}] . \tag{8.7c}$$

From (8.7a) we immediately know that the mapping $\gamma$ is unitary,

$$\gamma(O^{\dagger}) = [\gamma(O)]^{\dagger} = \hat{K}[\Gamma(O)]^{\dagger}\hat{K}^{-1} . \tag{8.7d}$$

(Notice that $[\Gamma(O)]^{\dagger}$ is defined in (7.5).) This is the conventional approach to the VCS theory. Therefore we have two alternative views of the hybrid mapping. It can be regarded either as unitary mapping $\gamma$ or a nonunitary mapping $\Gamma$, as shown schematically below:

$$\begin{array}{llll}
\text{Fermion space:} & \langle \psi_{j'}|O|\psi_j \rangle, & |\psi_j\rangle = \sum_{n=1}^{\mathbb{N}_b} (K_L^{-1})^*_{j,n} |\phi_n\rangle, & O|\phi_n\rangle \\
 & \updownarrow \text{ unitary} & |\psi_j\rangle \longleftarrow \text{nonunitary} \longrightarrow |\phi_n\rangle & \updownarrow \text{ nonunitary,} \\
\text{Hybrid space:} & (\psi_{j'}|\gamma(O)|\psi_j), & |\psi_j) = \sum_{n=1}^{\mathbb{N}_b} V_n^{(j)} |\phi_n), & [\Gamma(O^{\dagger})]^{\dagger}|\phi_n) \\
 & & |\psi_j) \longleftarrow \text{unitary} \longrightarrow |\phi_n). &
\end{array}$$

As is seen from the above we either have the unitary mapping $|\psi_j\rangle \to^{\gamma} |\psi_j)$, or nonunitary mapping $|\phi_n\rangle \to^{\mathbb{H}} |\phi_n)$. The mapping $|\phi_n) \to |\psi_j\rangle$ [see Eq. (28) of Ref. 9], or $|\phi_n\rangle \to K|\phi_n)$ (see the third equation of Ref. 2) is rather misleading.

Equation (8.7b) along with $\hat{K}^\dagger = \hat{K}$, agree exactly with the original form[1,2,5–8,10] of the VCS theory. Notice that the simplicity of the operator $\hat{K}$, i.e., $\hat{K} = \hat{K}^\dagger$ and $\hat{K}_L^{-1} = \hat{K}_R^{-1} = \hat{K}^{-1}$, is deceiving. It conceals the fact that $\hat{K}^{-1}$ is not *really* the inverse of $\hat{K}$, as is seen in (8.4b) and (8.5c) as well as the discussion following (4.8). Therefore it is dangerous to abuse the operator $\hat{K}^{-1}$. As shown in Sec. IV all the results of the VCS theory can be obtained in a straightforward way without introducing the operator $\hat{K}$ *and* $\hat{K}^{-1}$.

### B. The operator $\hat{K}$ versus the matrix $K$

The matrix $K$ or operator $\hat{K}$ is central in the VCS theory. In some of the literature,[1–12] however, no differentiation between the matrix and operator was made leading to some confusion. At first it was taken for granted that $K^\dagger = K$. Then it was pointed out that $K^\dagger \neq K$ in Ref. 9. Now these two statements are both correct and incorrect, depending on whether the $K$ refers to operator or matrix. Therefore it is crucial to differentiate these two. The following points should clear up all the confusions.

1. $\hat{K}$ is an operator in the $\mathbb{N}_b$-dimensional hybrid space, and thus its representation matrices in any of the three representations are *necessarily* a *square* matrix of dimension $\mathbb{N}_b$, while $K$ is a $\mathbb{N}_b \times \mathbb{N}_f$ rectangular matrix.

2. $\hat{K}$ is Hermitian and its representation matrices in the $j$- or $n$-representation are also Hermitian, but its representation matrix in the $n$ representation, which is essentially the matrix $K$, is not. The reason is that the bra and ket belong to two *different* representations. We can also find the representative matrix of the self-adjoint operator $\hat{K} = \hat{K}^\dagger$ in the $j$-$n$ representation,

$$(\hat{K})_{jn} = (\hat{K})^*_{nj} = K^*_{nj} \ . \tag{8.8a}$$

If we define a new matrix $\bar{K}$ by $(\bar{K})_{jn} \equiv (\hat{K})_{jn}$, then (8.8a) becomes

$$\bar{K} = K^\dagger \ , \tag{8.8b}$$

i.e., the matrices of the self-adjoint operator $\hat{K}$ in the $j$-$n$ and $n$-$j$ representations are Hermitian conjugate to one another, but each of them is not Hermitian.

3. $K_L^{-1}$ is the left inverse of $K$, while Eq. (8.4b) shows that $\hat{K}^{-1}$ is the inverse of $\hat{K}$ only in the physical subspace but not in the full hybrid space (an equivalent statement is that the operator $P$ is a unit operator only in the physical subspace).

4. Equation (8.5b) shows that in the $n$-representation where the physical and unphysical spaces are mixed up, $\hat{K}^{-1}$ *is not the inverse* of $\hat{K}$.

