Convergence of the nucleus-nucleus Glauber multiple scattering series

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The Glauber S-matrix operator for nucleus-nucleus scattering is expressed as a finite series of matrix elements involving Bell's polynomials. Analyzing α^{4}He elastic-scattering data at the incident momentum of 4.32 GeV/c, we infer that our expansion is appreciably converging. Further, by applying closure over target and projectile states and neglecting a certain class of terms involving intermediate excitations, we arrive at a recurrence relation for nucleus-nucleus multiple scattering series terms, which invites further study as it seems to provide a simple method for calculating the nucleus-nucleus elastic-scattering cross section.

I. INTRODUCTION

It is generally known that the entire Glauber multiple scattering series for nucleus-nucleus scattering is difficult to evaluate, even for the simplest nuclear models, and its expansion in terms of the NN profile function converges very slowly. However, for a realistic calculation one is compelled to incorporate higher-order terms of the series whose evaluation is beset with serious computational difficulties because of the occurrence of multidimensional integrals and the many-body densities of the two colliding nuclei. This naturally has been one of the reasons for the appearance of several expansions of the series in the literature,[1-4] each as an attempt to have a more rapidly convergent series so that the entire series could be approximated by the first few dominant terms.

Franco and Varma[1] were the first to undertake a detailed study of the problem of the evaluation of nucleus-nucleus elastic-scattering amplitude in the Glauber model using a generalization of the phase expansion method for N-nucleus scattering and the independent particle model for the two colliding nuclei, although for a practical point of view their expansion may be truncated to a finite order on the cast of the remaining higher-order terms which are not worth considering: the shortcoming, in principle, lies in its infinite nature, which makes it difficult to provide a direct interpretation to the expansion in terms of multiple scattering processes.

Alkhazov's approach,[2] a generalization of his correlation expansion for N-nucleus scattering,[5] was structured along the aforementioned lines, and consequently it also suffers from the weaknesses mentioned above. This led Ahmad[3] to propose an effective profile approach in which the various terms in the expansion are amenable to a direct interpretation of single, double, etc., scatterings of the nucleons of the colliding nuclei which involve virtual excitations of the target or projectile, or both, through an effective NN interaction. One of the merits of this approach is that it involves only a finite number of terms as against the consideration of an infinite number of terms present in the expansions proposed earlier. Ahmad has applied this expansion to study α-^{40}Ca and α-^{12}C elastic scatterings at 1.37 GeV with encouraging results. However, because of the computational difficulties, he could not go beyond the first two terms of the expansion. Therefore, not much can be said about the convergence properties of the effective profile expansion from his study.

In this work, we present a finite series expansion of the Glauber model S-matrix operator for nucleus-nucleus scattering. The expansion gives the various terms in a nicely arranged form, which also results in a recurrence relation for the entire nucleus-nucleus multiple scattering series under certain approximations. This seems to be reasonably potent enough for easy applicability of the method to heavy nuclear systems. However, this work is directed toward the convergence test of the proposed expansion rather than its application to calculate nucleus-nucleus elastic scattering.

II. FORMULATION

According to the Glauber multiple scattering model,[6] the elastic-scattering amplitude for the scattering of a projectile nucleus of mass number B on a target nucleus of mass number A (disregarding Coulomb scattering for simplicity of discussion) may be expressed as

$$F_{BA}(\mathbf{q})=\frac{iK}{2\pi}\int e^{i\mathbf{q}\cdot\mathbf{b}}[1-S_{BA}(\mathbf{b})]d^2b \ , \tag{2.1a}$$

where K is the incident momentum in center-of-mass system, $\mathbf{b}$ is the impact parameter, and $\mathbf{q}$ is the momentum transfer. The quantity $S_{BA}(b)$ is the elastic S-matrix element and is given by

$$S_{BA}(\mathbf{b})=\langle \psi_A\psi_B|\hat{S}(\mathbf{b})|\psi_A\psi_B\rangle \ , \tag{2.1b}$$

$$\hat{S}(\mathbf{b})=\prod_{i=1}^{A}\prod_{j=1}^{B}[1-\Gamma_{NN}(\mathbf{b}-\mathbf{s}_i+\mathbf{s}'_j)] \ , \tag{2.1c}$$

where ψ_A and ψ_B are the ground-state wave functions of the target and projectile nuclei, and $\mathbf{s}_i$ ($\mathbf{s}'_j$) are the projections of the target (projectile) nucleon position vectors $\mathbf{r}_i$ ($\mathbf{r}'_j$) on the plane perpendicular to incident momentum, respectively. $\hat{S}(\mathbf{b})$ is the S-matrix operator.

