Spontaneous breaking of Elliott symmetry in nuclear systems and the s and d Nambu-Goldstone bosons

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When the Elliott U(3) symmetry is spontaneously broken in a nonrelativistic nuclear system, a type of superconductivity order parameter leads to six Nambu-Goldstone bosons with spin zero and two. The rotational symmetry is not broken.

During the past two decades, it has been established that a variety of low-lying nuclear states can be successfully described in terms of interacting s- and d-boson degrees of freedom.[1] There must be a simple and fundamental reason for the existence of these bosonic degrees of freedom,[2] since they play such a basic role among a wide class of experimental data. Probably it is not an accident that similar fundamental bosonic degrees of freedom emerge in quantum gauge-field theories and condensed-matter physics. When the vacuum of a system with infinite spatial extent is not an eigenstate of certain continuous symmetry operators which commute with the Hamiltonian of the system, then the action of the symmetry generators creates a continuous set of infinitely degenerate vacuum. When this happens, we say that the symmetry is spontaneously broken. The whole system moving along the line in the degenerate direction is characterized by the Nambu-Goldstone (NG) bosons[3-5] whose lowest energy levels are degenerate with the vacuum energy. Then the bosonic excitations of the system are called gapless, i.e., energy of NG bosons $\omega_k \to 0$ as $k \to 0$. Often it has been stated in standard textbooks that the NG bosons are "massless," "spinless," and only associated with the system with infinite degrees of freedom. None of these characteristics seem to fit a consideration of the substructure of the bosons inside the nucleus to be NG bosons. Therefore, it becomes necessary, before we enter a mathematical consideration, to clarify this physical situation by recounting the arguments made to justify such conclusions.

In relativistic quantum field theories, the mass and the helicity are Casimir operators of the Poincaré group and relativistic invariance is not spontaneously broken, so any gapless Goldstone bosons must be massless and spinless. However, in nonrelativistic theories, the mass does not have such a significant feature related to the relativistic invariance. In this case energy and momentum are related by $\omega_k = k^2/2m$, where m is a parameter. Then the NG bosons are gapless without being massless. Next, in a nonrelativistic theory it is not unusual to find that a Lagrangian is invariant under a larger symmetry group, G, than R(3), the rotational invariance in three-dimensional space. Some of the generators of the larger symmetry group can be spherical tensors with a higher rank. Suppose this symmetry is spontaneously broken; then the corresponding NG bosons can carry nonvanishing spins. In this case if the rotational symmetry, the subgroup of the larger symmetry, is not broken the spin is a good quantum number and the NG bosons may have a definite spin instead of being spinless.

To analyze collective modes in nuclei, we make use of the quantum field theory (QFT). QFT has become a standard method for many-body problems. Even in such a simple system as the hydrogen atom, the Lamb shift exhibited the radiative corrections, indicating need of QFT. The finite size of nuclei indicates a view like the bag model in QCD; a QFT system with a given nucleon number self-consistently chooses a finite domain in which the wave functions are confined. A better analogy is a crystal with a finite size. In this paper we study the collective modes in nuclei by starting with a system of nucleon fields with infinite volume in order to search for a possible mechanism for the origin of the interacting-boson model (IBM). As it is well known, collective modes in such a system are maintained by certain Goldstone modes, whose energy is gapless. We point out here that the finite-size effects can permit energy gap of these Goldstone modes with order of magnitude of inverse of the nuclear radius. In this paper this small energy gap will be called gapless. More comment about the finite-size effects will be given at the end of the paper.

In this paper, we provide a nonrelativistic model Lagrangian that is invariant under the Elliott U(3) transformation[6] for a nuclear system. When the U(3) symmetry is spontaneously broken such that rotational invariance is preserved, six NG bosons are created, which are characterized by spin zero and two. The interesting fact is that the mechanism is closely related to the superconductivity model which is based on the spontaneous breakdown of U(1) symmetry and is characterized by a nonvanishing vacuum expectation value of an electron pair. We prove that in the Elliott U(3)-invariant Lagrangian the same kind of order parameter as superconductivity theory must create spin-0 and spin-2 NG bosons at the same time.

