PHYSICAL REVIEW C VOLUME 43, NUMBER 2 FEBRUARY 1991

Nonlocality and the radius of the deuteron

M.W. Kermode
Department of Applied Mathematics and Theoretical Physics, University of Liverpool, P.O. Box 147, Liverpool L69 3BX, United Kingdom

S.A. Moszkowski
Department of Physics, University of California at Los Angeles, Los Angeles, California 90024

M.M. Mustafa
Faculty of Science (Sohag), Assiut University, Sohag, Egypt

W. van Dijk
Redeemer College, Ancaster, Ontario L9G 3N6, Canada
and Department of Physics, McMaster University, Hamilton, Ontario L8S 4M1, Canada
(Received 27 August 1990)

We consider the effects of nonlocality on the determination of the deuteron radius for model potentials. We show that the relation between the model values for the radius and the scattering length which holds for local potentials is not necessarily valid for nonlocal potentials. It is possible for nonlocal potential models to produce a deuteron of smaller radius as required by experimental data. In particular, we show that the Reid model can be amended with the inclusion of an attractive separable term to produce a change in the radius which is of the correct order of magnitude.

I. INTRODUCTION

The low energy behavior of the two-nucleon system is an important testing ground for proposed realistic nucleon-nucleon potential models. The deuteron properties and the low-energy scattering parameters are known from experiment to a high precision (see, for example Refs. 1 and 2) and consequently any interaction suggested to describe the internucleon force ought to predict these quantities correctly. It is found, by studying a number of realistic interactions, that there exist important empirical relationships between some of these quantities. The origin and the nature of such relationships is still being investigated.[1,3,4]

Since nuclear force models are fitted to the elastic scattering data as well as the deuteron properties, they are approximately equivalent on the energy shell. However, since there are only a limited number of measured properties that depend on the nuclear wave function within the potential region, the various nuclear potentials differ in their off-shell properties. It is necessary therefore to study those properties which depend on the details of the wave function in order to distinguish realistic interactions.

A particular relationship between a pair of low-energy quantities, first studied by Klarsfeld *et al.*[3], is that of the triplet scattering length, a_t, and the deuteron matter radius, r_D. When these quantities, obtained from the various realistic potentials available at the time of their study, are plotted on a scattering length versus matter radius graph, one obtains practically a straight line which passes below and to the right of the experimental region. In other words the predicted matter radius of the deuteron is too large. The underlying reason for this relationship, which also holds for simple S-state models, is not yet known. The recent Bonn potential[5] gives a point which lies off the line on the side opposite to the experimental region and produces an even larger discrepancy between the predicted and experimental radii of the deuteron.

Explanations for the discrepancy in terms of meson-exchange-current effects[3] or relativistic effects[6] have proved unsuccessful. The potential models considered by Klarsfeld *et al.*[3] are predominantly local; it is natural therefore to investigate the same properties using nonlocal potentials or potentials with a mixture of local and nonlocal terms. A study of the model (in)dependence of the a_t-r_D relationship[7] suggests that one may be able to fit simultaneously the triplet scattering length and the deuteron radius, if the potential includes nonlocal components. By considering a class of phase equivalent potentials (and thus fixed a_t), generated by making unitary transformations on the Reid hard core potential[8], Mustafa and Hassan[9] reach a similar conclusion. Further and more recent work on simple potential models by Bhaduri *et al.*[10] confirms the necessity of nonlocality in the nuclear force. The latter authors also suggest that it is unlikely that the Δ-Δ component of the nuclear interaction is the source of the nonlocality that will resolve the issue.

In this paper we investigate whether it is indeed possible to obtain the experimental values for the scattering length and the root-mean-square radius of the deuteron with realistic potentials. It is well established that the long-range part of the interaction is properly represented by the one-pion-exchange potential, fixing the ratio of the asymptotic D to S states of the deuteron wave function. The potential model that we choose for the investigation is one which has both local and nonlocal components. The functional form of the local part of the potential is similar to the Reid hard core potential and the nonlocal component is a short-range separable term in the triplet S state. Since the S-state wave function is dominant in the determination of the deuteron radius, the nonlocality is introduced, initially, in the triplet S state only. Potentials of this type have been studied by McKerrell, Kermode, and Mustafa[11] and Mustafa and Zahran.[12]

The graph of the a_t-r_D relation shows that varying the strength of the nonlocal component with appropriate adjustments to the short-range part of the local potential produces a zero-slope line, because the scattering length is fixed. It is clear therefore that such nonlocality in a realistic force will permit a simultaneous fit to the scattering length and the matter radius. Furthermore it is seen from the position of the experimental point that the short-range nonlocality is attractive.

In Sec. II, we consider a general relation between the effective-range parameters and the deuteron radius. Here we develop an approach initiated by Bhaduri *et al.*[10] and Sprung *et al.*[13] and consider a number of particular models. For this investigation it is necessary to consider only an S-state model deuteron. The purpose of this study is to see whether simple nonlocal potentials lead to a reduction in the value of r_D/a_t compared to local potentials as required by experiment. In Sec. III we introduce a local-plus-separable model, which required extensive computation, the results of which are given in Sec. IV.

II. THE RELATION BETWEEN THE DEUTERON RADIUS AND THE EFFECTIVE-RANGE PARAMETERS

In this section we analyze the effect of certain properties of the two-nucleon system on the deuteron radius. In order to be able to obtain analytic results we will consider S-state interactions only. In the following sections we discuss a "realistic" interaction model giving S- and D-state components in the deuteron wave function. For notational convenience we assume that in the following the units are chosen so that $\hbar^2/2m = 1$ where m is the reduced two-nucleon mass.