### C. The FDSM versus IBFM

In Ref. 22 it was shown that when $\Omega \to \infty$, the FDSM wave functions with $u=0 \to$ IBM wave function. Now we can generalize it to the $u \neq 0$ case. In this section we use angular momentum representation. The fermion pairs are the $S(L=0)$ and $D(L=2)$ pairs defined in (2.1a), and the mapping (3.2a) becomes

$$\mathbb{H}(S^\dagger) = s^\dagger, \quad \mathbb{H}(D_m^\dagger) = d_m^\dagger \ . \tag{8.9}$$

With $S^\dagger$ and $D_m^\dagger$ instead of $A^\dagger_{mm'}$, the $K$ matrix given in Ref. 9 has to be multiplied by the factor $(\sqrt{2/3})^N$. For simplicity we only consider the simple states (actually, the IBFM studied so far corresponds only to the $u=1$ case, which is of course simple), for which $K_L^{-1} = K^{-1}$. The normalized fermion state is

$$|\sigma\omega j = \mathbf{n}, \alpha\rangle = K^{-1}(\sigma\omega)_{\mathbf{n},\mathbf{n}} (\sqrt{2/3})^N [Z^{\mathbf{n}}(S^\dagger, D^\dagger) \times |\sigma\rangle]^{[\omega]}_\alpha \ . \tag{8.10}$$

Using (5.1) one has

$$|\sigma\omega j = \mathbf{n}, \alpha\rangle \xrightarrow[\Omega \to \infty]{} \sqrt{1/\Omega^N} [Z^{\mathbf{n}}(S^\dagger, D^\dagger) \times |\sigma\rangle]^{[\omega]}_\alpha \xrightarrow{\mathbb{H}} [Z^{\mathbf{n}}(s^\dagger, d^\dagger) \times |\sigma]^{[\omega]}_\alpha \ , \tag{8.11}$$

where we have used the identification:[22] $S^\dagger/\sqrt{\Omega} \to s^\dagger$, and $D_m^\dagger/\sqrt{\Omega} \to d_m^\dagger$. Equation (8.11) shows that when $\Omega \to \infty$, the FDSM wave function goes over to the IBFM wave function.

According to Sec. VII we know that when $\Omega \to \infty$, the hybrid operator $\Gamma(O)$ consists of boson operaors and *true* fermion operators, i.e., the fermion operators of the FDSM become the fermion and boson operators of the IBFM. Of course here we have only outlined the relation between the FDSM and IBM. The details about the correspondence between the two remain to be worked out.

### D. The essential difference between the hybrid mapping and other mapping

The boson mapping is more than a quarter of a century old and even a brief review is certainly beyond the scope of the present paper. An exhaustive review article[20] on this subject by Klein and Marshalek is to appear. The hybrid mapping differs from all the others in the following aspects.

1. The mapping is not our goal, and it only serves as an intermediate step for computing the fermion matrix elements $O_{mn}$.

2. It is an exact mapping and the Pauli effects are fully taken into account by the $K$ matrix.

3. It is free of the spurious-state problem.

In mapping the $\mathbb{N}_b$ *linear dependent* states $|\phi_n\rangle$ to the $\mathbb{N}_b$ *orthonormal* states $|\phi_n)$, it is important to note that the $\mathbb{N}_b$ states $|\phi_n)$ include no redundant states, even though the original fermion states $|\phi_n\rangle$ included $\mathbb{N}_b - \mathbb{N}_f$ redundant states. Stating it differently: It is wrong to first pick out a set of linearly independent states $|\phi_n\rangle$, $n = 1, 2, \ldots, \mathbb{N}_f$, and then map them to the corresponding $\mathbb{N}_f$ states $|\phi_n)$ in the hybrid space. The reason is that although the $\mathbb{N}_f$ states $|\phi_n\rangle$ form a complete set in the fermion space, the $\mathbb{N}_f$ states $|\phi_n\rangle$ do not form a complete set in the hybrid space, i.e., they are not closed under the operation $\mathbb{H}(O)$ in (3.4b).

Very recently we have noticed a preprint by Klein and Walet[23] which is closely related to this work but with the essential difference that the realization of the intrinsic space in Ref. 23 is in terms of a set of quasifermions rather than the true fermions used here.

### E. Perspective of the hybrid mapping

Since the hybrid mapping does not depend on any group structure, it can be applied to the fermion system which does not have a dynamical symmetry, which is believed to be the case for the majority of nuclei. The FDSM is elegant in the sense that it is a fermion model which in the symmetry limit cases can give analytic solutions which describe the qualitative behavior of nuclear collective motion very nicely. The simplicity of the FDSM lies in the mathematically simple structure of the $S$ and $D$ pairs. There are indications[24] that the FDSM $S,D$ subspace may not be a good subspace for odd-$A$ nuclei with large deformation. Therefore it might be better to use "realistic" $S$ and $D$ pairs instead of the FDSM $S$ and $D$ pairs as the building blocks, while keeping the assumption intact that only $u=0$ and $u=1$ subspaces are important for describing the low-lying collective motions for even and odd nuclei, respectively. According to Talmi[25] and Iachello[26] the realistic $S$ and $D$ fermion pairs may be chosen as

$$S^\dagger=\sum_j c_j S_j^\dagger,\quad D_m^\dagger=[T^{(2)},S^\dagger]\ , \tag{8.12}$$

where $S_j^\dagger$ is the $S$-pair creation operator for a single $j$, $c_j$ are coefficients to be determined, and $T^{(2)}$ is the second quantized form of the quadrupole operator $\sum_j r_j^2 Y_{2\mu}(j)$. Applying the hybrid mapping to the $S,D$ pair, one is able to calculate the matrix elements of a shell model Hamiltonian in the $u=0$ or 1 subspace. The coefficients $c_j$ can be determined by minimizing the energy expectation value. If it is feasible, then the hybrid mapping may open a way for a realistic shell model calculation. If the variational calculation is too complicated, one might take as our $c_j$ the occupation amplitude $v_j$ from a BCS-type calculation, and diagonalize our model Hamiltonian in the $u=0$, or 1 subspace. Work along this line is in progress.

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