The U(3)-invariant nonrelativistic Lagrangian for nucleon fields can be written as

$$L(t)=\int d^3x\, \phi^*(x)\dot{\phi}(x)-H\ , \tag{1}$$

with

$$H=\int d^3x\, \phi^*(x)(\mathbf{x}^2-\nabla_x^2)\phi(x) + \frac{v_0}{2}\sum_{kq}(-1)^q\int d^3x\, d^3y\, \phi^*(x)\phi^*(y)T_q^k(\mathbf{x},\nabla_x) \times T_{-q}^k(\mathbf{y},\nabla_y)\phi(x)\phi(y)\ . \tag{2}$$

Here

$$T_q^k(\mathbf{x},\nabla_x)=\sum_{\alpha\beta}\langle 1\alpha 1-\beta|kq\rangle(-1)^{1-\beta}a_\alpha^\dagger a_\beta\ , \tag{3}$$

with $a_\alpha^\dagger=(1/\sqrt{2})(x_\alpha+\nabla_\alpha)$ and $a_\alpha=(1/\sqrt{2})(x_\alpha-\nabla_\alpha)$. It is easy to show that

$$T_0^0=\frac{1}{2\sqrt{3}}(\mathbf{x}^2-\nabla_x^2)\ ,$$
$$T_q^1=\frac{i}{\sqrt{2}}(\mathbf{x}\otimes\nabla_x)_q\ , \tag{4}$$
$$T_q^2=\frac{1}{\sqrt{6}}\left[\frac{4\pi}{5}\right]^{1/2}[x^2y_{2q}(\theta_x\phi_x)+\nabla^2y_{2q}(\theta_\nabla\phi_\nabla)]\ .$$

Nine operators in (4) are the complete set of generators for the U(3) group with the three operators, T_q^1, forming its rotational subgroup R(3). The Heisenberg operators, $\phi(x)$ and $\phi^*(x)$ in (1) and (2), are the nucleon fields which, for simplicity, are assumed to be scalar fields

$$[\phi(x),\phi^*(x)]_{t=t'}=\delta^3(x-x')\ . \tag{5}$$

$\phi(x)$ may be expanded in terms of a complete set of three-dimensional harmonic-oscillator solutions

$$\phi(x)=\sum_{nlm}H_{nlm}(\mathbf{x})b_{nlm} \tag{6}$$

and

$$[b_{nlm},b_{n'l'm'}^\dagger]=\delta_{nn'}\delta_{ll'}\delta_{mm'}\ . \tag{7}$$

The Hamiltonian [Eq. (2)] was first suggested[6] for N-body Schrödinger equation of quantum mechanics for studying the rotational spectra in nuclei. To be aware of the physical difference to the internal SU(3) symmetry in quark model, we call it, with no confusion, Elliott symmetry. Let us ask how the Elliott symmetry is realized in the field Lagrangian (1) and (2). Since the Elliott symmetry concerns both the coordinates space **x** and its differential operator ∇, it will be convenient to show the symmetry in the three-dimensional coherent-state representation. For example, the first term in (2), harmonic oscillator, can be put in the form

$$H_0=\int d\mu(\mathbf{z})\langle\phi^*|z\rangle z_a^*\partial_{z_a^*}\langle\mathbf{z}|\phi\rangle\ , \tag{8}$$

where $|z\rangle$ and $\langle z|$ are the biorthogonal set of coherent states. They are the left and right eigenstates of operators a and $a^\dagger$ with complex eigenvalues z and z^*, respectively: $a_\alpha|z_\alpha\rangle=z_\alpha|z_\alpha\rangle$, $\langle z_\alpha|a_\alpha^\dagger=z_\alpha^*\langle z_\alpha|$, $a_\alpha^\dagger|z_\alpha\rangle=\partial_{z_\alpha}|z_\alpha\rangle$, and $\langle z_\alpha|a_\alpha=\partial_{z_\alpha^*}\langle z_\alpha|$. The biorthogonal set is overcomplete, however, satisfying a closure relation in Fock space, $\int d\mu(\mathbf{z})|z_\alpha\rangle\langle z_\alpha|=1$, with $d\mu(\mathbf{z})=(dz_\alpha dz_\alpha^*/2\pi i)e^{-z_\alpha^*z_\alpha}$. In terms of this notation, the Elliott symmetry can be represented by the following transformation:

$$\mathbf{z}\to\mathbf{z}'=e^{i\theta_i\Lambda_i}\mathbf{z},\quad \mathbf{z}^*\to\mathbf{z}^*=e^{-i\theta_i\Lambda_i}\mathbf{z}^*\ , \tag{9}$$

where Λ_i are the nine independent infinitesimal generators of U(3). The explicit representation of Λ_i is not important. For convenience we can choose it to be Hermitian (eight Gellmann matrices plus one unit 3×3 matrix). Under the transformation given by (9), $d\mu(z)$ and $z_a^*\partial_{z_a^*}$ in (8) are invariant and

$$\langle\mathbf{z}|\phi\rangle=\phi(\mathbf{z})\to e^{-i\theta_i z_a^*\Lambda_{ab}^i\partial_{z_b^*}}\phi(z^*) =\langle z|e^{-i\theta_i a^\dagger\Lambda^i a}|\phi\rangle=\langle z|e^{-i\theta_{kq}T_q^k}|\phi\rangle\ , \tag{10}$$

where T_q^k have been defined in (3) and (4). The Lagrangians (1) and (2) are invariant under the Elliott transformation (9), when the transformation of field operators is given by (10). The Noether theorem may be written as

$$\frac{d}{dt}N_q^k(t)=0\ , \tag{11}$$

where

$$N_q^k(t)=\int d\mu(z)\,\phi^*(x)T_q^k(\partial_z^*z^*)\phi(z) =\int d^3x\,\phi^*(x)T_q^k(\mathbf{x},\nabla)\phi(x)\ . \tag{12}$$

N_q^k have the same algebraic structure as T_q^k, and N_q^k are the generators of the field transformations such that $[N_q^k(t),\phi(x)]=\delta_q^k\phi(x)$. The Hamiltonian (2) is invariant under the U(3) transformation so that

$$[H(t),N_q^k(t)]=0\ . \tag{13}$$

In classical nuclear physics, these symmetries, like all other symmetries of the system, are realized by the energy spectrum of the nucleus so that the energy spectrum is classified by the irreducible representations of the symmetry group. For example, the Elliott symmetry is used to classify the low-lying collective states in light nuclei. This makes a strong assumption about the vacuum. Since the vacuum is empty of particles, it belongs to a singlet representation of the symmetry group. In other words, the vacuum is assumed to be invariant under the transformation: $N_q^k|0\rangle=0$. This situation has changed. The vacuum, nowadays, is permitted to contain particles with nontrivial quantum numbers through boson condensation. Thus it is possible to have $N_q^k|0\rangle\neq0$ for certain choices of (k,q). When this happens, the U(3) symmetry is spontaneously broken and the nuclear spectrum does not exhibit the U(3) symmetry. At the same time the Goldstone theorem states that certain NG bosons should appear.

Let us start with the spontaneously broken U(1) symmetry, $N_0^0|0\rangle \neq 0$. This case is well established in the superconductivity theory. An equivalent statement to $N_0^0|0\rangle \neq 0$ in the superconductivity model is that the pair of fermion operators can have nonvanishing vacuum expectation value.[4,5] Define

$$\Phi = \sum_{nl}\sum_{m}(-1)^{l-m}(b_{nlm}^{\dagger}b_{nl\bar{m}}^{\dagger}+b_{nlm}b_{nl\bar{m}})$$
$$=\sum_{nl}\sqrt{2l+1}[(b_{nl}^{\dagger}b_{bl}^{\dagger})_{00}+(b_{nl}b_{nl})_{00}] , \quad (14)$$

where $(b_{ne}^{+}b_{ne}^{+})_{00}$ indicates the pair has total spin and its projection zero and

$$\langle 0|\Phi|0\rangle = \sum_{nl} V_{nl} \neq 0 . \quad (15)$$

Here, the summation l runs from $n, n-2, \ldots, 0$ or 1 and V_{nl} is defined by taking vacuum expectation value of the operator defined in (14). For simplicity, we do not put any structure with respect to n and l in this pair operator. Any variation would not change the following discussion.