The quantity $\Gamma_{NN}(b)$ is the NN profile function, and it is related to the basic NN amplitude as

$$\Gamma_{NN}(b)=\left[\frac{1}{2\pi i \mathbf{k}_N}\right]\int d^2q\, e^{-i\mathbf{q}\cdot\mathbf{b}} f_{NN}(q) , \quad (2.2)$$

where $\mathbf{k}_N$ is the incident nucleon momentum corresponding to the projectile energy per nucleon.

Assuming the usual Gaussian parametrization for the NN amplitude,

$$f_{NN}(q)=\frac{ik_N\sigma_{NN}(1-i\rho_{NN})}{4\pi}e^{-\beta_{NN}q^2/2} , \quad (2.3)$$

we get

$$\Gamma_{NN}(b)=\frac{\sigma_{NN}(1-i\rho_{NN})}{4\pi\beta_{NN}}e^{-b^2/2\beta_{NN}} , \quad (2.4)$$

where σ_{NN} is the NN total cross section, ρ_{NN} is the ratio of the real to the imaginary parts of the forward-scattering amplitude, and β_{NN} is the slope parameter.

Next we introduce an effective profile function γ_{ij} as

$$\gamma_{ij}=\frac{\Gamma_{NN}(\mathbf{b}-\mathbf{s}_i+\mathbf{s}'_j)-\Gamma_{00}(b)}{1-\Gamma_{00}(b)} , \quad (2.5)$$

where

$$\Gamma_{00}(b)=\langle\psi_A\psi_B|\Gamma_{NN}|\psi_A\psi_B\rangle , \quad (2.6)$$

and using it in Eq. (2.1c) to write the S-matrix operator as

$$\hat{S}(b)=[1-\Gamma_{00}(b)]^{AB}\prod_{i=1}^{A}\prod_{j=1}^{B}(1-\gamma_{ij}) . \quad (2.7)$$

Now, in order to develop a simple method for expanding $\hat{S}(b)$. We consider the expression

$$\hat{G}(t)=\prod_{i=1}^{A}\prod_{j=1}^{B}(1-\gamma_{ij}t) , \quad (2.8)$$

and expand it in ascending powers of t as

$$\hat{G}(t)=1-G_1t+G_2t^2-G_3t^3+\cdots+(-1)^N G_N t^N$$
$$=\sum_{r=0}^{N}(-1)^r G_r t^r , \quad (2.9)$$

where $N=AB$.

Obviously, the entire S-matrix operator is

$$\hat{S}(b)=[1-\Gamma_{00}(b)]^{AB}\hat{G}(1) , \quad (2.10)$$

with

$$\hat{G}(1)=\sum_{r=0}^{N}(-1)^r G_r . \quad (2.11)$$

Using Eqs. (2.10) and (2.11), we arrive at

$$\hat{S}(b)=\sum_{r=0}^{N}\hat{S}^{(r)}(b) , \quad (2.12)$$

where

$$\hat{S}^{(r)}(b)=[1-\Gamma_{00}(b)]^{AB}(-1)^r G_r . \quad (2.13)$$

From Eq. (2.8) it is straightforward to see that

$$\left[\frac{d^n\hat{G}}{dt^n}\right]_{t=0}=-\sum_{r=0}^{n-1}\begin{bmatrix}n-1\\ r\end{bmatrix}\left[\frac{d^{n-r-1}\hat{G}}{dt^{n-r-1}}\right]_{t=0}$$
$$\times r!\sum_{i=1}^{A}\sum_{j=1}^{B}(\gamma_{ij})^{r+1} . \quad (2.14)$$