A. Sticky core potential

A sticky core potential has an infinite repulsion inside a core of radius c and an infinitely deep and narrow attraction just outside, at a distance $d = c+\eta$, and there is no potential at larger distances. Assuming η to approach zero, we write the narrow attraction as

$$V = \frac{\lambda}{d-c}\delta(r-d) \ . \tag{2.1}$$

At any energy, the S-state radial wave $u(r)$ vanishes inside the core, but the logarithmic derivative of the wave function at the outside edge of the potential,

$$F = \left(\frac{d\ln u}{dr}\right)_{d+} = \left(\frac{u'}{u}\right)_{d+} , \tag{2.2}$$

is independent of energy. In terms of the potential parameters

$$F = -(1+\lambda)/(d-c) \ . \tag{2.3}$$

The potential can be characterized by its scattering length, a_t. Let the zero-energy wave function outside the core be $u = 1 - r/a_t$, then

$$F = 1/(c - a_t) \ . \tag{2.4}$$

Suppose that the potential is strong enough to give a bound state at energy $-\alpha^2$. Then the wave function outside the potential region is proportional to $\exp(-\alpha r)$, so that $F = -\alpha$ and $\alpha a_t = 1 + \alpha c$.

For the sticky core potential, we can calculate the deuteron matter radius exactly. We obtain

$$\begin{aligned}\langle r^2\rangle &= \frac{1}{4}\int_c^\infty e^{-2\alpha r} r^2 dr \Big/ \int_c^\infty e^{-2\alpha r} dr \\ &= (1 + 2\alpha c + 2\alpha^2 c^2)/8\alpha^2 \ .\end{aligned} \tag{2.5}$$

Now it is convenient (Bhaduri *et al.*,[10] Sprung *et al.*[13]) to work with the ratio $\sqrt{8}r_D/a_t$, where $r_D = \sqrt{\langle r^2\rangle}$. Then

$$\begin{aligned}\frac{\sqrt{8}r_D}{a_t} &= \frac{\sqrt{1+2\alpha c+2\alpha^2c^2}}{1+\alpha c} \\ &= 1 + \tfrac{1}{2}(\alpha c)^2 - (\alpha c)^3 + \tfrac{11}{8}(\alpha c)^4 - \cdots \ .\end{aligned} \tag{2.6}$$

Finally, let us use the effective range, instead of the core radius, as the independent variable. Using one (r_m) of the three possible effective ranges given in the next section, for which

$$\alpha = a_t^{-1} + \tfrac{1}{2}\alpha^2 r_m \ , \tag{2.7}$$

we can express the core radius in terms of the effective range r_m to obtain

$$\alpha c = \frac{\frac{1}{2}\alpha r_m}{1 - \frac{1}{2}\alpha r_m} \ . \tag{2.8}$$

Writing $x \equiv \frac{1}{2}\alpha r_m$ we find

$$\frac{\sqrt{8}r_D}{a_t} = \sqrt{1+x^2} = \sum_i a_i x^i = 1 + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 + \cdots \ . \tag{2.9}$$

with $a_1 = 0, a_2 = \frac{1}{2}$, and $a_3 = 0$. We shall see that the vanishing of the linear term and the coefficient $\frac{1}{2}$ of the quadratic term is a general result, valid for any potential. The vanishing of a_3, and of any a_i where i is odd, is a specific property of the sticky core potential.

B. Deuteron radius in terms of generalized effective-range expansion

We follow here the derivation given by Sprung *et al.*[13] with some slight changes. There are at least three different ways to define the effective range. The first one is the usual effective-range expansion,

$$y(k^2) \equiv k \cot \delta_0 = -\frac{1}{a_t} + \tfrac{1}{2} r_0 k^2 - P r_0^3 k^4 + \cdots . \tag{2.10}$$

A slightly different definition is to expand about the deuteron pole,

$$y(k^2) = -\alpha + \tfrac{1}{2}(k^2 + \alpha^2) r_d + \cdots . \tag{2.11}$$

A third way, which is the one we used in the previous section, and in which the effective range is determined solely by the scattering length, is

$$(\alpha a_t)^{-1} = 1 - \tfrac{1}{2}\alpha r_m . \tag{2.12}$$

The three effective ranges differ only to order α^2 due to the shape-dependent term P,

$$r_d = r_0 + 4P(\alpha^2 r_0^3) + \cdots , \tag{2.13}$$

$$r_m = r_0 + 2P(\alpha^2 r_0^3) + \cdots . \tag{2.14}$$

The matter radius can be obtained from the expressions

$$\langle r^2 \rangle = \frac{1}{4} \frac{\int_0^\infty u^2(\alpha, r) r^2 dr}{\int_0^\infty u^2(\alpha, r) dr} \tag{2.15}$$

$$= \frac{1}{4} \frac{\int_0^\infty e^{-2\alpha r} r^2 dr - I_2(\alpha)}{\int_0^\infty e^{-2\alpha r} dr - I_0(\alpha)} \tag{2.16}$$

$$= \frac{1}{8\alpha^2} [1 - 4\alpha^3 I_2(\alpha)]/[1 - 2\alpha I_0(\alpha)] , \tag{2.17}$$

where

$$I_0(\alpha) = \int_0^\infty \left[e^{-2\alpha r} - u^2(\alpha, r)\right] dr, \tag{2.18}$$

$$I_2(\alpha) = \int_0^\infty r^2 \left[e^{-2\alpha r} - u^2(\alpha, r)\right] dr , \tag{2.19}$$

where $u(\alpha, r)$ is normalized so that $u(\alpha, r) = e^{-\alpha r}$ for r greater than the range of the potential. Evidently, $I_0(\alpha) = \frac{1}{2} r_d$.

Thus

$$\frac{\sqrt{8} r_D}{a_t} = \frac{\sqrt{1 - 4\alpha^3 I_2(\alpha)}}{\alpha a_t \sqrt{1 - 2\alpha I_0(\alpha)}} . \tag{2.20}$$

This expression is exact and approximations to it have recently been derived by Bhaduri *et al.*[10] and Sprung *et al.*[13].