The infinitesimal transformation of Φ along the direction associated with U(1) group is

$$\delta_0^0\Phi = [N_0^0,\Phi] \equiv B_{00}$$
$$= \frac{1}{2\sqrt{3}}\sum_{nl} n\sqrt{2l+1}[(b_{nl}^{\dagger}b_{nl}^{\dagger})_{00}-(b_{nl}b_{nl})_{00}] , \quad (16)$$

with B_{00} satisfying

$$\langle 0|[N_0^0,B_{00}]|0\rangle = \frac{1}{12}\sum_{nl} n^2 V_{nl} \neq 0 . \quad (17)$$

The relation (17) is one version of the Goldstone theorem, which means that a nonvanishing vacuum expectation value of the pair operator is equivalent to the statement that the vacuum is not invariant under the U(1) transformation. The action of N_0^0 on the vacuum does not create any energy because this generator is independent of time (11). This proves that the generator acting on the vacuum creates zero-energy states which are not vacuum. In other words, they create the NG bosons. On the other hand, (17) provides a kind of creation-annihilation conjugate relation between $N_0^0|0\rangle$ and $B_{00}|0\rangle$. So the Goldstone boson has the pair structure in terms of nucleon operators according to (16).

In our case, U(1) is the subgroup of U(3), we have more generators, and there appear more Goldstone bosons induced by the assumption (15). The operator Φ defined by (14) is rotationally invariant. The order parameter $\langle 0|\Phi|0\rangle$ does not lead to the spontaneously broken R(3) symmetry. We can make the assumption that $N_q^1|0\rangle = 0$, therefore $B_{00}|0\rangle$ can be characterized by the spin quantum number zero, since $[N_q^1, B_{00}] = 0$. However, the variation of Φ along the direction associated with generators N_q^2 will lead to five extra NG bosons

$$\delta_q^2\Phi = [N_q^2,\Phi] \equiv B_{2q}$$
$$= \sum_{nll'} g_{ll'}^{n}[(b_{nl}^{\dagger}b_{nl'}^{\dagger})_{2q}-(-1)^{q}(b_{bl}b_{nl'})_{2\bar{q}}] , \quad (18)$$

where

$$g_{ll'}^{n} = \frac{2\sqrt{2l+1}}{\sqrt{5}}\langle nl\|T^2\|nl'\rangle$$
$$= \frac{4}{5\sqrt{6}}\sqrt{2l+1}\sqrt{2l'+1}(-1)^{l}\langle l0l'0|20\rangle R_{ll'}^{n} . \quad (19)$$

Here, $R_{ll'}^{n}$ is the integral of the quadrupole operator between radial functions of the three-dimensional harmonic-oscillator wave functions, and we have used the fact that the matrix elements within the same principal quantum number n of the momentum function are the same as those of the corresponding coordinate functions.

In analogy with what we have calculated in (17), we have

$$\langle 0|[N_{q'}^2,B_{2q}]|0\rangle = (-1)^{q}\delta_{q'\bar{q}}\sum_{nl} h_{nl}V_{nl} \neq 0 , \quad (20)$$

where $h_{nl} = [1/(2l+1)]\sum_{l'}(g_{ll'}^{n})^2$. The commutation on the left-hand side of (20), of course, includes terms with the pair coupled to a higher spin. However, only the term with zero spin has nonvanishing vacuum expectation value, according to our assumption for the order parameter, $\langle 0|\Phi|0\rangle$, defined in (15). It is seen from (20) that $N_q^2|0\rangle \neq 0$.