Noting that

$$G_n=(-1)^n\frac{1}{n!}\left[\frac{d^n\hat{G}}{dt^n}\right]_{t=0} , \quad (2.15)$$

we get the following recurrence relation for G_n:

$$G_n=\frac{1}{n}\sum_{r=0}^{n-1}(-1)^r G_{n-r-1}S_{r+1} \quad \text{for } n>0 , \quad (2.16a)$$

$$G_0=1 , \quad (2.16b)$$

where

$$S_r=\sum_{i=1}^{A}\sum_{j=1}^{B}(\gamma_{ij})^r . \quad (2.17)$$

Using Eq. (2.16), the series $\hat{G}(1)$ may be written as

$$\hat{G}(1)=1-G_0S_1+(G_1S_1-G_0S_2)/2-(G_2S_1-G_1S_2+G_0S_3)/3$$
$$+\cdots+(-1)^N[G_{N-1}S_1-G_{N-2}S_2+\cdots+(-1)^{N-1}G_0S_N]/N . \quad (2.18)$$

The successive terms of Eq. (2.18) may be written as

$$G_0S_1=-Y_1(-S_1) , \quad (2.19)$$

$$(G_1S_1-G_0S_2)/2=Y_2(-S_1,-S_2)/2! , \quad (2.20)$$

$$(G_2S_1-G_1S_2+G_0S_3)/3=-Y_3(-S_1,-S_2,-2!S_3)/3! , \quad (2.21)$$

$$\vdots$$

$$[G_{N-1}S_1-\cdots+(-1)^{N-1}G_0S_N]=(-1)^N Y_N(-S_1,-S_2,-2!S_3,\ldots,-(N-1)!S_N)/N! , \quad (2.22)$$

where Y_N is the Bell's polynomial of order N.

Using the expressions as given by Eqs. (2.19)–(2.22), Eq. (2.10) for the S-matrix operator may be written as

$$\hat{S}(b)=\sum_{n=0}^{N}\hat{S}^{(n)}(b)=[1-\Gamma_{00}(b)]^{AB}\left[\sum_{n=0}^{N}\frac{Y_n(-S_1,-S_2,-2S_3,\ldots,-(n-1)!S_n)}{n!}\right]. \tag{2.23}$$

Substituting Eq. (2.23) into Eq. (2.1b), one may write the entire S-matrix series as

$$S_{BA}(b)=\left\langle\psi_A\psi_B\left|\sum_{n=0}^{N}\hat{S}^{(n)}(b)\right|\psi_A\psi_B\right\rangle . \tag{2.24}$$

By applying the closure over the target and projectile states, Eq. (2.16) may be written as

$$\langle\psi_A\psi_B|G_n|\psi_A\psi_B\rangle=\frac{1}{n}\sum_{r=0}^{n-1}(-1)^r(\langle\psi_A\psi_B|G_{n-r-1}|\psi_A\psi_B\rangle\langle\psi_A\psi_B|S_{r+1}|\psi_A\psi_B\rangle$$
$$+\sum_{m\neq 0}\langle\psi_A\psi_B|G_{n-r-1}|\psi_A^{(m)}\psi_B^{(m')}\rangle\langle\psi_B^{(m')}\psi_A^{(m)}|S_{r+1}|\psi_A\psi_B\rangle)\quad\text{for } n>0\ , \tag{2.25}$$

where $\psi_A^{(m)}$ and $\psi_B^{(m')}$ denote the excited states of the target and projectile.