Define the dimensionless quantity

$$\mathcal{J}(\alpha) \equiv I_2(\alpha)/[I_0^3(\alpha)] . \tag{2.21}$$

We obtain

$$\frac{\sqrt{8} r_D}{a_t} = \frac{\sqrt{1 - \frac{1}{2}\mathcal{J}(\alpha)\alpha^3 r_d^3}\,(1 - \frac{1}{2}\alpha r_m)}{\sqrt{1 - \alpha r_d}} , \tag{2.22}$$

which, in terms of r_0, is

$$\frac{\sqrt{8} r_D}{a_t} = \frac{[1 - \frac{1}{4} J(\alpha r_0)^3 + \cdots]}{\sqrt{1 - \alpha r_0 - 4P(\alpha r_0)^3 + \cdots}} \times [1 - \tfrac{1}{2}\alpha r_0 - P(\alpha r_0)^3 + \cdots] . \tag{2.23}$$

In the last equation we have defined $J \equiv \mathcal{J}(0)$. Expanding in powers of α up to order α^3 we obtain our final result,

$$\frac{\sqrt{8} r_D}{a_t} = 1 + \tfrac{1}{2}(\tfrac{1}{2}\alpha r_0)^2 + a_3(\tfrac{1}{2}\alpha r_0)^3 + \cdots , \tag{2.24}$$

where $a_3 = 1 - 2J + 8P$.

Note that J involves the zero-energy wave function only, while P can be obtained only from an expansion of the S-wave phase shift about small k when the potential type, viz., local or nonlocal, is not specified. No explicit knowledge of the wave function, other than that for zero energy is required. This will be discussed in the following sections, not only for the sticky core, but for a number of other potentials as well. It will be seen that two potentials which are phase equivalent, can nevertheless have different values of the deuteron radius. This difference can be traced to differences in the zero-energy wave function, leading to different values of $I_2(0)$, and thus J.

C. Delta shell potential

As our next example we consider, in some detail, the delta shell potential

$$V = -(1 + \epsilon)\delta(r - c)/c . \tag{2.25}$$

For $\epsilon = 0$, we get a bound state at zero energy. The scattering length is given by

$$\frac{a_t}{c} = \frac{1 + \epsilon}{\epsilon} . \tag{2.26}$$

The bound-state wave function is given by

$$u(\alpha, r) = \frac{\sinh \alpha r}{\sinh \alpha c} e^{-\alpha c}, \quad r \le c , \qquad = e^{-\alpha r}, \quad r \ge c . \tag{2.27}$$

Note that u is continuous at c, but its derivative is not and we have

$$\alpha c (\coth \alpha c + 1) = 1 + \epsilon . \tag{2.28}$$

The quantity αc can be expanded in terms of ϵ. For $\epsilon \ll 1$,

$$\alpha c = \epsilon \left(1 - \tfrac{1}{3}\epsilon + \tfrac{2}{9}\epsilon^2 - \tfrac{22}{135}\epsilon^3 + \cdots\right) . \tag{2.29}$$

Furthermore

$$\alpha a_t = 1 + \tfrac{2}{3}\epsilon - \tfrac{1}{9}\epsilon^2 + \tfrac{8}{135}\epsilon^3 + \cdots , \tag{2.30}$$

and

$$(\alpha a_t)^{-1} = 1 - \tfrac{2}{3}\epsilon + \tfrac{5}{9}\epsilon^2 - \tfrac{68}{135}\epsilon^3 + \cdots . \tag{2.31}$$

In order to evaluate Eq. (2.22) we make the following expansions,

$$I_0(\alpha) = \tfrac{2}{3}c[1 - \tfrac{1}{2}\alpha c + \tfrac{1}{15}(\alpha c)^2 + \cdots] , \tag{2.32}$$

$$\alpha I_0(\alpha) = \tfrac{2}{3}\epsilon(1 - \tfrac{5}{6}\epsilon + \tfrac{28}{45}\epsilon^2 + \cdots) , \tag{2.33}$$

and

$$I_2(0) = \tfrac{2}{15}c^3 \quad \text{and} \quad J = \tfrac{9}{20}, \quad \text{exactly} . \tag{2.34}$$

Thus, up to order ϵ^3,

$$\frac{\sqrt{8}r_D}{a_t} = 1 + \tfrac{2}{9}\epsilon^2 - \tfrac{58}{135}\epsilon^3 + \cdots . \tag{2.35}$$

We now wish to express the right hand side in terms $x = \frac{1}{2}\alpha r_m$, rather than ϵ. From Eq. (2.12),

$$x = 1 - (\alpha a_t)^{-1} = \tfrac{2}{3}\epsilon - \tfrac{5}{9}\epsilon^2 + \tfrac{68}{135}\epsilon^3 - \cdots , \tag{2.36}$$

from which we obtain

$$\epsilon = \tfrac{3}{2}x + \tfrac{15}{8}x^2 + \cdots \tag{2.37}$$

and

$$\frac{\sqrt{8}r_D}{a_t} = 1 + \tfrac{1}{2}x^2 - \tfrac{1}{5}x^3 + \cdots . \tag{2.38}$$

As expected, there is no linear term, and the coefficient of the quadratic term is $\frac{1}{2}$. The coefficient $a_3 = -\frac{1}{5}$.

In order to calculate the shape dependent parameter, we consider the special case $\epsilon = 0$, i.e., a bound state at zero energy. For this case we have

$$\cot\delta_0 = kc\csc^2 kc - \cot kc = \tfrac{2}{3}kc + \tfrac{4}{45}(kc)^3 + \cdots . \tag{2.39}$$

Comparing this with the effective-range expansion,

$$k\cot\delta_0 = -\frac{1}{a_t} + \tfrac{1}{2}r_0k^2 - Pr_0^3k^4 + \cdots , \tag{2.40}$$

we see that $a_t = \infty$, $r_0 = 4c/3$, and $P = -\frac{3}{80}$. Substituting our values for J and P into the expression for a_3, $a_3 = 1 - 2J + 8P$, we reproduce our value $a_3 = -\frac{1}{5}$.