By applying the same argument [below Eq. (17)] about the Goldstone theorem to the relation (20), one finds the important fact that a nonvanishing vacuum expectation value of the pair operator, *à la* superconductivity, not only leads to a spin-zero boson, but also to a spin-2 boson, i.e., five additional NG bosons. Vacuum is not invariant under the action of N_q^2. The generators N_q^2 acting on the vacuum create five gapless NG bosons. The conjugate state $B_{2\bar{q}}|0\rangle$ appears to have a pair structure and is described by Eq. (18). It is trivial to prove that B_{2q} are the components of a spherical tensor with rank 2 by calculating the commutator $[N_q^1, B_{2q'}]$. Notice that vacuum is R(3) invariant; therefore $B_{2q}|0\rangle$ are the states with spin 2. So it is clear that preserving the rotational symmetry does not mean that NG bosons are spinless; it only leads to the result that NG bosons have definite spin.

Since the order parameter is the same as that in the superconductivity theory, the vacuum of the nuclear system should have the BCS structure.[4] By minimizing the expectation value of the Hamiltonian [Eq. (2)] in the BCS vacuum, we have the gap equation

$$\Delta_{nl} = 2v_0\sum_{l'}\frac{[\frac{1}{3}n^2-l(l+1)]\delta_{ll'}+\frac{1}{6}(g_{ll'}^{n})^2}{2\sqrt{\varepsilon_{nl'}^2+\Delta_{nl'}^2}}\Delta_{nl'} , \quad (21)$$

where ε_{nl} is the single-particle energy and $\Delta_{nl'}$, gap matrix, relates to our order parameters V_{nl} by

$$\Delta_{nl} = 2v_0\sum_{l'}\{[(\tfrac{1}{3}n^2-l(l+1)]\delta_{ll'}+\tfrac{1}{6}(g_{ll'}^{n})^2\}V_{nl'} . \quad (22)$$

In conclusion, when Elliott U(3) symmetry is spontaneously broken to R(3), a nonvanishing vacuum expectation value of the superconducting pair induces six zero-energy NG bosons $B_{00}|0\rangle$ and $B_{2q}|0\rangle$, (16) and (18), that are conjugate to the states obtained with six generators of U(3), N_0^0 and $N_{\bar{q}}^2$ acting on the vacuum (the conjugate means the relation of annihilation and creation). These bosons are characterized by spin 0 and 2, associated with the unbroken R(3) symmetry. It is to be emphasized that the appearance of six NG bosons is true in either bosonic [commutator Eq. (5)] or fermionic (anticommutator) field ϕ because they are composite particle pairs created by spontaneous breakdown of the Elliott symmetry.

For realistic nuclear Hamiltonians, the spins of the particles are involved and the SU(3) symmetry is explictly broken. In such a case the NG bosons will acquire an energy in analogy with the pions gaining mass when the chiral symmetry is explicitly broken. It will be of great interest to derive the low-energy theorem for the nuclear systems involving *s* and *d* NG bosons. The derivation is similar to the soft pion relations in the broken $SU(2)\times SU(2)$-chiral symmetry with pions being the NG bosons.

Finally, we would like to comment about the analogy to a finite crystal. There, one may start with a crystal of infinite size and create collective modes such as the phonons. The condensation of acoustic phonons creates defects such as dislocations and surface boundaries. When we break such a crystal, it breaks along the surface boundary. The surface itself then carries quantum modes such as surface phonons, as a nuclear surface does. The surface itself then carries an infinite number of degrees of freedom because it can assume any shape, any oscillation, any surface spin, and so on. This is important because spontaneous breakdown of symmetry does not necessarily need an infinite volume, but demands presence of an infinite number of degrees of freedom. A system with a finite volume with a boundary surface carrying infinite degrees of freedom can cause spontaneous breakdown of symmetries. In such a case with a relatively large size the theoretical analysis for a system with infinite volume can describe the conditions in the inner domain of the system, though the situation around the surface can deviate from the description for infinite volume. This deviation is important when we consider global quantum numbers such as the baryon number and the electric charge of the whole nucleus. Even though the spontaneous breakdown of certain phase symmetries changes these quantum numbers accumulated in the inner domain, this deviation is patched up by the quantum number accumulated by the domain near the surface so that the quantum number of the whole system is not modified by the spontaneous breakdown of symmetries. To understand this readers are advised to recall the case of superconductivity in a metal of a finite size. Although the Bogoliubov transformation modifies the charge in the finite domain of the metal, the missing charge is accumulated around the boundary surface.

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