Our study[7] based on microscopic calculations demonstrates that the single-step excitation processes along with the center-of-mass pair correlation contribute little to the elastic scattering at low momentum transfers, and whatever significance it assumes is at large momentum transfers. Thus it is worth trying to neglect the excited-state terms on the right-hand side of the above expression in order to get a simple result for the entire series. Subject to the limitation of this approximation, we may use Eqs. (2.13) and (2.25) to express the various terms of the elastic scattering matrix as

$$S_{BA}^{(n)}(b)=[1-\Gamma_{00}(b)]^{AB}\frac{(-1)^n}{n}\sum_{r=0}^{n-1}(-1)^r(\langle\psi_A\psi_B|G_{n-r-1}|\psi_A\psi_B\rangle\langle\psi_A\psi_B|S_{r+1}|\psi_A\psi_B\rangle) . \tag{2.26}$$

Again, using Eq. (2.13), we may write

$$\langle\psi_A\psi_B|G_{n-r-1}|\psi_A\psi_B\rangle=\frac{(-1)^{n-r-1}}{[1-\Gamma_{00}(b)]^{AB}}\langle\psi_A\psi_B|\hat{S}^{(n-r-1)}(b)|\psi_A\psi_B\rangle\ , \tag{2.27}$$

which, when substituted into Eq. (2.26), gives, after some rearrangement,

$$S_{BA}^{(n)}(b)=\frac{(-1)^{2n-1}}{n}\sum_{r=0}^{n-1}S_{BA}^{(n-r-1)}(b)\langle\psi_A\psi_B|S_{r+1}|\psi_A\psi_B\rangle\quad\text{for } n>0\ , \tag{2.28}$$

and

$$S_{BA}^{(0)}(b)=[1-\Gamma_{00}(b)]^{AB} . \tag{2.29a}$$

The expression (2.29a) corresponds to the optical limit result.[8]

Obviously, the first few terms of the expression (2.24) are

$$S_{BA}^{(1)}(b)=[1-\Gamma_{00}(b)]^{AB}\langle\psi_A\psi_B|Y_1(-S_1)|\psi_A\psi_B\rangle\ , \tag{2.29b}$$

$$S_{BA}^{(2)}(b)=[1-\Gamma_{00}(b)]^{AB}\left\langle\psi_A\psi_B\left|\frac{Y_2(-S_1,-S_2)}{2}\right|\psi_A\psi_B\right\rangle\ , \tag{2.29c}$$

$$S_{BA}^{(3)}(b)=[1-\Gamma_{00}(b)]^{AB}\left\langle\psi_A\psi_B\left|\frac{Y_3(-S_1,-S_2,-2S_3)}{3!}\right|\psi_A\psi_B\right\rangle . \tag{2.29d}$$

The various terms of the expression (2.24) are amenable to a direct interpretation of multiple scattering processes involving the effective interaction.

Equation (2.28) provides one of the possible simplifications in the evaluation of the total nucleus-nucleus multiple scattering series using the recurrence relation derived here. However, the usefulness of this approach has not been investigated in this work.

III. SAMPLING TECHNIQUE AND CALCULATION

Using the powerful sampling technique of Metropolis *et al.*,[9] the scattering matrix $S_{BA}(b)$ for the scattering of a projectile nucleus B on a target nucleus A is given by

$$S_{BA}(b)=\frac{\int\rho_A(\mathbf{r})\rho_B(\mathbf{r}')\{\hat{S}(b)\}d\mathbf{r}_1\cdots d\mathbf{r}_A d\mathbf{r}_1\cdots d\mathbf{r}'_B}{\int\rho_A(\mathbf{r})\rho_B(\mathbf{r}')d\mathbf{r}_1\cdots d\mathbf{r}_A d\mathbf{r}'_1\cdots d\mathbf{r}'_B}\ , \tag{3.1}$$

where the position coordinates of the nuclei are generated randomly subject to the center-of-mass constraint with the weight

$$W=|\psi_A^*\psi_A||\psi_B^*\psi_B|=\rho_A(\mathbf{r})\rho_B(\mathbf{r}') . \tag{3.2}$$

This implies evaluation of 24-dimensional integrals in the case of ^{4}He-^{4}He elastic scattering. Needless to say, for heavier nuclear systems the dimensionality is much larger. We consider Eq. (2.23) for $\hat{S}(b)$, which has been truncated at different terms in order to test the convergence and to assess the number of terms required to retain the full multiple scattering nature over a given momentum-transfer region.