Note that all the examples, except the sticky core (and the determination of J and P for the delta shell) deal with potentials that have infinite scattering lengths. In fact, the validity of our approximations, i.e., small deuteron binding, requires that we are not far from this limit. It turns out that for infinite a_t, the expressions for J and P are simpler than otherwise. Of course, a_t is finite, else we would get an infinite r_D. However, our expression for this quantity only needs J and P calculated for infinite a_t (at least up to order α^3). Thus, in order to calculate a_3 we do not need to consider the actual potential, which has a state at finite binding energy, but can scale the potential so as to give a zero-energy bound state. In fact, the quantities r_0 and J can be obtained in terms of the zero-energy wave function. To calculate P, we need to consider the scattering wave function at finite k^2. However, recently Kermode and van Dijk[14] have shown that, for the special case of a local potential, this quantity can also be obtained from the zero-energy wave function.

D. Results for simple local and separable potentials

In this section, we shall list some results for a few simple potentials. All the potentials are attractive and have a well depth adjusted so as to give a bound state at zero energy, i.e., infinite scattering length. Thus in each case, the zero-energy wave function equals unity outside the range of the potential. The results are given in Table I.

Note that the results for a_3 vary considerably. In particular, a_3 is much more negative for the exponential and Hulthén potentials than for the others. Physically, this is due to the fact that these potentials pull in the wave function toward the origin more strongly than do the other potentials. The sticky core is at the other extreme. Here the wave function is pushed outward as much as possible.

Next we give results for some simple separable potentials (Table II). One of these is the delta shell, which is *both* local and separable and may be regarded as providing the dividing line between local and nonlocal potentials particularly with regard to the quantity $J - 16P$. This combination of J and P is chosen because, as we will see in the next section, it has a remarkably simple form for the core plus delta shell potential. Note that the square-well equivalent potential is a particular separable potential whose form is chosen so that the zero-energy

TABLE I. Summary of results for local potentials [$\Gamma = 2.4048$ is the first root of $J_0(\Gamma) = 0$].

Potentials	$u(r < c)$	r_0	J	P	a_3	$J - 16P$
Delta shell	r/c	$4c/3$	0.4500	−0.0375	−0.2000	1.0500
Square well	$\sin(\pi r/2c)$	c	0.5228	−0.0327	−0.3069	1.0460
Sticky core	0	$2c$	0.3333	−0.0417	0.0000	1.0000
Exponential	$J_0(\Gamma e^{-\mu r/2})$	$5/2\mu$	0.9207	0.0119	−0.7461	0.7303
Hulthén	$1 - e^{-\mu r}$	$3/\mu$	1.1111	0.0381	−0.9173	0.5013

TABLE II. Summary of results for separable potentials.

Potentials	$u(r<c)$	r_0	J	P	a_3	$J-16P$
Delta shell	r/c	$4c/3$	0.4500	−0.0375	−0.2000	1.0500
SW equiv. sep.	$\sin(\pi r/2c)$	c	0.5228	−0.0360	−0.3333	1.0988
Square well	$2r/c-(r/c)^2$	$14c/15$	0.5623	−0.0351	−0.4057	1.1239
Exponential	$1-e^{-\mu r}$	$3/\mu$	1.1111	−0.0185	−1.3704	1.4071

wave function is identical to that for a local square well with infinite scattering length. Thus, although the two potentials have the same values of r_0 and J they are not phase equivalent. The separable potential has a more negative value of P, which means that the phase shift is slightly smaller than for the local potential. Thus a_3 is slightly smaller and $J-16P$ is slightly larger. Turning this argument around, we can say that for two phase equivalent potentials, one local and the other separable, the latter will give a slightly smaller value of a_3, i.e., a smaller deuteron radius. (See also Ref. 7.)

Finally, the quantity $J-16P$, which is also listed, is seen to be much less sensitive to the form of the potential than J and P separately. It is interesting to note that it is consistently larger for the separable potentials than for the local potentials.

E. Potentials with core plus attraction

We give here some results for two cases: (i) hard core plus delta shell attraction, and (ii) hard core plus exponential attraction. For both cases, the potential is assumed to have infinite scattering length.

1. Core plus delta shell potential

For a core radius c and a delta shell radius d, the S-wave phase shift is given by the analytic expression

$$\cot\delta_0=\frac{\{\cot k(d-c)-1/[k(d-c)]\}\cot kd+1}{\cot kd-\cot k(d-c)+1/[k(d-c)]}\,. \tag{2.41}$$

For $kd\ll 1$, we get

$$\cot\delta_0=\tfrac{1}{3}(2d+c)k+\tfrac{1}{45}(4d^3+8cd^2+2c^2d+c^3)k^3+\cdots\,, \tag{2.42}$$

$$r_0=\tfrac{4}{3}d+\tfrac{2}{3}c\,, \tag{2.43}$$

$$P=-\frac{3}{40}\frac{4d^3+8cd^2+2c^2d+c^3}{(2d+c)^3}\,, \tag{2.44}$$

$$J=\frac{9}{10}\frac{4d^3+3cd^2+2cd^2+c^3}{(2d+c)^3}\,, \tag{2.45}$$

$$a_3=1-2J+8P=-\frac{1}{5}\frac{\zeta^2(7\zeta-15)}{(\zeta-3)^3}\,, \tag{2.46}$$

where $c=d(1-\zeta)$ and

$$J-16P=1+\frac{\zeta^2(11\zeta-15)}{10(\zeta-3)^3}\,. \tag{2.47}$$

We observe that in Eq. (2.47) the terms linear in ζ cancel for the combination $J-16P$, whereas they do not cancel for J and P separately. Furthermore, the value of $J-16P$ for the sticky core potential ($\zeta=0$ corresponds to a sticky core) is exactly equal to 1. Both a_3 and $J-16P$ are straightforward to evaluate for particular values of the ratio c/d. As c/d increases from 0.0 to 0.5 to 1.0, a_3 ranges from −0.2000 to −0.0368 to 0.0, respectively, and $J-16P$ changes very little, decreasing slightly from 1.050 to 1.015 to 1.000, respectively.

2. Core plus exponential potential

For the core plus exponential potential we also hold the effective range constant at 2.5 fm. Thus c has a maximum value of 1.25 fm. We note that the shape parameter P (Table III) changes sign for a core radius as small as 0.1 fm. We note also that unlike the previous case, the value of $J-16P$ first increases, to a maximum of 1.048 which is almost the value for the delta shell potential, before decreasing again to the sticky core value of 1.000.

For a hard core plus attractive square well and r_0 fixed, J has a particularly simple form,

$$J=0.5228-4x(1-x)(1-8/\pi^2) \tag{2.48}$$

where $x=c/d<\frac{1}{2}$. Then $J<0.5228$.

Thus local potentials with hard cores behave like those without, i.e., $J-16P$ is *less* than 1.0500. On the other hand, nonlocal potentials appear to have values *greater* than 1.0500. It would appear that if a_3 and P could be determined experimentally then $J-16P$ could be calculated thus indicating, together with r_0, whether or not a potential is nonlocal.

TABLE III. a_3 and $J-16P$ for the core plus exponential potential.

c	P	J	a_3	$J-16P$
0.0	0.012	0.921	−0.746	0.730
0.1	−0.001	0.821	−0.650	0.836
0.2	−0.010	0.732	−0.543	0.892
0.4	−0.026	0.584	−0.375	1.000
0.6	−0.036	0.473	−0.234	1.048
0.8	−0.040	0.397	−0.114	1.036
1.0	−0.042	0.351	−0.038	1.024
1.25	−0.042	0.334	−0.000	1.000

III. THE NONLOCAL POTENTIAL MODEL

We now turn to a nonlocal interaction model which includes S- and D-state components in the deuteron wave function. One method of generating a nonlocal potential model is to perform a unitary transformation on a wave function obtained from a local potential.[15] This has been considered in connection with the matter radius of the deuteron by van Dijk[7] and Mustafa and Hassan.[9] They show that it is possible to reduce the value of r_D from the local potential model value for particular transformations. However, there is not a gradual change between the local and nonlocal potentials—the unitary transformation is either on or off; it cannot be turned on gradually. With the unitary transformation parameter ϵ (see van Dijk[7]), it is possible to make gradual changes between the wave functions for the nonlocal potentials.

In the remainder of this section, we consider an approach that allows the nonlocal component to be switched on gradually. The nonlocal potential model we use consists of a Reid-type local component, i.e., a sum of Yukawa terms for the central (C), spin-orbit (LS), and part of the tensor (T) interactions plus a regularized one-pion-exchange tensor term to avoid difficulties in the absence of a hard core. Thus,

$$V_i(r) = r^{-1} \sum_{n=1}^{N_i} A_{(i)n} e^{-n\mu r} \ (i = C, LS, T), \tag{3.1}$$

$$V_T^{(2)}(r) = Br^{-1} \left[\left(1 + \frac{3}{\mu r} + \frac{3}{(\mu r)^2}\right) e^{-\mu r} - \nu^2 \left(1 + \frac{3}{(\nu\mu r)} + \frac{3}{(\nu\mu r)^2}\right) e^{-\nu\mu r} \right], \tag{3.2}$$

where μ^{-1} is the pion Compton wavelength. The coefficients $A_{(LS),1} = 0$, $A_{(T),1} = 0$, $A_{(C),1}$, and B are fixed by the pion-nucleon coupling constant (-14.947 MeV fm) and the remaining parameters are determined by fitting the appropriate data. The constant ν is chosen arbitrar-