We generate trial configurations for the two colliding nuclei $X_A''=(\mathbf{r}_1'', \ldots, \mathbf{r}_A'')$ and $X_B'''=(\mathbf{r}_1''', \ldots, \mathbf{r}_B''')$ from initial configurations $X_A=(\mathbf{r}_1, \ldots, \mathbf{r}_A)$ and $X_B=(\mathbf{r}_1', \ldots, \mathbf{r}_B')$ by adding to each component m of $\mathbf{r}_i$ (i.e., x, y, and z coordinates) a random shift:

$$r_{i,m}''=r_{i,m}+h(u_i-0.5) , \tag{3.3}$$

and, similarly of $\mathbf{r}_i'$,

$$r_{i,m}'''=r_{i,m}'+h(u_i'-0.5) , \tag{3.4}$$

where h is a chosen step length, and u_i and u_i' are random numbers between 0 and 1. Before accepting the new configurations as successful trials, we impose the following checks: (1) If $W'>W$, the move is accepted, and X_A'' and X_B''' are used as the initial configurations for the next move. (2) If $W'/W>u_i$, the move is accepted; otherwise, it is rejected, and X_A and X_B' are kept as the initial configurations for the next move.

We start the calculation with $r_{i,m}=r_{i,m}'=0$ and $h=1.5$ and allow to move the system 100 times before S-matrix calculations begin. In order to reduce the correlation, we moved the system 10 times between two consecutive S-matrix calculations.

For each successful trial the real and imaginary parts of the S-matrix element along with the error bars are calculated as a function of impact parameter. In this study, the large momentum transfers show the limitations of the Monte Carlo method, particularly beyond the second minimum where the calculations are 50–80% precise. However, the method has an advantage of taking the c.m. constraint exactly, even when the series is truncated at some point where center of mass usually leads to diverging results (see, for example, Ref. 3). Further details about precision of the Monte Carlo calculation are available in Ref. 10. The emphasis is on the convergence test of the proposed expansion, and therefore, the results so obtained do not alter the spirit of the paper, which is independent of the method employed. The basic ingredients of the calculation are the NN parameters $\sigma_{NN}=32.3$ mb, $\rho_{NN}=-0.02$, and $\beta_{NN}=1.86$ (GeV/c)$^{-2}$ taken from the paper of Franco and Yin[11] and the single Gaussian density model for the ^{4}He nucleus:[12]

$$\rho_{^4\mathrm{He}}^{(\mathrm{SG})}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)=N_0\prod_{i=1}^{4} e^{-\alpha^2 r_j^2}\delta^{(3)}\left[\frac{\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3+\mathbf{r}_4}{4}\right] . \tag{3.5}$$

with $\alpha^2=0.535$ fm^{-2}. The scattering data are taken from Ref. 13.

IV. RESULTS AND DISCUSSION

Assuming the independent particle model for the two colliding nuclei and taking the center-of-mass constraints into account, we calculate α^4He elastic-scattering differential cross sections at the incident momentum of 4.32 GeV/c in the framework of the expansion developed in Sec. II by truncating it at the first, second, fourth, and sixth terms. Since our main objective is to examine the convergence of the expansion, we use the single Gaussian model for ^{4}He density and adopt the Monte Carlo technique for convenience. Undoubtedly, the computational technique adapted here consumes relatively more computer time, but it treats the c.m. constraint exactly. The results so obtained are shown in Fig. 1 for the aforementioned calculational processes.

The crossed and solid curves represent the two extremum calculations corresponding to the first term and full series, respectively. It may be seen that the first term is quite inadequate in getting any quantitative agreement with the solid curve even at too small $\mathbf{q}$ values. Further, the positions of the minima and maxima are shifted toward the forward-scattering region. It may also be noted that for $-t>0.4$ (GeV/c)^2, $S_{BA}^{(0)}$ does not contribute significantly.