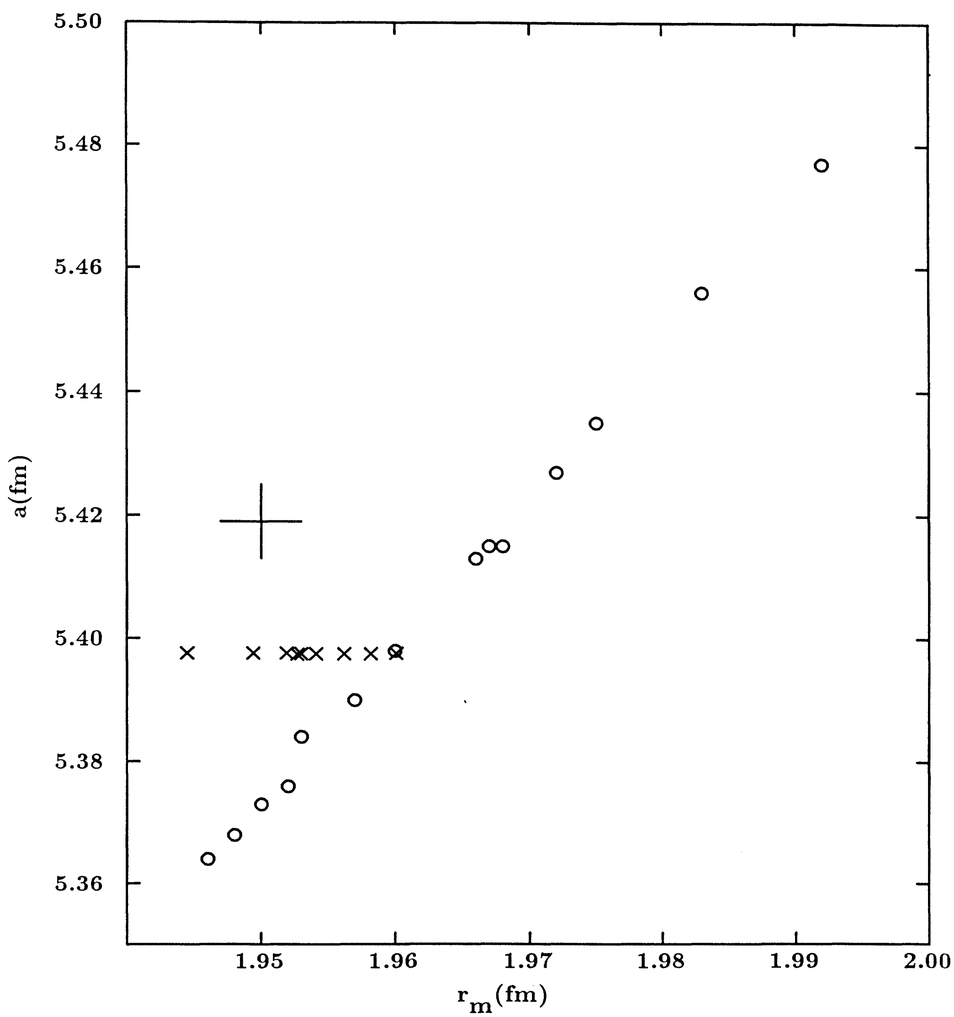

FIG. 1. The scattering length as a function of the deuteron matter radius. Circles give results of realistic potentials; see Ref. 3. Crosses indicate results when a separable nonlocal component is included in the interaction.

ily and in this paper we take it to have the value 5.

In addition there is a rank one separable S-wave central potential, which we write in the form $\lambda_u f(r)f(s)$. It is also possible to introduce separable potential components into the central D state, and the spin-orbit and tensor interactions. The latter is particularly interesting,[16] but for simplicity we shall concentrate only on the S-wave component in this paper.

With these potentials, the coupled, radial Schrödinger equation for the ${}^3S_1$-${}^3D_1$ state may be written

$$-u''(r) + V_{11}u(r) + V_{12}w(r) + \lambda_u f(r) \int_c^{\infty} f(s)u(s)ds = k^2 u(r) \tag{3.3}$$

$$-w''(r) + V_{12}u(r) + V_{22}w(r) = k^2 w(r), \quad r > c\,, \tag{3.4}$$

where c is the hard core radius.

The local potentials $V_{11}(r)$, $V_{12}(r)$, and $V_{22}(r)$, in units of fm^{-2}, are related to the customary central potential V_C, tensor potential V_T, and spin-orbit potential V_{LS} by

$$V_{11}(r) = V_C(r), \quad V_{12}(r) = 2\sqrt{2}V_T(r)\,, \tag{3.5}$$

$$V_{22}(r) = \frac{6}{r^2} + V_C(r) - 2V_T(r) - 3V_{LS}(r)\,. \tag{3.6}$$

TABLE IV. The coefficients $A_{(i)n}$ for the local potential component of the interaction, in units of MeV fm, for the case when $\lambda_u = -300\ \mathrm{fm}^{-3}$. (Also, $B = -14.947$ Mev fm.)

n	$A_{(C)n}$	$A_{(LS)n}$	$A_{(T)n}$
1	−1.4947(1)	0.0	0.0
2	−1.9660(3)	−1.1596(2)	−3.1274(2)
3	3.9874(4)	3.2574(3)	4.2246(3)
4	−2.5692(5)	−1.9346(4)	−1.2056(4)
5	7.2581(5)	6.4184(4)	1.0239(2)
6	−6.7975(5)	−6.3857(4)	1.8421(4)

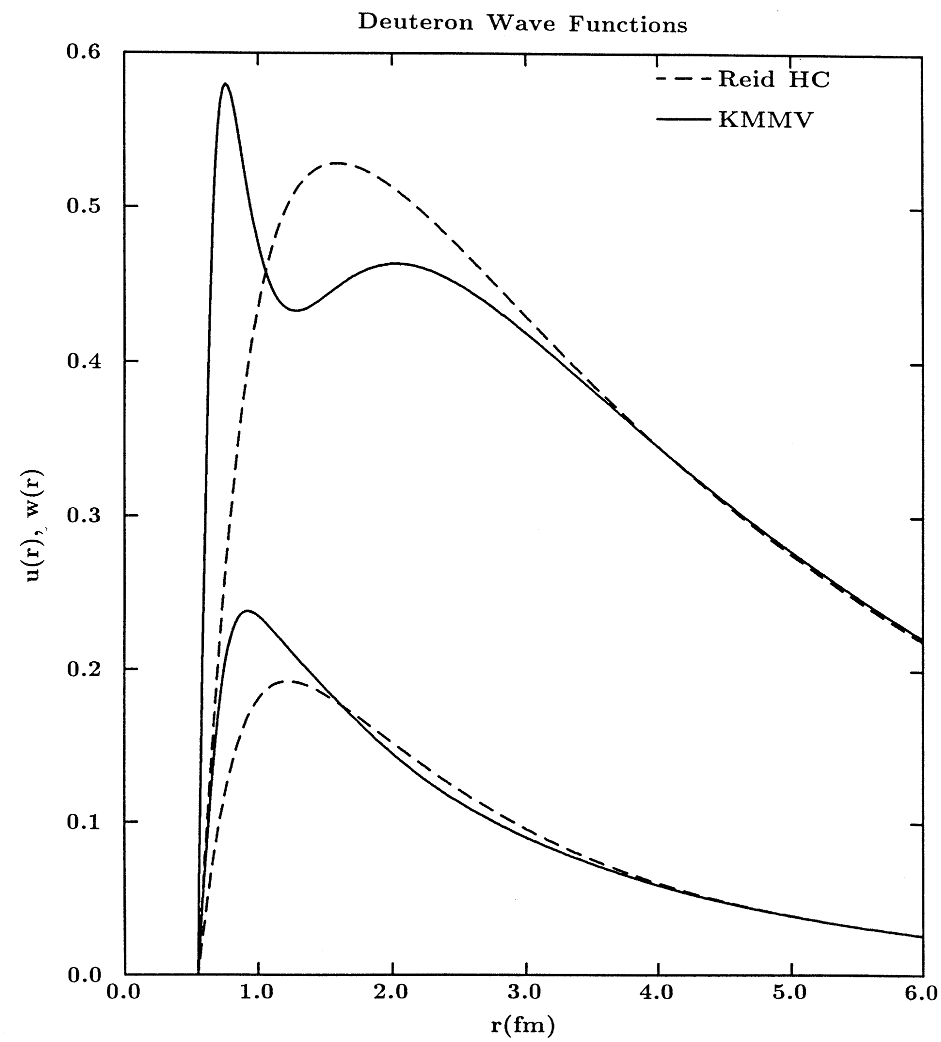

FIG. 2. Deuteron S- (upper) and D- (lower) state wave functions. The solid curves show the wave function for the potential with the nonlocal component; the dashed curves show the Reid hard core wave function.

The strength λ_u of the nonlocal term is measured in units of fm^{-3}. We take $f(r) = e^{-\beta r}$, with $\beta = 2.1$ fm^{-1}, as previously[9]. Because of the separability of rank 1, this equation may be solved numerically as a constrained third-order difference equation. The details of the method have been previously described[9].

Since this investigation is only to see what effects nonlocality may have rather than to obtain a detailed fit to the experimental point, we choose the Reid hard core potential model to generate the ${}^3S_1$-${}^3D_1$ phase parameters, which we then take to be the actual phase parameters. Keeping the hard core fixed and choosing various values for the sign and strength of the nonlocality λ_u, we adjust the force coefficients $A_{(i)n}$ to reproduce these phase parameters. Accordingly, the scattering length a_t is fixed but the deuteron radius may change. It is this change that is one of the basic considerations of this paper. In particular, can the change be sufficiently large to give a reasonable deviation from the a_t-r_D line?

IV. RESULTS

For a fixed value of λ_u, the free coefficients $A_{(i)n}$ were varied until the deuteron binding energy (α^2) and the phase parameters (i.e., two phase shifts and one mixing parameter for each energy) for the Reid potential were reproduced as closely as possible, using the appropriate Numerical Algorithm library least-squares routine. Then the resulting value of $A_{(i)2}$ was slightly adjusted to reproduce the scattering length for the Reid potential to an accuracy of four significant figures. We note that the Reid phase parameters cannot be reproduced exactly because of the fairly small dimension of the parameter space for Eq. (3.1) since the N_i's were restricted to 6.

It was found that for positive values of λ_u, i.e., repulsive nonlocality in that the local part alone supports a deeper bound state than that of the deuteron, the value of the deuteron radius increased as λ_u increased. For negative values of λ_u (attractive nonlocality), the radius

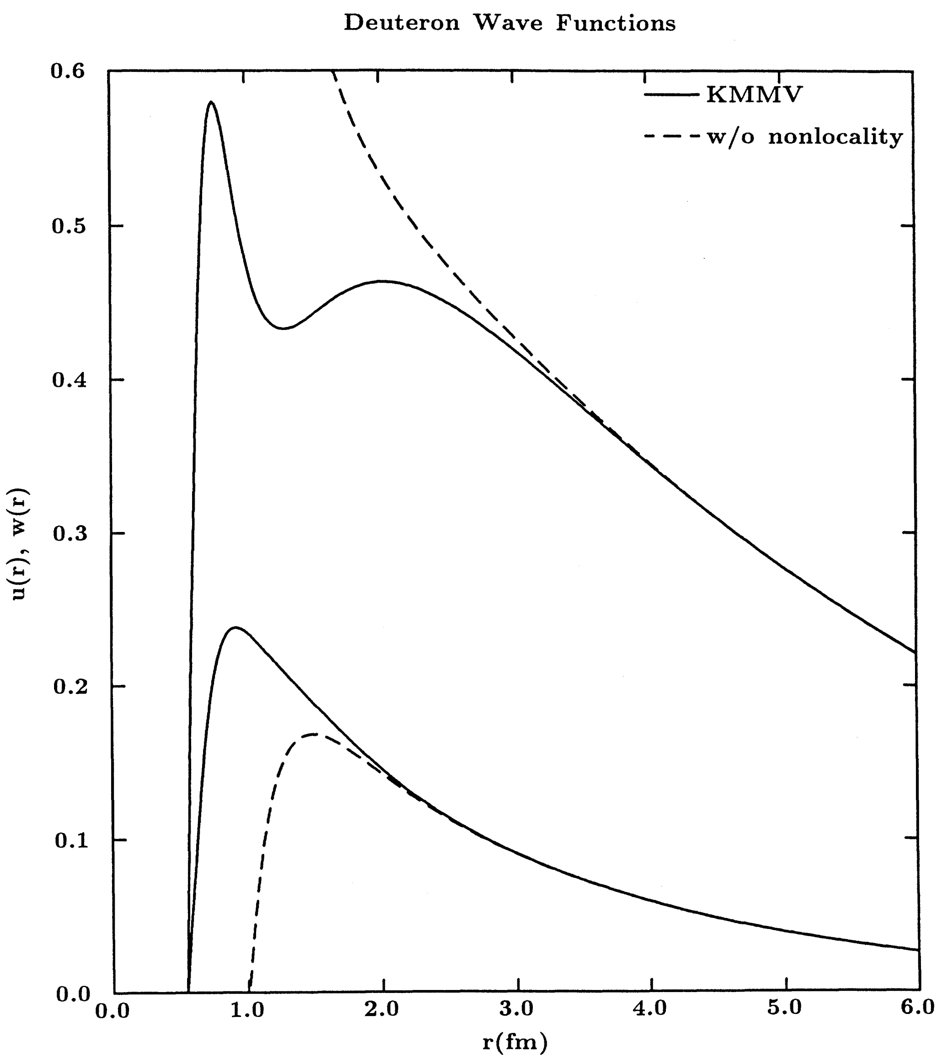

FIG. 3. Solution of the inward integration of the Schrödinger equation with the separable term switched off (dashed curve). The solid curve is the same as that of Fig. 2.