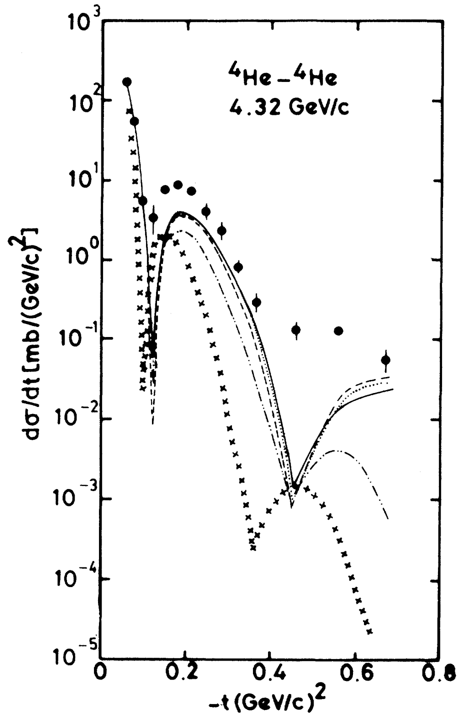

FIG. 1. Differential cross section for α^4He elastic scattering at 4.32 GeV/c incident momentum. The crossed and solid curves represent the first term and full series calculations, respectively. The dot-dashed, dashed, and dotted curves are obtained with the evaluation of the series up to the second, fourth, and sixth terms, respectively. The data are taken from Ref. 13.

The subsequent inclusion of the second term, $S_{BA}^{(2)}$, gives significant improvement in the theoretical situation by enhancing the differential cross section throughout the momentum-transfer region covered by experiment (dot-dashed curve). As a result, not only the experimental position of minima and maxima are well located, but also the solid curve is reproduced at low **q** values (almost up to the first minimum). The results of the calculations when the series is truncated on the fourth and sixth terms are shown by the dashed and dotted curves, respectively. It is seen that the dotted curve almost coincides with the full curve for $-t<0.4$ (GeV/c)^2. This implies that in the **q** region covered in this work, the series converges at the sixth term and that the contributions of the higher terms are too small to cause any noticeable error up to the second minimum.

V. CONCLUSIONS

It may be concluded that although our expansion provides remarkably satisfactory results, the convergence is still not as rapid as to make it of much practical utility especially when the data covers a wider momentum-transfer region. Still the present study amply demonstrates that the first six terms of the expansion are adequate for a semiqualitative study of the data for $-t\leq 0.4$ (GeV/c)^2. An interesting though no unexpected feature of the proposed expansion is that consideration of successive terms gradually reproduces the full series character for greater and greater momentum transfers.

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[1]V. Franco and G. K. Varma, Phys. Rev. C **18**, 349 (1978).
[2]G. D. Alkhazov (unpublished).
[3]I. Ahmad, J. Phys. G **6**, 947 (1980).
[4]Y Yin, Z. Tan, and K. Chen, Nucl. Phys. **A440**, 685 (1985).
[5]G. D. Alkhazov, Nucl. Phys. **A280**, 330 (1977).
[6]R. J. Glauber, in *Lectures in Theoretical Physics,* edited by W. E. Britten and L. C. Dunham (Wiley Interscience, New York, 1959), Vol. 1, p. 315; in *High Energy Physics and Nuclear Structure,* edited by S. Devons (Plenum, New York, 1970).
[7]A. A. Usmani, M. A. Alvi, and I. Ahmad, J. Phys. G **15**, 1667 (1989).
[8]W. Czyz and L. C. Maximon, Ann. Phys. (N.Y.) **52**, 59 (1969).
[9]N. Metropolis, A. W. Rosenbluth, M. N. Rosenbluth, A. H. Teller, and E. Teller, J. Chem. Phys. **21**, 1087 (1953); see, K. Binder, in *Monte Carlo Methods in Statistical Physics,* edited by K. Binder (Springer-Verlag, Berlin, 1979).
[10]A. A. Usmani, I. Ahmad, and Q. N. Usmani, Phys. Rev. C **39**, 1182 (1989).
[11]V. Franco and Y. Yin, Phys. Rev. C **34**, 608 (1986).
[12]R. H. Bassel and C. Wilkins, Phys. Rev. **174**, 1179 (1968).
[13]J. Berger *et al.*, Nucl. Phys. **A338**, 421 (1980).