decreased from that for the Reid potential itself. For the value of $\lambda_u = -300$ fm^{-3}, the "experimental" value of the deuteron radius is obtained although, of course, the experimental value of the scattering length is not reproduced because that for the Reid potential it is too small. The results of the least-squares fits are shown in Fig. 1 where the λ_u range in values from 0 to -350 fm^{-3}.

For the value of $\lambda_u = -300$ fm^{-3}, the coefficients for the local potential were adjusted in order to fit the phase shifts of Arndt *et al.*[17] instead of those of Reid. In Table IV we give the values of the coefficients $A_{(i)n}$ for this potential, which yields a deuteron radius of 1.955 fm. In Fig. 2, we show the deuteron wave function obtained with this potential together with the Reid wave function. The structure of the new wave function is particularly interesting and is a consequence of the nonlocality of the interaction. This shape would not be obtained from a local potential unless it was particularly pathological.

In view of the range of the separable term ($\beta^{-1} = \mu^{-1}/3$, i.e., one third the pion wavelength) it is a little surprising that the wave function for the nonlocal potential deviates from that of the local potential as far out as 3 fm. To see whether this is a consequence mainly of the separable term or of the change in the local part, we integrated the Schrödinger equation inwards from large distances with the separable term switched off (i.e., $\lambda_u = 0$). For the starting values, we used the asymptotic values $u = A_S e^{-\alpha r}$ and $w = A_D[1+3/(\alpha r)+3/(\alpha r)^2]e^{-\alpha r}$ with $r = 12$ fm and A_S, A_D from the Reid wave functions. In Fig. 3, we show the results of this calculation. We note that the solutions start to deviate from the Reid wave functions at about 2.5 fm, which is smaller than the 3 fm mentioned above but larger than the 1.4 fm suggested by Sprung.[13] The increase from the latter is probably due to the $n = 2$ terms in Eq. (3.1).

We do not regard the apparently long range of the nonlocal wave functions particularly important at this stage. The important results of this work are that (i) the model value of r_D may be reduced with an *attractive* nonlocal component, (ii) the wave function is increased at smaller r and reduced at slightly larger r (to maintain the normalization), and (iii) the wave function contains more structure than local wave functions.

We see from Fig. 2 that by increasing β, but with considerable further computation, it should be possible to construct a nonlocal potential that gives a wave function closer to the Reid's wave function at about 2 fm and smaller than our wave function from about 1 to 1.5 fm. This is an important point and is the indicator towards the solution of the a_t-r_D problem raised by Klarsfeld *et al.*[3] The nonlocal model in this paper produces wave functions that behave exactly the way one would expect if a local model radius of the deuteron is to be reduced.

ACKNOWLEDGMENTS

Support from the Natural Sciences and Engineering Research Council (NSERC) under operating Grant No. A8672 is gratefully acknowledged. M.W.K. is also grateful to the Royal Society and NSERC for support under the Anglo-Canadian Scientific Exchange Scheme, to the British Council for support under its Anglo-Egyptian scheme, and to the University of Liverpool for support from its Research Development Fund. M.W.K. also thanks the Physics Departments of McMaster University and Assiut University (Sohag) for their hospitality.

[1] T.E.O. Ericson, Nucl. Phys. **A416**, 281c (1984).
[2] L. Mathelitsch and B.J. VerWest, Phys. Rev. C **29**, 739 (1984).
[3] S. Klarsfeld, J. Martorell, J.A. Oteo, M. Nishimura, and D.W.L. Sprung, Nucl. Phys. **A456**, 373 (1986).
[4] J.L. Ballot, A.M. Eiró, and M.R. Robilotta, Phys. Rev. C **40**, 1459 (1989).
[5] R. Machleidt, K. Holinde, and Ch. Elster, Phys. Rep. **149**, 1 (1987).
[6] F.M. Toyama and Y. Nogami, Phys. Rev. C **38**, 2881 (1988).
[7] W. van Dijk, Phys. Rev. C **40**, 1437 (1989).
[8] R. V. Reid, Ann. Phys. **50**, 411 (1968).
[9] M.M. Mustafa and E.M. Hassan, Phys. Scr. **39**, 552 (1989).
[10] R.K. Bhaduri, W. Leidemann, G. Orlandini, and E.L. Tomusiak, Phys. Rev. C **42**, 1867 (1990).
[11] A. McKerrell, M.W. Kermode, and M.M. Mustafa, J. Phys. G **3**, 1349 (1977).
[12] M.M. Mustafa and E.S. Zahran, Phys. Rev. C **38**, 2416 (1988); see also (Ref. 15).
[13] D.W.L. Sprung, H. Wu, and J. Martorell, Phys. Rev. C **42**, 863 (1990).
[14] M. W. Kermode and W. van Dijk, Phys. Rev. C **42**, 1891 (1990).
[15] In this paper we use the term "nonlocal" in its widest sense so that it may have a local component. For "nonlocal component" we mean a nonlocal potential less its local component.
[16] M.W. Kermode and M.M. Mustafa (presented at the Few Body Conference XII , Vancouver, 1989) (unpublished).
[17] R.A. Arndt, L.D. Roper, R.A. Bryan, R. B. Clark, B.J. VerWest, and P. Signell, Phys. Rev. D **28**, 97 (1983).